

Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials II: Airy random point field

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Abstract

We give a new sufficient condition of the quasi-Gibbs property. This result is a refinement of one given in a previous paper (Osada (in press) [18]), and will be used in a forthcoming paper to prove the quasi-Gibbs property of Airy random point fields (RPFs) and other RPFs appearing under soft-edge scaling. The quasi-Gibbs property of RPFs is one of the key ingredients to solve the associated infinite-dimensional stochastic differential equation (ISDE). Because of the divergence of the free potentials and the interactions of the finite particle approximation under soft-edge scaling, the result of the previous paper excludes the Airy RPFs, although Airy RPFs are the most significant RPFs appearing in random matrix theory. We will use the result of the present paper to solve the ISDE for which the unlabeled equilibrium state is the Airy $_{\beta}$ RPF with $\beta = 1, 2, 4$.

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1. Introduction

Let $\beta = 1, 2, 4$. The Airy $_{\beta}$ random point field (RPF), denoted by $\mu_{\text{Ai},\beta}$, is a probability measure on the configuration space over \mathbb{R} , for which the n -correlation function $\rho_{\text{Ai},2}^n$ is

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given by

$$\rho_{\text{Ai},2}^n(x_1, \dots, x_n) = \det[K_{\text{Ai},2}(x_i, x_j)]_{i,j=1}^n \quad \text{for } \beta = 2. \quad (1.1)$$

Here $K_{\text{Ai},2}(x, y)$ is a continuous kernel on \mathbb{R}^2 defined by

$$K_{\text{Ai},2}(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y} \quad (x \neq y),$$

where we set $\text{Ai}'(x) = d\text{Ai}(x)/dx$ with $\text{Ai}(\cdot)$ denoting the Airy function

$$\text{Ai}(z) = \frac{1}{2\pi} \int_{\mathbb{R}} dk e^{i(zk + k^3/3)}, \quad z \in \mathbb{C}. \quad (1.2)$$

The correlation functions of Airy $_{\beta}$ RPFs for $\beta = 1, 4$ are given similarly by using the quaternion determinant or Pfaffians (see [2,12,3]).

It is well known that $\mu_{\text{Ai},\beta}$ results in the thermodynamic limit of the distributions for the Gaussian ensembles ($\beta = 1, 2, 4$). Indeed, the distribution of eigenvalues of the Gaussian ensembles with size $n \times n$ is given by

$$m_{\text{Gauss},\beta}^n(d\mathbf{x}_n) = \frac{1}{Z} \prod_{i < j}^n |x_i - x_j|^{\beta} \exp \left\{ -\frac{\beta}{4} \sum_{i=1}^n |x_i|^2 \right\} d\mathbf{x}_n, \quad (1.3)$$

where $\mathbf{x}_n = (x_1, \dots, x_n) \in \mathbb{R}^n$. Here $\beta = 1, 2$ and 4 correspond respectively to the Gaussian orthogonal (GOE), unitary (GUE), and symplectic (GSE) ensembles. Thus, the probability density coincides with the Boltzmann factor for log-gas systems at three special values of the inverse temperature, i.e., $\beta = 1, 2$ and 4 .

Let $\mu_{\text{Gauss},\beta}^n$ be the distribution of $n^{-1} \sum \delta_{x_i}$ under $m_{\text{Gauss},\beta}^n(d\mathbf{x}_n)$. Then the celebrated semi-circle law states that $\mu_{\text{Gauss},\beta}^n$ converges to the nonrandom $\sigma(x)dx$ weakly in the space of Radon measures over \mathbb{R} endowed with the vague topology. Here

$$\sigma(x) = \frac{1}{2\pi} 1_{[-2,2]}(x) \sqrt{4 - x^2}. \quad (1.4)$$

There exist two typical thermodynamic scalings in (1.3), called bulk and soft-edge. The former (centered at the origin) is given by the correspondence $x \mapsto x/\sqrt{n}$, which yields the RPF $\mu_{\text{bulk},\beta}^n$ with labeled density $m_{\text{bulk},\beta}^n$ such that

$$m_{\text{bulk},\beta}^n(d\mathbf{x}_n) = \frac{1}{Z} \prod_{i < j}^n |x_i - x_j|^{\beta} \exp \left\{ -\frac{\beta}{4n} \sum_{i=1}^n |x_i|^2 \right\} d\mathbf{x}_n, \quad (1.5)$$

and $\mu_{\text{bulk},\beta}^n$ converges weakly to $\mu_{\text{bulk},\beta}$, the Sine $_{\beta}$ RPF. The latter, in contrast, is centered at $2\sqrt{n}$ given by the correspondence $x \mapsto 2\sqrt{n} + xn^{-1/6}$ with labeled density $m_{\text{Ai},\beta}^n$ such that

$$m_{\text{Ai},\beta}^n(d\mathbf{x}_n) = \frac{1}{Z} \prod_{i < j}^n |x_i - x_j|^{\beta} \exp \left\{ -\frac{\beta}{4} \sum_{i=1}^n |2\sqrt{n} + xn^{-1/6}|^2 \right\}. \quad (1.6)$$

The Airy RPF $\mu_{\text{Ai},\beta}$ is the weak limit of $\mu_{\text{Ai},\beta}^n$ given by $m_{\text{Ai},\beta}^n$ as $n \rightarrow \infty$. The finite particle approximation $\{\mu_{\text{Ai},\beta}^n\}$ will be used in a forthcoming paper to prove the quasi-Gibbs property for $\mu_{\text{Ai},\beta}$.

