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# Empirical distributions in marked point processes

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**Abstract**

We study the asymptotic behaviour of the empirical distribution function derived from a stationary marked point process when a convex sampling window is expanding without bounds in all directions. We consider a random field model which assumes that the marks and the points are independent and admits dependencies between the marks. The main result is the weak convergence of the empirical process under strong mixing conditions on both independent components of the model. Applying an approximation principle weak convergence can be also shown for appropriately weighted empirical process defined from a stationary $d$-dimensional germ-grain process with dependent grains.


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## 1. Introduction

Marked point process models are often used when dealing with spatial data which consist of measurements at irregularly scattered spatial locations (see e.g. [5] or [15]). The locations form a spatial point process and the associated measurements are the marks which may depend

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on the locations. If the marks are independent identically distributed and independent of the points, we speak about independently marked (or randomly labelled) point process. This model consists of two independent random components, a point process and a sequence of i.i.d. marks. The case when the marks are generated by a random field (i.e. may be dependent) was introduced in [17]. It is usually referred to as the geostatistical (or external) marking or the random field model in the literature (see [23] and [15]). This model assumes independence between the observations and the locations. If the marks and the points may be dependent, then we speak about the case of non-geostatistical marking. Geostatistical marking was used to model observed gauge measurements for rainfalls in [18]. Other practical examples, where a random field model is suitable, involve e.g. precipitation data with raingauge measurements at the finitely many locations (see [17]) or measures of tree size in sparse forest (see [23]). Methods to test for geostatistical marking against non-geostatistical marking were proposed in [9,23].

We observe a single realization of a stationary marked point process with geostatistical marking in a convex compact window. Our aim is to estimate the distribution of the function of typical mark and we investigate asymptotic properties of this estimator as a sampling window grows without bounds in all directions. We consider the empirical distribution function and define the corresponding empirical process. Non-parametric estimators of the mean and covariance function of the underlying random field were studied in [17]. In the case of Poisson point process of the locations it was shown that these estimators are consistent and asymptotically normal under mild conditions on the random field. Asymptotic consistency and normality of a non-parametric kernel estimator of the mark variogram is proved in [10] also for the case of non-geostatistical marking. In order to establish the asymptotic normality the conditions on the strength of dependence in the random field are given in terms of strong mixing coefficients.

Central limit theorems for stationary marked point processes are well developed in the case of independently marked point processes; see [12,14] or [20]. Applying the blocking method from [10] we prove the central limit theorem for the mark sum derived from a stationary marked point process with geostatistical marking satisfying $\alpha$-mixing conditions on both points and marks. This result is used to show the convergence of the finite-dimensional distributions of the empirical process. Weak convergence of the empirical process is then established by verifying the tightness.

There are many papers dealing with the empirical processes of mixing random variables. Weak convergence was first shown in [2] for functions of a stationary sequence of $\varphi$-mixing random variables. This result was generalized in [7] and later extended in [24] for a class of $\alpha$-mixing sequences. On the other hand, the study of empirical processes in spatial statistics (see e.g. [1] and [13]) is not so frequent. In this paper we generalize the results from [13] where independently marked point processes were considered. We establish the weak convergence of the empirical processes for a class of marked point processes satisfying certain $\alpha$-mixing conditions. In addition to the classical empirical distribution function we also treat spatial Horvitz–Thompson style estimator for germ-grain processes. We can view the germ-grain process as a marked point process on $\mathbb{R}^d$ where the mark space is the space of compact sets. In this case not all marks are completely observable from a realization in a given window.

The paper is organized as follows. In Section 2 we define an empirical distribution function and recall strong convergence results for stationary ergodic marked point processes. We formulate the central limit theorem for stationary marked point process with geostatistical marking in Section 3. Weak convergence results are proved in Section 4. We conclude with a brief discussion in Section 5.

## 2. Empirical distribution function

Let $\Phi_m = \{(X_i, M_i)\}$ be a stationary marked point process on the $d$-dimensional Euclidean space $\mathbb{R}^d$ with some measurable mark space $\mathbb{M}$. Its intensity measure is defined as

$$\Lambda_m(B \times U) = \mathbb{E} \sum_i \mathbf{1}\{X_i \in B, M_i \in U\}, \quad B \in \mathcal{B}(\mathbb{R}^d), U \in \mathcal{B}(\mathbb{M}),$$

where $\mathbf{1}\{\cdot\}$ denotes the indicator function and $\mathcal{B}$ is the Borel $\sigma$-algebra. The intensity measure can be factorized into the multiple of Lebesgue measure and a probability distribution $\Lambda_0$,

$$\Lambda_m(B \times U) = \lambda |B| \Lambda_0(U).$$

Here, $\lambda$ is the intensity of the point process $\Phi = \{X_i\}$, $|B|$ stands for the Lebesgue measure of $B$ and $\Lambda_0$ is called the mark distribution. By $M_0$ we will denote a random variable with distribution $\Lambda_0$, it is called typical mark. For basic notions and results concerning stationary marked point processes we refer to [6,15].

The $k$th order factorial moment measure of the unmarked point process $\Phi$ is defined by

$$\alpha^{(k)}(B_1 \times \cdots \times B_k) = \mathbb{E} \sum_{i_1,\ldots,i_k}^{\neq} \mathbf{1}\{X_{i_1} \in B_1, \ldots, X_{i_k} \in B_k\},$$

where the symbol $\sum^{\neq}$ designates the summation over pairwise distinct indices. The well-known Campbell theorem has the form

$$\mathbb{E} \sum_{i_1,\ldots,i_k}^{\neq} h(X_{i_1}, \ldots, X_{i_k}) = \int \cdots \int h(x_1, \ldots, x_k) \alpha^{(k)}(\mathrm{d}x_1, \ldots, \mathrm{d}x_k) \tag{1}$$

for any measurable integrable function $h$. The first four factorial cumulant measures $\gamma^{(k)}$ can be recursively defined as follows (see [11]): $\gamma^{(1)}(B_1) = \alpha^{(1)}(B_1) = \lambda |B_1|$,

$$\begin{aligned}
&\gamma^{(2)}(B_1 \times B_2) = \alpha^{(2)}(B_1 \times B_2) - \lambda^2 |B_1||B_2|, \\
&\gamma^{(3)}(B_1 \times B_2 \times B_3) \;=\; \alpha^{(3)}(B_1 \times B_2 \times B_3) - \lambda |B_1| \alpha^{(2)}(B_2 \times B_3) \\
&\qquad\qquad\qquad\qquad\qquad - \lambda |B_2| \gamma^{(2)}(B_1 \times B_3) - \lambda |B_3| \gamma^{(2)}(B_1 \times B_2), \\
&\gamma^{(4)}(B_1 \times B_2 \times B_3 \times B_4) = \alpha^{(4)}(B_1 \times B_2 \times B_3 \times B_4) - \lambda |B_1| \alpha^{(3)}(B_2 \times B_3 \times B_4) \\
&\qquad - \lambda |B_2| \gamma^{(3)}(B_1 \times B_3 \times B_4) - \lambda |B_3| \gamma^{(3)}(B_1 \times B_2 \times B_4) \\
&\qquad - \lambda |B_4| \gamma^{(3)}(B_1 \times B_2 \times B_3) - \gamma^{(2)}(B_1 \times B_2) \alpha^{(2)}(B_3 \times B_4) \\
&\qquad - \gamma^{(2)}(B_1 \times B_3) \alpha^{(2)}(B_2 \times B_4) - \gamma^{(2)}(B_1 \times B_4) \alpha^{(2)}(B_2 \times B_3).
\end{aligned}$$

Since $\Phi$ is stationary, $\gamma^{(k)}$ can be written as

$$\gamma^{(k)}(B_1 \times \cdots \times B_k) = \lambda \int_{B_1} \gamma_{red}^{(k)}((B_2 - x) \times \cdots \times (B_k - x)) \, \mathrm{d}x,$$

where $\gamma_{red}^{(k)}$ is called the $k$th order reduced factorial cumulant measure of $\Phi$. It is a signed measure on $(\mathbb{R}^d)^{k-1}$. Let $|\gamma_{red}^{(k)}|(B_1 \times \cdots \times B_{k-1})$ be the total variation of $\gamma_{red}^{(k)}$ over $B_1 \times \cdots \times B_{k-1}$.

