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Strong approximation of spatial random walk in random scenery

Pál Révész[a,*], Zhan Shi[b]

[a] *Institut für Statistik und Wahrscheinlichkeitstheorie, Technische Universität Wien, Wiedner Hauptstrasse 8-10/107, A-1040 Vienna, Austria*

[b] *Laboratoire de Probabilités UMR 7599, Université Paris VI, 4 Place Jussieu, F-75252 Paris Cedex 05, France*

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Abstract

We prove a strong approximation for the spatial Kesten–Spitzer random walk in random scenery by a Wiener process. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $\{S_t,\ t=0,1,2,\ldots\}$ be a simple, symmetric random walk in $\mathbb{Z}^d$ (in this paper, $d\geqslant 3$) and let $Y(x)$ ($x\in\mathbb{Z}^d$) be an array of i.i.d. r.v.'s with

$$\boldsymbol{E}Y(x)=0,\quad \boldsymbol{E}Y^2(x)=1,\quad \boldsymbol{E}|Y(x)|^p<\infty,\ p>2.$$

We also assume that the processes $\{S_t\}$ and $\{Y(x)\}$ are independent.

The process

$$X(T)=\sum_{t=0}^{T} Y(S_t)\quad (T=0,1,2,\ldots) \tag{1.1}$$

is called *random walk in random scenery*. The study of this model was initiated by Kesten and Spitzer (1979). They investigated the case $d=1$, and proved that in this case, as $n\to\infty$, non-Gaussian limit laws appear (see (1.5) of Kesten and Spitzer 1979) as limits of $\{n^{-3/4}X(\lfloor nt\rfloor),\ t\geqslant 0\}$. Moreover, in general, they studied the limiting behaviour of $n^{-\delta}X(\lfloor nt\rfloor)$ with $\delta=1-\alpha^{-1}+(\alpha\beta)^{-1}$, in case of the $\mathbb{Z}$-valued random walk $\{S_k\}$ being asymptotically stable of index $\alpha\in(1,2]$ and $\{Y(x),\ x\in\mathbb{Z}^d\}$ being asymptotically stable of index $\beta\in(0,2]$. They showed that with the indicated positive δ, $n^{-\delta}X(\lfloor nt\rfloor)$ converges weakly as $n\to\infty$ to a self-similar process with stationary

* Corresponding author.

E-mail addresses: revesz@ci.tuwien.ac.at (P. Révész), zhan@proba.jussieu.fr (Z. Shi).

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increments, which depends on (α, β). Recently, the weak convergence was strengthened into strong approximations by Khoshnevisan and Lewis (1998) for Gaussian sceneries, and by Csáki et al. (1999) for general sceneries.

Kesten and Spitzer (1979) also conjectured that in case of $\{S_k\}$ being a simple random walk on the planar lattice $\mathbb{Z}^2$, then $(n \log n)^{-1/2} X(\lfloor nt \rfloor)$ converges weakly as $n \to \infty$ to a Brownian motion on $[0, \infty)$. This conjecture was proved by Bolthausen (1989) (see also Borodin, 1980), and a strong approximation was obtained by Csáki et al. (2000). Bolthausen (1989) also noted that in dimensions greater than or equal to 3, $T^{-1/2} X(T)$ is asymptotically normal. (Continuous-time analogues of these can be found in Rémillard and Dawson, 1991.)

The present paper is devoted to the case $d \geqslant 3$. We prove that, in this case, $X(T)$ can be approximated with probability 1 by a Wiener process.

Theorem 1.1. *Let $d \geqslant 3$. Let $\{S_t\}$ and $\{Y(x)\}$ be defined on a rich enough probability space $\{\Omega, \mathscr{F}, \boldsymbol{P}\}$. Then there exists a real-valued Wiener process $\{W(t),\ t \geqslant 0\}$ on Ω such that for any $\varepsilon > 0$ we have*

$$\left| X(T) - \left(\frac{2-\gamma}{\gamma} \right)^{1/2} W(T) \right| = \mathrm{o}(T^{\vartheta + \varepsilon}) \quad a.s.,$$

where

$$\vartheta = \max\left(\frac{1}{p}, \frac{5}{12} \right)$$

and the constant $\gamma = \gamma(d)$ is defined in Lemma 2.3.

Note that Theorem 1.1 readily implies

$$\lim_{T \to \infty} \boldsymbol{P} \left\{ \left(\frac{\gamma}{2-\gamma} \right)^{1/2} \frac{X(T)}{\sqrt{T}} < y \right\} = \Phi(y), \quad y \in \mathbb{R},$$

(where $\Phi(\cdot)$ is the standard normal distribution function) and

$$\limsup_{T \to \infty} \left(\frac{\gamma}{2-\gamma} \right)^{1/2} (2T \log\log T)^{-1/2} X(T) = 1 \quad \text{a.s.}$$

Similar strong or weak laws can also be obtained.

The rest of the paper is organized as follows. In Section 2 a few results on random walk are given. In Section 3 an invariance principle, strongly related to Theorem 1.1 is provided. Theorem 1.1 is proved in Section 4.

Throughout the paper, we assume $d \geqslant 3$, and write $\log x = \log \max(x, e)$.

2. Random walk

Introduce the following notations:

$$\xi(x, T) = \#\{t:\ 0 < t \leqslant T,\ S_t = x\}, \quad (x \in \mathbb{Z}^d,\ 0 < T \leq \infty),$$

$$I(k, t) = \begin{cases} 1 & \text{if } S_t \neq S_{t-j}\ (j = 1, 2, \ldots, t) \text{ and } \xi(S_t, \infty) = k, \\ 0 & \text{otherwise,} \end{cases}$$

$$A(k,T)=\{x\colon\ \xi(x,T)>0,\ \xi(x,\infty)=k\},$$

$$a(k,T)=\#\{x\colon\ x\in A(k,T)\}=\sum_{t=1}^{T} I(k,t)$$
$$=\#\{t\colon\ 0<t\leqslant T,\ S_t\neq S_{t-j}\ (j=1,2,\ldots,t),\ \xi(S_t,\infty)=k\},$$

$$\gamma_t=\gamma_t(d)=\boldsymbol{P}\{\xi(0,t-1)=0\},$$

$$N(t)=\{S_t\neq S_{t-j},\ j=1,2,\ldots,t\},$$

$$R(t,k)=\{\xi(S_t,\infty)-\xi(S_t,t)=k\},$$

$$V(u,v,\ell)=\{\xi(S_u,v)-\xi(S_u,u)=\ell\}\quad (0\leqslant u<v).$$

Now we recall a few known lemmas.

