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On tail probability of local times of Gaussian processes

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Abstract

We study the tail probability of the local time at the origin of Gaussian processes with stationary increments. The order of infinitesimal is obtained. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $X = \{X_t; t \geqslant 0\}$ be a Gaussian process with mean zero, stationary increments and $X_0 = 0$. Put

$$\sigma^2(h) = E[(X_{t+h} - X_t)^2], \quad t, h \geqslant 0.$$

Throughout the paper we further assume that $\sigma^2(h)$ is continuous and satisfies that

$$\int_0^1 \frac{\mathrm{d}t}{\sigma(t)} < \infty, \sigma(h) > 0 \quad (h > 0). \tag{1.1}$$

Then it is known that X has continuous local time $l(t,x)$;

$$l(t,x) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t I_{[x-\epsilon, x+\epsilon]}(X_s)\, \mathrm{d}s,$$

(see Berman, 1969). In what follows, we are interested in the law of $l(t,0)$, since it appears in some limit theorems for occupation times of Gaussian processes (see Kôno, 1996). As far as the authors know, we have little knowledge about the explicit distribution of $l(t,0)$ and so it would be of interest to study the relationship between the incremental covariance function $\sigma^2(\cdot)$ and the law of $l(t,0)$. As a typical example

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let us consider the case of the standard Brownian motion. It is well known that the law of $l(1,0)$ is the truncated normal distribution and therefore

$$-\log P[l(1,0) > x] \sim x^2/2 \quad \text{as } x \to \infty, \tag{1.2}$$

holds, where the sign $\sim$ indicates that the ratio of the two sides tends to 1. The aim of the present paper is to extend this fact to evaluate the tail probability $P[l(1,0) > x]$ as $x \to \infty$ in terms of $\sigma(\cdot)$ for more general Gaussian processes. Our main result is that, under some conditions

$$-\log P[l(1,0) > x] \asymp 1/\sigma^{-1}(1/x) \quad \text{as } x \to \infty, \tag{1.3}$$

holds, where $f \asymp g$ means $0 < \liminf f(x)/g(x) \leqslant \limsup f(x)/g(x) < \infty$ and σ^{-1} is the inverse function of $\sigma(t)$ (if exists). The details of the conditions and the proof will be given in Section 2, and we only remark that (1.3) is compatible with (1.2). Indeed, in the case of Brownian motion, $\sigma^2(t) = t$ and hence $1/\sigma^{-1}(1/x) = x^2$.

2. Results and proofs

For every $0 = t_0 < t_1 < \cdots, t_n < 1$, let $C_n = C_n(t_1, \ldots, t_n)$ denote the covariance matrix of the n-dimensional random vector

$$(X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})).$$

The diagonal elements of $C_n(t_1, \ldots, t_n)$ are $\sigma^2(t_j - t_{j-1})$ ($j = 1, \ldots, n$). Since C_n is positive definite, it is an elementary fact of linear algebra that

$$\det C_n(t_1, \ldots, t_n) \leqslant \prod_{j=1}^{n} \sigma^2(t_j - t_{j-1}). \tag{2.1}$$

In the sequel we shall assume the following condition, which may be regarded as a kind of 'local nondeterminism'. (See Nolan, 1989; Kôno and Shieh, 1993.)

Assumption (A). *There exist positive constants δ and c such that*

$$\det C_n(t_1, \ldots, t_n) \geqslant c\, \delta^n \prod_{j=1}^{n} \sigma^2(t_j - t_{j-1}) \tag{2.2}$$

for all $0 = t_0 < t_1 < \cdots < t_n < 1$ and for all sufficiently large n.

Remark. A sufficient condition for (A) is that $\sigma^2(t)$ is concave. Indeed, by Lemma 3.3 of Csörgő et al. (1995), the concavity of $\sigma^2(t)$ implies

$$\det C_n(t_1, \ldots, t_n) \geqslant 2^{-n} \prod_{j=1}^{n} \sigma^2(t_j - t_{j-1}). \tag{2.3}$$

Thus (2.2) holds with $c = 1$, $\delta = \frac{1}{2}$.