We shall denote by $W_n$ a bounded window where a realization of $\Phi_m$ is observed. Let $f : \mathbb{M} \to \mathbb{R}$ be some measurable real-valued function and we are interested in the estimation of

$$F(t) = \Lambda_0(\{m \in \mathbb{M} : f(m) \leq t\}) = \mathbb{P}(f(M_0) \leq t), \quad t \in \mathbb{R}. \tag{2}$$

If $\lambda$ is known, a natural estimator of (2) is

$$\tilde{F}_n(t) = \frac{1}{\lambda |W_n|} \sum_{X_i \in W_n} \mathbf{1}\{f(M_i) \le t\}. \tag{3}$$

As a consequence of the Campbell theorem for stationary marked point processes, (3) is an unbiased estimator of $F(t)$. In practice, $\lambda$ is usually unknown and must be estimated. Replacing it in (3) by a common estimator $\Phi(W_n)/|W_n|$, leads to

$$\hat{F}_n(t) = \frac{1}{\Phi(W_n)} \sum_{X_i \in W_n} \mathbf{1}\{f(M_i) \le t\}. \tag{4}$$

Since we are going to study the asymptotic properties of the estimators as the observation window is increasing, we have to put the assumptions on the growth of the window. Let $\{W_n\}$ be a convex averaging sequence (see [6]), i.e. $W_n$ is a convex compact subset of $\mathbb{R}^d$, $W_n \subseteq W_{n+1}$ for all $n$ and $\rho(W_n) = \sup\{r \ge 0 : b(x,r) \subseteq W_n, x \in W_n\} \underset{n\to\infty}{\longrightarrow} \infty$, where $b(x,r)$ denotes a $d$-dimensional ball of radius $r$ with centre $x$. By $o$ we denote the origin of $\mathbb{R}^d$. Later we will use the following auxiliary results,

$$\frac{|W_n \cap (W_n - x)|}{|W_n|} \underset{n\to\infty}{\longrightarrow} 1 \quad \text{for any } x \in \mathbb{R}^d, \tag{5}$$

$$|W_n \setminus (W_n \ominus b(o,r))| \le r H^{d-1}(\partial W_n) \quad \text{for any } r > 0 \tag{6}$$

and

$$\frac{H^{d-1}(\partial W_n)}{|W_n|} \le \frac{d}{\rho(W_n)} \underset{n\to\infty}{\longrightarrow} 0, \tag{7}$$

(see e.g. [13]). Here, $H^{d-1}(\partial W_n)$ is the $(d-1)$-dimensional Hausdorff measure of the boundary of $W_n$.

If $\Phi_m$ is a stationary ergodic marked point process the conditions put on the sequence $\{W_n\}$ are sufficient for the spatial individual ergodic theorem, see [6] or [19]. It implies $\tilde{F}_n(t) \underset{n\to\infty}{\longrightarrow} F(t)$ and $\hat{F}_n(t) \underset{n\to\infty}{\longrightarrow} F(t)$ $\mathbb{P}$-a.s. for any $t \in \mathbb{R}$. The uniform $\mathbb{P}$-a.s. convergence is obtained by applying a standard technique based on the monotonicity of the empirical distribution functions, see [16], Proposition 4.24. In other words, Glivenko–Cantelli theorem for stationary ergodic marked point processes holds,

$$\sup_{t\in\mathbb{R}} |\tilde{F}_n(t) - F(t)| \underset{n\to\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-a.s.}, \qquad \sup_{t\in\mathbb{R}} |\hat{F}_n(t) - F(t)| \underset{n\to\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-a.s.}$$

In the rest of the paper we are concerned with the weak convergence of the empirical distribution functions. We define the empirical processes

$$Y_n(t) = \sqrt{\Phi(W_n)} \left(\hat{F}_n(t) - F(t)\right), \quad t \in \mathbb{R}, n \in \mathbb{N}. \tag{8}$$

If $\Phi_m$ is an independently marked point process defined by a weakly stationary point process $\Phi$ with $\gamma^{(2)}_{red}$ of finite total variation, then Lemma 2 in [13] (combining with Slutsky type arguments) gives that $Y_n$ converges weakly in the Skorohod space $D(\mathbb{R})$ to the zero mean Gaussian process $Y$ with covariance function $\mathbb{E}Y(s)Y(t) = F(s \wedge t) - F(s)F(t)$.

In the present paper we consider stationary marked point processes with geostatistical marking. They can be described by the following random field model. Let $\{M(x), x \in \mathbb{R}^d\}$ be a stationary random field in $\mathbb{R}^d$, i.e. for each $y \in \mathbb{R}^d$, $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathbb{R}^d$ the distribution of $(M(x_1 + y), \ldots, M(x_n + y))$ coincides with the distribution of $(M(x_1), \ldots, M(x_n))$. Let $\Phi = \{X_i\}$ be a stationary point process on $\mathbb{R}^d$, independent of $\{M(x), x \in \mathbb{R}^d\}$. Then $\Phi_m = \{(X_i, M(X_i))\}$ defines a stationary marked point process, i.e. each point of $\Phi$ is marked by the corresponding value of the random field. Denote $\mu = \mathbb{E} f(M(x))$ and $R(x) = \operatorname{cov}(f(M(o)), f(M(x)))$. Asymptotic properties of non-parametric estimators of $\mu$ and $R(x)$ were developed in [17] under the assumption that $\Phi$ is a Poisson point process. We will be interested in the non-parametric estimator (4) of the distribution function $F(t) = \mathbb{P}(f(M(o)) \le t)$, $t \in \mathbb{R}$. We do not assume that $\Phi$ is a Poisson point process. Our main result is the weak convergence of (8). Before we formulate it we state the central limit theorem for an $\alpha$-mixing random field observed at the points of a stationary point process which also follows certain strong mixing conditions.

## 3. Central limit theorem

We will impose some weak dependence conditions on both the marks and the point process. A useful measure of dependence between two arbitrary $\sigma$-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ defined on the same probability space is the strong mixing ($\alpha$-mixing) coefficient

$$\alpha(\mathcal{F}_1, \mathcal{F}_2) = \sup\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| : A \in \mathcal{F}_1, B \in \mathcal{F}_2\}$$

which was introduced in [22]. Another possible quantification of dependence is provided by the absolute regularity coefficient (also known as $\beta$-mixing), which was introduced in [25],

$$\beta(\mathcal{F}_1, \mathcal{F}_2) = \sup \frac{1}{2} \sum_{i=1}^{I} \sum_{j=1}^{J} |\mathbb{P}(A_i \cap B_j) - P(A_i)P(B_j)|,$$

where the supremum is taken over all pairs of finite partitions $\{A_1, \ldots, A_I\}$ and $\{B_1, \ldots, B_J\}$ of $\Omega$ such that $A_i \in \mathcal{F}_1$ for each $i$ and $B_j \in \mathcal{F}_2$ for each $j$. More details about various mixing coefficients can be found in the monograph [8].