Lemma 2.1 (Pólya, 1921).

$$\boldsymbol{P}\{S_{2t}=0\}\sim 2\left(\frac{\mathrm{d}}{4t\pi}\right)^{d/2}\qquad (t\to\infty).$$

Lemma 2.2 (Dvoretzky and Erdős, 1951).

$$\boldsymbol{P}\{N(t)\}=\gamma_t.$$

Lemma 2.3 (Dvoretzky and Erdős, 1951). $\{\gamma_t\}$ *is a decreasing sequence with*

$$0<\gamma_t-\gamma=\mathrm{O}(t^{1-d/2}),$$

where

$$\gamma=\lim_{t\to\infty}\gamma_t.$$

Lemma 2.4 (Erdős and Taylor, 1960).

$$\boldsymbol{P}\{\xi(0,\infty)=k\}=(1-\gamma)^k\gamma.$$

The next lemmas are simple consequences of the above-presented lemmas.

Lemma 2.5.

$$\boldsymbol{P}\{\xi(0,\infty)-\xi(0,T)>0\}\leqslant\mathrm{O}(T^{1-d/2}).$$

Proof. Clearly

$$\boldsymbol{P}\{\xi(0,\infty)-\xi(0,T)>0\}\leqslant\sum_{t=T+1}^{\infty}\boldsymbol{P}\{S_t=0\}.$$

Hence we have Lemma 2.5 by Lemma 2.1. □

Lemma 2.6.

$$\boldsymbol{P}\{\xi(0,T)=k\}\leqslant\boldsymbol{P}\{\xi(0,\infty)=k\}+\mathrm{O}(T^{1-d/2})=\gamma(1-\gamma)^k+\mathrm{O}(T^{1-d/2}).$$

Proof. Since

$$\{\xi(0,T)=k\}\subset\{\xi(0,\infty)=k\}\cup\{\xi(0,\infty)-\xi(0,T)>0\},$$

we have Lemma 2.6 by Lemmas 2.5 and 2.4. $\square$

Lemma 2.7.

$$\boldsymbol{P}\{I(k,t)=1\}=\boldsymbol{E}I(k,t)=\gamma^2(1-\gamma)^{k-1}+(1-\gamma)^{k-1}\mathrm{O}(t^{1-d/2}),$$

where $\mathrm{O}(\cdot)$ *does not depend on* k.

Proof. Observe that

$$\{I(k,t)=1\}=N(t)\cap\{\xi(S_t,\infty)=k\}$$

and

$$\boldsymbol{P}\{N(t)\cap\xi(S_t,\infty)=k\}=\boldsymbol{P}\{N(t)\}\boldsymbol{P}\{\xi(0,\infty)=k-1\}.$$

Hence we have Lemma 2.7 by Lemmas 2.2–2.4. $\square$

Lemma 2.8. *For any* $0<u<v$ *and* $k=1,2,\ldots$ *we have*

$$\begin{aligned}\boldsymbol{P}\{I(k,u)I(k,v)=1\}=\gamma^4(1-\gamma)^{2k-2}+(1-\gamma)^{2k-2}\mathrm{O}(u^{1-d/2})\\ +(1-\gamma)^{k-1}\mathrm{O}((v-u)^{1-d/2}).\end{aligned}$$

Proof. Write $r=r(u,v)=\lfloor(u+v)/2\rfloor$. We have

$$\{I(k,u)I(k,v)=1\}=N(u)R(u,k-1)N(v)R(v,k-1),$$

$$R(u,k-1)\subset V(u,r,k-1)\cup\{\xi(S_u,\infty)-\xi(S_u,r)>0\}.$$

Hence by Lemmas 2.2–2.5,

$$\begin{aligned}&\boldsymbol{P}\{I(k,u)I(k,v)=1\}\\ &\quad\leqslant\boldsymbol{P}\{N(u)V(u,r,k-1)\}\boldsymbol{P}\{\{S_v\neq S_j,\ r\leqslant j\leqslant v-1\}\cap R(v,k-1)\}\\ &\qquad+\boldsymbol{P}\{\{\xi(S_u,\infty)-\xi(S_u,r)>0\}\cap R(v,k-1)t\}\\ &\quad\leqslant\boldsymbol{P}\{N(u)V(u,r,k-1)\}\gamma_{v-r}(1-\gamma)^{k-1}\gamma\\ &\qquad+\boldsymbol{P}\{\xi(0,\infty)-\xi(0,r-u)>0\}(1-\gamma)^{k-1}\gamma\\ &\quad=\boldsymbol{P}\{N(u)V(u,r,k-1)\}\gamma^2(1-\gamma)^{k-1}+(1-\gamma)^{k-1}\mathrm{O}((v-u)^{1-d/2}).\end{aligned}$$

Now we have Lemma 2.8 by Lemmas 2.2, 2.3 and 2.6. $\square$

The following lemma is a trivial consequence of Lemma 2.7.

Lemma 2.9.

$$\boldsymbol{E}a(k,T)=T\gamma^2(1-\gamma)^{k-1}+(1-\gamma)^{k-1}M(T),$$

where

$$M(T)=M_d(T)=\begin{cases}\mathrm{O}(T^{1/2}) & \text{if } d=3,\\ \mathrm{O}(\log T) & \text{if } d=4,\\ \mathrm{O}(1) & \text{if } d\geqslant 5.\end{cases}$$

Lemma 2.10.

$$\operatorname{Var} a(k,T) \leqslant (1-\gamma)^{k-1} T M(T).$$

Proof. By Lemma 2.8 we have

$$\sum_{0<u<v\leqslant T} \boldsymbol{E} I(k,u) I(k,v) \leqslant \binom{T}{2} \gamma^4 (1-\gamma)^{2k-2} + (1-\gamma)^{k-1} T M(T).$$

By Lemma 2.9

$$(\boldsymbol{E} a(k,T))^2 = T^2 \gamma^4 (1-\gamma)^{2k-2} + (1-\gamma)^{2k-2} T M(T).$$

Hence we have Lemma 2.10. □

Let

$$\mathscr{A}(k,T,\alpha) = \{|a(k,T) - T\gamma^2(1-\gamma)^{k-1}| \geqslant T^{3/4+\alpha}(1-\gamma)^{(k-1)/4}\}, \tag{2.1}$$

$$\mathscr{B}(T,\alpha) = \bigcup_{k=1}^{\infty} \mathscr{A}(k,T,\alpha). \tag{2.2}$$

Then we have

Lemma 2.11.

$$\boldsymbol{P}\{\mathscr{A}(k,T,\alpha)\} \leqslant c_1 (1-\gamma)^{(k-1)/2} T^{-2\alpha} \quad (0 < \alpha < \tfrac{1}{4}),$$

where $c_1 = c_1(d)$ is a constant depending only on d.

