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# Interface fluctuations in the two-dimensional weakly asymmetric simple exclusion process

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We consider the two-dimensional weakly asymmetric simple exclusion process, where the asymmetry is along the $X$-axis. The generator for such a process can be written as $\varepsilon^{-2}L_0+\varepsilon^{-1}L_\alpha$, $\varepsilon>0$, where $L_0$ and $L_\alpha$ are the generators for the nearest neighbor symmetric simple exclusion and totally asymmetric simple exclusion, respectively. We prove propagation of chaos and convergence to Burgers equation with viscosity in the limit as $\varepsilon$ goes to zero. The density fluctuation field converges to a generalized Ornstein-Uhlenbeck process. The covariance kernel for a class of travelling wave solutions is consistent with a phase boundary which fluctuates according to a linear stochastic partial differential equation.

Infinite particle systems * simple exclusion * Burgers equation * Ornstein-Uhlenbeck process

## Introduction

The one-dimensional Burgers equation,

$$\frac{\partial\rho}{\partial t}+\frac{1}{2}\frac{\partial}{\partial r}(\rho(1-\rho))=\lambda\frac{\partial^2}{\partial r^2}\rho,$$

$0\leqslant\rho\leqslant 1$, $r\in\mathbb{Z}$, $t\geqslant 0$, $\lambda\geqslant 0$, has been studied in great detail [1-9]. This equation has been used to model a wide variety of phenomena. Of particular interest are the class of travelling wave solutions of this equation. In the last decade a great deal of work has been devoted to the understanding of the macroscopic structure of Burgers equation. The approach has been along the direction suggested by McKean [14], namely that of using stochastic microscopic description. While there exists a number of works [3, 20] where interacting Brownian motions are used as a model at the microscopic level, we focus our attention in this paper on interacting infinite particle systems, particularly the exclusion process. A great deal is known about the hydrodynamic behaviour of asymmetric simple exclusion. In particular, the microscopic stability of the shock has been intensively studied in the last few years [1, 2, 7, 10, 24]. The recent work suggests that at the microscopic level the shock

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fluctuates rigidly, maintaining its shape, the translations being described by standard Brownian motion. While this has not been proved, there is considerable evidence to support this conjecture. Recent work of Ferrari [8] shows that the fluctuations of the shock come from the initial condition and not from the dynamics, and gives a precise characterization of the relation.

In this paper we consider the two dimensional Burgers equation. Very little seems to be known about this model at the microscopic level. There are the recent results of Landim [12] proving the convergence of two-dimensional asymmetric simple exclusion to Burgers equation in the hydrodynamic limit. We consider the two-dimensional weakly asymmetric simple exclusion, the asymmetry being along the $X$-axis. We prove propagation of chaos and convergence in the kinetic limit to Burgers equation with viscosity. We study the density fluctuation field and show that it converges to a mean zero generalized Ornstein–Uhlenbeck process. Then we consider the travelling wave solution of the Burgers equation obtained from the one-dimensional travelling wave solution by making the density uniform along the $Y$-direction. We show that the covariance of the fluctuation field is consistent with the following picture for the evolution of the boundary of the travelling wave solution (phase boundary). Let $h^\varepsilon(y, t)$ be the height of fluctuation (in the $X$-direction) of the boundary at time $t$ and level $y$. Then

$$\frac{\partial h^\varepsilon}{\partial t} = \Delta h^\varepsilon + \varepsilon D Z_{y,t},$$

where $Z_{y,t}$ is the space–time white noise. A discrete model at the microscopic level is

$$\mathrm{d}h^\varepsilon(y, t) = \varepsilon^{-2}\Delta_{\mathrm{d}} h^\varepsilon\,\mathrm{d}t + (\varepsilon D)\,\mathrm{d}B_t(y),$$

where $y \in \mathbb{Z}$, $\Delta_{\mathrm{d}}$ is the discrete Laplacian and $(B_t(y))_{y\in\mathbb{Z}}$ are a countable collection of independent Brownian motions. $D$ is a constant determined by the shape of the travelling wave solution.

We point out that models of the type described above appear in the physics literature [22]. Of course, the goal in such a context usually is to start from the model and obtain some information about the phenomenon. We have considered the problem of obtaining the model starting from an appropriate stochastic microscopic description. The linearity of the equation we have obtained originates from the assumption (of the model) that the asymmetry is always along a fixed direction ($X$) at every point on the phase boundary. A more general situation would be where the boundary grows along a direction normal to the boundary at every point on the boundary. In this case one would expect to obtain a nonlinear equation.

We use the correlation function technique [5] to obtain our results. This model was studied in one dimension by DeMasi, Presutti and Scacciatelli [4]. While we follow the approach in their paper there are particular technical problems which arise because of the two-dimensional nature of the problem. One of the important estimates in the correlation function technique involves the time integral of the probability of two random walks being nearest neighbors (in $\mathbb{Z}^2$). Since this probability goes like $1/t$ in two dimensions, one has to deal with logarithmic divergences, which are not present in one dimension. We note that this problem appears even in the study of symmetric simple exclusion in dimension greater than one and was solved using elementary methods in a recent paper [17].

## 1. Description of the model and results

The symmetric simple exclusion process on $\mathbb{Z}^2$ is a Markov process whose state space is the set of configurations of particles in $\mathbb{Z}^2$, i.e., $\{0,1\}^{\mathbb{Z}^2}$. (0 denotes an empty site while 1 denotes an occupied site.) A particle at a site attempts to jump to one of its nearest neighbor sites with probability $\frac{1}{4}$ after an exponential waiting time of mean one. The jump does not take place if the site it wants to jump to is occupied. Let $\eta \in \{0,1\}^{\mathbb{Z}^2}$ be a configuration (an element of the state space). We denote by $\eta(x)$ the coordinate projection of $\eta$ onto site $x$, i.e., $\eta(x)$ is the occupation number of the site $x \in \mathbb{Z}^2$ for the configuration $\eta$. It is well known that the symmetric simple exclusion process can be constructed with the generator $L_0$, where the action of $L_0$ on cylinder functions is defined as follows [13].

$$\begin{aligned} L_0 &= \frac{1}{4} \sum_{x \in \mathbb{Z}^2} \sum_{|x-y|=1} (f(\eta^{(x,y)}) - f(\eta))(\eta(x)(1-\eta(y))) \\ &= \frac{1}{8} \sum_{x \in \mathbb{Z}^2} \sum_{|x-y|=1} (f(\eta^{(x,y)}) - f(\eta)) \\ &= \frac{1}{4} \sum_{x \in \mathbb{Z}^2} \sum_{\alpha=1}^{2} (f(\eta^{(x,x+\varepsilon_\alpha)}) - f(\eta)), \end{aligned}$$

where

$$\eta^{(x,y)}(z) = \eta(z) \quad \text{if } z \neq x, \qquad \eta^{(x,y)}(x) = \eta(y), \qquad \eta^{(x,y)}(y) = \eta(x),$$

$$e_1 = (1,0) \quad \text{and} \quad e_2 = (0,1).$$

We denote by $L_\alpha$ the generator of the totally asymmetric simple exclusion process on $\mathbb{Z}^2$. This is the process where a particle in a configuration attempts to jump to the site one unit to its right. The jump takes place if the site is not occupied. The generator $L_\alpha$ is defined on cylinder functions as follows:

$$L_\alpha f(\eta) = \sum_{z \in \mathbb{Z}^2} (f(\eta^{(x,x+e_1)}) - f(\eta))(\eta(x)(1-\eta(x+e_1))).$$

By weakly asymmetric simple exclusion, we mean the family of processes with generators $L_\varepsilon = L_0 + \varepsilon L_\alpha$, $\varepsilon > 0$. We are interested in studying the process obtained when $\varepsilon \to 0$ and space is scaled like $\varepsilon^{-1}$ and the time is scaled like $\varepsilon^{-2}$. The typical displacement of a particle under $L_0$ (root mean square displacement) in a time $\varepsilon^{-2}$ is of the order $\varepsilon^{-1}$, while under $L_\alpha$, whch is a pure drift, the typical displacement in the same time is of the order of $\varepsilon^{-2}$. If we did not weaken $L_\alpha$ by multiplying it by $\varepsilon$, the drift would totally dominate the diffusion as $\varepsilon \to 0$. Since we are interested in studying the microscopic structure of Burgers equation with viscosity, we want both the drift and diffusion terms to survive in the limit. Therefore we have weakened the drift by multiplying it by $\varepsilon$, i.e., scaled down the rate of jumps by $\varepsilon$.

**Theorem 1.** *Let $\rho_0$ be a bounded $C^\infty$ function on $\mathbb{R}^2$ with bounded derivatives and values in $[0,1]$. For $\varepsilon > 0$, let $\mu^\varepsilon$ be the product measure on $\{0,1\}^{\mathbb{Z}^2}$ such that $\mu^\varepsilon(\eta \mid \eta(x)=1) = \rho_0(\varepsilon x)$. Then for $r \in \mathbb{R}^2$ and $t > 0$ uniformly in compacts of $\mathbb{R}^2 \times \mathbb{R}_+$,*

$$\lim_{\varepsilon \to 0} |E^\varepsilon_{\mu^\varepsilon}(\eta[\varepsilon^{-1} r], \varepsilon^{-2} t) - \rho(r,t)| = 0,$$

*where $\rho(r,t)$ is the solution of the Burgers equation*

$$\frac{\partial \rho}{\partial t} = \tfrac{1}{4}\Delta \rho - \frac{\partial}{\partial x}\rho(1-\rho), \tag{1.1}$$

*and $\rho(r,0) = \rho_0(r)$. $[x]$ denotes the integral part of $x$.*

This theorem states that the average value of particle density at $[\varepsilon^{-1} r]$ at time $\varepsilon^{-2} t$, with respect to the measure $\mu^\varepsilon P^\varepsilon_t$ (where $P^\varepsilon_t$ is the transition operator for the weakly asymmetric simple exclusion, WASEP), is well approximated by $\rho(r,t)$, the solution to Burgers equation with initial condition $\rho_0$. We indicate why this is a reasonble expectation by the following heuristic argument.

First note that if $f(\eta) = \eta(x)$, then

$$L_\varepsilon \eta(x) = \frac{1}{4} \sum_{|x-y|=1} (\eta(y) - \eta(x)) \\ + \varepsilon[\eta(x-e_1)(1-\eta(x)) - \eta(x)(1-\eta(x+e_1))].$$

If we assume the pair correlation function of the density factors (mean field) and denote $E^\varepsilon_{\mu^\varepsilon}(\eta(x,t))$ by $\bar{\rho}_\varepsilon(x,t)$, we obtain

$$\frac{\partial \bar{\rho}_\varepsilon}{\partial t}(x,t) = \frac{1}{4} \sum_{|x-y|=1} (\bar{\rho}_\varepsilon(y,t) - \bar{\rho}_\varepsilon(x,t)) \\ -\varepsilon[\bar{\rho}_\varepsilon(x,t)(1-\bar{\rho}_\varepsilon(x+e_1,t)) - \bar{\rho}_\varepsilon(x-e_1,t)(1-\bar{\rho}_\varepsilon(x,t))]. \tag{1.2}$$

It is easy to see that (1.2) is a discretization of the rescaled Burgers equation where the scaling is $\bar{\rho}_\varepsilon(x,t) = \rho(\varepsilon x, \varepsilon^2 t)$. Therefore we see that with the assumption that the correlation functions for density factor, the forward equation for density in the WASEP under diffusive scaling gives a discretized Burgers equation. This suggests that WASEP is an appropriate model for the Burgers equation with viscosity and that one might be able to make the heuristic arguments rigorous, if one is able to show that the discrepancy in the factoring property goes to zero fast enough as $\varepsilon$ goes to zero. This turns out to be a feasible strategy and motivates the next proposition. We need a few definitions before stating the proposition.