Interacting Brownian motions (IBMs) in infinite dimensions are diffusions $\mathbf{X}_t = (X_t^i)_{i \in \mathbb{Z}}$ consisting of infinitely many particles moving in \mathbb{R}^d with the effect of the external force coming from a self-potential $\Phi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$ and that of the mutual interaction coming from an interacting potential $\Psi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$ such that $\Psi(x, y) = \Psi(y, x)$.

Roughly speaking, an IBM is the stochastic dynamics of infinitely many particles described by the infinite-dimensional stochastic differential equation (ISDE) of the form

$$dX_t^i = dB_t^i - \frac{1}{2} \nabla \Phi(X_t^i) dt - \frac{1}{2} \sum_{j \in \mathbb{Z}, j \neq i} \nabla \Psi(X_t^i, X_t^j) dt \quad (i \in \mathbb{Z}). \quad (1.7)$$

The state space of the process $\mathbf{X}_t = (X_t^i)_{i \in \mathbb{Z}} \in (\mathbb{R}^d)^{\mathbb{Z}}$ by construction. Let \mathbf{X} be the configuration-valued process given by

$$\mathbf{X}_t = \sum_{i \in \mathbb{Z}} \delta_{X_t^i}. \quad (1.8)$$

Here δ_a denotes the delta measure at a and a configuration is a Radon measure consisting of a sum of delta measures. We call \mathbf{X} the labeled dynamics and \mathbf{X} the unlabeled dynamics.

The ISDE (1.7) was initiated by Lang [10,11], who studied the case $\Phi = 0$, and $\Psi(x, y) = \Psi(x - y)$, where Ψ is in $C_0^3(\mathbb{R}^d)$, superstable and regular according to Ruelle [21]. With the last two assumptions, the corresponding unlabeled dynamics \mathbf{X} has Gibbsian equilibrium states. See [22,4,24] for other works concerning the SDE (1.7).

In [13], the unlabeled diffusion was constructed using the Dirichlet form. The advantage of this method is that it gives a general and simple proof of construction. This work was followed by Alberverio et al. [1], Osada [15,14], Tanemura [25], Yoo [26] and Yoshida [27], and others. In all these, except [26] and some parts of [13], the equilibrium states are supposed to be Gibbs measures with Ruelle's class interaction potentials Ψ . Thus, the equilibrium states are described by the Dobrushin–Lanford–Ruelle (DLR) equations (see (2.9)), the usage of which plays a pivotal role in previous works.

The interaction potentials appearing in random matrix theory become logarithmic interaction potentials (2D Coulomb potentials):

$$\Psi(x, y) = -\beta \log |x - y|, \quad 0 < \beta < \infty. \quad (1.9)$$

Clearly these are not Ruelle's class potentials and the DLR equations would make no sense.

In [16–19], we have developed a general theory applicable to log potentials and solved the ISDE (1.7) with log interaction potentials. The key ingredients are two geometric properties of RPFs such that “the quasi-Gibbs property” and “the log derivative”. Although we checked these for Sine $_{\beta}$ RPFs ($\beta = 1, 2, 4$) and the Ginibre RPF in [18,19], the Airy $_{\beta}$ RPFs remain.

The purpose of this paper is to give a sufficient condition for the quasi-Gibbs property applicable to RPFs appearing under soft-edge scaling, in particular, the Airy $_{\beta}$ RPFs. We will do this in the main theorems (Theorems 2.1 and 2.2).

Let us briefly explain the main idea. The quasi-Gibbs property is a kind of existence of a locally bounded density conditioned outside (see Definition 2.1). We will prove this by uniform estimates of suitable, finite particle approximations. This finite particle system is (1.6) for the

Airy $_{\beta}$ RPFs. Note that the exponent in (1.6) is given by

$$-\frac{\beta}{4} \sum_{i=1}^n |2\sqrt{n} + n^{-1/6}x_i|^2 = -\frac{\beta}{4} \sum_{i=1}^n \{4n + n^{-1/3}|x_i|^2 + 4n^{1/3}x_i\}. \quad (1.10)$$

The term $4n$ can be absorbed in the normalizing constant, and the term $n^{-1/3}|x_i|^2$ can be neglected as $n \rightarrow \infty$. We have to prove, however, a rather precise cancellation between $e^{-\frac{\beta}{4} \sum_{i=1}^n 4n^{1/3}x_i}$ and the interaction term $\prod_{i \neq j}^n |x_i - x_j|^{\beta}$. This yields the main difficulty for the Airy $_{\beta}$ RPFs, and other RPFs under soft-edge scaling. Note that the term $4n^{1/3}x_i$ is linear in x_i ; from this, we arrive at the formulation in (2.17).

The organization of the paper is as follows. In Section 2, we describe the set-up and state the main results (Theorems 2.1 and 2.2). Sections 3–5 are devoted to the proof of Theorem 2.1. In Section 6, we give a sufficient condition for (H.3), which is the most important condition in Theorem 2.1. In Section 7, we prove Theorem 2.2, which is the special case $d = 1$ in Theorem 2.1, and we will give a convenient sufficient condition for (H.3) in this case.

2. Set-up and main results

Let S be a closed set in \mathbb{R}^d such that $0 \in S$ and $\overline{S^{\text{int}}} = S$, where S^{int} means the interior of S . Let $\mathbf{S} = \{\mathbf{s} = \sum_i \delta_{s_i}; \mathbf{s}(K) < \infty \text{ for any compact set } K\}$, where $\{s_i\}$ is a sequence in S . Then \mathbf{S} is the set of configurations on S by definition. We endow \mathbf{S} with the vague topology, under which \mathbf{S} is a Polish space.