Proof. By Lemmas 2.9, 2.10 and the Chebyshev inequality we have

$$\boldsymbol{P}\{|a(k,T) - T\gamma^2(1-\gamma)^{k-1}| \geqslant \lambda (1-\gamma)^{(k-1)/2} T^{3/4}\} \leqslant \frac{c_2}{\lambda^2}$$

for some $c_2 = c_2(d)$. Choosing

$$\lambda = T^{\alpha}(1-\gamma)^{-(k-1)/4}$$

we have Lemma 2.11. □

Lemma 2.11 easily implies

Lemma 2.12.

$$\boldsymbol{P}\{\mathscr{B}(T,\alpha)\} \leqslant c_1 (1-(1-\gamma)^{1/2})^{-1} T^{-2\alpha}.$$

Lemma 2.13. *There exists a sufficiently large constant $c_3 = c_3(d)$ such that for all $k \geqslant c_3 \log T$,*

$$\overline{\mathscr{A}(k,T,\alpha)} \subset \{a(k,T) = 0\},$$

where $\bar{\mathscr{A}}$ denotes the complementer of $\mathscr{A}$.

Proof. Trivial. □

Now, we present the main result of this section.

Theorem 2.1. *Among the events* $\mathscr{A}(T,\alpha)$ $(T=1,2,\dots)$ *only finitely many might occur with probability* 1 *if* $1/12<\alpha<1/4$.

Proof. Let

$$T_j=\lfloor j^K\rfloor,\quad 2K\alpha>1,\quad j=1,2,\dots\ .$$

Then by Lemma 2.12 among the events $\mathscr{B}(T_j,\alpha)$ $(j=1,2,\dots)$ only finitely many might occur with probability 1. Let $T_j\leqslant t<T_{j+1}$. Then

$$\begin{aligned}a(k,t)&\leqslant a(k,T_{j+1})\leqslant T_{j+1}\gamma^2(1-\gamma)^{k-1}+T_{j+1}^{3/4+\alpha}(1-\gamma)^{(k-1)/4}\\&\leqslant T_j\gamma^2(1-\gamma)^{k-1}+T_j^{3/4+\alpha+\varepsilon}(1-\gamma)^{(k-1)/4}\end{aligned}$$

for any $\varepsilon>0$ if j is big enough and

$$(2\alpha)^{-1}<K<\left(\frac{1}{4}-\alpha\right)^{-1}.$$

Similarly

$$\begin{aligned}a(k,t)&\geqslant a(k,T_j)\geqslant T_j\gamma^2(1-\gamma)^{k-1}-T_j^{3/4+\alpha}(1-\gamma)^{(k-1)/4}\\&\geqslant T_{j+1}\gamma^2(1-\gamma)^{k-1}-T_j^{3/4+\alpha+\varepsilon}(1-\gamma)^{(k-1)/4}\end{aligned}$$

for any $\varepsilon>0$ if j is big enough and K satisfies the above inequality. Hence we have Theorem 2.1. □

3. An invariance principle

Throughout this section, $Z_1,Z_2,\dots$ denote a sequence of i.i.d. r.v.'s with

$$\boldsymbol{E}Z_1=0,\qquad \boldsymbol{E}Z_1^2=1,\qquad \boldsymbol{E}|Z_1|^p<\infty,\quad p>2.$$

We first recall two useful results.

Lemma 3.1 (Komlós et al., 1976; Major, 1976). *Let* $\{Z_n\}$ *be defined on a rich enough probability space. Then there exists a Wiener process* $\{W(t),\ t\geqslant 0\}$ *such that*

$$\boldsymbol{P}\left\{\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^k Z_i-W(k)\right|>z\right\}<Cnz^{-p}$$

if

$$0<z<(n\log n)^{1/2},$$

where C *is a positive constant depending only on the distribution of* Z_1.

Lemma 3.2 (Csörgő and Révész, 1981, p. 24). *Let* $\{W(t),\ t\geqslant 0\}$ *be a standard Wiener process. For any* $\varepsilon>0$, *there exists a constant* $c_4=c_4(\varepsilon)$ *such that for all* $v>0$ *and all* $h\in(0,1)$,

$$\boldsymbol{P}\left(\sup_{0\leqslant s,t\leqslant 1,\,|t-s|<h}|W(s)-W(t)|\geqslant v\sqrt{h}\right)\leqslant\frac{c_4}{h}\exp\left(-\frac{v^2}{2+\varepsilon}\right).$$

The condition "$z < (n \log n)^{1/2}$" in Lemma 3.1 can be removed, as long as we replace Cnz^{-p} by a larger term. Indeed, when $n > z^2$, we have $z < (n \log n)^{1/2}$, thus Lemma 3.1 applies. On the other hand, if $n \leqslant z^2$, then $n \leqslant n^*$ where $n^* = \lfloor z^2 \rfloor + 1$. Since $z < (n^* \log n^*)^{1/2}$, we can apply the lemma to (n^*, z) in place of (n, z), to see that

$$\boldsymbol{P}\left\{ \max_{1 \leqslant k \leqslant n} \left| \sum_{i=1}^{k} Z_i - W(k) \right| > z \right\} \leqslant \boldsymbol{P}\left\{ \max_{1 \leqslant k \leqslant n^*} \left| \sum_{i=1}^{k} Z_i - W(k) \right| > z \right\}$$
$$\leqslant \frac{Cn^*}{z^p} \leqslant \frac{C(z^2+1)}{z^p}.$$

Therefore, whenever $n \geqslant 1$ and $z > 0$,

$$\boldsymbol{P}\left\{ \max_{1 \leqslant k \leqslant n} \left| \sum_{i=1}^{k} Z_i - W(k) \right| > z \right\} \leqslant c_5 \frac{\max(n, z^2)}{z^p}.$$

So we arrive at the following form of Lemm 3.1 which is more appropriate for applications later.

Lemma 3.3. *After possible redefinitions of variables, there exists a Wiener process $\{W(t),\ t \geqslant 0\}$ such that for all $n \geqslant 1$ and all $z > 0$,*

$$\boldsymbol{P}\left\{ \max_{1 \leqslant k \leqslant n} \left| \sum_{i=1}^{k} Z_i - W(k) \right| > z \right\} \leqslant c_5 \frac{n + z^2}{z^p},$$

where c_5 is a finite constant depending on the law of Z_1.