Before we state our main theorem, we recall the notion of *regular variation*, although we refer to Feller (1971), Seneta (1976) or Bingham et al. (1987) for details. A function $\varphi(x)$ $(x>0)$ is said to *vary regularly* at 0 [or ∞] (with index α) if and only if

$$\lim_{x\to 0[\infty]} \frac{\varphi(\lambda x)}{\varphi(x)} = \lambda^{\alpha} \quad \text{for every } \lambda > 0$$

holds. Clearly a function $\varphi(x)$ varies regularly at 0 with index α if and only if $\varphi(x) = x^{\alpha}L(1/x)$ for some function L which varies slowly at ∞ (i.e., $\lim_{x\to\infty} L(\lambda x)/L(x) = 1$, for every $\lambda > 0$). Our main result is:

Theorem 1. *Suppose that* (1.1) *and Assumption* (A) *are satisfied. If* $\sigma(t)$ *is continuous, strictly increasing on the interval* $[0,1]$ *and varies regularly at* 0 *with exponent* $0<\alpha<1$, *then*

$$-\log P[l(1,0) > x] \asymp 1/\sigma^{-1}(1/x) \quad \textit{as } x\to\infty.$$

Here $f \asymp g$ denotes $0 < \liminf f(x)/g(x) \leqslant \limsup f(x)/g(x) < \infty$ as before.

As a typical example of Theorem 1, let us mention the case of *fractional Brownian motion*, (i.e., $\sigma(t) = t^H$) in the form of Corollary: If $0 < H \leqslant \frac{1}{2}$, then $\sigma^2(t)$ is concave (and hence Assumption (A) is satisfied). Therefore, we have:

Corollary. *If* X *is the fractional Brownian motion with index* $0 < H \leqslant \frac{1}{2}$, *then*

$$-\log P[l(1,0) > x] \asymp x^{1/H} \quad \textit{as } x\to\infty. \tag{2.4}$$

The authors believe that $0 < H \leqslant \frac{1}{2}$ may be replaced by $0 < H < 1$ although, in the case where $\frac{1}{2} < H < 1$, they do not know whether Assumption (A) is satisfied or not. We further believe that (2.4) may be strengthened as

$$-\log P[l(1,0) > x] \sim C_H x^{1/H} \quad \text{as } x\to\infty,$$

for some suitable constant $C_H > 0$. However, it is still open.

Our basic tool for the proof of Theorem 1 is the following Tauberian theorem obtained by one of the authors (Kasahara, 1978, Theorem 4).

Theorem A. *Let* ξ *be a positive random variable and let* $\varphi(x)$ *be a function which varies regularly at* ∞ *with exponent* $0<\alpha<1$. *Then*,

$$-\log P[\xi > \varphi(x)] \asymp x \quad \textit{as } x\to\infty$$

holds if and only if

$$E[\xi^n]^{1/n} \asymp \varphi(n) \quad \textit{as } n\to\infty.$$

Therefore, let $\varphi(x)=1/\sigma(x)$. Since $\varphi^{-1}(x)=1/\sigma^{-1}(1/x)$, Theorem 1 is equivalent to:

Theorem 2. *Suppose that* (1.1) *and Assumption* (A) *are satisfied. If* $\sigma(t)$ *is continuous, strictly increasing and varies regularly at* 0 *with exponent* $0<\alpha<1$, *then*

$$E[l(1,0)^n]^{1/n} \asymp \frac{1}{\sigma(1/n)}, \quad n \to \infty.$$

Thus, in what follows we shall prove Theorem 2 instead of 1 and to this end let us start with expressing the moments $E[l(1,0)^n]$ $(n \geqslant 1)$ as multiple integrals. Let $0=t_0<t_1<\cdots t_n<1$ and recall that $C_n = C_n(t_1,\dots,t_n)$ denote the covariance matrix of the n-dimensional random vector

$$(X(t_1)-X(t_0), X(t_2)-X(t_1),\dots,X(t_n)-X(t_{n-1})).$$

Therefore, its Gaussian kernel is

$$g(t_1,\dots,t_n;x)=\frac{1}{(2\pi)^{n/2}\sqrt{\det C_n}}\exp\{-(C_n^{-1}x,x)/2\}, \quad x \in \mathbb{R}^n.$$