Let μ be a probability measure on $(\mathbf{S}, \mathcal{B}(\mathbf{S}))$. We call a function ρ^n the n -correlation function of μ with respect to (w.r.t.) the Lebesgue measure if $\rho^n : S^n \rightarrow \mathbb{R}$ is a permutation invariant function such that

$$\int_{A_1^{k_1} \times \cdots \times A_m^{k_m}} \rho^n(x_1, \dots, x_n) dx_1 \cdots dx_n = \int_{\mathbf{S}} \prod_{i=1}^m \frac{\mathbf{s}(A_i)!}{(\mathbf{s}(A_i) - k_i)!} d\mu \quad (2.1)$$

for any sequence of disjoint bounded measurable subsets $A_1, \dots, A_m \subset S$ and a sequence of natural numbers k_1, \dots, k_m satisfying $k_1 + \cdots + k_m = n$. Here we set $(\mathbf{s}(A_i) - k_i)! = \infty$ if $(\mathbf{s}(A_i) - k_i) < 0$.

We assume μ satisfies the following.

(H.1) The measure μ has a locally bounded, n -correlation function ρ^n for each $n \in \mathbb{N}$.

We introduce a Hamiltonian on a bounded Borel set A as follows. For Borel measurable functions $\Phi : S \rightarrow \mathbb{R} \cup \{\infty\}$ and $\Psi : S \times S \rightarrow \mathbb{R} \cup \{\infty\}$ with $\Psi(x, y) = \Psi(y, x)$, let

$$\mathcal{H}_A^{\Phi, \Psi}(\mathbf{x}) = \sum_{x_i \in A} \Phi(x_i) + \sum_{x_i, x_j \in A, i < j} \Psi(x_i, x_j), \quad \text{where } \mathbf{x} = \sum_i \delta_{x_i}. \quad (2.2)$$

We assume $\Phi < \infty$ almost everywhere (a.e.) to avoid triviality.

For two measures ν_1, ν_2 on a measurable space (Ω, \mathcal{B}) , we write $\nu_1 \leq \nu_2$ if $\nu_1(A) \leq \nu_2(A)$, for all $A \in \mathcal{B}$. We say a sequence of finite Radon measures $\{\nu^n\}$ on a Polish space Ω converges weakly to a finite Radon measure ν if $\lim_{n \rightarrow \infty} \int f d\nu^n = \int f d\nu$, for all $f \in C_b(\Omega)$.

Throughout this paper, $\{b_r\}$ denotes an increasing sequence of natural numbers. We set

$$S_r = \{s \in S; |s| < b_r\}, \quad \mathbf{S}_r^m = \{\mathbf{s} \in \mathbf{S}; \mathbf{s}(S_r) = m\}. \quad (2.3)$$

For notational brevity, we suppress the dependence of S_r on $\{b_r\}$. We will later introduce $\tilde{S}_r = \{x \in S; |x| < r\}$ in (2.11). By definition $S_r = \tilde{S}_{b_r}$. In the proof of the main theorems, we will use S_r more frequently than \tilde{S}_r , which is the reason we have assigned the more complicated notation \tilde{S}_r to the simpler object $\{x \in S; |x| < r\}$. We set

$$\mathcal{H}_r(x) = \mathcal{H}_{S_r}^{\Phi, \Psi}(x). \quad (2.4)$$

For a subset $A \subset S$, we define the map $\pi_A : S \rightarrow S$ by $\pi_A(s) = s(A \cap \cdot)$.

Let Λ be the Poisson RPF for which the intensity is the Lebesgue measure on S . We set $\Lambda_A = \Lambda \circ \pi_A^{-1}$. By construction, Λ_A is the Poisson RPF with intensity $1_A dx$. We set $\Lambda_r = \Lambda_{S_r}$.

Definition 2.1. A probability measure μ is said to be a (Φ, Ψ) -quasi-Gibbs measure if the following holds.

(1) There exists an increasing sequence $\{b_r\}$ of natural numbers such that, for each $r, m \in \mathbb{N}$, there exists a sequence of Borel subsets $S_{r,k}^m$ satisfying

$$S_{r,k}^m \subset S_{r,k+1}^m \subset S_r^m \quad \text{for all } k, \quad \lim_{k \rightarrow \infty} \mu_{r,k}^m = \mu_r^m \quad \text{weakly}, \quad (2.5)$$

where $\mu_{r,k}^m = \mu(\cdot \cap S_{r,k}^m)$ and $\mu_r^m = \mu(\cdot \cap S_r^m)$.

(2) For all $r, m, k \in \mathbb{N}$ and $\mu_{r,k}^m$ -a.e. $s \in S$,

$$\frac{1}{c_1} e^{-\mathcal{H}_r(x)} 1_{S_r^m}(x) \Lambda_r(dx) \leq \mu_{r,k,s}^m(dx) \leq c_1 e^{-\mathcal{H}_r(x)} 1_{S_r^m}(x) \Lambda_r(dx). \quad (2.6)$$

Here, $c_1 = c_1(r, m, k, \pi_{S_r^c}(s))$ is a positive constant and $\mu_{r,k,s}^m$ is the regular conditional probability measure of $\mu_{r,k}^m$ defined by

$$\mu_{r,k,s}^m(dx) = \mu_{r,k}^m(\pi_{S_r} \in dx | \pi_{S_r^c}(s)). \quad (2.7)$$

We remark that the original definition of the quasi-Gibbs property in [18] is slightly more general than the above.