Now we turn to the study of $X(T) = \sum_{t=0}^{T} Y(S_t)$, the random walk in random scenery which was introduced in (1.1). We recall that the simple random walk $\{S_i,\ i = 0, 1, 2, \ldots\}$ is independent of the random scenery $\{Y(x),\ x \in \mathbb{Z}^d\}$.

We first look at $\sum_{x \in A(k,T)} Y(x)$ (for the definition of $A(k, T)$, see Section 2). According to Lemma 3.3, conditionally on $\{S_i,\ i = 0, 1, 2, \ldots\}$, there exists a sequence of independent Wiener processes (depending on $S_i,\ i=0, 1, 2, \ldots$), denoted by $\{W_k(\cdot)\}_{k \geqslant 1}$, such that for all $y > 0$,

$$\boldsymbol{P}\left\{ \max_{1 \leqslant t \leqslant T} \left| \sum_{x \in A(k,t)} Y(x) - W_k(a(k,t)) \right| > y \,\middle|\, S_i,\ i = 0, 1, 2, \ldots \right\}$$
$$\leqslant c_5 \frac{a(k,T) + y^2}{y^p}. \tag{3.1}$$

We mention that both $A(k, t)$ and $a(k, t)$ are measurable with respect to $\sigma\{S_i,\ i = 0, 1, 2, \ldots\}$.

Let $\alpha \in (0, \frac{1}{4})$. Recall $\mathscr{A}(k, T, \alpha)$ from (2.1) in Section 2, and write $\mathscr{B}(T, \alpha) = \bigcup_{k=1}^{\infty} \mathscr{A}(k, T, \alpha)$ as before.

Fix $r > 1/p$ and let

$$B_k = \left\{ \max_{1 \leqslant t \leqslant T} \left| \sum_{x \in A(k,t)} Y(x) - W_k(a(k,t)) \right| > T^r \right\}.$$

It is clear that

$$\boldsymbol{P}\left\{\bigcup_{k=1}^{\infty} B_k\right\} \leqslant \boldsymbol{P}\{\mathscr{B}(T,\alpha)\} + \sum_{k=1}^{\infty} \boldsymbol{P}\{\overline{\mathscr{A}(k,T,\alpha)} \cap B_k\}.$$

By Lemma 2.13, when $k \geqslant c_3 \log T$, on $\overline{\mathscr{A}(k,T,\alpha)}$ we have $a(k,T)=0$ (thus $A(k,t)=\emptyset$ for any $t \leqslant T$), which implies $B_k=\emptyset$. Therefore $\overline{\mathscr{A}(k,T,\alpha)} \cap B_k = \emptyset$ for all $k \geqslant c_3 \log T$. On the other hand, Lemma 2.12 guarantees $\boldsymbol{P}\{\mathscr{B}(T,\alpha)\} \leqslant c_6/T^{2\alpha}$. Accordingly,

$$\boldsymbol{P}\left\{\bigcup_{k=1}^{\infty} B_k\right\} \leqslant \frac{c_6}{T^{2\alpha}} + \sum_{k=1}^{\lfloor c_3 \log T\rfloor} \boldsymbol{P}\{\overline{\mathscr{A}(k,T,\alpha)} \cap B_k\}. \tag{3.2}$$

Observe that each $\mathscr{A}(k,T,\alpha)$ is measurable with respect to $\sigma\{S_i,\ i=0,1,2,\ldots\}$. Moreover, on $\overline{\mathscr{A}(k,T,\alpha)}$, we have

$$a(k,T) \leqslant T\gamma^2(1-\gamma)^{k-1} + T^{3/4+\alpha}(1-\gamma)^{(k-1)/4} \leqslant 2T(1-\gamma)^{(k-1)/4}.$$

In view of inequality (3.1), we have

$$\begin{aligned}\boldsymbol{P}\{\overline{\mathscr{A}(k,T,\alpha)} \cap B_k \mid S_i,\ i=0,1,2,\ldots\} &\leqslant c_5 \frac{2T(1-\gamma)^{(k-1)/4} + T^{2r}}{T^{rp}} \\ &= \frac{2c_5(1-\gamma)^{(k-1)/4}}{T^{rp-1}} + \frac{c_5}{T^{r(p-2)}}.\end{aligned}$$

Going back to (3.2), we obtain

$$\begin{aligned}\boldsymbol{P}\left\{\bigcup_{k=1}^{\infty} B_k\right\} &\leqslant \frac{c_6}{T^{2\alpha}} + \frac{2c_5}{T^{rp-1}} \sum_{k=1}^{\infty} (1-\gamma)^{(k-1)/4} + \frac{c_5 c_3 \log T}{T^{r(p-2)}} \\ &\leqslant \frac{c_6}{T^{2\alpha}} + \frac{c_7}{T^{rp-1}} + \frac{c_8 \log T}{T^{r(p-2)}}.\end{aligned}$$

Since $r > 1/p$, we can choose a constant b such that

$$b > \max\left(\frac{1}{2\alpha}, \frac{1}{rp-1}, \frac{1}{r(p-2)}\right).$$

Take $T = T_\ell = \lfloor \ell^b \rfloor$ (for $\ell = 1,2,\ldots$). By the Borel–Cantelli lemma, almost surely for all large ℓ and all $k \geqslant 1$,

$$\max_{1 \leqslant t \leqslant T_\ell} \left| \sum_{x \in A(k,t)} Y(x) - W_k(a(k,t)) \right| \leqslant (T_\ell)^r.$$

Let $T \in [T_\ell, T_{\ell+1}]$. For large T and all $k \geqslant 1$,

$$\begin{aligned}\max_{1 \leqslant t \leqslant T} \left| \sum_{x \in A(k,t)} Y(x) - W_k(a(k,t)) \right| &\leqslant \max_{1 \leqslant t \leqslant T_{\ell+1}} \left| \sum_{x \in A(k,t)} Y(x) - W_k(a(k,t)) \right| \\ &\leqslant (T_{\ell+1})^r \leqslant 2T^r.\end{aligned}$$

Since $\{W_k(\cdot)\}_{k \geqslant 1}$ are independent Wiener processes under the conditional probability $\boldsymbol{P}\{\cdot \mid S_i,\ i=0,1,2,\ldots\}$, they are also independent Wiener processes under the absolute probability $\boldsymbol{P}$. As $r > 1/p$ is arbitrary, we have succeeded in proving the following: for any $\varepsilon > 0$, when T goes to infinity,

$$\sup_{k\geqslant 1}\left|\sum_{x\in A(k,T)} Y(x) - W_k(a(k,T))\right| = \mathrm{o}(T^{1/p+\varepsilon}) \quad \text{a.s.} \tag{3.3}$$

