Stochastic Processes and their Applications 90 (2000) 263–275
stochastic processes and their applications
www.elsevier.com/locate/spa

Individual behaviors of oriented walks

Aihua Fan
Department of Mathematics, University of Picardie, 33 Rue Saint Leu, 80039 Amiens Cedex 1, France

Received 14 December 1999; received in revised form 2 May 2000; accepted 4 May 2000

Abstract

Given an infinite sequence $t=(\varepsilon_k)_k$ of -1 and $+1$, we consider the oriented walk defined by $S_n(t)=\sum_{k=1}^{n}\varepsilon_1\varepsilon_2\ldots\varepsilon_k$. The set of t's whose behaviors satisfy $S_n(t)\sim bn^{\tau}$ is considered ($b\in\mathbb{R}$ and $0<\tau\leqslant 1$ being fixed) and its Hausdorff dimension is calculated. A two-dimensional model is also studied. A three-dimensional model is described, but the problem remains open. © 2000 Elsevier Science B.V. All rights reserved.

MSC: 28A78; 42A55; 60G50

Keywords: Oriented walk; Random walk; Hausdorff dimension; Riesz product

1. Introduction and results

Let $t=(\varepsilon_n(t))_{n\geqslant 1}\in\mathbb{D}:=\{-1,+1\}^{\mathbb{N}}$. Consider

$$S_n(t)=\sum_{k=1}^{n}\varepsilon_1(t)\varepsilon_2(t)\ldots\varepsilon_k(t).$$

We would like to study the behavior of $S_n(t)$ as $n\to\infty$ for different $t\in\mathbb{D}$.

We regard $S_n(t)$ as an oriented walk on $\mathbb{Z}$ of an individual following the signals of t: suppose that at time 0, the individual is at the origin of $\mathbb{Z}$ and keeps the orientation to the right. If the signal $\varepsilon_1(t)=1$, he forwards one step in the orientation he kept, i.e., to the right and if the signal $\varepsilon_1(t)=-1$, he returns back and then forwards one step (in the orientation opposite to that he kept, i.e., to the left). We say that the *state* of the individual at time 1 is (S_1,ξ_1) where $\xi_1=\varepsilon_1$ means the *orientation* kept by the individual at time 1 ("-1"= *left*, "$+1$" = *right*) and $S_1(t)=\varepsilon_1$ means the *position* of the individual. Suppose that the state of the individual at time n is (S_n,ξ_n). The state of the next time $n+1$ is determined as follows. If the signal $\varepsilon_{n+1}(t)=1$ (resp. -1), the individual forwards one step in the orientation ξ_n (resp. in the orientation $-\xi_n$). Thus we get the recursive relation

$$S_{n+1}=S_n+\varepsilon_{n+1}\xi_n,\quad \xi_{n+1}=\varepsilon_{n+1}\xi_n.$$

E-mail address: ai-hua.fan@mathinfo.u-picardie.fr (A. Fan).

PII: S0304-4149(00)00044-2

By induction, we get that

$$\xi_n = \varepsilon_1\varepsilon_2\ldots\varepsilon_n, \quad S_n = \sum_{k=1}^{n} \varepsilon_1\varepsilon_2\ldots\varepsilon_k.$$

This walk S_n is different from the classical random walk $\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_n$, where "random" means that $\mathbb{D}$ is equipped with the Lebesgue probability measure (Bernoulli probability measures, even arbitrary probability measures may also be considered). For the classical random walk, the steps at different times are independent. But for the present oriented walk, the step at time $n+1$ depend not only on the signal at time $n+1$ but also on the orientation kept at the time n. Actually we will consider the present walk from the deterministic point of view, because no prior probability measure will be imposed. For convenience, we may also think of t as an individual and $(\varepsilon_n(t))$ as his thoughts at different times.

The space $\mathbb{D}$ is a compact metric space and we will take the usual metric on it, which is defined as $d(t,s)=2^{-n}$ for $t=(t_k)$ and $s=(s_k)$ in $\mathbb{D}$ with $n=\sup\{k: t_k=s_k\}$. Thus different notions of dimensions are defined on $\mathbb{D}$. We will talk about Hausdorff dimension $\dim_H$, packing dimension $\dim_{\mathrm{P}}$ and upper box dimension $\overline{\dim}_{\mathrm{B}}$ (see, Kahane, 1985; Mattila, 1995 for a general account of dimensions). The closed 2^{-n} ball containing $t=(\varepsilon_k)$ will be denoted by $I(\varepsilon_1,\ldots,\varepsilon_n)$. It is also called an n-cylinder.

Let $b\in\mathbb{R}$. Introduce the set (or population)

$$E_b = \left\{ t\in\mathbb{D} : \lim_{n\to\infty} \frac{1}{n} S_n(t) = b \right\}.$$

Theorem 1. *If $b\notin[-1,1]$, we have $E_b=\emptyset$. If $b\in[-1,1]$, we have*

$$\dim_H E_b = \dim_{\mathrm{P}} E_b = H\left(\frac{1+b}{2}\right)$$

where $H(x)=-x\log_2 x-(1-x)\log_2(1-x)$.