Let $\rho_\varepsilon(x, t)$ be the solution of the following integral equation with $\rho_\varepsilon(x, 0) = \rho_0(\varepsilon x)$, $\rho_\varepsilon(x, t) = \sum_z \bar{P}_t(x \to z)\rho_\varepsilon(z, 0)$.

$$-\varepsilon \int_0^t \mathrm{d}s \sum_z \bar{P}_{t-s}(x \to z)[\rho_\varepsilon(z, s)(1-\rho_\varepsilon(z+e_1)) - \rho_\varepsilon(z-e_1, s)(1-\rho_\varepsilon(z, s))], \tag{1.3}$$

$\bar{P}_t(x \to z)$ denotes the transition kernel for the symmetric random walk in $\mathbb{Z}^2$.

**Definition.** For any $n \geq 1$, let $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$, $x_i \in \mathbb{Z}^2$, be an $n$-tuple of distinct points in $\mathbb{Z}^2$.

$$V_n^\varepsilon(\underset{\sim}{x}, t) = E_{\mu^\varepsilon}^\varepsilon\left(\prod_{i=1}^n \eta(x_i, t) - \rho_\varepsilon(x_i, t)\right),$$

where $E_{\mu^\varepsilon}^\varepsilon$ denotes expectation with respect to WASEP with parameter $\varepsilon$.

**Proposition.** *Let $T > 0$, $\gamma > 0$. Then for each $n \geq 3$, there is $C_n$ such that*

$$\sup_{\underset{\sim}{x}} |V_n^\varepsilon(\underset{\sim}{x}, t)| < C_n \varepsilon^{n-\gamma}, \quad \forall t \leq \varepsilon^{-2}T,\ \forall \varepsilon > 0, \tag{1.4}$$

*and for $n = 1, 2$, there exists $a > 0$ such that*

$$|V_n^\varepsilon(\underset{\sim}{x}, t)| < a\varepsilon^2 g(x_1, x_2, t), \quad \forall t \leq \varepsilon^{-1}T,\ \forall \varepsilon > 0, \tag{1.5}$$

*where $g(x_1, x_2, t) = 1 + \int_0^t \sum_z P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z}) 1\{|z_1 - z_2| = 1\}\, \mathrm{d}s$ and $P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z})$ is the transition probability kernel for two symmetric simple exclusion particles.*

Let $Y_t^\varepsilon(\phi) = \varepsilon^2 \sum_x \phi(\varepsilon x)\eta(x, \varepsilon^{-2}t)$, where $\phi \in S(\mathbb{R}^2)$, be the density field for the WASEP. It is easy to establish that $\lim_{\varepsilon \to 0} Y_t^\varepsilon(\phi) = \int_{\mathbb{R}^2} \phi(x)\rho(x, t)\, \mathrm{d}x$, $\forall t \in [0, T]$, $T < \infty$. Given this law of large numbers, we next define the fluctuation field and state a central limit theorem. The fluctuation field $X_t^\varepsilon(\phi)$ for $\phi \in S(\mathbb{R}^2)$ is defined as

$$X_t^\varepsilon(\phi) = \varepsilon \sum_x \phi(\varepsilon x)(\eta(x, \varepsilon^{-2}t) - E_{\mu^\varepsilon}(\eta(x, \varepsilon^{-2}t))). \tag{1.6}$$

$S(\mathbb{R}^2)$ denotes the space of $C^\infty$ functions which, along with their derivatives, decay exponentially fast at infinity.

Given an initial measure $\mu^\varepsilon$, we have thus defined a random linear functional on $S(\mathbb{R}^2)$ at each time $t$. Let $\mathcal{P}^\varepsilon$ be the law of this process with paths in $D(\mathbb{R}_+ \to S'(\mathbb{R}^2))$, right-continuous left-limited paths taking values in the space of linear functionals on $S(\mathbb{R}^2)$.

**Theorem 2.** *The law of $\mathcal{P}^\varepsilon$ converges weakly to $\mathcal{P}$, the law of a mean zero generalized Ornstein–Uhlenbeck process (O–U process). $\mathcal{P}$ solves the following Martingale problem. For any $\phi \in S(\mathbb{R}^2)$ and $F \in C^\infty(S(\mathbb{R}^2))$,*

$$F(X_t(\phi)) - \int_0^t \mathrm{d}s\, F'(X_s(\phi))X_s(A_s\phi) - \int_0^t \mathrm{d}s\, \tfrac{1}{2}\|B_s\phi\|^2 F''(X_s(\phi)) \tag{1.7}$$

*is a $\mathcal{P}$ Martingale on $C^{\infty}(\mathbb{R}\to S'(\mathbb{R}^2))$. Operators $A_s$ and $B_s$ are defined as follows,*

$$(A_s\phi)(r)=\tfrac{1}{4}\Delta\phi(r)+(1-2\rho(r,s))\frac{\partial}{\partial x}\phi(r), \tag{1.8}$$

*where $\rho$ is the solution to Burgers equation, and*

$$\|B_s\phi\|^2=\tfrac{1}{2}\int_{\mathbb{R}^2}\rho(r,s)(1-\rho(r,s))\left(\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right)\mathrm{d}r. \tag{1.9}$$

**Remark.** We note that if we considered the density field $Y_t^{\varepsilon}(\phi)$, it would be easy to check the Martingale condition stated in Theorem 2, with $A_s=\frac{1}{4}\Delta-(\partial/\partial x)(\rho_s(1-\rho_s))$ and $B_s=0$, thus proving the weak convergence of the density field to the deterministic functional given by $\int\rho(\cdot,t)$.

It is well known that there exists travelling wave solutions to the Burgers equation in one dimension [21]. If we consider an initial profile which is in the shape of a travelling wave solution to the one-dimensional Burgers equation along the $X$-direction and uniform along the $Y$-direction, then it is easy to see that this will be a travelling wave solution to the two-dimensional Burgers equation. Let us consider such a solution $\rho((x,y),t)=\bar{\rho}(x-ct)$, where $c$ is the velocity of the travelling wave. Specifically, let $\bar{\rho}$ be monotonically increasing in $x$ with $\lim_{x\to\pm\infty}\bar{\rho}(x)=\rho_{\pm}$, and $\bar{\rho}_0(0)=\frac{1}{2}(\rho_{+}+\rho_{-})$. The microscopic stability of this profile in one dimension was considered in [4]. There it was shown that the covariance of the fluctuation field is consistent with the following picture. At time $t$ define a (random) local equilibrium profile $\rho_{\mathbb{R}}^{\varepsilon}(r,t)=\bar{\rho}(r-ct+B_t^{\varepsilon})$, where $B_t^{\varepsilon}$ is a Brownian motion with a diffusion coefficient $D\varepsilon$. The vaue of $D$ depends on the shape of $\rho$. Let $\phi_t(r)=\phi(r+ct)$ and $E_{B_t}$ the expectation with respect to Brownian motion. Let $\mu_{\mathbb{R},\varepsilon,t}$ be the random product measure defined by $\rho_{\mathbb{R}}^{\varepsilon}(r,t)$, i.e., $\mu_{\mathbb{R},\varepsilon,t}(\eta\,|\,\eta(x)=1)=\rho_{\mathbb{R}}^{\varepsilon}(\varepsilon x,t)$ for all $x$ in $\mathbb{Z}$. It was proved in [4] that

$$\lim_{t\to 0}\frac{1}{t}\lim_{\varepsilon\to 0}E_{B_t}E_{\mu_{\mathbb{R},\varepsilon,t}}(X_0^{\varepsilon}(\phi)X_0^{\varepsilon}(\psi))=\lim_{t\to 0}\frac{1}{t}\lim_{\varepsilon\to 0}E_{\mu^{\varepsilon}}(X_t^{\varepsilon}(\phi_t)X_t^{\varepsilon}(\psi_t)).$$

This supports the conjecture that even at the microscopic level, the profile of the travelling wave solution is stable except for rigid translations.

In two dimensions, the picture for which we argue is a bit more complicated. Here too the profile seems to be shifted rigidly at the microscopic level. In this model, in addition to the drift along the $X$-direction, there is also diffusion along the $Y$-direction. The picture which emerges is the following. Microscopically, the profile is shifted along the $X$-direction randomly. If we denote the shift for given values of $y$ and $t$ and scaling parameter $\varepsilon$ by $h^{\varepsilon}(y,t)$, then

$$\frac{\partial h^{\varepsilon}}{\partial t}(y,t)=\Delta h^{\varepsilon}(y,t)+\varepsilon DZ_{yt}, \tag{1.10}$$

where $Z_{yt}$ is the space–time white noise [22] and $D>0$ is determined by $\bar{\rho}$. Equation (1.10) may be thought of as a model of fluctuations of a phase boundary. We can think of the straight line $l(t)=(ct, y)$ as an approximate macroscopic phase boundary, separating the occupied sites (to the right) from empty ones (to the left). The boundary would be sharp if the profile $\bar{\rho}$ was a step function instead of being smooth. At time $t$, $l(t)$ describes the macroscopic average position of the corresponding microscopic boundary. The microscopic fluctuations around this line can be obtained from the fluctuation field of the WASEP. We now describe a different model for the fluctuations of the phase boundary which will be shown to be consistent with the WASEP at the level of covariance. Let $l^\varepsilon(t)=(ct+h^\varepsilon(y,t), y)$, where $h^\varepsilon$ is defined by equation (1.10). $l^\varepsilon(t)$ is the position of the microscopic phase boundary in a shifted profile $\rho^\varepsilon_{\mathbb{R},t}(x,y)=\bar{\rho}(x-ct+h^\varepsilon(y,t))$. Expected value of $l^\varepsilon(t)$ with respect to $h^\varepsilon(t)$ is $l(t)$. We show that the microscopic fluctuations of WASEP are asymptotically equal to the microscopic fluctuations produced by the random shifting of the macroscopic phase boundary described above.

Let $\mu^\varepsilon_{\mathbb{R},t}$ be the random measure on configurations in $\{0,1\}^{\mathbb{Z}^2}$, given by the random profile $\rho^\varepsilon_{\mathbb{R},t}(x,y)=\bar{\rho}(x-ct+h^\varepsilon(y,t))$. That is, $\mu^\varepsilon_{\mathbb{R},t}=\prod \nu^\varepsilon_{(x,y),t}$, where $\nu^\varepsilon_{(x,y),t}(\eta(x,y)\,|\,\eta(x,y)=1)=\rho^\varepsilon_{\mathbb{R},t}(\varepsilon x,\varepsilon y)$. Then

$$\lim_{t\to 0}\frac{1}{\sqrt{t}}\lim_{\varepsilon\to 0} E_{h^\varepsilon}E_{\mu_{\mathbb{R},\varepsilon,t}}(X^\varepsilon_0(\phi))(X^\varepsilon_0(\psi))=\lim_{t\to 0}\frac{1}{\sqrt{t}}\lim_{\varepsilon\to 0} E_{\mu^\varepsilon}(X^\varepsilon_t(\phi_t))(X^\varepsilon_t(\psi_t)),$$

for all $\phi,\psi\in S(\mathbb{R}^2)$, where $E_{h^\varepsilon}$ is expectation with respect to $h^\varepsilon$.