The following correlation inequality (9) plays an essential role in the proofs of limit theorems for mixing processes. Let $X$ and $Y$ be random variables that are measurable with respect to $\sigma$-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. Let $1 \le p, q, r \le \infty$ be such that $1/p + 1/q + 1/r = 1$ and $\|X\|_p < \infty$, $\|Y\|_q < \infty$. Then

$$|\operatorname{cov}(X, Y)| \le 4\alpha(\mathcal{F}_1, \mathcal{F}_2)^{1/r} \|X\|_p \|Y\|_q, \tag{9}$$

(see [4] or [8]). The inequality (9) was proved in [7] with constant 10. Moreover, due to the inequality $2\alpha(\mathcal{F}_1, \mathcal{F}_2) \le \beta(\mathcal{F}_1, \mathcal{F}_2)$, the correlation inequality (9) also holds with $\alpha$ replaced by $\beta$.

Let $\Phi$ be a stationary point process on $\mathbb{R}^d$ and $\{Z(x), x \in \mathbb{R}^d\}$ be a stationary random field. For each $A \in \mathcal{B}^d$ write $\mathcal{F}^Z(A)$ for the $\sigma$-algebra generated by $\{Z(x), x \in A\}$ and $\mathcal{F}^\Phi(A)$ for the $\sigma$-algebra generated by $\Phi \cap A$. We define for $s > 0$ and $a > 0$,

$$\alpha_a^Z(s) = \sup\{\alpha(\mathcal{F}^Z(A), \mathcal{F}^Z(B)) : B = A + x, |A| = |B| \le a, d(A, B) \ge s\},$$

and

$$\alpha_a^\Phi(s) = \sup\{\alpha(\mathcal{F}^\Phi(A), \mathcal{F}^\Phi(B)) : B = A + x, |A| = |B| \le a, d(A, B) \ge s\},$$

where $d(A, B) = \inf\{\|x - y\|_\infty : x \in A, y \in B\}$ and the supremum is taken over all compact and convex sets $A$ and over all $x \in \mathbb{R}^d$. For $a = 0$ let $\alpha_0^Z(s) = \sup\{\alpha(\mathcal{F}^Z(A), \mathcal{F}^Z(B)) : |A| = |B| = 0, d(A, B) \geq s\}$ and $\alpha_0^\Phi(s) = \sup\{\alpha(\mathcal{F}^\Phi(A), \mathcal{F}^\Phi(B)) : |A| = |B| = 0, d(A, B) \geq s\}$.

**Theorem 1.** *Let $\Phi_m = \{(X_i, M(X_i))\}$ be a stationary marked point process with geostatistical marking. Consider a measurable function $f : \mathbb{M} \to \mathbb{R}$ and put $Z(x) = f(M(x))$. Assume the following mixing conditions on $Z$ and $\Phi$:*

$$\sup_{a \geq 0} \frac{\alpha_a^Z(s)}{a \vee 1} = O(s^{-d-\varepsilon}) \quad \text{for some } \varepsilon > 0, \tag{10}$$

$$\sup_{a \geq 0} \frac{\alpha_a^\Phi(s)}{a \vee 1} = O(s^{-d-\varepsilon}) \quad \text{for some } \varepsilon > 0. \tag{11}$$

*Further, assume that the second order reduced factorial cumulant measure $\gamma_{red}^{(2)}$ of the unmarked point process $\Phi = \{X_i\}$ satisfies $|\gamma_{red}^{(2)}|(\mathbb{R}^d) < \infty$ (i.e. it is of bounded total variation). Let $h : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function such that $\mathbb{E}h(Z(o)) = 0$. Assume that $\{W_n\}$ is a convex averaging sequence which fulfills $|W_n| = O(\rho(W_n)^d)$. Finally, assume the following mild moment condition*

$$\sup_{n \in \mathbb{N}} \mathbb{E}|S_n|^q \leq C_q < \infty \quad \text{for some } q > 2, \tag{12}$$

*where $S_n = \frac{1}{\sqrt{|W_n|}} \sum_{X_i \in W_n} h(Z(X_i))$. Then the integral*

$$\int_{\mathbb{R}^d} \mathbb{E}h(Z(o))h(Z(x))\,\mathrm{d}x$$

*converges absolutely and*

$$S_n \underset{n\to\infty}{\Longrightarrow} N(0, \sigma_h^2),$$

*where $\sigma_h^2 = \lambda \mathbb{E}h(Z(o))^2 + \lambda \int \mathbb{E}h(Z(o))h(Z(x))\,\gamma_{red}^{(2)}(\mathrm{d}x) + \lambda^2 \int \mathbb{E}h(Z(o))h(Z(x))\,\mathrm{d}x$.*

**Proof.** From (9) we get

$$\begin{aligned}\int_{\mathbb{R}^d} |\mathbb{E}h(Z(o))h(Z(x))|\,\mathrm{d}x &\leq 4\|h(Z(o))\|_\infty^2 \int_{\mathbb{R}^d} \alpha_0^Z(\|x\|_\infty)\,\mathrm{d}x \\ &= 8d2^{d-1}\|h(Z(o))\|_\infty^2 \int_0^\infty s^{d-1}\alpha_0^Z(s)\,\mathrm{d}s < \infty.\end{aligned}$$

In the rest of the proof we follow [10]. We divide $W_n$ into $k_n$ non-overlapping subcubes $W_{l(n)}^i \subseteq W_n$, $i = 1, \dots, k_n$, of side length $l(n) = \rho(W_n)^\alpha$ for some $\frac{2d}{2d+\varepsilon} < \alpha < 1$. Let $m(n) = \rho(W_n)^\alpha - \rho(W_n)^\eta$ for some $\frac{2d}{2d+\varepsilon} < \eta < \alpha$. Further, let $W_{m(n)}^i$, $i = 1, \dots, k_n$ be the subcubes of side length $m(n)$, with the same centre as $W_{l(n)}^i$. Thus, $d(W_{m(n)}^i, W_{m(n)}^j) \geq \rho(W_n)^\eta$ for $i \neq j$. Define

$$T_n = \frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} T_{ni}, \qquad T_n' = \frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} T_{ni}',$$

where

$$T_{ni} = \frac{1}{\sqrt{|W^i_{m(n)}|}} \sum_{X_j \in W^i_{m(n)}} h(Z(X_j))$$

and $T'_{ni}$, $i = 1, \ldots, k_n$, have the same marginal distribution as $T_{ni}$ but are independent.

First we show that $R_n = S_n - T_n$ is asymptotically negligible. Denoting $W_n^* = \bigcup_{i=1}^{k_n} W^i_{m(n)}$ we have

$$\begin{aligned} R_n &= \frac{1}{\sqrt{|W_n|}} \sum_{X_i \in W_n} h(Z(X_i)) - \frac{1}{\sqrt{|W_n^*|}} \sum_{X_i \in W_n^*} h(Z(X_i)) \\ &= \frac{1}{\sqrt{|W_n|}} \sum_{X_i \in W_n \setminus W_n^*} h(Z(X_i)) - \frac{\sqrt{|W_n|} - \sqrt{|W_n^*|}}{\sqrt{|W_n||W_n^*|}} \sum_{X_i \in W_n^*} h(Z(X_i)) = R_n^{(1)} + R_n^{(2)}. \end{aligned}$$