So, for any non-negative, continuous function $F(x)\,(x\in\mathbb{R})$,

$$\begin{aligned} E\left[\left(\int_0^1 F(X_s)\,\mathrm{d}s\right)^n\right] &= n!\int\cdots\int_{0<t_1<\cdots<t_n<1} E[F(X_{t_1})\cdots F(X_{t_n})]\,\mathrm{d}t_1\cdots\mathrm{d}t_n \\ &= n!\int\cdots\int_{0<t_1<\cdots<t_n<1} \mathrm{d}t_1\cdots\mathrm{d}t_n\int_{\mathbb{R}^n} g(t_1,\dots,t_n;x) \\ &\quad \times F(x_1)\cdots F(x_1+\cdots+x_n)\,\mathrm{d}x. \end{aligned}$$

By a standard argument we may let $F(x)$ approach the Dirac function $\delta(x)$ to have

$$E[(l(1,0))^n] = n!\int\cdots\int_{0<t_1<\cdots<t_n<1} g(t_1,\dots,t_n;0)\,\mathrm{d}t_1\cdots\mathrm{d}t_n.$$

Thus we obtain:

Lemma 1.

$$E[(l(1,0))^n]=\frac{n!}{\sqrt{2\pi}^n}\int\cdots\int_{0<t_1<\cdots<t_n<1}\frac{\mathrm{d}t_1\cdots\mathrm{d}t_n}{\sqrt{\det C_n(t_1,\dots,t_n)}}, \quad n\geqslant 1.$$

Now combining Lemma 1 with (2.1) and (2.2), we obtain:

Lemma 2.

$$E[(l(1,0))^n]^{1/n} \asymp \left(n!\int\cdots\int_{0<t_1<\cdots<t_n<1}\frac{\mathrm{d}t_1\cdots\mathrm{d}t_n}{\prod_{j=1}^n \sigma(t_j-t_{j-1})}\right)^{1/n} \quad \textit{as } n\to\infty.$$

In order to evaluate the right-hand side of the above relation we shall prepare an auxiliary result, which is the key lemma of this paper.

Lemma 3. *Let $U(x)$ be a right continuous, non-decreasing function defined on $I=[0,\infty)$ such that $U(0)=0$. If $U(x)$ varies regularly at 0 with index $\beta>0$, then*

$$\left(\int\cdots\int_{I^n, 0<t_1+\cdots+t_n<1} \mathrm{d}U(t_1)\cdots \mathrm{d}U(t_n)\right)^{1/n} \asymp U(1/n) \quad as\ n\to\infty.$$

Proof. The lower estimate is easy;

$$\int\cdots\int_{I^n, 0<t_1+\cdots+t_n<1} \mathrm{d}U(t_1)\cdots \mathrm{d}U(t_n)$$
$$\geqslant \int\cdots\int_{0<t_k<1/n, k=1,\dots,n} \mathrm{d}U(t_1)\cdots \mathrm{d}U(t_n) = (U(1/n))^n.$$

To prove the upper inequality, define

$$F_n(t)=\int\cdots\int_{I^n,\,0<t_1+\cdots+t_n<t} \mathrm{d}U(t_1)\cdots \mathrm{d}U(t_n), \quad n=1,2,\dots\ .$$

It is $F_n(1)$ that we need to evaluate. Now notice that

$$\int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}F_n(t)=\left(\int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}U(t)\right)^n.$$