Now we need to control the increment $W_k(a(k,T)) - W_k(T\gamma^2(1-\gamma)^{k-1})$. Fix any $\frac{1}{12} < \alpha < \frac{1}{4}$. On $\overline{\mathscr{B}(T,\alpha)}$, $|a(k,T) - T\gamma^2(1-\gamma)^{k-1}| < T^{3/4+\alpha}$, which yields

$$|W_k(a(k,T)) - W_k(T\gamma^2(1-\gamma)^{k-1})| \leqslant \Delta_k(T),$$

where

$$\Delta_k(T) = \sup_{0\leqslant s,t\leqslant 2T, |t-s| < T^{3/4+\alpha}} |W_k(s) - W_k(t)|.$$

Now using the scaling property of the Wiener process, and applying Lemma 3.2 to $h = T^{\alpha-1/4}/2$, $\varepsilon = 1$ and $v = 3\sqrt{\log T}$,

$$\begin{aligned}
&\boldsymbol{P}(\Delta_k(T) \geqslant 3T^{\alpha/2+3/8}(\log T)^{1/2}) \\
&\quad = \boldsymbol{P}\left(\sup_{0\leqslant s,t\leqslant 1, |t-s| < T^{\alpha-1/4}/2} |W_k(s) - W_k(t)| \geqslant \frac{3}{\sqrt{2}} T^{\alpha/2-1/8}(\log T)^{1/2}\right) \\
&\quad \leqslant c_9 T^{-\alpha-11/4},
\end{aligned}$$

where c_9 is an absolute constant. As a consequence,

$$\boldsymbol{P}\left(\max_{1\leqslant k\leqslant T} \Delta_k(T) \geqslant 3T^{\alpha/2+3/8}(\log T)^{1/2}\right) \leqslant c_9 T^{-\alpha-7/4},$$

which, by virtue of the Borel–Cantelli lemma, yields that, for any $\frac{1}{12} < \alpha < \frac{1}{4}$,

$$\max_{1\leqslant k\leqslant T} \Delta_k(T) = \mathrm{O}(T^{\alpha/2+3/8}(\log T)^{1/2}) \quad \text{a.s.}$$

Since $|W_k(a(k,T)) - W_k(T\gamma^2(1-\gamma)^{k-1})| \leqslant \Delta_k(T)$ on $\overline{\mathscr{B}(T,\alpha)}$, and since by Theorem 2.1 almost surely only finitely many events among $\{\mathscr{B}(T,\alpha)\}_{T=1,2,\dots}$ can occur, we can go back to (3.3) to see that, for any $\varepsilon > 0$, with probability one,

$$\max_{1\leqslant k\leqslant T}\left|\sum_{x\in A(k,T)} Y(x) - W_k(T\gamma^2(1-\gamma)^{k-1})\right| = \mathrm{o}(T^{\vartheta+\varepsilon}), \tag{3.4}$$

where $\vartheta = \max(1/p, \frac{5}{12})$.

For large k, it is easy to estimate the difference between $\sum_{x\in A(k,T)} Y(x)$ and $W_k(T\gamma^2(1-\gamma)^{k-1})$. More precisely, we now prove that both terms are very small when k is a large constant multiple of $\log T$. We first look at $\sum_{x\in A(k,T)} Y(x)$. Let $\frac{1}{12} < \alpha < \frac{1}{4}$. Again, owing to Theorem 2.1, we only have to work on the set $\overline{\mathscr{B}(T,\alpha)} = \bigcap_{k=1}^{\infty} \overline{\mathscr{A}(k,T,\alpha)}$. Recall that by Lemma 2.13, for $k \geqslant c_3 \log T$, on $\overline{\mathscr{A}(k,T,\alpha)}$, we have $a(k,T) = 0$, i.e., $A(k,T) = \emptyset$. Consequently, with probability one, for all large T,

$$\sum_{x\in A(k,T)} Y(x) = 0, \quad k \geqslant c_3 \log T. \tag{3.5}$$

On the other hand, we can choose $c_{10}=c_{10}(d)$ sufficiently large so that $(1-\gamma)^{(k-1)/12}\leqslant 1/T$ for all $k\geqslant c_{10}\log T$. Accordingly,

$$\begin{aligned}
&\boldsymbol{P}\left(\sup_{k\geqslant c_{10}\log T}\frac{|W_k(T\gamma^2(1-\gamma)^{k-1})|}{(1-\gamma)^{(k-1)/4}}\geqslant 1\right)\\
&\leqslant \boldsymbol{P}\left(\sup_{k\geqslant c_{10}\log T}\frac{|W_k(T\gamma^2(1-\gamma)^{k-1})|}{T(1-\gamma)^{(k-1)/3}}\geqslant 1\right)\\
&\leqslant \sum_{k\geqslant c_{10}\log T}\boldsymbol{P}(|W_k(T\gamma^2(1-\gamma)^{k-1})|>T(1-\gamma)^{(k-1)/3})\\
&\leqslant \sum_{k\geqslant c_{10}\log T}\exp\left(-\frac{T}{2\gamma^2(1-\gamma)^{(k-1)/3}}\right),
\end{aligned}$$

which is summable for $T\geqslant 1$. By the Borel–Cantelli lemma, uniformly for all $k\geqslant c_{10}\log T$,

$$W_k(T\gamma^2(1-\gamma)^{k-1})=\mathrm{O}((1-\gamma)^{(k-1)/4})\quad \text{a.s.}$$

Let $c_{11}=c_{10}+c_3$. Taking (3.5) into account, we obtain: uniformly for $k\geqslant c_{11}\log T$,

$$\sum_{x\in A(k,T)}Y(x)-W_k(T\gamma^2(1-\gamma)^{k-1})=\mathrm{O}((1-\gamma)^{(k-1)/4})\quad \text{a.s.}$$

This, jointly considered with (3.4), yields

$$\sum_{k=1}^{\infty}k\sum_{x\in A(k,T)}Y(x)-\sum_{k=1}^{\infty}kW_k(T\gamma^2(1-\gamma)^{k-1})=\mathrm{o}(T^{\vartheta+\varepsilon})\quad \text{a.s.}$$

Take

$$W(T)=\left(\frac{\gamma}{2-\gamma}\right)^{1/2}\sum_{k=1}^{\infty}kW_k(T\gamma^2(1-\gamma)^{k-1}),\quad T\geqslant 0,$$

which is again a standard Wiener process. We have therefore proved the following:

Theorem 3.1. *There exists a Wiener process $\{W(t),\ t\geqslant 0\}$ such that for any $\varepsilon>0$*

$$\sum_{k=1}^{\infty}k\sum_{x\in A(k,T)}Y(x)-\left(\frac{2-\gamma}{\gamma}\right)^{1/2}W(T)=\mathrm{o}(T^{\vartheta+\varepsilon})\quad \textit{a.s.},$$

where $\vartheta=\max(1/p,\frac{5}{12})$.