Since $\mathbb{E}R_n = \mathbb{E}R_n^{(1)} = \mathbb{E}R_n^{(2)} = 0$, we get from the Campbell theorem (1)

$$\begin{aligned} \operatorname{var} R_n^{(1)} &= \frac{\lambda}{|W_n|} \left( \mathbb{E}h(Z(o))^2 |W_n \setminus W_n^*| + \int_{W_n \setminus W_n^*} \int_{(W_n \setminus W_n^*) - x} \mathbb{E}h(Z(o))h(Z(y))\, \gamma^{(2)}_{red}(\mathrm{d}y)\, \mathrm{d}x \right) \\ &\quad + \frac{\lambda^2}{|W_n|} \int_{W_n \setminus W_n^*} |(W_n \setminus W_n^*) \cap ((W_n \setminus W_n^*) - x)|\, \mathbb{E}h(Z(o))h(Z(x))\, \mathrm{d}x \\ &\le \left( \lambda(1 + |\gamma^{(2)}_{red}|(\mathbb{R}^d))\mathbb{E}h(Z(o))^2 + \lambda^2 \int \mathbb{E}h(Z(o))h(Z(x))\, \mathrm{d}x \right) \frac{|W_n \setminus W_n^*|}{|W_n|} \end{aligned}$$

and

$$\begin{aligned} \operatorname{var} R_n^{(2)} &= \frac{\lambda |W_n \setminus W_n^*|}{|W_n||W_n^*|} \left( \mathbb{E}h(Z(o))^2 |W_n^*| + \int_{W_n^*} \int_{W_n^* - x} \mathbb{E}h(Z(o))h(Z(y))\, \gamma^{(2)}_{red}(\mathrm{d}y)\, \mathrm{d}x \right) \\ &\quad + \frac{\lambda^2 |W_n \setminus W_n^*|}{|W_n||W_n^*|} \int_{W_n^*} |W_n^* \cap (W_n^* - x)|\, \mathbb{E}h(Z(o))h(Z(x))\, \mathrm{d}x \\ &\le \left( \lambda(1 + |\gamma^{(2)}_{red}|(\mathbb{R}^d))\mathbb{E}h(Z(o))^2 + \lambda^2 \int \mathbb{E}h(Z(o))h(Z(x))\, \mathrm{d}x \right) \frac{|W_n \setminus W_n^*|}{|W_n|}. \end{aligned}$$

It follows from (6) and (7) that $|W_n \setminus W_n^*|/|W_n|$ goes to 0 and consequently $R_n$ converges in $L^2$ to 0. Thus, using a standard approximation Slutsky type principle (see [2]), it suffices to prove $T_n \underset{n\to\infty}{\Longrightarrow} N(0, \sigma_h^2)$.

Let $\phi_n(t)$ and $\phi'_n(t)$ be the characteristic functions of $T_n$ and $T'_n$, respectively. We show that $\phi_n(t) - \phi'_n(t) \underset{n\to\infty}{\longrightarrow} 0$ for any $t \in \mathbb{R}$. Fix $t \in \mathbb{R}$ and define $U_i = \exp\{\mathrm{i}tT_{ni}/\sqrt{k_n}\}$. Then

$$\phi_n(t) = \mathbb{E}\prod_{i=1}^{k_n} U_i, \qquad \phi'_n(t) = \prod_{i=1}^{k_n} \mathbb{E}U_i$$

and

$$|\phi_n(t) - \phi'_n(t)| \le \sum_{j=1}^{k_n - 1} \left| \mathbb{E}\prod_{i=1}^{j+1} U_i - \mathbb{E}\prod_{i=1}^{j} U_i \mathbb{E}U_{j+1} \right| = \sum_{j=1}^{k_n-1} \left| \operatorname{cov}\left( \prod_{i=1}^{j} U_i, U_{j+1} \right) \right|. \qquad (13)$$

Conditioning on $\Phi$ we get

$$\operatorname{cov}\left(\prod_{i=1}^{j} U_i, U_{j+1}\right) = \mathbb{E}\left[\operatorname{cov}\left(\prod_{i=1}^{j} U_i, U_{j+1} \mid \Phi\right)\right] + \operatorname{cov}\left(\mathbb{E}\left[\prod_{i=1}^{j} U_i \mid \Phi\right], \mathbb{E}[U_{j+1} \mid \Phi]\right).$$

Since $|U_i| \le 1$ and $|W_{m(n)}^{j+1}| \le |\bigcup_{i=1}^{j} W_{m(n)}^{i}| = jm(n)^d$, we have from (9) and (10)

$$\operatorname{cov}\left(\prod_{i=1}^{j} U_i, U_{j+1}\right) \le Cj(\rho(W_n)^\alpha - \rho(W_n)^\eta)^d \rho(W_n)^{-\eta(d+\varepsilon)}.$$

Combining with (13) and $k_n = O(\rho(W_n)^{d(1-\alpha)})$ we get

$$|\phi_n(t) - \phi_n'(t)| \le O(\rho(W_n)^{2d-\alpha d-\eta(d+\varepsilon)}).$$

The proof is completed by observing that $T_n' \underset{n\to\infty}{\Longrightarrow} N(0, \sigma_h^2)$ due to the Lyapunov central limit theorem. The form of the asymptotic variance $\sigma_h^2$ follows from (1) and (5). $\square$

**Remark 1.** The proof reveals that the condition on boundedness of $h$ can be weakened to $\mathbb{E}|h(Z(o))|^{2+\delta} < \infty$ for some $\delta > 0$ if we moreover assume

$$\int_0^\infty s^{d-1}\alpha_0^Z(s)^{\delta/(2+\delta)}\,\mathrm{d}s < \infty.$$

Another possibility how to prove Theorem 1 is based on the central limit theorem for stationary $\alpha$-mixing random fields (see [3]). This would lead to slightly different mixing conditions.

## 4. Weak convergence of empirical process

Now we are ready to prove our main results.

### 4.1. Marked point process

**Theorem 2.** *Let $\Phi_m = \{(X_i, M(X_i))\}$ be a stationary marked point process with geostatistical marking. Consider a measurable function $f : \mathbb{M} \to \mathbb{R}$ and put $Z(x) = f(M(x))$. Assume that (10) holds and*

$$\int_0^\infty s^{2d-2}\alpha_0^Z(s)^{1/2-\tau}\,\mathrm{d}s < \infty$$

*for some $0 < \tau < 1/2$. Further, assume that the unmarked point process $\Phi = \{X_i\}$ satisfies (11) and that first four reduced factorial cumulant measures of $\Phi$ are of bounded total variation, i.e. $|\gamma_{red}^{(2)}|(\mathbb{R}^d) < \infty$, $|\gamma_{red}^{(3)}|(\mathbb{R}^d \times \mathbb{R}^d) < \infty$, $|\gamma_{red}^{(4)}|(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d) < \infty$. Let $\{W_n\}$ be a convex averaging sequence such that $|W_n| = O(\rho(W_n)^d)$. Then the empirical process defined by (8) converges weakly in $D(\mathbb{R})$ (as $n \to \infty$) to the Gaussian process $Y$ with zero mean and covariance function*

$$\mathbb{E}Y(s)Y(t) = F(s \wedge t) - F(s)F(t) + \int \mathbb{E}g_s(o)g_t(x)\,\gamma_{red}^{(2)}(\mathrm{d}x) + \lambda \int \mathbb{E}g_s(o)g_t(x)\,\mathrm{d}x,$$

*where $g_t(x) = \mathbf{1}\{Z(x) \le t\} - F(t)$.*

**Proof.** Since $\frac{\Phi(W_n)}{\lambda|W_n|} \underset{n\to\infty}{\longrightarrow} 1$ in probability (see [13]), the weak limit (if it exists) of $Y_n$ coincides with that of

$$\tilde{Y}_n(t) = \sqrt{\frac{\Phi(W_n)}{\lambda|W_n|}}\, Y_n(t) = \frac{1}{\sqrt{\lambda|W_n|}} \sum_{X_i \in W_n} g_t(X_i), \quad t \in \mathbb{R}. \tag{14}$$