We call Φ (resp. Ψ) a free (interaction) potential. When Ψ is an interaction potential, we implicitly assume that $\Psi(x, y) = \Psi(y, x)$.

Remark 2.1. (1) By definition, $\mu_{r,k}^m((S_r^m)^c) = 0$. Since $\mu_{r,k,s}^m$ is $\sigma[\pi_{S_r^c}]$ -measurable in s , we have the disintegration of the measure $\mu_{r,k}^m$

$$\mu_{r,k}^m \circ \pi_{S_r}^{-1}(dx) = \int_S \mu_{r,k,s}^m(dx) \mu_{r,k}^m(ds). \quad (2.8)$$

(2) Let $\mu_{r,s}^m(dx) = \mu_r^m(\pi_{S_r}(s) \in dx | \pi_{S_r^c}(s))$. Recall that a probability measure μ is said to be a (Φ, Ψ) -canonical Gibbs measure if μ satisfies the DLR equation (2.9); that is, for each $r, m \in \mathbb{N}$, the conditional probability $\mu_{r,s}^m$ satisfies

$$\mu_{r,s}^m(dx) = \frac{1}{c_2} e^{-\mathcal{H}_r(x) - \Psi_r(x,s)} 1_{S_r^m}(x) \Lambda_r(dx) \quad \text{for } \mu_r^m\text{-a.e. } s. \quad (2.9)$$

Here, $0 < c_2 < \infty$ is the normalization and, for $x = \sum_i \delta_{x_i}$ and $s = \sum_j \delta_{s_j}$, we set

$$\Psi_r(x, s) = \sum_{x_i \in S_r, s_j \in S_r^c} \Psi(x_i, s_j). \quad (2.10)$$

(3) (Φ, Ψ) -canonical Gibbs measures are (Φ, Ψ) -quasi-Gibbs measures. The converse is, however, not true. When $\Psi(x, y) = -\beta \log |x - y|$ and the μ are translation invariant, the μ are not (Φ, Ψ) -canonical Gibbs measures. This is because the DLR equation does not make sense. Indeed, $|\Psi_r(x, s)| = \infty$ for μ -almost surely (a.s.) s . The point is that one can expect a cancellation between c_2 and $e^{-\Psi_r(x, s)}$ even if $|\Psi_r(x, s)| = \infty$.

(4) Unlike canonical Gibbs measures, the notion of quasi-Gibbs measures is quite flexible for free potentials. Indeed, if μ is a (Φ, Ψ) -quasi-Gibbs measure, then μ is also a $(\Phi + F, \Psi)$ -quasi-Gibbs measure for any locally bounded measurable function F . Thus, we write μ a Ψ -quasi-Gibbs measure if μ is a $(0, \Psi)$ -quasi-Gibbs measure.

We give a pair of conditions for the quasi-Gibbs property. These conditions guarantee that μ has a good finite-particle approximation $\{\mu^n\}_{n \in \mathbb{N}}$ that enables us to prove the quasi-Gibbs property. We set

$$\tilde{S}_r = \{x \in S; |x| < r\}, \quad \tilde{S}_r^n = \prod_{m=1}^n \{|x_m| < r\}. \quad (2.11)$$

(H.2) There exists a sequence of probability measures $\{\mu^n\}_{n \in \mathbb{N}}$ on S satisfying the following.

(1) The n -correlation functions ρ_n^n of μ^n satisfy

$$\lim_{n \rightarrow \infty} \rho_n^n(\mathbf{x}_n) = \rho^n(\mathbf{x}_n) \quad \text{a.e. for all } n \in \mathbb{N}, \quad (2.12)$$

$$\sup\{\rho_n^n(\mathbf{x}_n); n \in \mathbb{N}, \mathbf{x}_n \in \tilde{S}_r^n\} \leq \{c_3 n^\delta\}^n \quad \text{for all } n, r \in \mathbb{N}, \quad (2.13)$$

where $\mathbf{x}_n = (x_1, \dots, x_n) \in S^n$, $c_3 = c_3(r) > 0$, and $\delta = \delta(r) < 1$ are constants depending on $r \in \mathbb{N}$.

(2) $\mu^n(S(S) = N_n) = 1$ for each n , where $N_n \in \mathbb{N}$ are strictly increasing.

(3) μ^n is a (Φ^n, Ψ^n) -canonical Gibbs measure.

(4) There exists a sequence $\{m_\infty^n\}_{n \in \mathbb{N}}$ in \mathbb{R}^d such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \{\Phi^n(x) - m_\infty^n \cdot x\} &= \Phi(x) \quad \text{for a.e. } x, \\ \inf_{n \in \mathbb{N}} \inf_{x \in S} \{\Phi^n(x) - m_\infty^n \cdot x\} &> -\infty. \end{aligned} \quad (2.14)$$

Here \cdot denotes the standard inner product in \mathbb{R}^d .