The behaviors of $S_n(t)$ described by E_b's are far from exhaustive. We will illustrate this by considering two other types of behavior. Let

$$E_{\mathrm{bd}} = \{t\in\mathbb{D} : S_n(t) = O(1) \text{ as } n\to\infty\}.$$

For $b\in\mathbb{R}$ and $0<\tau<1$, let

$$E_{b,\tau} = \left\{ t\in\mathbb{D} : \lim_{n\to\infty} \frac{S_n(t)}{n^\tau} = b \right\}.$$

Theorem 2. *We have* $\dim_H E_{\mathrm{bd}}=1$. *For $b\in\mathbb{R}$ and $\frac{1}{2}<\tau<1$, we have* $\dim_H E_{b,\tau}=1$.

Now let us consider an oriented walk on the lattice $\mathbb{Z}^2$. We insist that on every point in the lattice there are four orientations which are, respectively, represented by $1, \mathrm{i}, -1$ and $-\mathrm{i}$ (rightward, upward, leftward and downward). At a given time, an individual not only has a position and but also keeps an orientation. Let (S_n,ξ_n) be the state of the individual $t=(t_n)$ at time n. We define its state (S_{n+1},ξ_{n+1}) at time $n+1$ as

$$S_{n+1} = S_n + \mathrm{e}^{\varepsilon_{n+1}(\pi/2)\mathrm{i}}\xi_n, \quad \xi_{n+1} = \mathrm{e}^{\varepsilon_{n+1}(\pi/2)\mathrm{i}}\xi_n.$$

This means when $\varepsilon_{n+1}=+1$ (resp. -1), the individual turns an angle $\pi/2$ (resp. $-\pi/2$). By using the relation $\mathrm{i}^{\varepsilon}=\varepsilon\mathrm{i}$ for $\varepsilon=-1$ or $+1$, we get the expression

$$S_n=\sum_{k=1}^{n}\mathrm{i}^k\varepsilon_1\varepsilon_2\ldots\varepsilon_k.$$

Thus we get a formula which is similar to that of the 1-dimensional case. The difference is that in the 2-dimensional case, $S_n(t)$ is a sum of $\varepsilon_1\ldots\varepsilon_k$ weighted by complex numbers i^k. For any complex number $z\in\mathbb{C}$, let

$$F_z=\left\{t\in\mathbb{D}:\ \lim_{n\to\infty}\frac{S_n(t)}{n}=z\right\}.$$

Theorem 3. *Let* $\Delta=\{Z=x+\mathrm{i}y:\ |x|\leqslant 1/2, |y|\leqslant 1/2\}$. *If* $z\notin\Delta$, *we have* $F_z=\emptyset$. *If* $z=x+\mathrm{i}y\in\Delta$, *we have*

$$\dim_H F_z=\dim_P F_z=\tfrac{1}{2}\left[H\left(\frac{1+2x}{2}\right)+H\left(\frac{1+2y}{2}\right)\right]$$

where $H(x)=-x\log_2 x-(1-x)\log_2(1-x)$.

Let

$$F_{\mathrm{bd}}=\{t\in\mathbb{D}: S_n(t)=\mathrm{O}(1)\text{ as } n\to\infty\}.$$

For $z\in\mathbb{C}$ and $0<\tau<1$, let

$$F_{z,\tau}=\left\{t\in\mathbb{D}:\ \lim_{n\to\infty}\frac{S_n(t)}{n^{\tau}}=z\right\}.$$

Theorem 4. *We have* $\dim_H F_{\mathrm{bd}}=1$. *For* $b\in\mathbb{C}$ *and* $1/2<\tau<1$, *we have* $\dim_H F_{b,\tau}=1$.

In Section 2 we will construct a class of probability measures on $\mathbb{D}$, called Riesz products, which are well known in harmonic analysis [Z]. The theorems will be proved in Sections 3–5. In the last Section 6, we give some remarks on the related works and some unsolved questions to be considered.

2. Riesz products

As a preliminary, we introduce and study in this section a class of probability measures defined on $\mathbb{D}$ which will be useful in our proofs of the theorems. Consider $\{-1,1\}$ as a (multiplicative) group and $\mathbb{D}$ as its infinite product group. The dual group $\hat{\mathbb{D}}$ of $\mathbb{D}$ consists of the constant function 1 and all possible products $\varepsilon_{n_1}(t)\varepsilon_{n_2}(t)\ldots\varepsilon_{n_k}(t)$ ($\forall k\geqslant 1,\forall 1\leqslant n_1<n_2<\cdots<n_k$). There is a convenient way to represent $\hat{\mathbb{D}}$. Let $w_0=1$. Let $n\geqslant 1$ be an integer. It has a unique representation $n=2^{n_1-1}+\cdots+2^{n_k-1}$. We define

$$w_n(t)=\varepsilon_{n_1}(t)\varepsilon_{n_2}(t)\ldots\varepsilon_{n_k}(t).$$

Then we have $\hat{\mathbb{D}}=\{w_n\}_{n\geqslant 0}$. The functions w_n are called Walsh functions. A finite sum of Walsh functions is called a Walsh polynomial, whose order is the largest index n of Walsh functions w_n contained in the sum defining it.