For any $c_1, \dots, c_p \in \mathbb{R}$ and $t_1, \dots, t_p \in \mathbb{R}$ denote $h(Z(x)) = \sum_{k=1}^p c_k(\mathbf{1}\{Z(x) \le t_k\} - F(t_k))$. It will be seen later that (12) is satisfied with $q = 4$. Applying Theorem 1 we get

$$\frac{1}{\sqrt{\lambda|W_n|}} \sum_{X_i \in W_n} h(Z(X_i)) = \sum_{k=1}^p c_k \tilde{Y}_n(t_k) \underset{n\to\infty}{\Longrightarrow} N(0, \sigma^2_{t_1,\dots,t_p}),$$

where

$$\begin{aligned}
\sigma^2_{t_1,\dots,t_p} &= \mathbb{E}h(Z(o))^2 + \int \mathbb{E}h(Z(o))h(Z(x))\, \gamma^{(2)}_{red}(\mathrm{d}x) + \lambda \int \mathbb{E}h(Z(o))h(Z(x))\,\mathrm{d}x \\
&= \sum_{k=1}^p \sum_{l=1}^p c_k c_l \Big[ F(t_k \wedge t_l) - F(t_k)F(t_l) + \int \mathbb{E}g_{t_k}(o)g_{t_l}(x)\, \gamma^{(2)}_{red}(\mathrm{d}x) \\
&\quad + \lambda \int \mathbb{E}g_{t_k}(o)g_{t_l}(x)\,\mathrm{d}x \Big].
\end{aligned}$$

Hence, $(\tilde{Y}_n(t_1), \dots, \tilde{Y}_n(t_p)) \underset{n\to\infty}{\Longrightarrow} N_p(0, \Sigma_{t_1,\dots,t_p})$, where $\Sigma_{t_1,\dots,t_p}(k,l) = F(t_k \wedge t_l) - F(t_k)F(t_l) + \int \mathbb{E}g_{t_k}(o)g_{t_l}(x)\, \gamma^{(2)}_{red}(\mathrm{d}x) + \lambda \int \mathbb{E}g_{t_k}(o)g_{t_l}(x)\,\mathrm{d}x$. We have shown the convergence of finite-dimensional distributions and now it remains to prove the tightness of $\tilde{Y}_n$, $n \in \mathbb{N}$. We will show tightness by bounding the mixed fourth moments of the increments.

For $u < v < w$ we have to verify

$$\mathbb{E}(\tilde{Y}_n(v) - \tilde{Y}_n(u))^2(\tilde{Y}_n(w) - \tilde{Y}_n(v))^2 \le C(F(v) - F(u))^\beta (F(w) - F(v))^\beta$$

with $\beta > 1/2$, see [2], p. 128. We shall use the short notation $F(u, v]$ for $F(v) - F(u)$ and $g_{u,v}(x)$ for $\mathbf{1}\{Z(x) \in (u, v]\} - F(u, v]$.

We obtain

$$\begin{aligned}
&\mathbb{E}(\tilde{Y}_n(v) - \tilde{Y}_n(u))^2(\tilde{Y}_n(w) - \tilde{Y}_n(v))^2 \\
&\quad = \frac{1}{\lambda^2|W_n|^2} \mathbb{E} \sum_{X_i, X_j, X_k, X_l \in W_n} g_{u,v}(X_i)g_{u,v}(X_j)g_{v,w}(X_k)g_{v,w}(X_l)
\end{aligned}$$

and rewrite the sum on the right-hand side by means of the following lemma.

**Lemma 3.** *For any measurable functions $f$ and $g$ we have*

$$\begin{aligned}
\sum_{i,j,k,l} f(x_i)f(x_j)g(x_k)g(x_l) &= \sum_{i,j,k,l}^{\neq} f(x_i)f(x_j)g(x_k)g(x_l) + \sum_{i,j,k}^{\neq} f(x_i)^2 g(x_j)g(x_k) \\
&\quad + 4\sum_{i,j,k}^{\neq} f(x_i)f(x_j)g(x_i)g(x_k) + \sum_{i,j,k}^{\neq} f(x_i)f(x_j)g(x_k)^2 \\
&\quad + 2\sum_{i,j}^{\neq} f(x_i)f(x_j)g(x_i)g(x_j) + 2\sum_{i,j}^{\neq} f(x_i)^2 g(x_i)g(x_j)
\end{aligned}$$

$$+2\sum_{i,j}^{\neq} f(x_i)f(x_j)g(x_j)^2 + \sum_{i,j}^{\neq} f(x_i)^2 g(x_j)^2 + \sum_i f(x_i)^2 g(x_i)^2.$$

Using Lemma 3, (1) and independence between the marks and the points we get

$$\mathbb{E}(\tilde{Y}_n(v) - \tilde{Y}_n(u))^2(\tilde{Y}_n(w) - \tilde{Y}_n(v))^2 = \int_{W_n}\int_{W_n}\int_{W_n}\int_{W_n} V_1\,\alpha^{(4)}(\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3, \mathrm{d}x_4)$$
$$+ \int_{W_n}\int_{W_n}\int_{W_n}(V_2 + 4V_3 + V_4)\,\alpha^{(3)}(\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3)$$
$$+ \int_{W_n}\int_{W_n}(2V_5 + 2V_6 + 2V_7 + V_8)\,\alpha^{(2)}(\mathrm{d}x_1, \mathrm{d}x_2) + \lambda\int_{W_n} V_9\,\mathrm{d}x_1,$$

where the terms $V_1, \ldots, V_9$ are

$$\begin{aligned}
V_1 &= \mathbb{E}g_{u,v}(x_1)g_{u,v}(x_2)g_{v,w}(x_3)g_{v,w}(x_4),\\
V_2 &= \mathbb{E}g_{u,v}(x_1)^2 g_{v,w}(x_2)g_{v,w}(x_3)\\
&= \mathbb{E}g_{u,v}(x_1)g_{v,w}(x_2)g_{v,w}(x_3)(1 - 2F(u,v])\\
&\quad + \mathbb{E}g_{v,w}(x_2)g_{v,w}(x_3)F(u,v](1 - F(u,v]),\\
V_3 &= \mathbb{E}g_{u,v}(x_1)g_{u,v}(x_2)g_{v,w}(x_1)g_{v,w}(x_3)\\
&= -\mathbb{E}g_{u,v}(x_1)g_{u,v}(x_2)g_{v,w}(x_3)F(v,w] - \mathbb{E}g_{v,w}(x_1)g_{u,v}(x_2)g_{v,w}(x_3)F(u,v]\\
&\quad - \mathbb{E}g_{u,v}(x_2)g_{v,w}(x_3)F(u,v]F(v,w],\\
V_4 &= \mathbb{E}g_{u,v}(x_1)g_{u,v}(x_2)g_{v,w}(x_3)^2\\
&= \mathbb{E}g_{u,v}(x_1)g_{u,v}(x_2)g_{v,w}(x_3)(1 - 2F(v,w])\\
&\quad + \mathbb{E}g_{u,v}(x_1)g_{u,v}(x_2)F(v,w](1 - F(v,w]),\\
V_5 &= \mathbb{E}g_{u,v}(x_1)g_{u,v}(x_2)g_{v,w}(x_1)g_{v,w}(x_2)\\
&= \mathbb{E}g_{u,v}(x_1)g_{u,v}(x_2)F(v,w]^2 + 2\mathbb{E}g_{u,v}(x_1)g_{v,w}(x_2)F(u,v]F(v,w]\\
&\quad + \mathbb{E}g_{v,w}(x_1)g_{v,w}(x_2)F(u,v]^2 + F(u,v]^2F(v,w]^2,\\
V_6 &= \mathbb{E}g_{u,v}(x_1)^2 g_{v,w}(x_1)g_{v,w}(x_2)\\
&= \mathbb{E}g_{v,w}(x_1)g_{v,w}(x_2)F(u,v]^2 - \mathbb{E}g_{u,v}(x_1)g_{v,w}(x_2)(1 - 2F(u,v]),\\
V_7 &= \mathbb{E}g_{u,v}(x_1)g_{u,v}(x_2)g_{v,w}(x_2)^2\\
&= \mathbb{E}g_{u,v}(x_1)g_{u,v}(x_2)F(v,w]^2 - \mathbb{E}g_{u,v}(x_1)g_{v,w}(x_2)F(u,v](1 - 2F(v,w]),\\
V_8 &= \mathbb{E}g_{u,v}(x_1)^2 g_{v,w}(x_2)^2 = F(u,v]F(v,w](1 - F(u,v])(1 - F(v,w])\\
&\quad + (1 - 2F(u,v])(1 - 2F(v,w])\mathbb{E}g_{u,v}(x_1)g_{v,w}(x_2),\\
V_9 &= \mathbb{E}g_{u,v}(x_1)^2 g_{v,w}(x_1)^2 = F(u,v]F(v,w](F(u,v] + F(v,w] - 3F(u,v]F(v,w]).
\end{aligned}$$