(5) The interaction potentials $\Psi^n : S \times S \rightarrow \mathbb{R} \cup \{\infty\}$ satisfy the following.

$$\begin{aligned} \lim_{n \rightarrow \infty} \Psi^n &= \Psi \quad \text{compactly and uniformly in } C^1(S \times S \setminus \{x = y\}), \\ \inf_{n \in \mathbb{N}} \inf_{x, y \in \tilde{S}_r} \Psi^n(x, y) &> -\infty \quad \text{for all } r \in \mathbb{N}. \end{aligned} \quad (2.15)$$

Remark 2.2. For the GUE soft-edge approximation of the Airy RPF, we take $m_\infty^n = n^{1/3}$. In fact, in this case, the limit of Φ^n diverges. Hence, we substitute $m_\infty^n \cdot x$ from $\Phi^n(x)$ to make the limit finite. In a forthcoming paper, we will see that the terms $m_\infty^n \cdot x$ are cancelled by the interaction terms.

The next assumption (H.3) is a tightness condition on $\{\mu^n\}$ according to the interaction Ψ^n . Indeed, (H.3) plays the most significant role in the proof of the quasi-Gibbs property of μ . To introduce (H.3), we establish some notations.

Let $\mathbf{x} = \sum \delta_{x_i}$ and $\mathbf{y} = \sum \delta_{y_j} \in \mathbf{S}$. For $\{S_r\}$ in (2.3), we set $S_{rs} = S_s \setminus S_r$ and $S_{r\infty} = S_r^c$. For $r < s \leq t < u \leq \infty$, we set

$$\Psi_{rs,tu}^n(\mathbf{x}, \mathbf{y}) = \sum_{x_i \in S_{rs}, y_j \in S_{tu}} \Psi^n(x_i, y_j). \quad (2.16)$$

We write $\Psi_{r,st}^n = \Psi_{0r,st}^n$ and $\Psi_{r,rs}^n(\mathbf{x}, \mathbf{y}) = \Psi_{r,rs}^n(x, y)$ if $\mathbf{x} = \delta_x$.

For $r < s \leq t < u \leq \infty$, let

$$\tilde{\Psi}_{rs,tu}^n(\mathbf{x}, \mathbf{y}) = \Psi_{rs,tu}^n(\mathbf{x}, \mathbf{y}) + \left\{ \sum_{x_i \in S_{rs}} x_i \right\} \cdot (m_t^n - m_u^n). \quad (2.17)$$

We set $\tilde{\Psi}_{r,st}^n = \tilde{\Psi}_{0r,st}^n$. For $\{\Psi^n\}$, $r, k \in \mathbb{N}$, and $\{m_s^n\}$ we define $H_{r,k}$ by

$$H_{r,k} = \sum_{n=1}^{\infty} H_{r,k}^n, \quad (2.18)$$

where \sum denotes the disjoint union, and $H_{r,k}^n$ is the set defined by

$$H_{r,k}^n = \left\{ \mathbf{y} \in \mathbf{S}; y(S) = N_n, \left\{ \sup_{r < s \in \mathbb{N}} \sup_{\substack{x, w \in S_r \\ x \neq w}} \frac{|\tilde{\Psi}_{r,rs}^n(x, \mathbf{y}) - \tilde{\Psi}_{r,rs}^n(w, \mathbf{y})|}{|x - w|} \leq k \right\} \right\}.$$

By construction and the assumption (H.2) (2), we see that $H_{r,k}^m \cap H_{r,k}^n = \emptyset$ for $m \neq n$. We note that the set $H_{r,k}$ depends on $\{m_s^n\}$, although for brevity, we suppress $\{m_s^n\}$ in denoting $H_{r,k}$. The functions $\{m_s^n\}$ in (2.18) are related to the sequence $\{m_\infty^n\}$ in (4) of (H.2) by the condition (2.20) below.

(H.3) There exists a sequence $\{m_s^n\}$ in \mathbb{R}^d such that the set $H_{r,k}$ satisfies the following:

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu^n(H_{r,k}^c) = 0 \quad \text{for all } r \in \mathbb{N}, \quad (2.19)$$

$$\lim_{s \rightarrow \infty} m_s^n = m_\infty^n, \quad \sup_{n \in \mathbb{N}} |m_s^n| < \infty \quad \text{for all } s \in \mathbb{N}. \quad (2.20)$$

Remark 2.3. When $m_s^n \equiv 0$, the set $H_{r,k}$ in (2.18) equals $H_{r,k}$ in [18]. Thus, this definition is a generalization of $H_{r,k}$ in [18]. The function m_∞^n compensates the sum $(\Psi_{r,rs}^n(x, \mathbf{y}) - \Psi_{r,rs}^n(w, \mathbf{y})) / (x - w)$. For the Airy RPFs, we have no hope to ensure (H.3) without this compensation.

Theorem 2.1. Assume (H.1)–(H.3). Then μ is a (Φ, Ψ) -quasi-Gibbs measure.

We next assume $d = 1, 2$. Thus, to unify these two cases, we set $S = \mathbb{C}$. Indeed, we regard here \mathbb{R}^2 as \mathbb{C} by the natural correspondence: $\mathbb{R}^2 \ni (x, y) \mapsto x + \sqrt{-1}y \in \mathbb{C}$, and \mathbb{R} as the real axis in \mathbb{C} . Hence, we view $m_r^n = (m_{r,1}^n, m_{r,2}^n) \in \mathbb{R}^2$ as $m_r^n = m_{r,1}^n + \sqrt{-1}m_{r,2}^n \in \mathbb{C}$.

We assume Ψ^n is independent of n and of the form

$$\Psi(x, y) := \Psi^n(x, y) = -\beta \log |x - y| \quad (\beta \in \mathbb{R}). \quad (2.21)$$

We will give a sufficient condition of (H.3) in terms of correlation functions.