Let $c = (c_k)_{k\geqslant 1}$ be a sequence of real numbers such that $|c_k| \leqslant 1$. The following infinite product

$$d\mu_c(t) = \prod_{k=1}^{\infty}(1 + \ c_k\varepsilon_1(t)\varepsilon_2(t)\dots\varepsilon_k(t))\,dt$$

defines a probability measure, in the sense that its partial products converge in the weak-$*$ topology to a measure μ_c. The measure μ_c is called a *Riesz product*. Recall that when $c_k = 0$ $(\forall k)$, we have $d\mu_c(t) = dt$, the Lebesgue measure on $\mathbb{D}$.

Lemma 1. *The above infinite product does define a probability measure μ_c. Furthermore, for a function f having its Taylor development*

$$f(x) = \sum_{n=1}^{\infty} f_n x^n \quad (|x| \leqslant 1), \qquad \sum_{n=1}^{\infty} |f_n| < \infty$$

we have

$$\mathbb{E}_{\mu_c} f(\varepsilon_1\varepsilon_2\dots\varepsilon_k) = U + c_k V$$

$$\mathrm{Cov}_{\mu_c}(f(\varepsilon_1\varepsilon_2\dots\varepsilon_k), f(\varepsilon_1\varepsilon_2\dots\varepsilon_\ell)) = 0, \quad (k \neq \ell)$$

where

$$U = \sum_{n=1}^{\infty} f_{2n}, \quad V = \sum_{n=1}^{\infty} f_{2n-1}.$$

Proof. Let $P_n(t)$ be the nth partial product of the infinite product. It is clear that P_n is a Walsh polynomial of order $2^n - 1$ (at most). It is clear that P_n are non-negative. Note that

$$P_{n+1} - P_n = c_{n+1}P_n\varepsilon_1\varepsilon_2\dots\varepsilon_{n+1}$$

is a Walsh polynomial, a sum of w_k with $2^n \leqslant k < 2^{n+1}$. It follows that $\int P_n(t)\,dt = 1$ and that the Fourier–Walsh coefficient $\hat{P}_n(w_k)$ is constant when n is sufficiently large. Therefore, the measures $P_n(t)\,dt$ converge weakly to a limit μ_c (see Zymund (1959) for details in the case of the circle). The above argument also shows that

$$\hat{\mu}_c(\varepsilon_1\dots\varepsilon_k) = c_k.$$

$$\hat{\mu}_c(\varepsilon_{k+1}\dots\varepsilon_\ell) = c_k c_\ell, \quad (\forall k < \ell).$$

Using these two equalities, we can obtain the general expressions stated in the lemma. Note first that

$$(\varepsilon_1\varepsilon_2\dots\varepsilon_k)^{2n} = 1, \quad (\varepsilon_1\varepsilon_2\dots\varepsilon_k)^{2n-1} = \varepsilon_1\varepsilon_2\dots\varepsilon_k.$$

We have

$$\begin{aligned}\mathbb{E}_{\mu_c} f(\varepsilon_1\varepsilon_2\dots\varepsilon_k) &= \sum_{n=1}^{\infty} f_n \mathbb{E}_{\mu_c}(\varepsilon_1\varepsilon_2\dots\varepsilon_k)^n \\ &= \sum_{n=1}^{\infty} f_{2n} + \sum_{n=1}^{\infty} f_{2n-1}\mathbb{E}_{\mu_c}\varepsilon_1\varepsilon_2\dots\varepsilon_k \\ &= U + c_k V.\end{aligned}$$

Suppose $k < \ell$. We have

$$\begin{aligned}\mathbb{E}_{\mu_c} f(\varepsilon_1 \dots \varepsilon_k)\bar{f}(\varepsilon_1 \dots \varepsilon_\ell) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f_n \bar{f}_m \mathbb{E}_{\mu_c}(\varepsilon_1 \dots \varepsilon_k)^n (\varepsilon_1 \dots \varepsilon_\ell)^m \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f_n \bar{f}_m \mathbb{E}_{\mu_c}(\varepsilon_1 \dots \varepsilon_k)^{n+m} (\varepsilon_{k+1} \dots \varepsilon_\ell)^m.\end{aligned}$$

Remark that

$$(\varepsilon_1 \dots \varepsilon_k)^{n+m}(\varepsilon_{k+1} \dots \varepsilon_\ell)^m = 1, \quad \varepsilon_{k+1} \dots \varepsilon_\ell,\ \varepsilon_1 \dots \varepsilon_k \quad \text{or} \quad \varepsilon_1 \dots \varepsilon_\ell$$

according to

$$(n+m, m) = (\text{even, even}), (\text{even, odd}), (\text{odd, even}), (\text{odd, odd}).$$