## 2. The proofs

**Proof of the proposition.** We proceed along the same line as in the one-dimensional case [4]. We obtain an integral equation for $V_n$. (We omit the superscript $\varepsilon$ to simplify notation.) This equation, as in the one-dimensional case, involves integrals of $V_n$, $V_{n-1}$, $V_{n-2}$, and $V_{n+1}$. That is, we obtain an infinite hierarchy of equations. To terminate the hierarchy, we need to control $V_n$ when $n$ is large. This is done by obtaining a bound of the type $\sup_{\underline{x}}|V_n(\underline{x},t)|<c_n\varepsilon^{n\delta}$, $\forall t\leqslant\varepsilon^{-2}T$, for some $\delta>0$. This was done in one dimension by first establishing this bound for a microscopic time $(t\leqslant\varepsilon^{-2+\gamma}T,\ \gamma>0)$ and then iterating to extend it to macroscopic times $(\varepsilon^{-2}T)$. There is no essential difference in this part of the argument for the two-dimensional case. Therefore, we assume the existence of such an a priori bound and proceed with the proof of the proposition.

The integral equation satisfied by $V_n(\underline{x},t)$, $n\geqslant 2$, can be written as follows. A detailed derivation of the integral equation can be found in [18].

$$V_n(\underline{x},t)=\int_0^t\left[\sum_x P_{t-s}(\underline{x}\to\underline{z})(Q_1(\underline{z},s)+Q_2(\underline{z},s))+\varepsilon R(\underline{x},\underline{z},t-s)\right]\mathrm{d}s, \tag{2.1}$$

where

$$Q_1(\underset{\sim}{z}, t) = \varepsilon\left[\sum_i 1\{x_i + e_1 \neq x_j, \forall j \neq i\} V_{n+1}(\underset{\sim}{x}, x_i + e_1, t)\right.$$

$$\left. - \sum_i 1\{x_i - e_1 \neq x_j, \forall j \neq i\} V_{n+1}(\underset{\sim}{x}, x_i - e_1, t)\right]$$

$$+ \sum_{i,j} 1\{x_j = x_i + e_1\}[(\rho_\varepsilon(x_i, t) - \rho_\varepsilon(x_j, t)) + (1 - \rho_\varepsilon(x_j, t))] V_n(\underset{\sim}{x}, t)$$

$$+ \varepsilon \sum_{i,j} 1\{x_j = x_i - e_i\}\rho_\varepsilon(x_i - e_1, t)(V_n(\underset{\sim}{x}, t)), \tag{2.2}$$

$$Q_2(\underset{\sim}{x}, t) = \sum_{\alpha=1}^{2} \frac{1}{4} \sum_{i,j} 1\{x_j = x_i + \varepsilon_\alpha\}$$

$$\times [2(\rho_\varepsilon(x_i, t) - \rho_\varepsilon(x_j, t))(V_{n-1}(\underset{\sim}{x}(i), t) - V_{n-1}(\underset{\sim}{x}(j), t))]$$

$$+ \varepsilon\left[\sum_{i,j} 1\{x_j = x_i + e_1\}\right.$$

$$\times (\rho_\varepsilon(x_i, t)(1 - \rho_\varepsilon(x_j, t)) V_{n-1}(\underset{\sim}{x}(i), t) - V_{n-1}(\underset{\sim}{x}(j), t))$$

$$+ \rho_\varepsilon(x_j, t) - \rho_\varepsilon(x_i, t)(1 - \rho_\varepsilon(x_j, t)) V_{n-1}(\underset{\sim}{x}(j), t)$$

$$\left. - \rho_\varepsilon(x_i, t) V_{n-1}(\underset{\sim}{x}(i), t))\right]$$

$$+ \varepsilon \sum_{i,j} 1\{x_j = x_i + e_1\}(\rho_\varepsilon(x_i, t) - \rho_\varepsilon(x_j, t))$$

$$\times \rho_\varepsilon(x_i, t)(\rho_\varepsilon(x_j, t) - 1) V_{n-2}(\underset{\sim}{x}(i, j), t)$$

$$- \sum_{\alpha=1}^{2} \frac{1}{4} \sum_{i,j} 1\{x_j = x_i + e_\alpha\}(\rho_\varepsilon(x_i, t) - \rho_\varepsilon(x_j, t))^2 V_{n-2}(\underset{\sim}{x}(i, j), t), \tag{2.3}$$

$$R(\underset{\sim}{x}, \underset{\sim}{z}, t - s) = \sum_i (P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z} + e_{i,1}) - P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z}))(1 - \rho_\varepsilon(z_i, s)) V_n(\underset{\sim}{z}, s)$$

$$+ (P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z} - e_{i,1}) - P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z}))\rho_\varepsilon(z_i, s) V_n(\underset{\sim}{z}, s)$$

$$+ P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z} + e_{i,1})(\rho_\varepsilon(z_i, s) - \rho_\varepsilon(z_i + e_1, s)) V_n(\underset{\sim}{z}, s)$$

$$+ P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z} - e_{i,1})(\rho_\varepsilon(z_i - e_1, s) - \rho_\varepsilon(z_i, s)) V_n(\underset{\sim}{z}, s)$$

$$+ P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z})(\rho_\varepsilon(z_i + e_1, s) - \rho_\varepsilon(z_i - e_1, s)) V_n(\underset{\sim}{z}, s). \tag{2.4}$$

$P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z})$ is the semi-group for the symmetric simple exclusion process, starting from a configuration with $n$ particles. We have identified a configuration with the set of occupied sites $(\underset{\sim}{x})$. $\sum_{i,j}$ indicates sum over all distinct ordered pairs. $\underset{\sim}{x}(i, j) = \underset{\sim}{x} \backslash \{x_i, x_j\}$, $\underset{\sim}{x}(i) = \underset{\sim}{x} \backslash \{x_i\}$.

Before proceeding with the proof of the proposition, we note that the above integral equation applies only for $n \geqslant 2$. It is easy to write down an integral equation for $V_1$, which we do now.

$$\frac{\mathrm{d}V_1}{\mathrm{d}t}=\left[\frac{1}{4}\sum_{|x-y|=1}(V_1(y,t)-V_1(x,t))\right]$$
$$+\varepsilon[(V_1(x+e_1,t)-V_1(x,t))+(V_2(x+e_1,x,t)-V_2(x,x-e_1,t))],$$
$$V_1(x,t)=\varepsilon\int_0^t\sum_z P_{t-s}(\underset{\sim}{x}\to\underset{\sim}{z})[(V_1(z+e_1,s)-V_1(z,s))$$
$$+(V_2(z+e_1,z,s)-V_2(z,z-e_1,s))]\,\mathrm{d}s$$
$$=\varepsilon\int_0^t\sum_z(P_{t-s}(\underset{\sim}{x}\to\underset{\sim}{z}-e_1)-P_{t-s}(\underset{\sim}{x}\to\underset{\sim}{z}))$$
$$\times(V_1(z,s)+V_2(z,z-e_1,s))\,\mathrm{d}s.$$

$P_t(\underset{\sim}{x}\to\underset{\sim}{z})$ in the above expression is the transition kernel for a symmetric simple random walk in $\mathbb{Z}^2$. By coupling two random walks, one starting from $x$ and the other starting from $x+e_1$, in such a way that they move independently until they meet and together afterwards, one can easily show that [5]

$$\sum_{\underset{\sim}{z}}|P_{t-s}(\underset{\sim}{x}\to\underset{\sim}{z}-e_1)-P_{t-s}(\underset{\sim}{x}\to\underset{\sim}{z})|<\frac{C}{\sqrt{t-s}},$$

where $C$ is independent of $t-s$.

Let $a(\varepsilon,k,u)=\sup_{\underset{\sim}{x},s\le u}|V_k(\underset{\sim}{x},s)|$. We now show that, given $\tau>0$, there is $M(\tau)$ such that

$$a(\varepsilon,1,t)<M(\tau)a(\varepsilon,2,u),\quad \forall t\le\varepsilon^{-2}\tau,\ \varepsilon>0. \tag{2.5}$$

Let $m\in N$ be such that $C\sqrt{\tau/m}\le\frac{1}{4}$. Divide the interval $[0,\varepsilon^{-2}\tau]$ into $m$ equal parts, that is, sub-intervals of the form $[\tau_k,\tau_{k+1}]$, where $\tau_k=(k/m)\varepsilon^{-2}\tau, 0\le k\le m-1$. Let $t\in[\tau_k,\tau_{k+1}]$.

$$V_1(x,t)\le a(\varepsilon,1,\tau_k)+\varepsilon\int_{\tau_k}^{\tau_{k+1}}\sum_z|P_{\tau_{k+1}-s}(x\to z-e_1)-P_{\tau_{k+1}-s}(x\to z)|$$
$$\times(a(\varepsilon,1,\tau_{k+1})+a(\varepsilon,2,\tau_{k+1}))\,\mathrm{d}s$$
$$<a(\varepsilon,1,\tau_k)+\tfrac{1}{2}(a(\varepsilon,1,\tau_{k+1})+a(\varepsilon,2,\tau_{k+1})).$$

Therefore,

$$a(\varepsilon,1,\tau_{k+1})\le 2a(\varepsilon,1,\tau_k)+a(\varepsilon,2,\tau_{k+1})\le 2a(\varepsilon,1,\tau_k)+a(\varepsilon,2,\tau_m).$$

From this, by iteration, it is easy to see that there is a constant $M(\tau)$ with $a(\varepsilon,1,\tau_m)<M(\tau)a(\varepsilon,2,\tau_m)$. Before proceeding with the iteration of (2.1), we rewrite the first two terms in $\sum_z P_{t-s}(\underset{\sim}{x}\to\underset{\sim}{z})Q_1(\underset{\sim}{z},s)$ by summation by parts. The idea is to obtain in place of the gradient of $V_{n+1}$ with the indicator functions, a gradient of the transition function and an indicator function for two exclusion particles to be nearest neighbors.