Let $x = \sum_i \delta_{x_i}$ and $\tilde{S}_{rs} = \tilde{S}_s \setminus \tilde{S}_r$, where $\tilde{S}_r = \{s \in S; |s| < r\}$, as before. For $1 \leq r < s \leq \infty$ let $v_{\ell,rs} : S \rightarrow \mathbb{C}$ such that

$$v_{\ell,rs}(x) = \beta \left\{ \sum_{x_i \in \tilde{S}_{rs}} \frac{1}{x_i^\ell} \right\} \quad (\ell \geq 2) \quad (2.22)$$

$$v_{1,rs}(x) = \beta \left\{ \sum_{x_i \in \tilde{S}_{rs}} \frac{1}{x_i} \right\} + \bar{m}_r^n - \bar{m}_s^n \quad (\ell = 1). \quad (2.23)$$

Here $\bar{m}_r^n = m_{r,1}^n - \sqrt{-1}m_{r,2}^n$ is the complex conjugate of m_r^n .

Note that the sum in (2.22) makes sense for μ^n -a.s. x even if $s = \infty$. Indeed, by (2) of (H.2), the total number of particles is N_n under μ^n . Hence, $v_{\ell,rs}(x)$ is well defined and finite for μ^n -a.s. x , for all $n \in \mathbb{N}$.

Now the key assumption is as follows.

(H.4) There exists an ℓ_0 such that $2 \leq \ell_0 \in \mathbb{N}$ and that

$$\sup_{n \in \mathbb{N}} \left\{ \int_{1 \leq |x| < \infty} \frac{1}{|x|^{\ell_0}} \rho_n^1(x) dx \right\} < \infty \quad (2.24)$$

and that, for each $1 \leq \ell < \ell_0$,

$$\lim_{s \rightarrow \infty} \sup_{n \in \mathbb{N}} \|v_{\ell,s\infty}\|_{L^1(S, \mu^n)} = 0. \quad (2.25)$$

Theorem 2.2. Assume (2.21) and $S = \mathbb{C}$. Assume (H.1), (H.2) and (H.4). Assume (2.20). Then μ is a (Φ, Ψ) -quasi-Gibbs measure.

In a forthcoming paper, we will prove the quasi-Gibbs property of Airy_β RPFs, and solve the associated ISDEs. Theorem 2.2 will be used there. Whenever we consider the RPFs appearing under soft-edge scaling, such as Tacknode [5], the divergence of the free potentials such as (1.10) always occurs, which causes a difficulty in treating soft-edge scaling. It is plausible that our results can resolve this.

Stochastic dynamics of infinitely many particle systems in \mathbb{R} related to random matrix theory have been constructed by explicit calculation based on space-time correlation functions (see [6–9,20,23], and others). In this body of work, the properties of dynamics from a viewpoint of stochastic analysis, such as the semi-martingale property, and Ito's formula, have not yet been well developed. Our method, together with the forthcoming paper, gives SDE representations of the dynamics, which enables us to use the stochastic analysis effectively.

In [26], Yoo proved that all determinantal RPFs with kernels K such that the spectrum $\text{Spec}(K)$ of the associated L^2 -operator satisfies $0 < \text{Spec}(K) < 1$ become a kind of Gibbs measure, and by using this, he constructed associated diffusions. However, the spectrum of kernels of determinantal RPFs appearing in random matrix theory in the infinite-volume limit usually contains 1. Hence, his result excludes RPFs related to random matrix theory such as

Dyson's model, the Bessel RPF, and, in particular, the Airy RPF. It is an interesting open problem to prove that all determinantal RPFs are quasi-Gibbs measures.

3. Proof of Theorem 2.1

In Sections 3–5, we will prove Theorem 2.1. In the present section, we first prepare a lemma from [18], and explain the strategy of the proof of Theorem 2.1. In fact, we divide the proof into two parts. We will prove the first step (3.17) in Section 4, and the second step (3.18) in Section 5.

We fix $r, m \in \mathbb{N}$ throughout Sections 3–5. Let S_r^m be as in (2.3). Using the set $H_{r,k}$ defined in (2.18), we introduce cut-off measures $\mu_{r,k}^{n,m}$:

$$\mu_{r,k}^{n,m} = \mu^n(\cdot \cap S_r^m \cap H_{r,k}). \quad (3.1)$$

Since $\mu^n(H_{r,k}^n) = 1$ and $H_{r,k}^n \subset H_{r,k}$, we have $\mu_{r,k}^{n,m} = \mu^n(\cdot \cap S_r^m \cap H_{r,k}^n)$.

We will prove Theorem 2.1 along the sequence $\{\mu_{r,k}^{n,m}\}$. For this purpose, we first note the following.