Consequently,

$$\begin{aligned}\mathbb{E}_{\mu_c} f(\varepsilon_1 \dots \varepsilon_k) f(\varepsilon_1 \dots \varepsilon_\ell) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f_{2n}\bar{f}_{2m} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f_{2n-1}\bar{f}_{2m} c_k \\ &\quad + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f_{2n-1}\bar{f}_{2m-1} c_k c_\ell \\ &\quad + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f_{2n}\bar{f}_{2m-1} c_\ell \\ &= |U|^2 + c_k V\bar{U} + c_\ell U\bar{V} + c_k c_\ell |V|^2.\end{aligned}$$

This, together with the formula $\mathbb{E}_{\mu_c} f(\varepsilon_1 \dots \varepsilon_k) = c_k$, implies that the covariance is zero. □

Let us give a remark, a direct consequence of the above lemma. Consider the orthogonal series in $L^2(\mu_c)$

$$\sum_{k=1}^{\infty} \alpha_k (f(\varepsilon_1 \dots \varepsilon_k) - \mathbb{E}_{\mu_c} f(\varepsilon_1 \dots \varepsilon_k)).$$

According to the Menchoff theorem [Z], the series converges μ_c almost everywhere under the condition $\sum_{k=1}^{\infty} |\alpha_k|^2 \log^2 n < \infty$.

3. Proof of Theorem 1

From the fact that $|S_n(t)| \leqslant n$ ($\forall n \geqslant 1$ and $\forall t \in \mathbb{D}$), we get $E_b = \emptyset$ for $b \notin [-1, 1]$. Consider the map $\tau : \mathbb{D} \to \mathbb{D}$, which changes the first coordinate t_1 of t to $-t_1$. Then $E_{-b} = \tau E_b$. Since τ is an isometry from $\mathbb{D}$ onto $\mathbb{D}$, E_b and E_{-b} have the same dimension (Hausdorff dimension, packing dimension or box dimension). So, we have only to prove the result for $0 \leqslant b \leqslant 1$. Note that E_0 is of full Lebesgue measure. This is a consequence of the law of large numbers (implied by the remark after Lemma 1).

Upper bound: Fix $0 < b \leqslant 1$. Take $\delta > 0$ such that $0 < b - \delta < b$. Consider the following set of n-cylinders:

$$\mathscr{C}_n = \left\{ I(\varepsilon_1, \ldots, \varepsilon_n) \colon \sum_{k=1}^{n} \varepsilon_1 \ldots \varepsilon_k \geqslant n(b - \delta) \right\}$$

and the set covered by these cylinders

$$G_n = \bigcup_{I \in \mathscr{C}_n} I.$$

It is obvious that

$$E_b \subset \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} G_n.$$

It follows that

$$\dim_P E_b \leqslant \sup_{N \geqslant 1} \dim_P \bigcap_{n=N}^{\infty} G_n \leqslant \sup_{N \geqslant 1} \overline{\dim}_{\mathrm{B}} \bigcap_{n=N}^{\infty} G_n.$$

However $\mathscr{C}_n$ is a cover of $\bigcap_{n=N}^{\infty} G_n$ by n-cylinders, when $n \geqslant N$. By considering (ε_k) as independent variables with respect to the Lebesgue probability measure P on $\mathbb{D}$, we get that for any $a > 0$

$$\frac{\mathrm{Card}\, \mathscr{C}_n}{2^n} = P\left(\sum_{k=1}^{n} \varepsilon_1 \ldots \varepsilon_k \geqslant n(b - \delta) \right) \leqslant \frac{\mathbb{E} a^{\sum_{k=1}^{n} \varepsilon_1 \ldots \varepsilon_k}}{a^{n(b-\delta)}}$$

where $\mathbb{E}$ denotes the expectation with respect to P. Note that by first integrating with respect to ε_1, we get

$$\mathbb{E} a^{\sum_{k=1}^{n} \varepsilon_1 \ldots \varepsilon_k} = \tfrac{1}{2}(a + a^{-1}) \mathbb{E} a^{\sum_{k=2}^{n} \varepsilon_2 \ldots \varepsilon_k}.$$

Then by induction, we get

$$\mathbb{E} a^{\sum_{k=1}^{n} \varepsilon_1 \ldots \varepsilon_k} = (\tfrac{1}{2}(a + a^{-1}))^n.$$

Thus we have

$$\mathrm{Card}\, \mathscr{C}_n \leqslant 2^{n[\log_2(a + a^{-1}) - (b - \delta)\log_2 a]} = 2^{nh(a)}$$

where

$$h(a) = \log_2(a + a^{-1}) - (b - \delta) \log_2 a.$$

It is easy to see that

$$\min h(a) = h\left(\sqrt{\frac{1 + (b - \delta)}{1 - (b - \delta)}} \right) = H\left(\frac{1 + b - \delta}{2} \right)$$

where

$$H(x) = -x \log_2 x - (1 - x) \log_2(1 - x).$$

It follows that

$$\overline{\dim}_{\mathrm{B}} \bigcap_{n=N}^{\infty} G_n \leqslant \limsup_{n\to\infty} \frac{\operatorname{Card} \mathscr{C}_n}{\log 2^n} \leqslant H\left(\frac{1+b-\delta}{2}\right).$$

Letting $\delta \to 0$, we get $\dim_P E_b \leqslant H(\frac{1+b}{2})$.