Consider

$$\sum_i\sum_z P_{t-s}(\underset{\sim}{x}\to\underset{\sim}{z})1\{z_i+e_1\ne z_j,\forall j\ne i\}V_{n+1}(\underset{\sim}{z},z_i+e_1,s).$$

Let $\underset{\sim}{z}' = \underset{\sim}{z} + e_{i,1}$. Then we can rewrite the above as

$$\sum_i \sum_z P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z}' - e_{i,1}) 1\{z_1' \neq z_j', \forall j \neq i\} V_{n+1}(\underset{\sim}{z}', z_i' - e_1, s).$$

Since there is a one-to-one correspondence between the set of $\underset{\sim}{z}'$'s and $\underset{\sim}{z}$'s, we can replace the sum over $\underset{\sim}{z}$ by the sum over $\underset{\sim}{z}'$. Thus we obtain

$$\sum_i \sum_z P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z})(1\{z_i + e_1 \neq z_j, \forall j \neq i\} V_{n+1}(\underset{\sim}{z}, z_i + e_1, s)$$

$$- 1\{z_i - e_1 \neq z_j, \forall j \neq i\} V_{n+1}(\underset{\sim}{z}, z_i - e_1, s))$$

$$= \sum_i \left( \sum_z \left[ (P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z} - e_{i,1}) - P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z})) 1\{z_i \neq z_j, \forall j \neq i\} \right.\right.$$

$$\left.\left. + \sum_{j \neq i} P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z}) 1\{z_i - e_1 = z_j\} \right] V_{n+1}(\underset{\sim}{z}, z_i - e_1, s) \right). \tag{2.6}$$

We use a set of inequalities in our argument. Hereafter $C$ is a constant whose value changes from line to line.

$$\sup_{\underset{\sim}{x}} |\rho_\varepsilon(\underset{\sim}{x} + e_\alpha, s) - \rho_\varepsilon(\underset{\sim}{x}, s)| < C\varepsilon, \quad \alpha = 1, 2, \tag{2.7}$$

where $C$ is a constant independent of $\varepsilon$ and $s$. This follows from the equation for $\rho_\varepsilon$ and the smoothness of $\mu_\varepsilon$ [5].

$$\sum_i P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z}) 1\{|z_i - z_j| = 1\} < \frac{C}{t - s + 1}, \tag{2.8}$$

$$\sum_z |P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z}) - P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z} + e_{i,\alpha})| < \frac{C}{\sqrt{t-s}}. \tag{2.9}$$

(2.8) and (2.9) can be proved using the two-dimensional version of the arguments given in [5]. We also note that $\rho_\varepsilon(\underset{\sim}{x}, s)$ is uniformly bounded in $\varepsilon$, $\underset{\sim}{x}$, and $s$. From (2.6) we get

$$\int_0^t \left| \sum_i \left( \sum_z P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z} - e_{i,1}) - P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z}) 1\{z_i - e_1 \neq z_j, \forall j \neq i\} \right.\right.$$

$$\left.\left. + \sum_{j \neq i} P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z}) 1\{z_i - e_1 = z_j\} \right) V_{n+1}(\underset{\sim}{z}, z_i - e_1, s) \right| \mathrm{d}s$$

$$\leqslant \int_0^t \sum_i \left( \sum_z |(P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z} - e_{i,1}) - P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z})) 1\{z_i - e_1 \neq z_j, \forall j \neq i\}| \right.$$

$$\left. + \sum_{j \neq i} P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z}) 1\{z_i - e_1 = z_j\} \right) a(\varepsilon, n+1, s)\, \mathrm{d}s$$

$$\leqslant \int_0^t \sum_i \left( \sum_z |P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z} - e_{i,1}) - P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z})| \right.$$

$$\left. + 2 \sum_{j \neq i} P_{t-s}(\underset{\sim}{x} \to \underset{\sim}{z}) 1\{z_i = e_1 = z_j\} \right) a(\varepsilon, n+1, s)\, \mathrm{d}s.$$

We prove equations (1.4) by induction. Let $0<\gamma<1$ be the number specified in the proposition. Let $K\in\mathbb{N}$ and $k\geqslant 2$.

*Claim 1.* If there exist positive real numbers $\{C_{K,n}\}_{n\geqslant K}$, $\{C_{n,n}\}_{0<n<K}$, and $\alpha_n>n-\gamma$, $1\leqslant n\leqslant K$ such that

$$a(\varepsilon,n,T)<\begin{cases}C_{K,n}\varepsilon^{\alpha_K}, & n\geqslant K,\\ C_{n,n}\varepsilon^{\alpha_n}, & n<K,\end{cases}$$

$\forall\varepsilon>0$, then there exist positive real numbers $\{C_{K+1,n}\}_{n\geqslant K+1}$ and $\alpha_{K+1}>(K+1)-\gamma$ such that

$$a(\varepsilon,n,T)<\begin{cases}C_{K+1,n}\varepsilon^{\alpha_{K+1}}, & n\geqslant K+1,\\ C_{n,n}\varepsilon^{\alpha_n}, & n<K+1,\end{cases}$$

$\forall\varepsilon>0$.

*Proof of Claim 1.* Let $0<u<t$, then it follows from (2, 3) that

$$\begin{aligned}|V_n(\underline{x},t)|\leqslant{}& a(\varepsilon,n,u)\\ &+c\int_u^t\Bigg[\frac{\varepsilon n}{\sqrt{t-s}}a(\varepsilon,n+1,s)\\ &\quad+\frac{2\varepsilon n(n-1)}{t-s+1}a(\varepsilon,n+1,s)+\frac{\varepsilon n}{t-s+1}a(\varepsilon,n,s)\\ &\quad+\frac{\varepsilon n}{t-s+1}(\varepsilon a(\varepsilon,n-1,s)+\varepsilon a(\varepsilon,n-2,s)+a(\varepsilon,n-1,s))\\ &\quad+\frac{\varepsilon n}{\sqrt{t-s}}a(\varepsilon,n,s)+n\varepsilon^2a(\varepsilon,n,s)\Bigg]\,\mathrm{d}s.\end{aligned}\tag{2.10}$$

Let $T=\varepsilon^{-2}\tau$. Let $m\in\mathbb{N}$ be such that $\max((n\sqrt{\tau/m})c,n\tau c/m)<\frac{1}{4}$. Divide the interval $[0,\varepsilon^{-2}\tau]$ into $m$ equal parts, that is, consider the set of intervals $[\tau_k,\tau_{k+1}]$, $0\leqslant k\leqslant m-1$, where $\tau_k=\varepsilon^{-2}\tau k/m$. Let $t\in[\tau_k,\tau_{k+1}]$. We use (2, 10) to estimate $V_n(\underline{x},t)$. Let $N=\min\{p\,|\,p\delta>K+1\}$. Thus we have $a(\varepsilon,n,T)<C_n\varepsilon^{n\delta}<C_n\varepsilon^{K+1}$, if $n\geqslant N$. If $K+1\geqslant N$, then we are done with the proof. Suppose $K<N-1$. From (2.10) we obtain

$$\begin{aligned}&a(\varepsilon,N-1,\tau_{k+1})\\ &\quad\leqslant a(\varepsilon,N-1,\tau_k)\\ &\qquad+c\int_{\tau_k}^{\tau_{k+1}}\Bigg[\frac{1}{\tau_{k+1}-s+1}(2\varepsilon(N-1)(N-2)a(\varepsilon,n,s)\\ &\qquad\qquad+\varepsilon(N-1)a(\varepsilon,N-1,s)+\varepsilon^2(N-1)a(\varepsilon,N-2,s)\\ &\qquad\qquad+(N-1)\varepsilon^2a(\varepsilon,N-3,s)+\varepsilon(N-1)a(\varepsilon,N-2,s))\\ &\qquad+\frac{\varepsilon(N-1)}{\sqrt{\tau_{k+1}-s}}a(\varepsilon,N,s)+\frac{\varepsilon(N-1)}{\sqrt{\tau_{k+1}-s}}a(\varepsilon,N-1,s)\\ &\qquad+(N-1)\varepsilon^2a(\varepsilon,N-1,s)\Bigg]\,\mathrm{d}s.\end{aligned}$$

Let $\lambda_K = \frac{1}{2}\min\{\alpha_K - (K-\gamma), \alpha_{K-1} - (K-1-\gamma)\}$. Let $C_{\lambda_K}$ be a positive real number such that

$$\int_{\tau_k}^{\tau_{k+1}} \frac{\mathrm{d}s}{\tau_{k+1}-s+1} = \ln\left(\frac{\varepsilon^{-2}\tau}{m}+1\right) < C_{\lambda_K}\varepsilon^{-\lambda/2},$$

$\forall \varepsilon > 0$. Using this estimate in the inequality above we get

$$a(\varepsilon, N-1, \tau_{k+1}) < a(\varepsilon, N-1, \tau_k) + \tfrac{1}{2}a(\varepsilon, N-1, \tau_{k+1}) + C\varepsilon^{\alpha_{K+1}},$$

where $\alpha_{K+1} > (K+1)-\gamma$ and $C$ is independent of $\varepsilon$. Thus we have

$$a(\varepsilon, N-1, \tau_{k+1}) \leqslant 2a(\varepsilon, N-1, \tau_k) + 2C\varepsilon^{\alpha_{K+1}}.$$

From this estimate it is easy to see that there exists a constant $C_{K+1,N-1}$ such that

$$a(\varepsilon, N-1, T) \leqslant C_{K+1,N-1}\varepsilon^{\alpha_{K+1}}.$$

Since $N-(K+1)$ is finite, repeating this argument a finite number of times gives us

$$a(\varepsilon, n, T) \leqslant C_{K+1,n}\varepsilon^{\alpha_{K+1}}, \quad n \geqslant K+1.$$

This completes the proof of Claim 1.

*Claim 2.* There exist positive real numbers $C_{1,1}\{C_{2,n}\}_{n\geqslant 2}$, $\alpha_2 > 2-\gamma$ and $\alpha_1 > 1-\gamma$ such that

$$a(\varepsilon, n, T) < C_{2,n}\varepsilon^{\alpha_2}, \quad n \geqslant 2, \qquad a(\varepsilon, 1, T) < C_{1,1}\varepsilon^{\alpha_1}.$$

*Proof of Claim 2.* We proceed as in the proof of claim 1. Let $M = \min\{p \mid p\delta > 2-\gamma\}$. If $n \geqslant M$, then $a(\varepsilon, n, T) < C_n\varepsilon^{n\delta} < C_n\varepsilon^{2-\gamma}$. We estimate $a(\varepsilon, M-1, T)$ using (2.10). Put $\Delta = \frac{1}{2}\min\{\delta, \delta M - (2-\gamma)\}$. Let $C_\Delta$ be a positive real number such that

$$\int_{\tau_k}^{\tau_{k+1}} \frac{\mathrm{d}s}{\tau_{K+1}-s+1} < C_\Delta \varepsilon^{-\Delta}.$$

Using this estimate and the a priori estimate $a(\varepsilon, n, T) < C_n\varepsilon^{n\delta}$ in (2.10), we easily obtain

$$a(\varepsilon, M-1, T) < C_{2,M-1}\varepsilon^{\alpha_2},$$

where $\alpha_2 > 2-\gamma$. Since $M-2$ is finite, repeating this argument a finite number of times we get

$$a(\varepsilon, n, T) < C_{2,n}\varepsilon^{\alpha_2}.$$

Since $a(\varepsilon, 1, T) < M(\tau)a(\varepsilon, 2, T)$, we easily obtain $a(\varepsilon, 1, T) < C_{1,1}\varepsilon^{\alpha_1}$ where $\alpha_1 > 1-\gamma$. This completes the proof of Claim 2.

Equation (1.4) follows from Claim 1 and 2 by induction.

We have proved equation (1.4) of the proposition, and now we proceed with the proof of equation (1.5). Note that the main difference between equations (1.4) and (1.5) is that in the former, the bound is in the uniform norm, while in the latter, it is pointwise. The implication for the proof is that we have to continue the iteration a few more steps before estimating the terms.