Lemma 3.1 (Lemma 4.2 in [18]). *There exists a weak convergent subsequence of $\{\mu_{r,k}^{n,m}\}$, denoted by the same symbol, with limit measures $\{\mu_{r,k}^m\}$ satisfying (2.5), for all r, k, m .*

Let $\mu_{r,k,S,rs}^{n,m}$ denote the conditional probability of $\mu_{r,k}^{n,m}$ defined by

$$\mu_{r,k,S,rs}^{n,m}(d\mathbf{x}) = \mu_{r,k}^{n,m}(\pi_{S_r} \in d\mathbf{x} | \pi_{S_{rs}}(\mathbf{S})). \quad (3.2)$$

We note that, although $\mu_{r,k}^{n,m}$ is not necessarily a probability measure, we normalize it in such a way that the conditional measure $\mu_{r,k,S,rs}^{n,m}$ is a probability measure. As a result, we have $\mu_{r,k,S,rs}^{n,m}(\mathbf{S}) = 1$ and

$$\mu_{r,k}^{n,m} \circ \pi_{S_r}^{-1}(d\mathbf{x}) = \int_{\mathbf{S}} \mu_{r,k,S,rs}^{n,m}(d\mathbf{x}) \mu_{r,k}^{n,m} \circ \pi_{S_{rs}}^{-1}(d\mathbf{S}). \quad (3.3)$$

Recall that by (H.2), μ^n is a (Φ^n, Ψ^n) -canonical Gibbs measure. Then μ^n satisfies the DLR equation (2.9). Hence, $\mu_{r,k,S,rs}^{n,m}$ is absolutely continuous w.r.t. $e^{-\mathcal{H}_r^n(\mathbf{x})} \Lambda_r(d\mathbf{x})$. Therefore, we denote its density by $\sigma_{r,k,S,rs}^{n,m}$. Then by definition, we have for $\mu_{r,k}^{n,m}$ -a.e. \mathbf{s}

$$\sigma_{r,k,S,rs}^{n,m}(\mathbf{x}) e^{-\mathcal{H}_r^n(\mathbf{x})} \Lambda_r(d\mathbf{x}) = \mu_{r,k,S,rs}^{n,m}(d\mathbf{x}), \quad \text{where } \mathcal{H}_r^n = \mathcal{H}_{S_r}^{\Phi^n, \Psi^n}. \quad (3.4)$$

We recall that the limit $\lim_{n \rightarrow \infty} \Phi^n$ diverges in general. Such a divergence implies that for \mathcal{H}_r^n . Thus, to prevent this, we consider the compensation constant m_∞^n in (2.14), and set

$$\tilde{\mathcal{H}}_r^n(\mathbf{x}) = \sum_{x_i \in S_r} \{\Phi^n(x_i) - m_\infty^n \cdot x_i\} + \sum_{x_i, x_j \in S_r, i < j} \Psi^n(x_i, x_j), \quad (3.5)$$

where $\mathbf{x} = \sum_i \delta_{x_i}$.

Lemma 3.2. $\tilde{\mathcal{H}}_r^n$ satisfy the following.

$$\lim_{n \rightarrow \infty} e^{-\tilde{\mathcal{H}}_r^n(\mathbf{x})} = e^{-\mathcal{H}_r(\mathbf{x})} \quad \text{for } \mu\text{-a.e. } \mathbf{x}, \quad (3.6)$$

$$\left\{ \sup_{n \in \mathbb{N}} \sup_{\mathbf{x} \in S_r^m} e^{-\tilde{\mathcal{H}}_r^n(\mathbf{x})} \right\} < \infty \quad \text{for each } m \in \mathbb{N}. \quad (3.7)$$

Proof. Lemma 3.2 follows immediately from (2.14) and (2.15) combined with (3.5). \square

We estimate the Boltzmann constants for the Hamiltonians $\tilde{\mathcal{H}}_r^n$.

Lemma 3.3. Let $c_4(n)$ be the constant defined by

$$c_4(n) = \sup_{n \in \mathbb{N}} \max \left\{ \int_{\mathbb{S}_r^m} e^{-\tilde{\mathcal{H}}_r^n(\mathbf{x})} \Lambda(d\mathbf{x}), \frac{1}{\int_{\mathbb{S}_r^m} e^{-\tilde{\mathcal{H}}_r^n(\mathbf{x})} \Lambda(d\mathbf{x})} \right\}.$$

Here $\mathbf{x} = \sum_{i=1}^m \delta_{x_i}$. Then there exists an n_0 such that $c_4(n_0) < \infty$.

Proof. We deduce from (3.6), (3.7), and the bounded convergence theorem that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{S}_r^m} e^{-\tilde{\mathcal{H}}_r^n(\mathbf{x})} \Lambda(d\mathbf{x}) = \int_{\mathbb{S}_r^m} e^{-\mathcal{H}_r(\mathbf{x})} \Lambda(d\mathbf{x}) < \infty.$$

Recall that $\Phi(x) < \infty$ a.e. by assumption (see the line after (2.2)) and $\Psi(x, y) < \infty$ a.e. by the first assumption of (2.15). Then we see that $\mathcal{H}_r(\mathbf{x}) < \infty$ a.e. Hence,

$$\int_{\mathbb{S}_r^m} e^{-\mathcal{H}_r(\mathbf{x})} \Lambda(d\mathbf{x}) > 0.$$

Combining these completes the proof. \square

Taking Lemmas 3.2 and 3.3 into account we consider the Radon–Nikodym density $\tilde{\sigma}_{r,k,s,rs}^{n,m}$ of $\mu_{r,k,s,rs}^{n,m}$ w.r.t. $e^{-\tilde{\mathcal{H}}_r^n(\mathbf{x})} \Lambda_r(d\mathbf{x})$. Namely, by definition we have

$$\tilde{\sigma}_{r,k,s,rs}^{n,m}(\mathbf{x}) e^{-\tilde{\mathcal{H}}_r^n(\mathbf{x})} \Lambda_r(d\mathbf{x}) = \mu_{r,k,s,rs}^{n,m}(d\mathbf{x}). \quad (3.8)$$

It is then clearly seen that with normalization c_5

$$\tilde{\sigma}_{r,k,s,rs}^{n,m}(\mathbf{x}) = \frac{1}{c_5} e^{-m_\infty^n \cdot \sum_{x_i \in S_r} x_i} \sigma_{r,k,s,rs}^{n,m}(\mathbf{x}) \quad \text{for } \mathbf{x} = \sum_i \delta_{x_i}. \quad (3.9)$$

We next consider the decomposition of $\tilde{\sigma}_{r,k,s,rs}^{n,m}$ in (3.9).