Lower bound: It suffices to consider $0<b<1$, because we have proved $\dim_P E_1 = 0$. Consider the Riesz product μ_b with $c_k = b$ ($\forall k \geqslant 1$). As a consequence of Lemma 1 and the Kronecker lemma, we get that

$$\lim_{n\to\infty} \frac{S_n(t)}{n} = b \quad \mu_b\text{-}a.e.$$

This implies $\dim_H E_b \geqslant \dim \mu_b$ where $\dim \mu_b$ denotes the dimension of the measure μ_b which is defined as the infimum of the Hausdorff dimensions of Borel sets with full μ_b-measure (Fan (1994)). It is known that (Fan (1994))

$$\dim \mu_b = \lim_{n\to\infty} \frac{\log \mu_b(I_n(x))}{\log 2^{-n}} \quad \mu_b\text{-}a.e.$$

We check that there are constants $0<A<B<\infty$ such that

$$\frac{A}{2^n}\prod_{k=1}^{n}(1+b\varepsilon_1\ldots\varepsilon_k) \leqslant \mu_b(I_n(x)) \leqslant \frac{B}{2^n}\prod_{k=1}^{n}(1+b\varepsilon_1\ldots\varepsilon_k)$$

for any $x=(\varepsilon_n)_{n\geqslant 1} \in \mathbb{D}$ and any $n \geqslant 1$ (see Fan (1997b)) for details). It follows that

$$\dim \mu_b = 1 - \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \log_2(1+b\varepsilon_1\ldots\varepsilon_k) \quad \mu_b\text{-}a.e.$$

Using once more the consequence of Lemma 1 applied to $\alpha_n = 1/n$ and $f(x) = \log_2(1+bx)$, we get

$$\dim \mu_b = 1 - \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}_{\mu_b} \log_2(1+b\varepsilon_1\ldots\varepsilon_k).$$

Remark that $\log(1+bx) = bx - (b^2/2)x^2 + (b^3/3)x^3 + \cdots$. By Lemma 1, we have

$$\begin{aligned}
\mathbb{E}_{\mu_b} \log(1+b\varepsilon_1\ldots\varepsilon_k) &= -\sum_{n=1}^{\infty}\frac{b^{2n}}{2n} + b\sum_{n=1}^{\infty}\frac{b^{2n-1}}{2n-1} \\
&= b\left(-\sum_{n=1}^{\infty}\frac{b^{2n}}{2n} + \sum_{n=1}^{\infty}\frac{b^{2n-1}}{2n-1}\right) + (b-1)\sum_{n=1}^{\infty}\frac{b^{2n}}{2n} \\
&= b\log(1+b) - \frac{b-1}{2}\log(1-b^2) \\
&= \frac{1+b}{2}\log(1+b) + \frac{1-b}{2}\log(1-b) \\
&= \frac{1+b}{2}\log\frac{1+b}{2} + \frac{1-b}{2}\log\frac{1-b}{2} + \log 2.
\end{aligned}$$

Finally we get $\dim \mu_b = H((1+b)/2)$. $\square$

4. Proof of Theorem 3

Notice first that $\mathrm{i}^k = 1, \mathrm{i}, -1, \text{or} -\mathrm{i}$ according to $k = 0, 1, 2$ or $3 \pmod 4$. Given two real numbers α, β, define a real sequence (a_k) as follows

$$a_k = \alpha, -\beta, -\alpha \text{ or } \beta,$$

according to $k = 0, 1, 2$ or $3 \pmod 4$.

Then we can write

$$\langle S_n(t), \alpha + \mathrm{i}\beta \rangle = \sum_{k=1}^{n} a_k \varepsilon_1 \varepsilon_2 \ldots \varepsilon_k,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of two complex numbers regarded as two points in the euclidean space $\mathbb{R}^2$.

In particular, if $\alpha = 1$ and $\beta = 0$, the last sum is just the real part of $S_n(t)$. It follows that the real part of S_n is bounded by $(n/2) + 1$. It is the same for the imaginary part of $S_n(t)$. So, we have $F_z = \emptyset$ for $z \notin \Delta$. Now we are going to prove the dimension formula.

Upper bound: For any real numbers α and β, we have $F_z \subset \mathscr{F}_{\alpha,\beta}$ where

$$\mathscr{F}_{\alpha,\beta} = \left\{ t \in \mathbb{D} : \lim_{n\to\infty} \frac{\langle S_n(t), \alpha + \mathrm{i}\beta \rangle}{n} = \alpha x + \beta y \right\}.$$