We call your attention to the following corollary of Theorems 1.1 and 3.1.

Corollary. *For any $\varepsilon>0$, as $T\to\infty$,*

$$X(T)-\sum_{k=1}^{\infty}k\sum_{x\in A(k,T)}Y(x)=\mathrm{o}(T^{\vartheta+\varepsilon})\quad \textit{a.s.},$$

where ϑ is as in Theorem 1.1 and

$$A(k,T)=\{x\colon\ \xi(x,T)>0,\ \xi(x,\infty)=k\}.$$

4. Proof of Theorem 1.1

Let $\{S_t,\ t=0,1,2,\ldots\}$ be a simple, symmetric random walk in $\mathbb{Z}^d (d \geqslant 3)$, independent of the scenery $\{Y(x),\ x \in \mathbb{Z}^d\}$ which is an array of i.i.d. r.v.'s with

$$\boldsymbol{E}Y(x) = 0, \qquad \boldsymbol{E}Y^2(x) = 1, \qquad \boldsymbol{E}|Y(x)|^p < \infty \quad \text{for some } p > 2.$$

To prove Theorem 1.1, we assume $p \leqslant \frac{12}{5}$ without loss of generality (otherwise, we replace p by $\min(p, \frac{12}{5})$).

Introduce the following notations:

$$J_T(k,t) = \begin{cases} 1 & \text{if } S_t \neq S_{t-j}\ (j = 1,2,\ldots t) \text{ and } \xi(S_t, T) = k \\ 0 & \text{otherwise,} \end{cases}$$

$$B(k,T) = \{x\colon \xi(x,T) = k\},$$

$$\xi^*(T) = \sup_{x \in \mathbb{Z}^d} \xi(x,T),$$

$$V(T) = \sum_{i,j=0}^{T} \mathbf{1}_{\{S_i = S_j\}},$$

where $\mathbf{1}_A$ is the indicator of A. Note that $T \leqslant V(T) \leqslant T\xi^*(T)$.

Lemma 4.1. *There exists a constant* $c_{12} = c_{12}(d) \in (0,\infty)$, *depending only on* d, *such that for all* $T \geqslant 1$ *and* $y \geqslant 1$,

$$\boldsymbol{P}\{\xi^*(T) \geqslant y\} \leqslant T \exp(-c_{12} y). \tag{4.1}$$

Proof. This was implicitly proved in Erdős and Taylor (1951). For any $x \in \mathbb{Z}^d$, $\xi(x,T)$ is stochastically smaller than or equal to $\xi(0,T)$. Therefore,

$$\boldsymbol{P}\{\xi^*(T) \geqslant y\} \leqslant T \sup_{x \in \mathbb{Z}^d} \boldsymbol{P}\{\xi(0,T) \geqslant y\}.$$

Now (4.1) follows from Lemma 2.4. $\square$

Lemma 4.2. *Let* $0 < t \leqslant T$. *Then*

$$\boldsymbol{P}\{I(k,t) \neq J_T(k,t)\} = (1-\gamma)^{k/2} \mathrm{O}((T-t)^{1-d/2}),$$

where as before $\mathrm{O}(\cdot)$ *does not depend on* k.

Proof. Let

$$v_1 = v_1(t,T) = \max\left\{m\colon t \leqslant m \leqslant \frac{T+t}{2},\ S_m = S_t\right\},$$

$$v_2 = v_2(t,T) = \min\left\{m\colon m > \frac{T+t}{2},\ S_m = S_t\right\}$$

with the usual convention that $\min \emptyset = \infty$. Note that $v_2 = \infty$ with positive probability. We have

$$\{I(k,t) \neq J_T(k,t)\} = E_1 \cup E_2, \tag{4.2}$$

where

$$E_1 = \left\{ I(k,t) \neq J_T(k,t),\ \xi(S_t, v_1) \geqslant \frac{k}{2} \right\},$$

$$E_2 = \left\{ I(k,t) \neq J_T(k,t),\ \xi(S_t, v_1) < \frac{k}{2} \right\}.$$

We call S_t to be a new site if $S_t \neq S_{t-j}$ for all $j = 1, 2, \ldots, t$. Observe that

$$E_1 \subseteq \left\{ S_t \text{ is a new site},\ \xi(S_t, v_1) \geqslant \frac{k}{2},\ \exists\, u \geqslant T,\ S_u = S_{v_1} = S_t \right\}$$

$$\subseteq \{S_t \text{ is a new site},\ S_t \text{ is visited at least } (k/2) \text{ times up to } v_1,$$

$$\text{and the random walk returns to } S_{v_1} \text{ at least once after } T\}.$$

By the strong Markov property and Lemmas 2.5 and 2.7 (noting that $T - v_1 \geqslant (T-t)/2$), we get

$$\boldsymbol{P}(E_1) = (1-\gamma)^{k/2} \mathrm{O}((T-t)^{1-d/2}). \tag{4.3}$$

On the other hand,

$$E_2 \subseteq \left\{ v_2 < \infty,\ \xi(S_t, \infty) - \xi(S_t, v_2) \geqslant \frac{k}{2} \right\}$$

$$\subseteq \{\text{the random walk returns to } S_t \text{ at } v_2 < \infty,$$

$$\text{the site } S_t = S_{v_2} \text{ is visited at least } (k/2) \text{ times after } v_2\},$$

which, in view of the strong Markov property and Lemmas 2.5 and 2.4 (noting that $v_2 - t \geqslant (T-t)/2$), yields

$$\boldsymbol{P}(E_2) = \mathrm{O}((T-t)^{1-d/2})(1-\gamma)^{k/2}. \tag{4.4}$$

Combining (4.2)–(4.4) implies Lemma 4.2. $\square$

Lemma 4.3. *Let*

$$Z(k,T) = \sum_{x \in A(k,T)} Y(x) - \sum_{x \in B(k,T)} Y(x) = \sum_{t=1}^{T} Y(S_t)(I(k,t) - J_T(k,t)).$$

Then we have

$$\boldsymbol{E}(Z(k,T))^2 = (1-\gamma)^{k/2} \mathrm{O}(M(T)),$$

where $M(T)$ is as in Lemma 2.9.