The integral of the first term is

$$\iiiint \mathbb{E}g_{u,v}(x_1)g_{u,v}(x_2)g_{v,w}(x_3)g_{v,w}(x_4)\,\alpha^{(4)}(\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3, \mathrm{d}x_4)$$
$$= \lambda^4 \iiiint \mathbb{E}g_{u,v}(x_1)g_{u,v}(x_2)g_{v,w}(x_3)g_{v,w}(x_4)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4$$
$$+ \lambda^2 \iiiint \mathbb{E}g_{u,v}(x_1)g_{u,v}(x_1 + x_2)g_{v,w}(x_3)g_{v,w}(x_3 + x_4)\,\mathrm{d}x_1\,\gamma_{red}^{(2)}(\mathrm{d}x_2)\,\mathrm{d}x_3\,\gamma_{red}^{(2)}(\mathrm{d}x_4)$$
$$+ 2\lambda^2 \iiiint \mathbb{E}g_{u,v}(x_1)g_{u,v}(x_2)g_{v,w}(x_1 + x_3)g_{v,w}(x_2 + x_4)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\gamma_{red}^{(2)}(\mathrm{d}x_3)\,\gamma_{red}^{(2)}(\mathrm{d}x_4)$$

$$
\begin{aligned}
&+\lambda^3 \iiiint \mathbb{E} g_{u,v}(x_1) g_{u,v}(x_2) g_{v,w}(x_3) g_{v,w}(x_3+x_4)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\gamma_{red}^{(2)}(\mathrm{d}x_4)\\
&+4\lambda^3 \iiiint \mathbb{E} g_{u,v}(x_1) g_{u,v}(x_2) g_{v,w}(x_3) g_{v,w}(x_1+x_4)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\gamma_{red}^{(2)}(\mathrm{d}x_4)\\
&+\lambda^3 \iiiint \mathbb{E} g_{u,v}(x_1+x_2) g_{u,v}(x_2) g_{v,w}(x_3) g_{v,w}(x_4)\,\mathrm{d}x_1\,\gamma_{red}^{(2)}(\mathrm{d}x_2)\,\mathrm{d}x_3\,\mathrm{d}x_4\\
&+2\lambda^2 \iiiint \mathbb{E} g_{u,v}(x_1) g_{u,v}(x_2) g_{v,w}(x_2+x_3) g_{v,w}(x_2+x_4)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\gamma_{red}^{(3)}(\mathrm{d}x_3,\mathrm{d}x_4)\\
&+2\lambda^2 \iiiint \mathbb{E} g_{u,v}(x_1+x_4) g_{u,v}(x_2+x_4) g_{v,w}(x_3) g_{v,w}(x_4)\,\gamma_{red}^{(3)}(\mathrm{d}x_1,\mathrm{d}x_2)\,\mathrm{d}x_3\,\mathrm{d}x_4\\
&+\lambda \iiiint \mathbb{E} g_{u,v}(x_1) g_{u,v}(x_1+x_2) g_{v,w}(x_1+x_3) g_{v,w}(x_1+x_4)\,\mathrm{d}x_1\,\gamma_{red}^{(4)}(\mathrm{d}x_2,\mathrm{d}x_3,\mathrm{d}x_4).
\end{aligned}
$$

We describe in detail the treatment of the first term. For each quadruple $(x_1, x_2, x_3, x_4)$ let $a = \max_{i=1,2,3,4} \min_{j\neq i} \|x_i - x_j\|_\infty$. Denote by $I$ the index at which this maximum is attained and let $J$ be the index of the nearest point of $x_I$. Thus, $\|x_I - x_J\|_\infty = \min_{j\neq I} \|x_I - x_j\| = a$. The other two points (denoted as $x_K$ and $x_L$) then have to satisfy $\|x_K - x_L\|_\infty \leq 2a$. Choose the notation such that $\|x_J - x_K\|_\infty \leq \|x_J - x_L\|_\infty$. If $\|x_J - x_K\|_\infty \geq a$ define $b = d(\{x_I, x_J\}, \{x_K, x_L\})$. Observe that $b \geq a$. We put $1/r = 1/2 - \varepsilon$ and $1/p = 1/4 + \varepsilon/2$. Then using (9) with $p = q$ we obtain

$$
\begin{aligned}
&|\mathbb{E} g_{u,v}(o) g_{v,w}(x)| \leq 4\alpha_0^Z(\|x\|_\infty)^{1/r} \|g_{u,v}(o)\|_p \|g_{v,w}(o)\|_p,\\
&|\mathbb{E} g_{u,v}(x_1) g_{u,v}(x_2) g_{v,w}(x_3) g_{v,w}(x_4)| \leq 4\alpha_0^Z(a)^{1/r} \|g_{u,v}(o)\|_p \|g_{v,w}(o)\|_p
\end{aligned}
$$

and if $\|x_J - x_K\|_\infty \geq a$,

$$
\begin{aligned}
&|\mathbb{E} g_{u,v}(x_1) g_{u,v}(x_2) g_{v,w}(x_3) g_{v,w}(x_4)|\\
&\quad\leq 16\alpha_0^Z(\|x_I - x_J\|_\infty)^{1/r} \alpha_0^Z(\|x_K - x_L\|_\infty)^{1/r} \|g_{u,v}(o)\|_p^2 \|g_{v,w}(o)\|_p^2\\
&\qquad+4\alpha_0^Z(b)^{1/r} \|g_{u,v}(o)\|_p \|g_{v,w}(o)\|_p.
\end{aligned}
$$