It follows that

$$\dim_P F_z \leqslant \inf_{\alpha,\beta} \dim_P \mathscr{F}_{\alpha,\beta}.$$

In order to estimate $\dim_P \mathscr{F}_{\alpha,\beta}$, we follow the same proof as in the case of 1-dimension. We shall use the expression given above for $\langle S_n(t), \alpha + \mathrm{i}\beta \rangle$. Without loss of generality, we assume that $0 < \alpha x + \beta y$. Take $\delta > 0$ such that $0 < \alpha x + \beta y - \delta$. Consider the following set of n-cylinders:

$$\mathscr{C}_n = \left\{ I(\varepsilon_1, \ldots, \varepsilon_n) \colon \sum_{k=1}^{n} a_k \varepsilon_1 \ldots \varepsilon_k \geqslant n(\alpha x + \beta y - \delta) \right\}.$$

We have

$$\dim_P E_{\alpha,\beta} \leqslant \limsup_{n\to\infty} \frac{\operatorname{Card} \mathscr{C}_n}{\log 2^n}.$$

It may be calculated that

$$\mathbb{E} \mathrm{e}^{\sum_{k=1}^{n} a_k \varepsilon_1 \ldots \varepsilon_k} \approx \left(\tfrac{1}{2}(\mathrm{e}^{\alpha} + \mathrm{e}^{-\alpha})\right)^{n/2} \left(\tfrac{1}{2}(\mathrm{e}^{\beta} + \mathrm{e}^{-\beta})\right)^{n/2}.$$

Then

$$\begin{aligned}\operatorname{Card} \mathscr{C}_n &= 2^n P\left(\sum_{k=1}^{n} a_k \varepsilon_1 \ldots \varepsilon_k \geqslant n(\alpha x + \beta y - \delta) \right) \\ &\leqslant 2^{n/\log 2[1/2 \log(\mathrm{e}^{\alpha}+\mathrm{e}^{-\alpha})+1/2 \log(\mathrm{e}^{\beta}+\mathrm{e}^{-\beta})-(\alpha x+\beta y-\delta)]}.\end{aligned}$$

Thus we get

$$\dim_{\mathrm{P}} \mathscr{F}_{\alpha,\beta} \leqslant \frac{h(\alpha,\beta)}{\log 2},$$

where

$$h(\alpha,\beta)=\tfrac{1}{2}\log(\mathrm{e}^{\alpha}+\mathrm{e}^{-\alpha})+\tfrac{1}{2}\log(\mathrm{e}^{\beta}+\mathrm{e}^{-\beta})-(\alpha x+\beta y-\delta).$$

Note that

$$\frac{\partial h}{\partial\alpha}=\frac{1}{2}\frac{\mathrm{e}^{\alpha}-\mathrm{e}^{-\alpha}}{\mathrm{e}^{\alpha}+\mathrm{e}^{-\alpha}}-x,\qquad \frac{\partial h}{\partial\beta}=\frac{1}{2}\frac{\mathrm{e}^{\beta}-\mathrm{e}^{-\beta}}{\mathrm{e}^{\beta}+\mathrm{e}^{-\beta}}-x.$$

It follows that the minimal point (α,β) of the function h is defined by

$$\mathrm{e}^{\alpha}=\sqrt{\frac{1+2x}{1-2x}},\quad \mathrm{e}^{\beta}=\sqrt{\frac{1+2y}{1-2y}}$$

Let $h_{\min}$ be the minimal value of the function h. We have

$$\begin{aligned}\frac{h_{\min}}{\log 2}&=\frac{1}{2\log 2}\left[\log\left(\sqrt{\frac{1+2x}{1-2x}}+\sqrt{\frac{1-2x}{1+2x}}\right)+\log\left(\sqrt{\frac{1+2x}{1-2x}}+\sqrt{\frac{1-2x}{1+2x}}\right)\right.\\&\qquad\left.-x\log\frac{1+2x}{1-2x}-y\log\frac{1+2y}{1-2y}+\delta\right]\\&=\frac{1}{2}\left(H\left(\frac{1+2x}{2}\right)+H\left(\frac{1+2y}{2}\right)\right)+\delta.\end{aligned}$$

Lower bound: In order to get the lower bound we consider the Riesz product μ (we denote it by μ instead of μ_c) defined in Section 2 with

$$c_k=a,b,c \text{ or } d$$

according to $k=0,1,2$ or $3\,(\mathrm{mod}\,4)$, where a,b,c and d are four real numbers of absolute value smaller than 1, which will be determined later. By the remark after Lemma 1 and the Kronecker lemma, we get that

$$\lim_{n\to\to\infty}\frac{S_n(t)}{n}=\frac{a-c}{4}+\mathrm{i}\frac{d-b}{4}\quad \mu\text{-a.e.}$$

Assume first that

$$a-c=4x,\quad d-b=4y.$$

Then $\mu(F_z)=1$. So $\dim_H F_z\geqslant\dim\mu$. However, it may be calculated that $\dim\mu=\phi(a,b,c,d)$ where

$$\phi(a,b,c,d)=\frac{1}{4}\left[H\left(\frac{1+a}{2}\right)+H\left(\frac{1+b}{2}\right)+H\left(\frac{1+c}{2}\right)+H\left(\frac{1+d}{2}\right)\right].$$