Proof. Write

$$C(k,T) = \{t\colon 0 < t \leqslant T,\ I(k,t) = 1,\ J_T(k,t) = 0\},$$

$$D(k,T) = \{t\colon 0 < t \leqslant T,\ I(k,t) = 0,\ J_T(k,t) = 1\},$$

$$c(k,T) = \#\{t\colon t \in C(k,T)\},$$

$$d(k,T) = \#\{t\colon t \in D(k,T)\}.$$

Then

$$Z(k,T) = \sum_{t \in C(k,T)} Y(S_t) - \sum_{t \in D(k,T)} Y(S_t).$$

Hence by Lemma 4.2

$$\begin{aligned}\boldsymbol{E}(Z(k,T))^2 &= \boldsymbol{EE}\{(Z(k,T))^2 \mid S_i,\ i=0,1,2,\ldots\}\\ &= \boldsymbol{E}(c(k,T)+d(k,T))\\ &= \mathrm{O}\left((1-\gamma)^{k/2}\sum_{t=1}^{T}(T-t)^{1-d/2}\right),\end{aligned}$$

which, in turn, implies Lemma 4.3. □

Lemma 4.4. *Let $X(T)$ be the random walk in random scenery defined in* (1.1), *and let $r>0$. If $T_n=\lfloor n^r\rfloor$, then for any $\varepsilon>0$,*

$$X(T_n)-\sum_{k=1}^{\infty}k\sum_{x\in A(k,T_n)}Y(x)=\mathrm{o}(T_n^{1/4+1/(2r)+\varepsilon}),\quad a.s.$$

Proof. We can write $X(T)$ as

$$X(T)=\sum_{k=1}^{\infty}k\sum_{x\in B(k,T)}Y(x).$$

According to Lemma 4.3,

$$\boldsymbol{E}\left[\left(X(T)-\sum_{k=1}^{\infty}k\sum_{x\in A(k,T)}Y(x)\right)^2\right]=\mathrm{O}(M(T)).$$

Since $M(T)=\mathrm{O}(T^{1/2})$, an application of Chebyshev's inequality and the Borel–Cantelli lemma immediately yields the desired result. □

Now we recall two known results. The first (Lemma 4.5), which can be found in Lewis (1993), is a maximal inequality for $X(T)$. See also Bolthausen (1989) for a detailed proof, formulated for dimension $d=2$ though valid for any dimension. The second result, borrowed from Shorack and Wellner (1986, p. 849), is a refinement of the classical Berry–Esseen inequality.

Lemma 4.5. *For $T\geqslant 1$ and $a>\sqrt{2}$,*

$$\boldsymbol{P}\left\{\max_{0\leqslant t\leqslant T}X(t)>a\sqrt{V(T)}\right\}\leqslant 2\boldsymbol{P}\{X(T)>(a-\sqrt{2})\sqrt{V(T)}\}.$$

Lemma 4.6. *Let $\{\eta_i\}_{i\geqslant 1}$ be a sequence of i.i.d. random variables with $\boldsymbol{E}(\eta_1)=0$, $\boldsymbol{E}(\eta_1^2)=1$ and $\boldsymbol{E}(|\eta_1|^p)<\infty$ for some $2<p\leqslant 3$. Then there exists a constant $c_{13}=c_{13}(p)\in(0,\infty)$, depending only on p, such that for all $x\neq 0$ and all $n\geqslant 1$,*

$$\left|\boldsymbol{P}\left\{\sum_{i=1}^{n}\eta_i>\sqrt{n}x\right\}-\boldsymbol{P}\{\mathcal{N}(0,1)>x\}\right|\leqslant c_{13}\frac{n}{(\sqrt{n}|x|)^p},$$

where $\mathcal{N}(0,1)$ denotes a standard Gaussian variable.

We present a few preliminary estimates.

Lemma 4.7. *For* $T \geqslant 1$, $v > 0$ *and* $k \geqslant 1$,

$$\boldsymbol{P}\left\{\left|\sum_{x\in B(k,T)} Y(x)\right| > v\right\} \leqslant \exp\left(-\frac{v^2}{2T}\right) + 2c_{13}\frac{T}{v^p}.$$

Proof. According to Lemma 4.6, for any $x > 0$,

$$\boldsymbol{P}\left\{\left|\sum_{i=1}^{n} \eta_i\right| > \sqrt{n}x\right\} \leqslant \boldsymbol{P}\{|\mathcal{N}(0,1)| > x\} + 2c_{13}\frac{n}{(\sqrt{n}|x|)^p}$$
$$\leqslant \exp\left(-\frac{x^2}{2}\right) + 2c_{13}\frac{n}{(\sqrt{n}|x|)^p}.$$

Let $b(k,T) = \#B(k,T)$. Conditioning on the random walk $\{S_i,\ i=0,1,2,\ldots\}$, $\sum_{x\in B(k,T)} Y(x)$ is the sum of $b(k,T)$ random variables which are i.i.d. Therefore,

$$\boldsymbol{P}\left\{\left|\sum_{x\in B(k,T)} Y(x)\right| > v\right\} \leqslant \boldsymbol{E}\exp\left(-\frac{v^2}{2b(k,T)}\right) + 2c_{13}\boldsymbol{E}\frac{b(k,T)}{v^p}.$$

This yields Lemma 4.7 by virtue of the fact that $b(k,T) \leqslant T$. $\square$

Lemma 4.8. *If* $T \geqslant 1$, $\lambda > 0$ *and* $y \geqslant 1$, *then*

$$\boldsymbol{P}\{|X(T)| > \lambda\} \leqslant T\mathrm{e}^{-c_{12}y} + y\exp\left(-\frac{\lambda^2}{2y^2T}\right) + 2c_{13}\frac{y^{p+1}T}{\lambda^p}.$$

Proof. We have

$$X(T) = \sum_{k=1}^{\xi^*(T)} k \sum_{x\in B(k,T)} Y(x).$$

Applying Lemma 4.1 implies

$$\boldsymbol{P}\{|X(T)| > \lambda\} \leqslant \boldsymbol{P}\{\xi^*(T) > y\} + \boldsymbol{P}\left\{\sum_{k=1}^{y} k\left|\sum_{x\in B(k,T)} Y(x)\right| > \lambda\right\}$$
$$\leqslant T\mathrm{e}^{-c_{12}y} + \sum_{k=1}^{y}\boldsymbol{P}\left\{\left|\sum_{x\in B(k,T)} Y(x)\right| > \frac{\lambda}{y}\right\},$$

which yields the desired estimate by applying Lemma 4.7 to $v = \lambda/y$. $\square$