We start with the integral equation for $V_2$. Recall from equation (2.1),

$$V_2(\underline{x}, t) = \int_0^t \left[ \sum_z P_{t-s}(\underline{x} \to \underline{z})(Q_1(\underline{z}, s) + Q_2(\underline{z}, s)) + \varepsilon R(\underline{x}, \underline{z}, t-s) \right] \mathrm{d}s \tag{2.11}$$

Note that $R(\underline{x}, \underline{z}, t-s)$ can be written as

$$R(\underline{x}, \underline{z}, t-s) = R_1(\underline{x}, \underline{z}, t-s) V_2(\underline{z}, s).$$

Therefore we can iterate equation (2.11) by substituting for $V_2$ in the expression for $R$. Iterating, we obtain

$$\begin{aligned} V_2(\underline{x}, t) = &\int_0^t \sum_z P_{t-s}(\underline{x} \to \underline{z})(Q_1 + Q_2)(\underline{z}_1, s_1)\, \mathrm{d}s \\ &+ \sum_{k=1}^{\infty} \int_0^t \int_0^{s_1} \cdots \int_0^{s_k} \Big( \sum_{\underline{z}_1} \cdots \sum_{\underline{z}_{k+1}} \varepsilon R_1(\underline{x}, \underline{z}_1, t-s_1) \varepsilon R_1(\underline{z}_1, \underline{z}_2, s_1 - s_2) \\ &\qquad \cdots \varepsilon R_1(\underline{z}_{k-1}, \underline{z}_k, s_{k-1} - s_k) \\ &\qquad \times \sum_{z_{k+1}} P_{s_k - s_{k+1}}(\underline{x} \to \underline{z})(Q_1 + Q_2) \\ &\qquad \times (\underline{z}_{k+1}, s_{k+1}) \Big) \mathrm{d}s_1\, \mathrm{d}s_2 \cdots \mathrm{d}s_{k+1}. \end{aligned} \tag{2.12}$$

Using the bounds used in the proofs of Claims 1 and 2, it is easy to see that the contribution of the $Q_1$ term in the first integral goes like $\varepsilon^{2+\beta}$ for some $\beta > 0$. Therefore we consider only the contribution from the $Q_2$ term in the first integral. Since we know $|V_1(x, t)| < C\varepsilon^{\alpha}$, for some $\alpha > 1$ and for all $t \leq \varepsilon^{-2}\tau$ and $x \in \mathbb{Z}^2$, it is easy to see that the first two terms in $Q_2$ contribute less than $C\varepsilon^{2+\beta}$, for some $\beta > 0$. Therefore, it is enough to consider the last term. The integral of the first term is bounded in absolute value by

$$C\varepsilon^2 \int_0^t \sum_{\underline{z}} P_{t-s}(\underline{x} \to \underline{z}) 1\{|z_1 - z_2| = 1\}\, \mathrm{d}s = C\varepsilon^2 g(x_1, x_2, \varepsilon, t).$$

Therefore, we see that the contribution from the first integral is of the right type. We now show that the second term on the right-hand side of (2.12) is bounded in absolute value by $C\varepsilon^2$. We do this in two steps. First, we show that the term obtained after the last two iterations is bounded in absolute value by $C\varepsilon^2$. Then we show that further iterations lead to a convergent infinite series in $k$ and show that the second term converges to a value bounded above by $C\varepsilon^2$.

We note that $R_1(\underline{x}, \underline{z}, t-s)$ contains two types of terms: (i) terms of the form $P_{t-s}(\underline{x} \to \underline{z} + e_{i,\alpha}) - P_{t-s}(\underline{x} \to \underline{z}) f(\underline{z}, s)$ and (ii) terms of the form $P_{t-s}(\underline{x} \to \underline{z}) \times (\rho_\varepsilon(z_i + e_\alpha) - \rho_\varepsilon(z_i))$. Consider the last two iterations for the $k$th term in the series. First we consider terms of the first type in $R_1$. A typical term of this type is

$$\begin{aligned} \varepsilon \int_0^{s_{k-1}} &\sum_{\underline{z}_k} (P_{s_{k-1}-s_k}(\underline{z}_{k-1} \to \underline{z}_k - e_{i,1}) - P_{s_{k-1}-s_k}(\underline{z}_{k-1} \to \underline{z}_k)) \rho_\varepsilon(\underline{z}_{k,i}, s_k) \\ &\times \int_0^{s_k} \sum_{\underline{z}_{k+1}} P_{s-s_{k+1}}(\underline{z} \to \underline{z}_{k+1})(Q_1 + Q_2)(\underline{z}_{k+1}, s_{k+1})\, \mathrm{d}s_k\, \mathrm{d}s_{k+1}, \end{aligned}$$

where $z_{k,i}$ is the $i$th coordinate projection of $z$. Using equations (2.7)-(2.10), it is easy to see that the contribution from the $Q_1$ term is bounded by $C\varepsilon^{2+\beta}$, for some $\beta > 0$. Therefore, we consider only the contributions from the $Q_2$ term.

$$\begin{aligned}
&\sum_{z_k} P_{s_{k-1}-s_k}(z_{k-1} \to z_k - e_{i,1})\rho_\varepsilon(z_{k,i}, s_k) \sum_{z_{k+1}} P_{s_k - s_{k+1}}(z_k \to z_{k+1}) Q_2(z_{k+1}, s_{k+1}) \\
&\quad = \sum_{z'_k} P_{(s_{k-1}-s_k)/2}(z_{k-1} + e_{i,1} \to z'_k) \sum_{z_k} P_{(s_{k-1}-s_k)/2}(z'_k \to z_k)\rho_\varepsilon(z_{k,i}, s_k) \\
&\qquad \times \sum_{z_{k+1}} P_{s_k - s_{k+1}}(z_k \to z_{k+1}) Q_2(z_{k+1}, s_{k+1}).
\end{aligned}$$

Here we have used the translation invaraiance property of the exclusion process and the Markov property. Thus

$$\begin{aligned}
&\sum_{z_k} (P_{s_{k-1}-s_k}(z_{k+1} \to z_k - e_{i,1}) - P_{s_{k-1}-s_k}(z_{k-1} \to z_k))\rho_\varepsilon(z_{k,i}, s_k) \\
&\qquad \times \sum_{z_{k+1}} P_{s_k - s_{k+1}}(z_k \to z_{k+1})(Q_2)(z_{k+1}, s_{k+1}) \\
&\quad = \sum_{z'_k} (P_{(s_{k-1}-s_k)/2}(z_{k-1} \to z'_k - e_{i,1}) - P_{(s_{k-1}-s_k)/2}(z_{k-1} \to z'_k)) \\
&\qquad \times \sum_{z_k} [P_{(s_{k-1}-s_k)/2}(z'_k \to z_k)\rho_\varepsilon(z_{k,i}, s_k) \\
&\qquad\qquad \times \sum_{z_{k+1}} P_{s_k - s_{k+1}}(z_k \to z_{k+1})(Q_2)(z_{k+1}, s_{k+1})].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\Bigg| \sum_{z'_k} (P_{(s_{k-1}-s_k)/2}(z_{k-1} \to z'_k - e_{i,1}) - P_{(s_{k-1}-s_k)/2}(z_{k-1} \to z'_k)) \\
&\qquad \times \sum_{z_k} \Bigg[ P_{(s_{k-1}-s_k)/2}(z'_k \to z_k)\rho_\varepsilon(z_{k,i}, s_k) \\
&\qquad\qquad \times \sum_{z_{k+1}} P_{s_k - s_{k+1}}(z_k \to z_{k+1})(Q_2)(z_{k+1}, s_{k+1}) \Bigg] \Bigg| \\
&\quad \leq \sum_{z'_k} |P_{(s_{k-1}-s_k)/2}(z_{k-1} \to z'_k - e_{i,1}) - P_{s_{k-1}-s)/2}(z_{k-1} \to z'_k)| \\
&\qquad \times \Bigg| \sum_{z_k} P_{(s_{k-1}-s_k)/2}(z'_k \to z_k)\rho_\varepsilon(z_{k,i}, s_k) \\
&\qquad\qquad \times \sum_{z_{k+1}} P_{s_k - s_{k+1}}(z_k \to z_{k+1})(Q_2)(z_{k+1}, s_{k+1}) \Bigg| \\
&\quad \leq \sum_{z'_k} \Big| P_{(s_{k-1}-s_k)/2}(z_{k-1} \to z'_k - e_{i,1}) - P_{(s_{k-1}-s_k)/2}(z_{k-1} \to z'_k) \Big| \\
&\qquad \times \sum_{z_k} P_{(s_{k-1}-s_k)/2}(z'_k \to z_k)\rho_\varepsilon(z_{k,i}, s_k) \\
&\qquad \times \sum_{z_{k+1}} P_{s_k - s_{k+1}}(z_k \to z_{k+1}) |(Q_2)(z_{k+1}, s_{k+1})|.
\end{aligned}$$

Each term in $Q_2$ is bounded in absolute value by $C\varepsilon^2 1\{|z_{k+1,i}-z_{k+1,j}|=1\}$ for an appropriate choice of $i,j$. Therefore, we have

$$\sum_{z_k'} |P_{(s_{k-1}-s_k)/2}(z_{k-1}\to z_k'-e_{i,1})-P_{(s_{k-1}-s_k)/2}(z_{k-1}\to z_k')|$$

$$\times \sum_{z_k} P_{(s_{k-1}-s_k)/2}(z_k'\to z_k)\rho_\varepsilon(z_{k,i},s_k)$$

$$\times \sum_{z_{k+1}} P_{s_k-s_{k+1}}(z_k\to z_{k+1})|(Q_2)(z_{k+1},s_{k+1})|$$

$$\leqslant C\varepsilon^2 \sum_{z_k'} |P_{(s_{k-1}-s_k)/2}(z_{k-1}\to z_k'-e_{i,1})-P_{(s_{k-1}-s_k)/2}(z_{k-1}\to z_k')|$$

$$\times P_{(s_{k-1}-s_k)/2-s_{k+1}}(z_k'\to z_{k+1})1\{|z_{k+1,i}-z_{k+1,j}|=1\}$$

(Here we have used the positivity and boundedness of $\rho_\varepsilon$, and the Markov property again)

$$< C\varepsilon^2 \frac{\sqrt{2}}{\sqrt{s_{k-1}-s_k}}\,\frac{2}{(s_k+s_{k-1})-2s_{k+1}},$$

where we have made use of inequalities (2.8) and (2.9). Therefore,

$$\left|\varepsilon\int_0^{s_{k-1}} \sum_{z_k} P_{s_{k-1}-s_k}(z_{k-1}\to z_k-e_{i,1})P_{s_{k-1}-s_k}(z_{k-1}\to z_k)\rho_\varepsilon(z_{k,i},s_k)\right.$$

$$\left.\times\int_0^{s_k} \sum_{z_{k+1}} P_{s_k-s_{k+1}}(z_k\to z_{k+1})(Q_2)(z_{k+1},s_{k+1})\,\mathrm{d}s_k\,\mathrm{d}s_{k+1}\right|$$

$$< C\varepsilon^3\int_0^{s_{k-1}}\int_0^{s_k} \frac{\sqrt{2}}{\sqrt{s_{k-1}-s_k}}\,\frac{2}{(s_k+s_{k-1})-2s_{k+1}}\,\mathrm{d}s_k\,\mathrm{d}s_{k+1}.$$