Furthermore,

$$
\|g_{u,v}(x)\|_p \leq C_0(F(v) - F(u))^{1/p}.
$$

Hence,

$$
\begin{aligned}
&\int_{W_n}\int_{W_n}\int_{W_n}\int_{W_n} \mathbb{E} g_{u,v}(x_1) g_{u,v}(x_2) g_{v,w}(x_3) g_{v,w}(x_4)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4\\
&\quad\leq 24\cdot 16\|g_{u,v}(o)\|_p^2 \|g_{v,w}(o)\|_p^2 \int_{W_n}\int_{W_n}\int_{W_n}\int_{W_n} \alpha_0^Z(\|x_I - x_J\|_\infty)^{1/r} \alpha_0^Z(\|x_K - x_L\|_\infty)^{1/r}\\
&\qquad\times \mathbf{1}\{\|x_J - x_K\|_\infty \geq \|x_I - x_J\|_\infty\}\,\mathrm{d}x_I\,\mathrm{d}x_J\,\mathrm{d}x_K\,\mathrm{d}x_L\\
&\qquad+24\cdot 4\|g_{u,v}(o)\|_p \|g_{v,w}(o)\|_p \int_{W_n}\int_{W_n}\int_{W_n}\int_{W_n} \alpha_0^Z(b)^{1/r} \mathbf{1}\{\|x_J - x_K\|_\infty \geq \|x_I - x_J\|_\infty\}\\
&\qquad\times \mathbf{1}\{\|x_I - x_J\|_\infty \leq b\} \mathbf{1}\{\|x_K - x_L\|_\infty \leq 2b\}\,\mathrm{d}x_I\,\mathrm{d}x_J\,\mathrm{d}x_K\,\mathrm{d}x_L\\
&\qquad+24\cdot 4\|g_{u,v}(o)\|_p \|g_{v,w}(o)\|_p \int_{W_n}\int_{W_n}\int_{W_n}\int_{W_n} \alpha_0^Z(a)^{1/r} \mathbf{1}\{\|x_J - x_K\|_\infty < a\}\\
&\qquad\times \mathbf{1}\{\|x_K - x_L\|_\infty \leq 2a\}\,\mathrm{d}x_I\,\mathrm{d}x_J\,\mathrm{d}x_K\,\mathrm{d}x_L\\
&\quad\leq 24\cdot 16\|g_{u,v}(o)\|_p^2 \|g_{v,w}(o)\|_p^2 \int_{W_n}\int_{W_n}\int\int \alpha_0^Z(\|x_1\|_\infty)^{1/r} \alpha_0^Z(\|x_2\|_\infty)^{1/r}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4
\end{aligned}
$$

$$
\begin{aligned}
&+24\cdot 4\|g_{u,v}(o)\|_p\|g_{v,w}(o)\|_p\int_{W_n}\int\int\int\alpha_0^Z(b)^{1/r}\mathbf{1}\{\|x_1\|_\infty=b\}\\
&\times\mathbf{1}\{\|x_2\|_\infty\le b\}\mathbf{1}\{\|x_3\|_\infty\le 2b\}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4\\
&+24\cdot 4\|g_{u,v}(o)\|_p\|g_{v,w}(o)\|_p\int_{W_n}\int\int\int\alpha_0^Z(a)^{1/r}\mathbf{1}\{\|x_1\|_\infty=a\}\\
&\times\mathbf{1}\{\|x_2\|_\infty\le a\}\mathbf{1}\{\|x_3\|_\infty\le 2a\}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4.
\end{aligned}
$$

We can find real constants $C_1$, $C_2$, $C_1^*$ and $C_2^*$ such that

$$
\begin{aligned}
&\int_{W_n}\int_{W_n}\int_{W_n}\int_{W_n}\mathbb{E}g_{u,v}(x_1)g_{u,v}(x_2)g_{v,w}(x_3)g_{v,w}(x_4)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4\\
&\quad\le C_1\|g_{u,v}(o)\|_p^2\|g_{v,w}(o)\|_p^2|W_n|^2\left(\int s^{d-1}\alpha_0^Z(s)^{1/r}\,\mathrm{d}s\right)^2\\
&\qquad+C_2\|g_{u,v}(o)\|_p\|g_{v,w}(o)\|_p|W_n|\int b^{2d-2}\alpha_0^Z(b)^{1/r}\,\mathrm{d}b\\
&\quad\le C_1^*|W_n|^2(F(v)-F(u))^{2/p}(F(w)-F(v))^{2/p}\\
&\qquad+C_2^*|W_n|(F(v)-F(u))^{1/p}(F(w)-F(v))^{1/p}.
\end{aligned}
$$

Similar calculations together with the assumption of bounded total variation of reduced factorial cumulant measures give the bound for

$$
\int_{W_n}\int_{W_n}\int_{W_n}\int_{W_n}\mathbb{E}g_{u,v}(x_1)g_{u,v}(x_2)g_{v,w}(x_3)g_{v,w}(x_4)\,\alpha^{(4)}(\mathrm{d}x_1,\mathrm{d}x_2,\mathrm{d}x_3,\mathrm{d}x_4).
$$

Similarly as above we get

$$
\begin{aligned}
&\int_{W_n}\int_{W_n}\int_{W_n}\mathbb{E}g_{u,v}(x_1)g_{v,w}(x_2)g_{v,w}(x_3)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\\
&\quad\le 6\cdot 4\|g_{u,v}(o)\|_p\|g_{v,w}(o)\|_p|W_n|\int s^{d-1}\alpha_0^Z(s)^{1/r}\,\mathrm{d}s\\
&\quad\le C_3|W_n|(F(v)-F(u))^{1/p}(F(w)-F(v))^{1/p}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{W_n}\int_{W_n}\mathbb{E}g_{u,v}(x_1)g_{u,v}(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2&\le 4\|g_{u,v}(o)\|_p^2|W_n|\int s^{d-1}\alpha_0^Z(s)^{1/r}\,\mathrm{d}s\\
&\le C_4|W_n|(F(w)-F(v))^{2/p}.
\end{aligned}
$$

These two relations can help in bounding the integrals of remaining 8 terms $V_2,\dots,V_9$ obtained from Lemma 3.

Summarizing, for $2/p=1/2+\varepsilon>1/2$, the following inequality holds for any $u<v<w$ and $n\in\mathbb{N}$,

$$
\begin{aligned}
&\mathbb{E}(\tilde{Y}_n(v)-\tilde{Y}_n(u))^2(\tilde{Y}_n(w)-\tilde{Y}_n(v))^2\\
&\quad\le\frac{1}{\lambda^2|W_n|^2}C\Big(|W_n|^2(F(v)-F(u))^{2/p}(F(w)-F(v))^{2/p}\\
&\qquad+|W_n|(F(v)-F(u))^{1/p}(F(w)-F(v))^{1/p}\Big)
\end{aligned}
$$

and this completes the verification of tightness.

The only step missing is to prove the moment inequality (12) for $q = 4$. Using Lemma 3 with $f = g = g_t$ and similar arguments as above we can show

$$\sup_{n\in\mathbb{N}} \mathbb{E}|\tilde{Y}_n(t)|^4 \le C_t < \infty$$

for any $t \in \mathbb{R}$. □

**Remark 2.** Theorem 2 remains true if we replace conditions on $\alpha$-mixing coefficients by analogous conditions on $\beta$-mixing coefficients which are slightly stronger but they are often easier to handle; see e.g. [11].

In [17] different estimation problem is considered if $\Phi$ is a stationary Poisson point process. For this case the assumptions on the unmarked point process can be relaxed.

**Corollary 4.** *Let $\Phi_m = \{(X_i, M(X_i))\}$ be a stationary marked point process with geostatistical marking such that the unmarked point process $\Phi = \{X_i\}$ is a stationary Poisson point process. Consider a measurable function $f : \mathbb{M} \to \mathbb{R}$ and put $Z(x) = f(M(x))$. Assume that (10) holds and*

$$\int_0^\infty s^{2d-2}\alpha_0^Z(s)^{1/2-\tau}\,\mathrm{d}s < \infty$$

*for some $0 < \tau < 1/2$. Let $\{W_n\}$ be a convex averaging sequence satisfying $|W_n| = O(\rho(W_n)^d)$. Then the empirical process defined by (8) converges weakly in $D(\mathbb{R})$ (as $n \to \infty$) to the Gaussian process $Y$ with zero mean and covariance function*

$$\mathbb{E}Y(s)Y(t) = F(s \wedge t) - F(s)F(t) + \lambda \int \mathbb{E}g_s(o)g_t(x)\,\mathrm{d}x,$$

*where $g_t(x) = \mathbf{1}\{Z(x) \le t\} - F(t)$.*

In Theorem 2 the underlying point process has to have first four reduced factorial cumulant measures with finite total variation. This is satisfied for any Brillinger mixing point process. Furthermore, we have put some strong mixing conditions on $\Phi$. A Poisson point process is the simplest example which fulfills both assumptions. A non-trivial example of feasible unmarked point processes is provided by a Poisson cluster point process with bounded clusters and finite moments of the number of points per cluster.