Lemma 4.9. *For* $T \geqslant 1$, $z > 0$, $y \geqslant 1$ *and* $a \geqslant \sqrt{2}$,

$$\boldsymbol{P}\left\{\max_{0\leqslant t\leqslant T} |X(t)| > z\right\} \leqslant 4T\mathrm{e}^{-c_{12}y} + 4y\exp\left(-\frac{a^2}{2y^2}\right)$$
$$+ \frac{8c_{13}y^{p+1}}{a^pT^{(p-2)/2}} + 2T\exp\left(-\frac{c_{12}z^2}{4Ta^2}\right).$$

Proof. Observe that

$$P\left\{\max_{0\leqslant t\leqslant T} X(t) > z\right\} \leqslant P\left\{\max_{0\leqslant t\leqslant T} X(t) > (a+\sqrt{2})\sqrt{V(T)}\right\} + P\left\{V(T) > \frac{z^2}{(a+\sqrt{2})^2}\right\},$$

which, in view of Lemma 4.5 and the fact that $T\leqslant V(T)\leqslant T\xi^*(T)$, is

$$\leqslant 2P\{X(T) > a\sqrt{V(T)}\} + P\left\{\xi^*(T) > \frac{z^2}{T(a+\sqrt{2})^2}\right\}$$

$$\leqslant 2P\{X(T) > a\sqrt{T}\} + P\left\{\xi^*(T) > \frac{z^2}{4Ta^2}\right\}.$$

By Lemma 4.8,

$$P\{X(T) > a\sqrt{T}\} \leqslant T\mathrm{e}^{-c_{12}y} + y\exp\left(-\frac{a^2}{2y^2}\right) + \frac{2c_{13}y^{p+1}}{a^p T^{(p-2)/2}},$$

whereas by Lemma 4.1,

$$P\left\{\xi^*(T) > \frac{z^2}{4Ta^2}\right\} \leqslant T\exp\left(-\frac{c_{12}z^2}{4Ta^2}\right).$$

Assembling these pieces gives that

$$P\left\{\max_{0\leqslant t\leqslant T} X(t) > z\right\} \leqslant 2T\mathrm{e}^{-c_{12}y} + 2y\exp\left(-\frac{a^2}{2y^2}\right) + \frac{4c_{13}y^{p+1}}{a^p T^{(p-2)/2}} + T\exp\left(-\frac{c_{12}z^2}{4Ta^2}\right).$$

The same estimate holds for $P\{\max_{0\leqslant t\leqslant T}(-X(t)) > z\}$, we have proved the lemma. $\square$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Fix an arbitrary $\vartheta_1 > \vartheta = \max(1/p, \frac{5}{12}) = 1/p$ (recalling that we assume $p\leqslant\frac{12}{5}$). Let

$$r := \frac{2p}{4-p},$$

which satisfies

$$\frac{1}{4} + \frac{1}{2r} < \vartheta_1. \tag{4.5}$$

Let $\beta > 0$ be such that

$$\frac{8-3p}{2(4-p)} < \beta < \frac{(2\vartheta_1 - 1)r + 1}{2}. \tag{4.6}$$

Since $2\beta p + (r-1)(p-2) > 2$, it is possible to choose $0 < \delta < \beta$ such that

$$\beta p + \frac{(r-1)(p-2)}{2} - (p+1)\delta > 1. \tag{4.7}$$

Let $T_n = \lfloor n^r \rfloor$ for $n \geqslant 1$. Then $T_{n+1} - T_n \sim rn^{r-1}$ as $n \to \infty$. Applying Lemma 4.9 to $T = T_{n+1} - T_n$, $z = T_n^{\vartheta_1}$, $y = n^\delta$ and $a = n^\beta$ gives

$$\begin{aligned} &\boldsymbol{P}\left\{ \max_{0 \leqslant t \leqslant T_{n+1} - T_n} |X(t)| > T_n^{\vartheta_1} \right\} \\ &\quad \leqslant c_{14} n^{r-1} \exp(-c_{12} n^\delta) + 4n^\delta \exp\left(-\frac{n^{2(\beta-\delta)}}{2} \right) \\ &\qquad + \frac{c_{15}}{n^{\beta p + (r-1)(p-2)/2 - (p+1)\delta}} + c_{16} n^{r-1} \exp(-c_{17} n^{(2\vartheta_1 - 1)r + 1 - 2\beta}), \end{aligned}$$

which, in view of (4.7) and (4.6), is summable in n. Since $\max_{T_n \leqslant t \leqslant T_{n+1}} |X(t) - X(T_n)|$ is distributed as $\max_{0 \leqslant t \leqslant T_{n+1} - T_n} |X(t)|$, we obtain

$$\sum_n \boldsymbol{P}\left\{ \max_{T_n \leqslant t \leqslant T_{n+1}} |X(t) - X(T_n)| > T_n^{\vartheta_1} \right\} < \infty.$$

By the Borel–Cantelli lemma, as $n \to \infty$,

$$\max_{T_n \leqslant t \leqslant T_{n+1}} |X(t) - X(T_n)| = \mathrm{O}(T_n^{\vartheta_1}), \quad \text{a.s.}$$

This, jointly considered with Lemma 4.4, (4.5) and Theorem 3.1, yields

$$\max_{T_n \leqslant t \leqslant T_{n+1}} \left| X(t) - \left(\frac{2-\gamma}{\gamma} \right)^{1/2} W(T_n) \right| = \mathrm{O}(T_n^{\vartheta_1}), \quad \text{a.s.}$$

On the other hand, the usual Brownian increment result (see Theorem 1.2.1 of Csörgő and Révész, 1981, p. 30) tells us that, for any $\varepsilon > 0$,

$$\max_{T_n \leqslant t \leqslant T_{n+1}} |W(t) - W(T_n)| = \mathrm{O}(T_n^{(r-1)/(2r) + \varepsilon}), \quad \text{a.s.}$$

Since $(r-1)/(2r) < \vartheta_1$, we have proved that, when $T \to \infty$,

$$X(T) - \left(\frac{2-\gamma}{\gamma} \right)^{1/2} W(T) = \mathrm{O}(T^{\vartheta_1}), \quad \text{a.s.}$$

This yields Theorem 1.1, as $\vartheta_1 > \vartheta$ is arbitrary. $\square$

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