Thus we get

$$\dim_H F_z\geqslant\sup_{a-c=4x,d-b=4y}\phi(a,b,c,d).$$

In order to apply the Lagrange multiplier method, we introduce

$$\Phi(a,b,c,d,\alpha,\beta)=\phi(a,b,c,d)-\alpha(a-c-4x)-\beta(d-b-4y).$$

We have

$$\frac{\partial \Phi}{\partial a} = \frac{1}{8}\log\frac{1-a}{1+a} - \alpha.$$

$$\frac{\partial \Phi}{\partial c} = \frac{1}{8}\log\frac{1-c}{1+c} + \alpha.$$

If $\partial\Phi/\partial a = \partial\Phi/\partial c = 0$, we have $(1-a)/(1+a) = (1+c)/(1-c)$. This, together with $a - c = 4x$, allows us to get $a = 2x, c = -2x$. Similarly, we get $d = 2y, b = -2y$. So, if a, b, c, d are chosen in this way, we get

$$\dim_H F_z \geqslant \phi(2x, -2y, -2x, +2y) = \frac{1}{2}\left[H\left(\frac{1+2x}{2}\right) + H\left(\frac{1+2y}{2}\right)\right].$$

(here we used the fact that $H(x) = H(1-x)$). $\square$

5. Proofs of Theorems 2 and 4

Proof of Theorem 2. Fix an integer $m \geqslant 1$. For $j \geqslant 1$, let

$$\sigma_j(t) = \sum_{k=2m(j-1)+1}^{2m(j-1)+2m} X_{j,k}$$

with

$$X_{j,k} = \varepsilon_{2m(j-1)+1}(t)\ldots\varepsilon_{k-1}(t)\varepsilon_k(t).$$

(We have cut $\sum_{k=1}^{\infty}\varepsilon_1\varepsilon_2\ldots\varepsilon_k$ into blocks of $2m$ terms. σ_j is just the jth block, but without the common factor $\varepsilon_1\varepsilon_2\ldots\varepsilon_{2m(j-1)}$). Then for $J \geqslant 1$, we have

$$S_{2mJ}(t) = \sum_{j=1}^{J} \varepsilon_1(t)\varepsilon_2(t)\ldots\varepsilon_{2m(j-1)}(t)\sigma_j(t).$$

Let

$$\mathscr{E}_m = \{t \in \mathbb{D} : \sigma_j(t) = 0, \ \forall j \geqslant 1\}.$$

It is clear that for any $m \geqslant 1$, we have

$$\mathscr{E}_m \subset \{t \in \mathbb{D} : |S_n(t)| \leqslant 2m, \ \forall n \geqslant 1\} \subset E_{\mathrm{bd}}.$$

It follows that

$$\dim_H E_{\mathrm{bd}} \geqslant \sup_m \dim_H \mathscr{E}_m.$$

Now let us estimate $\dim_H \mathscr{E}_m$. Let

$$\mathscr{D}_j = \{(\varepsilon_{2m(j-1)+1}, \ldots, \varepsilon_{2m(j-1)+2m} : X_{j,1} + \cdots + X_{j,2m} = 0\}.$$

$$\mathscr{D} = \{(\varepsilon_1, \ldots, \varepsilon_{2m}) \colon \varepsilon_1 + \cdots + \varepsilon_{2m} = 0\}.$$

We claim that Card $\mathscr{D}_j$ = Card $\mathscr{D}$ ($\forall j \geqslant 1$). In fact, the calculation in the proof of Theorem 1 shows that the random variables $X_{j,1} + \cdots + X_{j,2m}$ and $\varepsilon_1 + \cdots + \varepsilon_{2m}$

(relative to the Lebesgue probability measure) have the same distribution because they have the same moment generating function $(1/2(a+a^{-1}))^{2m}$. So,

$$\begin{aligned}\operatorname{Card}\mathscr{D}_j &= 2^{2m}P(X_{j,1}+\cdots+X_{j,2m}=0)\\ &= 2^{2m}P(\varepsilon_1+\cdots+\varepsilon_{2m}=0)=\operatorname{Card}\,\mathscr{D}=C_{2m}^m.\end{aligned}$$

This implies that $\mathscr{E}_m$ is a homogeneous Cantor set and then its dimension equals

$$\dim_H \mathscr{E}_m = \log\frac{C_{2m}^m}{\log 2^{2m}} = 1+\mathrm{O}\left(\frac{\log m}{m}\right).$$

Thus we have $\dim_H E_{\mathrm{bd}} = 1$.