Now using the fact that $s_k<\varepsilon^{-2}\tau$, $\forall k\in\mathbb{N}$, one can easily show that the term above is bounded above by $C\varepsilon^2$, where $C$ is independent of $\varepsilon$. Now we consider the contribution from the second type of term

$$\left|\varepsilon\int_0^{s_{k-1}} \sum_{z_k} P_{s_{k-1}-s_k}(z_{k-1}\to z_k)(\rho_\varepsilon(z_{k,i}+\varepsilon_\alpha,s_k)-\rho_\varepsilon(z_{k,i},s_k))\right.$$

$$\left.\times\rho_\varepsilon(z_{k,i},s_k)\int_0^{s_k} \sum_{z_{k+1}} P_{s_k-s_{k+1}}(z_k\to z_{k+1})(Q_2)(z_{k+1},s_{k+1})\,\mathrm{d}s_k\,\mathrm{d}s_{k+1}\right|$$

$$<\varepsilon\int_0^{s_{k-1}} \sum_{z_k} P_{s_{k-1}-s_k}(z_{k-1}\to z_k)|\rho_\varepsilon(z_{k,i}+\varepsilon_\alpha,s_k)-\rho_\varepsilon(z_{k,i},s_k)|$$

$$\times\rho_\varepsilon(z_{k,i},s_k)\int_0^{s_k} \sum_{z_{k+1}} P_{s_k-s_{k+1}}(z_k\to z_{k+1})|(Q_2)(z_{k+1},s_{k+1})|\,\mathrm{d}s_k\,\mathrm{d}s_{k+1}$$

$$< C\varepsilon^4\int_0^{s_{k-1}}\int_0^{s_k}\mathrm{d}s_k\,\mathrm{d}s_{k+1}P_{s_{k-1}-s_{k+1}}(z_{k-1}\to z_{k+1})1\{|z_{k+1,i}-z_{k+1,j}|=1\}$$

(where we have again made use of the Markov property and the uniform boundedness of $\rho_\varepsilon$ by a constant)

$$< C\varepsilon^4 \int_0^{s_{k+1}} \mathrm{d}s_k \int_0^{s_k} \frac{\mathrm{d}s_{k+1}}{s_{k-1}-s_{k+1}}.$$

Again using the fact that $s_k < \varepsilon^{-2}\tau$, $\forall k \in \mathbb{N}$, we easily obtain the desired estimate.

Call

$$A_k = \left| \int_0^{s_0} \cdots \int_0^{s_{k-2}} \sum_{z_1} \cdots \sum_{z_{k+1}} \varepsilon R_1(\underset{\sim}{x}, \underset{\sim}{z}_1, t-s_1)\varepsilon R_1(\underset{\sim}{z}_1, \underset{\sim}{z}_2, s_1-s_2) \right.$$
$$\left. \cdots \varepsilon R_1(\underset{\sim}{z}_{k-1}, \underset{\sim}{z}_k, s_{k-1}-s_k)\, \mathrm{d}s_1 \cdots \mathrm{d}s_{k-1} \right|.$$

To complete the proof of the proposition, we now show that

$$\sum_{k=2}^{\infty} A_k < K < \infty,$$

where $K$ is independent of $\varepsilon$, and $s_0 = t$.

$$A_k < \int_0^t \left( C_1\varepsilon^2 + \frac{C_2\varepsilon}{\sqrt{t-s_1}} \right) \mathrm{d}s_1 \cdots \int_0^{s_k-2} \left( C_1\varepsilon^2 + \frac{C_2\varepsilon}{\sqrt{s_{k-2}+s_{k-1}}} \right) \mathrm{d}s_{k-1}, \qquad (2.13)$$

where $C_1$ and $C_2$ are constants independent of $\varepsilon$. Since $s_k - s_{k+1} \leqslant t \leqslant \varepsilon^{-2}\tau$, we have

$$C_1\varepsilon^2 \leqslant C_3 \frac{\varepsilon}{\sqrt{s_k - s_{k+1}}},$$

for some constant $C_3$. Therefore,

$$A_k < C \int_0^t \frac{\varepsilon}{\sqrt{t-s_1}} \mathrm{d}s_1 \cdots C \int_0^{s_{k-2}} \frac{\varepsilon}{\sqrt{s_{k-2}-s_{k-1}}} \mathrm{d}s_{k-1} < \frac{C^{k-1}}{\frac{1}{2}(k-1)!}$$

[4]. This proves (2.13), and completes the proof of the proposition. □

**Proof of Theorem 1.** From the proposition, we know that

$$|V_1([\varepsilon^{-1}, r], \varepsilon^{-2}t)| = |E_{\mu^\varepsilon}(\eta([\varepsilon^{-1}r], \varepsilon^{-2}t) - \rho_\varepsilon([\varepsilon^{-1}r], \varepsilon^{-2}t))| \to 0,$$

as $\varepsilon \to 0$, for all $r \in \mathbb{R}^2$, uniformly on compact intervals of time. Therefore, the proof of the theorem will be complete once we show that $\rho_\varepsilon([\varepsilon^{-1}r], \varepsilon^{-2}t)$ converges uniformly on compacts to the solution $\rho(r, t)$ of the Burgers equation with initial condition $\rho_0(x)$. This can be proved easily as was done in the one-dimensional case [4]. Propagation of chaos can be proved in a similar manner, using the estimates for $V_n$. □

**Proof of Theorem 2.** We now prove that the fluctuation field $X_t^\varepsilon(\phi)$ converges to a mean zero O–U process in the space $D(\mathbb{R}_+ \to S'(\mathbb{R}^2))$. From the Holley-Stroock [11, 16, 19, 15] characterization of the O–U process, it follows that it is sufficient to establish the following criteria.

(a) The set of measures $\{\mathcal{P}^\varepsilon, 0<\varepsilon\leqslant 1\}$ for the WASEP is tight in $D(\mathbb{R}_+ \to S'(\mathbb{R}^2))$, and any limit point of the set has support in $C^0(\mathbb{R}_+ \to S'(\mathbb{R}^2))$.

(b) Any limit point $\mathcal{P}$ (in the weak-$*$-topology) of the set $\{\mathcal{P}^\varepsilon, 0<\varepsilon\leqslant 1\}$ solves the following Martingale problem. For any $\phi \in S(\mathbb{R}^2)$ and $F\in C^\infty(\mathbb{R}^2)$,

$$F(X_t(\phi)) - \int_0^t \mathrm{d}s\, F'(X_t(\phi))X_s(A_s\phi) - \int_0^t \mathrm{d}s\, \tfrac{1}{2}\|B_s\phi\|^2 F''(X_t(\phi))$$

is a $\mathcal{P}$-Martingale, with respect to the canonical filtration in $C(\mathbb{R}_+ \to S'(\mathbb{R}^2))$. The operators $A_s$ and $B_s$ are defined as follows:

$$(A_s\phi)(r) = \tfrac{1}{4}\Delta\phi(r) + [1-2\rho(r,s)]\frac{\partial}{\partial x}\phi(r), \tag{2.14}$$

where $r=(x,y)$ and $\rho$ is the solution to Burgers equation.

$$\|B_s\phi\|^2 = \frac{1}{2}\int_{\mathbb{R}^2} \rho(r,s)(1-\rho(r,s))\left(\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right)\mathrm{d}r. \tag{2.15}$$

*Remark.* Note that $A_s$ is the dual of the linearization of the operator on the right-hand side of Burgers equation. We may interpret the Martingale condition as saying that the limiting fluctuation field $X_t(\phi)$ formally satisfies the stochastic differential equation

$$\mathrm{d}X_t(\phi) = X_t(A_t\phi)\,\mathrm{d}t + \mathrm{d}W_t(B_t\phi).$$

(c) The law of the limiting process at time zero is Gaussian with mean zero and a covariance kernel

$$C_0(r,r') = \rho_0(r)(1-\rho_0(r))\delta(r-r').$$

It is easy to check condition (c). The support properties of the process follow from the observation that jumps of $X_t^\varepsilon(\phi)$ are bounded by $C\varepsilon$. We now verify the conditions (a) and (b).

We observe that

$$\begin{aligned} &F(X_t^\varepsilon(\phi)) - \varepsilon^{-2}\int_0^t \mathrm{d}s\, L^\varepsilon(F(X_s^\varepsilon(\phi))) \\ &\quad + \varepsilon^{-2}\int_0^t \mathrm{d}s\, F'(X_t^\varepsilon(\phi))E_{\mu^\varepsilon}L^\varepsilon(F(X_s^\varepsilon(\phi))) \end{aligned} \tag{2.16}$$

is a $E_{\mu^\varepsilon}$-Martingale, where $L^\varepsilon$ is the generator of the WASEP. From the definition of the WASEP, it follows that

$$\begin{aligned} &L^\varepsilon(F(X_t^\varepsilon(\phi))) - F'(X_t^\varepsilon(\phi))E_{\mu^\varepsilon}L^\varepsilon(F(X_t^\varepsilon(\phi))) \\ &\quad = F'(X_t^\varepsilon(\phi))\gamma_1^\varepsilon(t,\phi) + F''(X_t^\varepsilon(\phi))\gamma_2^\varepsilon(t,\phi) + R^\varepsilon(t,\phi), \end{aligned} \tag{2.17}$$

where $\varepsilon^{-2}R^{\varepsilon}(t,\phi)\to 0$ as $\varepsilon\to 0$,

$$\varepsilon^{-2}\gamma_1(t,\phi)=X_t^{\varepsilon}(\tfrac{1}{4}\Delta\phi)+Y_t^{\varepsilon}\left(\frac{\partial\phi}{\partial x},\eta\right),$$

$$Y_t^{\varepsilon}\left(\frac{\partial\phi}{\partial x},\eta\right)=\varepsilon\sum_{\underline{x}\in\mathbb{Z}^2}\frac{\partial\phi}{\partial x}(\varepsilon\underline{x})\{\eta(\underline{x},\varepsilon^{-1}t)(1-\eta(\underline{x}+e_1,\varepsilon^{-2}t))$$
$$-E_{\mu^{\varepsilon}}(\eta(\underline{x},\varepsilon^{-2}t)(1-\eta(\underline{x}+e_1,\varepsilon^{-2}t)))\},\tag{2.18}$$

$$\varepsilon^{-2}\gamma_2^{\varepsilon}(t,\phi)=\tfrac{1}{8}\varepsilon^2\sum_{\underline{x}\in\mathbb{Z}^2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2(\varepsilon\underline{x})(\eta(x,\varepsilon^{-2}t)-\eta(\underline{x}+e_1,\varepsilon^{-2}t))^2\right.$$
$$\left.+\left(\frac{\partial\phi}{\partial y}\right)^2(\varepsilon\underline{x})(\eta(\underline{x},\varepsilon^{-2}t)-\eta(\underline{x}+e_1,\varepsilon^{-2}t))^2\right].\tag{2.19}$$

Using this observation we first establish tightness for the process. It is well known [16] that a family of processes $(X_t^{\varepsilon}(\cdot)\,|\,0<\varepsilon\leqslant 1)$, with values in $D([0,T],S'(\mathbb{R}^2))$ is tight if, for each test function $\phi$, the family of processes $(X_t^{\varepsilon}(\phi)\,|\,0<\varepsilon\leqslant 1)$ with values in $D([0,T],\mathbb{R}^2)$ is tight. The Martingale conditions for tightness of $(X_t^{\varepsilon}(\phi)\,|\,0<\varepsilon\leqslant 1)$ are as follows:

(i) For all $\varepsilon$ with $0<\varepsilon\leqslant 1$,

$$\sup_{0\leqslant t\leqslant T}E_{\mu_\varepsilon}(X_t^{\varepsilon}(\phi))^2<\infty.$$

(ii) There exist non-anticipating functions $\bar\gamma_1^{\varepsilon}(t,\phi)$, $\bar\gamma_t^{\varepsilon}(t,\phi)$ such that for $t\in[0,T]$,

$$M^{\varepsilon}(t)\equiv(X_t^{\varepsilon}(\phi))-\int_0^t\bar\gamma_1^{\varepsilon}(s,\phi)\,\mathrm{d}s,$$

$$N^{\varepsilon}\equiv(M^{\varepsilon}(t))^2-\int_0^t\bar\gamma_2^{\varepsilon}(s,\phi)\,\mathrm{d}s$$

are $E_{\mu^{\varepsilon}}$-Martingales, and

$$\sup_{\varepsilon}\sup_{0\leqslant t\leqslant T}E_{\mu^{\varepsilon}}(|\bar\gamma_i^{\varepsilon}(s,\phi)|^2)<\infty,\quad i=1,2.$$

It can be shown that $M^{\varepsilon}$ and $N^{\varepsilon}$ are Martingales if we choose

$$\bar\gamma_1^{\varepsilon}(s,\phi)=\varepsilon^{-2}L^{\varepsilon}X_s^{\varepsilon}(\phi)-E_{\mu^{\varepsilon}}(\varepsilon^{-2}L^{\varepsilon}X_s^{\varepsilon}(\phi))$$

and

$$\bar\gamma_2^{\varepsilon}(s,\phi)=\varepsilon^{-2}(L^{\varepsilon}(X_s^{\varepsilon}(\phi))^2-2X_s^{\varepsilon}(\phi)L^{\varepsilon}X_s^{\varepsilon}(\phi))$$

[6, 11]. Now let $F(z)=z$ and $z^2$ in (2, 17). Then we obtain

$$L^{\varepsilon}X_t^{\varepsilon}(\phi)-E_{\mu^{\varepsilon}}(L^{\varepsilon}X_t^{\varepsilon}(\phi))=\gamma_1^{\varepsilon}(t,\phi)+R_1^{\varepsilon}(t,\phi),$$
$$L^{\varepsilon}(X_t^{\varepsilon}(\phi))^2-2X_t^{\varepsilon}(\phi)E_{\mu^{\varepsilon}}(L^{\varepsilon}X_t^{\varepsilon}(\phi))$$
$$=2X_t^{\varepsilon}(\phi)\gamma_1^{\varepsilon}(t,\phi)+2\gamma_2^{\varepsilon}(t,\phi)+R_2^{\varepsilon}(t,\phi).$$

Thus

$$\varepsilon^{-2}\gamma_1^2(t,\phi)=\bar{\gamma}_1^\varepsilon(t,\phi)-\varepsilon^{-2}R_1^\varepsilon(t,\phi),$$

$$\varepsilon^{-2}\gamma_2^\varepsilon(t,\phi)=\tfrac{1}{8}[\bar{\gamma}_2^\varepsilon(t,\phi)+2\varepsilon^{-2}R_1^\varepsilon(t,\phi)X_t^\varepsilon(\phi)-\varepsilon^{-2}R_2^\varepsilon(t,\phi)].$$

Since $\varepsilon^{-2}R_i^\varepsilon(t,\phi)$ is uniformly bounded in $\varepsilon$ and $t$, it is easy to see that $E_{\mu^\varepsilon}(\varepsilon^{-2}R_t^\varepsilon(t,\phi))^2$ is uniformly bounded for $i=1,2$. We show that

$$\sup_\varepsilon \sup_{0\le t\le T} E_{\mu^\varepsilon}(|X_t^\varepsilon(\phi)|^2)<\infty.$$

Therefore, it follows that it is sufficient to show that

$$\sup_\varepsilon \sup_{0\le t\le T} E_{\mu^\varepsilon}(\varepsilon^{-2}|\gamma_i^\varepsilon(s,\phi)|^2)<\infty \tag{2.20}$$

in order to check condition (ii). Condition (2.10) for $\gamma_2^\varepsilon(t,\phi)$ follows easily from the boundedness of $\eta(x,t)$ and the integrability of $(\partial\phi/\partial x)^2$ and $(\partial\phi/\partial y)^2$. It can be easily seen that

$$E_{\mu^\varepsilon}(|X_t^\varepsilon(\phi)|^2)<C\varepsilon^2\sum_{x_1,x_2}|V_2(x_1,x_2,\varepsilon^{-2}t)|+R_\varepsilon. \tag{2.21}$$

The sum on the right-hand side of (2.21) can be shown to be uniformly bounded in $\varepsilon$ and $t$ [17]. To verify the condition for $\gamma_1^\varepsilon(t,\phi)$, we note that $E_{\mu^\varepsilon}(Y_t^\varepsilon)^2$ can be seen to be uniformly bounded in $\varepsilon$ and $t$ by expressing it in terms of $V_1$ and $V_2$. With this observation, it is easy to see from equation (2.18) that condition (2.20) is true for $\gamma_1^\varepsilon(t,\phi)$. This completes the proof of tightness (a).

Verification of the Martingale condition (b): The term on the right-hand side of equation (2.17) is a Martingale for every $\varepsilon>0$. Now if we take the limit $\varepsilon\to 0$, then we see that the second term converges to the third ($F''$) term in condition (b). The first term on the right-hand side of (2.16) has two parts. The $X_t^\varepsilon(\frac{1}{4}\Delta\phi)$ term converges to the $X_t(\frac{1}{4}\Delta\phi)$ term in condition (b). The Martingale condition (b) will be proven if we show that

$$\lim_{\varepsilon\to 0} E_{\mu^\varepsilon}\left(\int_0^\varepsilon Y_s^\varepsilon(\phi)-X_s^\varepsilon\left([1-2\rho(\cdot,s)]\frac{\partial\phi}{\partial x}(\cdot)\right)\mathrm{d}s\right)^2=0. \tag{2.22}$$

From the known results of the Boltzmann–Gibbs principle [6, 9], we have

$$\lim_{T\to\infty}\lim_{\varepsilon\to 0} E_{\mu^\varepsilon}\left(\frac{1}{T}\int_t^{t+T} Y_{\varepsilon^2 s}^\varepsilon(\phi)-X_{\varepsilon^2 s}^\varepsilon\left([1-2\rho(\cdot,s)]\frac{\partial\phi}{\partial x}(\cdot)\right)\mathrm{d}s\right)^2=0. \tag{2.23}$$

(2.22) follows from (2.23). This completes the verification of condition (b), and completes the proof of Theorem 2. □

We now obtain an integral equation for the equal time covariance kernel for the limiting process of the fluctuation field. The covariance kernel is defined by the following relation.

$$E(X_t(\phi)X_t(\psi))=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\phi(r)C_t^*(r,r')\psi(r')\,\mathrm{d}r\,\mathrm{d}r',$$

where $X_t$ is the limiting O–U process. If we let $F(z) = z^2$ in the Martingale condition (b), we obtain

$$E(X_t(\phi)X_t(\phi)) - E\int_0^t 2X_s(\phi)X_s(A_s\phi)\,\mathrm{d}s - \int_0^t \|B_s\phi\|^2\,\mathrm{d}s = 0.$$

Using the definition of the covariance kernel, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \phi(r)C_t^*(r, r')\psi(r')\,\mathrm{d}r\,\mathrm{d}r' \\
&\quad= \int_0^t \int\int 2\phi(r)C_s^*(r, r')\,\mathrm{d}r\,\mathrm{d}r'\,\mathrm{d}s \\
&\qquad+ \int\int_0^t \tfrac{1}{2}\rho(r, s)(1-\rho(r, s))\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]\mathrm{d}s\,\mathrm{d}r.
\end{aligned} \tag{2.24}$$

We now define a modified covariance kernel $C_t(r, r')$ by

$$C_t^*(r, r') = C_t(r, r') + \rho(r, t)(1-\rho(r, t))\delta(r-r').$$

This splitting of $C_t^*$ into diagonal and non-diagonal parts is a natural one, as can be seen from the form of the expected value of the square of the fluctuation field. With this definition from (2.24), we obtain

$$\begin{aligned}
&\int\int C_t(r, r')\phi(r)\phi(r')\,\mathrm{d}r\,\mathrm{d}r' \\
&\quad= -\int \phi(r)^2\rho(r, t)(1-\rho(r, t))\,\mathrm{d}r \\
&\qquad+ \int_0^t \int\int 2\phi(r)C_s(r, r')(A_s\phi(r'))\,\mathrm{d}r\,\mathrm{d}r'\,\mathrm{d}s \\
&\qquad+ \int_0^t \int 2\phi(r)A_s\phi(r)\rho(r, s)(1-\rho(r, s))\,\mathrm{d}r\,\mathrm{d}s \\
&\qquad+ \frac{1}{2}\int_0^t \int \rho(r, s)(1-\rho(r, s))\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]\mathrm{d}r\,\mathrm{d}s.
\end{aligned} \tag{2.25}$$

We integrate the second term on the right-hand side by parts to transfer the derivatives in $A_s$ on to $C_r(r, r')$ to obtain $\int_0^t\iint 2\phi(r)\phi(r')L_sC_s(r, r')\,\mathrm{d}r\,\mathrm{d}r'$ where $L_s$ is the linearization of the operator on the right-hand side of Burgers equation, i.e.,

$$L_sC_s(r, r') = \tfrac{1}{4}\Delta_{r'}C_s - \frac{\partial}{\partial x'}(1-2\rho(r', s))C_s(r, r').$$

We observe that

$$\begin{aligned}
&\int \tfrac{1}{2}\phi^2\Delta(\rho(r, t)(1-\rho(r, t)))\,\mathrm{d}r \\
&\quad= \int \phi(r)(\Delta\phi)(r)\rho(r, t)(1-\rho(r, t))\,\mathrm{d}r \\
&\qquad+ \int \rho(r, t)(1-\rho(r, t))\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]\mathrm{d}r.
\end{aligned}$$

Therefore, if we differentiate equation (2.25) with respect to $t$, we get

$$\begin{aligned}
&\int \frac{\partial C_t(r,r')}{\partial t}\phi(r)\phi(r')\,\mathrm{d}r\,\mathrm{d}r' \\
&= -\int \phi^2(r)(1-2\rho(r,t))\left[\tfrac{1}{4}\Delta\rho(r,t) - \frac{\partial}{\partial x}(\rho(r,t)(1-\rho(r,t)))\right]\mathrm{d}r \\
&\quad +2\iint \phi(r)\phi(r')L_tC_t(r,r')\,\mathrm{d}r\,\mathrm{d}r' \\
&\quad +2\int \phi(r)\left[\left(\tfrac{1}{4}\Delta\phi(r) + \left(1-2\rho(r,t)\frac{\partial\phi}{\partial x}\right)\right)\rho(r,t)(1-\rho(r,t))\right]\mathrm{d}r \\
&\quad -\frac{1}{2}\int \phi(r)\Delta\phi(r)\rho(r,t)(1-\rho(r,t))\,\mathrm{d}r \\
&\quad +\frac{1}{4}\int \phi^2(r)\left[(1-2\rho(r,t))\Delta\rho(r,t) - 2\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]\right]\mathrm{d}r \\
&= 2\iint \phi(r)\phi(r')L_tC_t(r,r')\,\mathrm{d}r\,\mathrm{d}r' \\
&\quad +2\int \phi^2(r)\frac{\partial\rho}{\partial x}(r,t)(\rho(r,t)(1-\rho(r,t))\,\mathrm{d}r \\
&\quad -\frac{1}{2}\int \phi^2(r)\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]\mathrm{d}r.
\end{aligned} \tag{2.26}$$

We now consider the special initial profile of the form $\rho_0(x,y)=\bar{\rho}_0(x)$, where $\bar{\rho}_0(x)$ is the profile of a travelling wave solution of the one-dimensional Burgers equation with viscosity. $\lim_{x\to\pm\infty}\bar{\rho}_0(x)=\rho_\pm$ and $0\le\rho_-<\rho_+\le 1$. $\bar{\rho}(x,t)=\bar{\rho}_0(x-ct)$ where $c=1-(\rho_-+\rho_+)$. Clearly, $\rho((x,y),t)=\bar{\rho}(x,t)$ is a travelling wave solution to the Burgers equation in two dimensions. We now obtain an integral representation for the equal time covariance kernel of the limiting fluctuation field in this special case. We are interested in the fluctuations of the shape of the travelling wave. Therefore, we consider the fluctuations at $r=(x+ct,y)=r_c$, $r'=(x'+ct,y')=r'_c$.