Therefore, we have

$$F_n(1)\leqslant \mathrm{e}^n\int_0^1 \mathrm{e}^{-nt}\,\mathrm{d}F_n(t)\leqslant \mathrm{e}^n\int_0^\infty \mathrm{e}^{-nt}\,\mathrm{d}F_n(t)\leqslant\left(e\int_0^\infty \mathrm{e}^{-nt}\,\mathrm{d}U(t)\right)^n,$$

which implies

$$\limsup_{n\to\infty}\frac{1}{U(1/n)}F_1(1)^{1/n}\leqslant \limsup_{n\to\infty}\frac{e}{U(1/n)}\int_0^\infty \mathrm{e}^{-nt}\,\mathrm{d}U(t).$$

By the well-known Karamata's Tauberian theorem (see e.g. Feller, 1971, p. 443) we see that the right-hand side equals $e\Gamma(\beta+1)$, which completes the proof of Lemma 1. □

We are now ready to prove Theorem 2. Let

$$U(t)=\int_{0\leqslant u\leqslant \min\{t,1\}}\frac{\mathrm{d}u}{\sigma(u)}, \quad t\geqslant 0.$$

Since σ is a regularly varying function by assumption, we have

$$U(t)\sim\frac{1}{1-\alpha}\frac{t}{\sigma(t)} \quad \text{as } t\to 0.$$

Therefore, $U(t)$ varies regularly as t goes to 0 with index $\beta = 1 - \alpha$ (> 0). Since

$$\left(n! \int \cdots \int_{I^n,\, 0<t_1<\cdots<t_n<1} \frac{\mathrm{d}t_1 \cdots \mathrm{d}t_n}{\prod_{j=1}^n \sigma(t_j - t_{j-1})}\right)^{1/n} = \left(n! \int \cdots \int_{I^n, 0<s_1+\cdots+s_n<1} \mathrm{d}U(s_1)\cdots \mathrm{d}U(s_n)\right)^{1/n}, \tag{2.5}$$

we can apply Lemma 3 and obtain

$$\left(n! \int \cdots \int_{0<t_1<\cdots<t_n<1} \frac{\mathrm{d}t_1 \cdots \mathrm{d}t_n}{\prod_{j=1}^n \sigma(t_j - t_{j-1})}\right)^{1/n} \asymp (n!)^{1/n} U(1/n) \asymp (n!)^{1/n} \frac{1}{n\sigma(1/n)}.$$

Here, we used (2.5). Since $(n!)^{1/n} \sim n/e$ by Stirling's formula, we conclude that

$$\left(n! \int \cdots \int_{0<t_1<\cdots<t_n<1} \frac{\mathrm{d}t_1 \cdots \mathrm{d}t_n}{\prod_{j=1}^n \sigma(t_j - t_{j-1})}\right)^{1/n} \asymp \frac{1}{\sigma(1/n)} \quad \text{as } n \to \infty.$$

Combining this with Lemma 2, we complete the proof of Theorem 2.

In the above, we did not actually use the regular variation of $\sigma(t)$ itself but that of $U(t) = \int_0^t 1/\sigma(u)\,\mathrm{d}u$. Therefore, we have

Theorem 3. *Suppose that* (1.1) *and Assumption* (A) *are satisfied. If* $U(t) = \int_0^t 1/\sigma(u)\,\mathrm{d}u$ *varies regularly at* 0 *with index* $0 < \beta < 1$, *then*

$$E[l(1,0)^n]^{1/n} \asymp nU(1/n) \quad \textit{as } n \to \infty$$

and

$$-\log P[l(1,0) > xU(1/x)] \asymp x \quad \textit{as } x \to \infty. \tag{2.6}$$

We also remark that, if $U(t)$ varies regularly at 0 with index $0 < \beta < 1$, (2.6) is equivalent to the following condition by Theorem 2 of Kasahara (1978):

$$\log E[\mathrm{e}^{\lambda l(1,0)}] \asymp 1/U^{-1}(1/\lambda), \quad \lambda \to \infty.$$

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