In order to prove $\dim_H E_{b,\tau}=1$, consider the Riesz product μ_c with $c_k=b(k^\tau-(k-1)^\tau)$ (it may be assumed that k is sufficiently large so that $|c_k|<1$ because $\tau<1$). By the remark after Lemma 1, the following series

$$\sum_k \frac{1}{k^\tau}(\varepsilon_1(t)\varepsilon_2(t)\ldots\varepsilon_k(t)-b(k^\tau-(k-1)^\tau)) \quad (\tau>\tfrac{1}{2})$$

converges μ_c-almost everywhere. By using the Kronecker lemma, we get that $\mu_c(E_{b,\tau})=1$. On the other hand, since $\tau<1$, we have $c_k\to 0$ then $\dim\mu_c=1$ (it may be proved as in the proof of Theorem 1). Finally we get $\dim_H E_{b,\tau}\geqslant\dim\mu_c=1$. □

Proof of Theorem 4. It suffices to follow the proof of Theorem 2. □

6. Remarks

1. Let us first give a remark concerning the 1-dimensional case. From the arguments given in the introduction, we can see that when $\mathbb{D}$ is equipped with the Lebesgue probability measure, S_n is not Markovian. However, if we write

$$X_n=\begin{pmatrix}S_n\\ \xi_n\end{pmatrix},\quad A_n=\begin{pmatrix}1 & \varepsilon_n\\ 0 & \varepsilon_n\end{pmatrix},$$

we have

$$X_n=A_nA_{n-1}\ldots A_1X_0,$$

where

$$X_0=\begin{pmatrix}0\\ 1\end{pmatrix}.$$

It is to be noted that the matrices A_n are independent and X_n is Markovian.

2. The following general question remains unsolved. Given an angle $0<\alpha<2\pi$. What is the behavior of

$$S_n=\mathrm{e}^{\varepsilon_1\alpha\mathrm{i}}+\mathrm{e}^{(\varepsilon_1+\varepsilon_2)\alpha\mathrm{i}}+\cdots+\mathrm{e}^{(\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_n)\alpha\mathrm{i}}?$$

The case we have studied above corresponds to $\alpha=\pi/2$. For an arbitrary angle α, in general, S_n may not stay on a lattice.

3. A 3-dimensional generalization is the following

$$S_n(t) = \sum_{k=1}^{n} R^{\varepsilon_1+\cdots+\varepsilon_k} v$$

where $v = (1,0,0)^{\mathrm{t}}$ and R is the rotation defined by the orthogonal matrix

$$R = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The walk stays on the lattice $\mathbb{Z}^3$. But all the same questions as in the 1-dimensional case remain unanswered. The simplest 3-dimensional model would be defined in the same way by replacing R by the following matrix:

$$R' = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

However, we have no results about it either. Nevertheless, the present method does work for $S_n(t) = \sum_{k=1}^{n} \varepsilon_1(t)\ldots\varepsilon_k(t)v_k$ where $\{v_k\}$ is a fixed sequence of vectors.

4. Let us mention some previous studies in similar situations. For the classical walk $\varepsilon_1 + \cdots + \varepsilon_n$, the result corresponding to Theorem 1 is a well known theorem due to Bescicovitch (1934) and Eggleston (1949) and the result corresponding to Theorem 2 is due to Wu (1998). Trigonometric sums are studied by Fan (1997a). Oriented walks on graphs guided substitutive sequences are considered by Wen and Wen (1992), and Dekking and Wen (1996).

5. By using the method in Wu (1998), we can relax the restriction $1/2 < \tau < 1$ to $0 < \tau < 1$.

6. Some of the properties of the Riesz products constructed in Section 2 may be deduced from Fan (1993). We wonder if there is a necessary and sufficient condition for the series at the end of Section 2 to converge almost everywhere with respect to the Riesz product.

Acknowledgements

The author thanks Professors K.S. Lau and Z.Y. Wen for valuable discussions and to Dr T. Hilberdink for careful reading of an earlier version. The author also thanks Department of Mathematics, CUHK for providing facilities. The research was partially supported by Zheng Ge Ru Foundation when the author was visiting IMS, CUHK.

References

Bescicovitch, A.S., 1934. On the sum of digits of real numbers represented in the dyadic system. Math. Ann. 110, 321–330.

Dekking, M., Wen, Z.Y., 1996. Boundedness of oriented walks generated by substitutions. Séminaire de Théorie des Nombres, Bordeaux, 8, 377–386.

Eggleston, H.G., 1949. The fractional dimension of a set defined by decimal properties. Q. J. Math. Oxford Ser. 20, 31–46.

Fan, A.H., 1993. Quelques propriétés des produits de Riesz. Bull. Sc. Math. 117, 421–439.

Fan, A.H., 1994. Sur les dimensions de mesures. Studia Math. 111, 1–17.

Fan, A.H., 1997a. A refinement of an ergodic theorem and its application to Hardy functions. C.R. Acad. Sci. Paris, t. 325, Séri I, 145–150.

Fan, A.H., 1997b. Multifractal analysis of infinite products. J. Stat. Phys. 86 (5/6), 1313–1336.

Kahane, J.P., 1985. Some random series of functions. Cambridge University Press, Cambridge.

Mattila, P., 1995. Geometry of sets and measures in Euclidean spaces, Fractals and rectifiability. Cambridge University Press, Cambridge.

Wu, J., 1998. Dimension of level sets of some Rademacher series C.R. Acad. Sci. Paris, t. 327, Séri I, 29–33.

Wen, Z.X., Wen, Z.Y., 1992. Marches sur les arbres homogènes suivant une suite substitutive. Séminaire de Théorie des Nombres, Bordeaux, 4, 155–186.

Zymund, A., 1959. Trigonometric series. Cambridge University Press, Cambridge.