It follows from (2.26) that

$$\begin{aligned}
\frac{\partial}{\partial t}C_t(r_c,r'_c,t) &= \left(\tfrac{1}{4}\Delta_r - \frac{\partial}{\partial x}(1-2\bar{\rho}_0(x)-c)\right)C_t(r_c,r'_c,t) \\
&\quad +\left(\tfrac{1}{4}\Delta_{r'} - \frac{\partial}{\partial x'}(1-2\bar{\rho}_0(x')-c)\right)C_t(r_c,r'_c,t) \\
&\quad +\delta(r-r')\left\{2\frac{\partial\bar{\rho}_0}{\partial x}(\bar{\rho}_0(1-\bar{\rho}_0))(x) - \frac{1}{2}\left(\frac{\partial\bar{\rho}_0}{\partial x}\right)^2\right\},\quad C_0(r,r')=0.
\end{aligned} \tag{2.27}$$

Let $K_t(z, r)$ be the transition kernel for a two-dimensional diffusion which is a product of two independent diffusions. Along the $X$-direction, we take a diffusion with drift $(1-2\bar{\rho}-c)$ and along the $Y$-direction, we take the standard Brownian motion. The transition kernel for this process can be written as

$$K_t(z, r) = \tfrac{1}{2} Q_t(z_1, x) \frac{1}{\sqrt{2\pi t}} e^{-(y-z_2)^2/(zt)}$$

$z = (z_1, z_2)$ and $r = (x, y)$, and $Q_t$ satisfies the forward equation

$$\frac{\partial Q_t}{\partial t}(z_1, x) = \frac{1}{2}\frac{\partial^2 Q_t}{\partial x^2} - \frac{\partial}{\partial x}(1 - 2\bar{\rho}_0(x) - c) Q_t,$$

$$Q_0(z_1, x) = \delta(z_1 - x).$$

With these definitions, $C_t(r_c, r_c')$ can be written as

$$\begin{aligned} C_t(r_c, r_c') &= \int_0^t \int_{\mathbb{R}^2} K_{t-s}(z, r) K_{t-s}(z, r') \\ &\qquad \times [2(\bar{\rho}_0'(z_1)(1-\bar{\rho}_0(z_1))\bar{\rho}_0(z_1)) - \tfrac{1}{2}[\rho_0'(z_1)]^2] \, dz \, ds \\ &= \int_0^t \int_{\mathbb{R}} \frac{1}{2\pi(t-s)} e^{(-(y-z_2)^2-(y'-z_2)^2)/(2(t-s))} \, dz_2 \, ds \\ &\quad \times \left[\frac{1}{4}\int_{\mathbb{R}} Q_{t-s}(z_1, x) Q_{t-s}(z_1, x') \right. \\ &\qquad \left. \times \{2(\bar{\rho}_0'(z_1)(1-\bar{\rho}_0(z_1))\bar{\rho}_0(z_1)) - \tfrac{1}{2}(\rho_0'(z_1))^2\} \, dz_1 \right], \end{aligned}$$

where $\bar{\rho}_0'(z_1)$ is the derivative of $\bar{\rho}_0(z_1)$. We note that except for the coefficient $\frac{1}{2}$ in front of the $(\bar{\rho}_0'(z_1))^2$ term, the term in the curly brackets is the same as the one which appeared in the one-dimensional case [4]. We have chosen the origin of the initial profile $\bar{\rho}_0$ such that $\bar{\rho}_0(0) = \frac{1}{2}(\rho_+ + \rho_-)$. Therefore $(1 - 2\bar{\rho}_0(r) - c)$ is negative if $r > 0$ and positive if $r < 0$. That is, $Q_t$ is the probability kernel for a diffusion with drift directed towards the origin. Therefore, the process with the transition kernel $Q_t$ has an invariant measure $\mu$. Since $\bar{\rho}_0'$ is a stationary solution of the linearized Burgers equation, the invariant measure can be written as

$$\mu(dr) = \frac{1}{\bar{\rho}_+ - \bar{\rho}_-} \bar{\rho}_0'(r) \, dr.$$

We note that

$$\int_{\mathbb{R}} \frac{1}{2\pi(t-s)} e^{(-(y-z_2)^2-(y'-z_2)^2)/(2(t-s))} \, dz_2 = \frac{1}{\sqrt{4\pi(t-s)}} e^{-(y-y')^2/(4(t-s))}.$$

Thus

$$C_t(r_c, r_c') = \frac{1}{\sqrt{4\pi(t-s)}} e^{-(y-y')^2/(4(t-s))}$$
$$\times \left[ \frac{1}{4} \int_{\mathbb{R}} Q_{t-s}(z_1, x) Q_{t-s}(z_1, x') \right.$$
$$\left. \times \{2(\bar{\rho}_0'(z_1)(1-\bar{\rho}_0(z_1))\bar{\rho}_0(z_1)) - \tfrac{1}{2}(\rho_0'(z_1))^2\} \, dz_1 \right]. \quad (2.28)$$

It is easy to see from (2.28) that

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} EX_t(\phi_t)X_t(\psi_t)$$
$$= \frac{1}{8\sqrt{\pi}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \phi(r)\psi(r')\bar{\rho}_0'(x)\bar{\rho}_0'(x')$$
$$\times \int \{2(\bar{\rho}_0'(z_1)(1-\bar{\rho}(z_1))\bar{\rho}_0(z_1)) - \tfrac{1}{2}(\rho_0'(z_1))^2\} \, dz_1 \, dr \, dr'$$
$$= D \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \phi(r)\psi(r')\bar{\rho}_0'(x)\bar{\rho}_0'(x') \, dr \, dr',$$

where $\phi_t(r) = \phi(x+ct, y)$. Now consider the stochastic partial differential equation

$$\frac{\partial h^\varepsilon}{\partial t}(y, t) = \Delta h^\varepsilon + \sqrt{D}\, \varepsilon Z_{y,t},$$

where $Z_{y,t}$ is the space-time white noise [22]. Then it is known that

$$h^\varepsilon(\phi, t) = \int \phi(y) h^\varepsilon(y, t) \, dy$$
$$= \varepsilon\sqrt{D} \int_0^t \int\int \phi(y) \frac{1}{\sqrt{2\pi(t-s)}} e^{-(y-y')^2/(2(t-s))} \, dZ_{y',s} \, dy$$

for all $\phi \in S(\mathbb{R})$. Therefore,

$$E_{h^\varepsilon}(h^\varepsilon(\phi, t) h^\varepsilon(\psi, t))$$
$$= \varepsilon^2 D \int_0^t \int\int\int \phi(y)\psi(y') \frac{1}{2\pi(t-s)} e^{(-(y-u)^2-(y'-u)^2)/(2(t-s))} \, dy \, dy' \, du \, ds,$$

where $E_{h\varepsilon}$ is the expectation for the process $h^\varepsilon$. Now we define a random set of profiles (for almost every path of $h^\varepsilon$) $\rho^\varepsilon_{\mathbb{R},t}$ as

$$\rho^\varepsilon_{\mathbb{R},t}(x, y, t) = \bar{\rho}_0(x - ct + h^\varepsilon(y, t)).$$

Now we define a (random) product measure on configurations in $\mathbb{Z}^2$, as

$$\mu^\varepsilon_{\mathbb{R},t} = \prod_{x,y} \nu^\varepsilon_{x,y},$$

where $\nu^{\varepsilon}_{x,y}(\eta(x,y)=1)=\rho^{\varepsilon}_{\mathbb{R},t}(\varepsilon x, \varepsilon y, t)$. Note that the random measure $\mu^{\varepsilon}_{\mathbb{R},t}$ describes the random shifts of the profile, where the shift at a given value $y$ of the $Y$-coordinate is $h^{\varepsilon}(y,t)$.

$$\lim_{\varepsilon\to 0} E_{h^{\varepsilon}} E_{\mu^{\varepsilon}_{\mathbb{R},t}}(X^{\varepsilon}_0(\phi)X^{\varepsilon}_0(\psi))$$

$$=\lim_{\varepsilon\to 0} E_{h^{\varepsilon}}\left[\left(\sum_{x,y}\sum_{x',y'}\phi(\varepsilon x+ct, \varepsilon y)\right.\right.$$

$$\left.\left.\times\,\psi(\varepsilon x'+ct, \varepsilon y')\bar{\rho}_0'(\varepsilon x')h^{\varepsilon}(y,t)h^{\varepsilon}(y',t)\right)+R_{\varepsilon}\right],$$

where $E_{h^{\varepsilon}}R_{\varepsilon}$ goes to zero as $\varepsilon$ tends to zero. Now we can see that

$$\lim_{t\to 0}\frac{1}{\sqrt{t}}\lim_{\varepsilon\to 0} E_{h^{\varepsilon}} E_{\mu^{\varepsilon}_{\mathbb{R},t}}(X^{\varepsilon}_0(\phi)X^{\varepsilon}_0(\psi))$$

$$=D\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\phi(r)\psi(r')\bar{\rho}_0'(x)\bar{\rho}_0'(x')\,\mathrm{d}r\,\mathrm{d}r'.$$

Thus we have shown that the covariance of the limiting fluctuation field is consistent with the picture of random rigid translation of the profile of the travelling wave we described in Section 1. We note that, instead of modelling the phase boundary at the macroscopic level by a stochastic partial differential equation, we could model it at the microscopic level by a (countable) system of stochastic differential equations, which will give us the same covariance in the limit. Such a model is given by

$$\mathrm{d}h^{\varepsilon}(y,t)=\varepsilon^{-2}\Delta_{\mathrm{d}}h^{\varepsilon}\,\mathrm{d}t+\varepsilon D\,\mathrm{d}B_t(y),$$

where $y\in\mathbb{Z}$, $\Delta_{\mathrm{d}}$ is the discrete Laplacian and $(B_t(y))_{y\in\mathbb{Z}}$ are a countable collection of independent standard Brownian motions.

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