We have also imposed strong mixing conditions on the random field which marks the points. The dependence must decrease at a polynomial rate sufficiently fast. In particular, any $m$-dependent random field (i.e. observations separated by a distance larger than $m$ are independent) satisfies these conditions. As an example of random field with non-finite range of dependence we mention stationary Gaussian random field such that its correlation decays exponentially, see [8].

### *4.2. Germ-grain process*

In the definition of the empirical distribution function (4) it is important that we have knowledge about the marks from the observation of $\Phi_m$ in $W_n$. But this does not have to be always true. For example, in stochastic geometry when we observe a germ-grain process (mark space $\mathbb{M}$ is the space of compact sets) through a bounded window some of the grains may not be completely observable because of edge effects. Taking into account only completely observable grains leads to the spatial bias in the estimator (4). This sampling procedure is called minus sampling and an appropriate estimator of $F(t)$ is then a weighted estimator of Horvitz–Thompson type.

To be more specific, let $\Phi_m = \{(X_i, \Xi_i)\}$ be a stationary germ-grain process which consists of two independent components: a stationary point process $\Phi = \{X_i\}$ on $\mathbb{R}^d$ with intensity $\lambda$ and a sequence of random compact sets $\{\Xi_i\}$ with the reference point at the origin $o$. We are interested in the distribution of $f(\Xi_0)$ for some function $f$ describing parameters of grains and typical grain $\Xi_0$. The Horvitz–Thompson type estimator of the distribution function $F(t) = \mathbb{P}(f(\Xi_0) \le t)$ is given by

$$\hat{F}_n^{HT}(t) = \frac{1}{\hat{\lambda}_n} \sum_i \frac{\mathbf{1}\{X_i + \Xi_i \subseteq W_n\}}{|W_n \ominus \check{\Xi}_i|} \mathbf{1}\{f(\Xi_i) \le t\}, \quad t \in \mathbb{R},$$

where

$$\hat{\lambda}_n = \sum_i \frac{\mathbf{1}\{X_i + \Xi_i \subseteq W_n\}}{|W_n \ominus \check{\Xi}_i|}$$

and $W_n \ominus \check{\Xi}_i = \{x \in \mathbb{R}^d : x + \Xi_i \subseteq W_n\}$ is the window $W_n$ eroded by the grain $\Xi_i$. If $\Phi_m$ is a stationary ergodic germ-grain process then Glivenko–Cantelli theorem for $\hat{F}_n^{HT}$ was shown in [13] under the additional assumption on the size of the grains and the windows $\{W_n\}$,

$$\mathbb{E}\|\Xi_0\|^q < \infty \quad \text{for some } q \ge d \quad \text{and} \quad \frac{H^{d-1}(\partial W_n)}{|W_n|^{1-1/q}} \le c_0 < \infty. \tag{15}$$

We define the empirical process

$$Y_n^{HT}(t) = \sqrt{\Phi(W_n)} \left( \hat{F}_n^{HT}(t) - F(t) \right), \quad t \in \mathbb{R}. \tag{16}$$

In [13] weak convergence of $Y_n^{HT}$ was proved under the assumption that $\{\Xi_i\}$ is a sequence of independent identically distributed grains (i.e. $\Phi_m$ is an independently marked point process). With the help of Theorem 2 we can establish weak convergence for the case of geostatistical marking.

**Theorem 5.** *Let $\{\Xi(x), x \in \mathbb{R}^d\}$ be a stationary random field with values in the space of compact sets on $\mathbb{R}^d$ with the reference point at the origin $o$. Let $\Phi = \{X_i\}$ be a stationary point process on $\mathbb{R}^d$, independent of $\{\Xi(x), x \in \mathbb{R}^d\}$. Consider a germ-grain process $\Phi_m = \{(X_i, \Xi_i(X_i))\}$ and a measurable function $f$. Assume that $Z(x) = f(\Xi(x))$ satisfies (10) and*

$$\int_0^\infty s^{2d-2} \alpha_0^Z(s)^{1/2-\tau} \, ds < \infty$$

*for some $0 < \tau < 1/2$. Further, assume that (11) holds and that first four reduced factorial cumulant measures of the unmarked point process $\Phi = \{X_i\}$ are of bounded total variation, i.e. $|\gamma_{red}^{(2)}|(\mathbb{R}^d) < \infty$, $|\gamma_{red}^{(3)}|(\mathbb{R}^d \times \mathbb{R}^d) < \infty$, $|\gamma_{red}^{(4)}|(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d) < \infty$. Let $\{W_n\}$ be a convex averaging sequence such that $|W_n| = O(\rho(W_n)^d)$. Assume that the condition (15) is fulfilled. Then the empirical process defined by (16) converges weakly in $D(\mathbb{R})$ (as $n \to \infty$) to the Gaussian process $Y$ with zero mean and covariance function*

$$\mathbb{E}Y(s)Y(t) = F(s \wedge t) - F(s)F(t) + \int \mathbb{E}g_s(o)g_t(x)\, \gamma_{red}^{(2)}(dx) + \lambda \int \mathbb{E}g_s(o)g_t(x)\, dx,$$

*where $g_t(x) = \mathbf{1}\{Z(x) \le t\} - F(t)$.*

**Proof.** From Theorem 2 we have weak convergence of (14). Lemma 1 in [13] states that $\hat{\lambda}_n \underset{n\to\infty}{\longrightarrow} \lambda$ in probability. Hence, applying Slutsky type arguments, the weak limit (if it exists) of $Y_n^{HT}(t)$, $t \in \mathbb{R}$, coincides with that of

$$\tilde{Y}_n^{HT}(t) = \frac{\hat{\lambda}_n \sqrt{|W_n|}}{\sqrt{\lambda \Phi(W_n)}} Y_n^{HT}(t), \quad t \in \mathbb{R}.$$

The proof will be completed if we verify

$$\sup_{u\in[s,t]} |\tilde{Y}_n(u) - \tilde{Y}_n^{HT}(u)| \underset{n\to\infty}{\longrightarrow} 0$$

in probability for any $s < t$. In order to do this we can follow the proof of Theorem 2 in [13] and consider the difference process $\Delta_n(t) = \sqrt{\lambda}(\tilde{Y}_n(t) - \tilde{Y}_n^{HT}(t))$. Then all the arguments remain unchanged except of bounding the mixed fourth moment which can be accomplished in the same manner as in the proof of Theorem 2. □

## 5. Concluding remarks

We have considered marked point processes with geostatistical marking. The independence between marks and points can be formally tested; see [9] or [23]. Our model allows correlated marks. In order to obtain asymptotic results we have imposed strong mixing conditions. In practice, these conditions can be difficult to verify. However, they are satisfied for any process with finite dependence range. We have shown weak convergence of the corresponding empirical processes. The covariance function of the limiting Gaussian process reflects the dependence structure of the model. It restricts the applicability of the results. Kolmogorov–Smirnov test which is used for testing the goodness of fit in the independent marking case (see [13] or [21]) is not plausible.

For the ease of presentation, all the results were formulated for the estimation of one-dimensional distribution function. The case of multivariate distribution function can be treated in the same way as it is carried out in [13].

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