Lemma 3.4. The density $\tilde{\sigma}_{r,k,s,rs}^{n,m}$ is expressed in such a way that

$$\tilde{\sigma}_{r,k,s,rs}^{n,m}(\mathbf{x}) = \frac{1}{c_6^n(\mathbf{s})} e^{-m_\infty^n \cdot \sum_{x_i \in S_r} x_i - \tilde{\Psi}_{r,rs}^n(\mathbf{x}, \mathbf{s})} \tilde{\tau}_{r,rs}^n(\mathbf{x}, \mathbf{s}) \quad \text{for } \mu_{r,k}^{n,m} \text{-a.e. } \mathbf{s}. \quad (3.10)$$

Here $\tilde{\Psi}_{r,rs}^n$ were given by (2.11), and $c_6^n(\mathbf{s})$ is the normalization

$$c_6^n(\mathbf{s}) = \int_{\mathbb{S}} e^{-m_\infty^n \cdot \sum_{x_i \in S_r} x_i - \tilde{\Psi}_{r,rs}^n(\mathbf{x}, \mathbf{s})} \tilde{\tau}_{r,rs}^n(\mathbf{x}, \mathbf{s}) e^{-\tilde{\mathcal{H}}_r^n(\mathbf{x})} \Lambda_r(d\mathbf{x}), \quad (3.11)$$

and $\tilde{\tau}_{r,rs}^n(\mathbf{x}, \mathbf{s})$ is defined by

$$\begin{aligned} \tilde{\tau}_{r,rs}^n(\mathbf{x}, \mathbf{s}) &= 1_{\mathbb{S}_r^m}(\mathbf{x}) \int_{\mathbb{S}} 1_{H_{r,k}}(\pi_{S_{rs}}(\mathbf{s}) + \mathbf{z}) \\ &\quad \cdot e^{-\tilde{\Psi}_{r,s\infty}^n(\mathbf{x}, \mathbf{z}) - \tilde{\Psi}_{rs,s\infty}^n(\mathbf{s}, \mathbf{z})} \mu_{r,k}^{n,m} \circ \pi_{S_{s\infty}}^{-1}(d\mathbf{z}). \end{aligned} \quad (3.12)$$

Proof. Lemma 3.4 is immediate from (3.2) and (3.4). Indeed, recall that μ^n is a (Φ^n, Ψ^n) -canonical Gibbs measure by the assumption (2) of (H.2). Then from this and noting (3.1), we deduce that the Radon–Nikodym density $\sigma_{r,k,s,rs}^{n,m}$ given by (3.4) satisfies

$$\sigma_{r,k,s,rs}^{n,m}(x) = \text{const. } e^{-\Psi_{r,rs}^n(x,s)} \tau_{r,rs}^n(x, s), \quad (3.13)$$

where $\tau_{r,rs}^n(x, s)$ is defined by

$$\begin{aligned} \tau_{r,rs}^n(x, s) &= 1_{S_r^m}(x) \int_S 1_{H_{r,k}}(\pi_{S_{rs}}(s) + z) \\ &\quad \cdot e^{-\Psi_{r,s\infty}^n(x,z) - \Psi_{rs,s\infty}^n(s,z)} \mu_{r,k}^{n,m} \circ \pi_{S_{s\infty}}^{-1}(dz). \end{aligned} \quad (3.14)$$

We deduce from (3.9) and (3.13) that, for $\mu_{r,k}^{n,m}$ -a.e. s ,

$$\tilde{\sigma}_{r,k,s,rs}^{n,m}(x) = \frac{1}{c_7^n(s)} e^{-m_\infty^n \cdot \sum_{x_i \in S_r} x_i} e^{-\Psi_{r,rs}^n(x,s)} \tau_{r,rs}^n(x, s). \quad (3.15)$$

Here $c_7^n(s)$ is the normalization

$$c_7^n(s) = \int_S e^{-m_\infty^n \cdot \sum_{x_i \in S_r} x_i - \Psi_{r,rs}^n(x,s)} \tau_{r,rs}^n(x, s) e^{-\tilde{\mathcal{H}}_r^n(x)} \Lambda_r(dx). \quad (3.16)$$

Therefore, we deduce (3.10) from (3.15) combined with (3.12) and (3.14), and the definition of $\tilde{\Psi}_{r,rs}^n$. \square

The quasi-Gibbs property consists of two conditions: (2.5) and (2.6). We have already proved (2.5) by Lemma 3.1. Therefore, it only remains to prove (2.6). This task will be carried out in the next two sections. We now explain the strategy of the proof of (2.6).

By taking the representation (3.3) into account, the proof consists of two kinds of limit procedure: (3.17) $n \rightarrow \infty$ and then (3.18) $s \rightarrow \infty$, which involve the following convergence.

$$\lim_{n \rightarrow \infty} \mu_{r,k,s,rs}^{n,m} = \mu_{r,k,s,rs}^m, \quad \lim_{n \rightarrow \infty} \mu_{r,k}^{n,m} \circ \pi_{S_{rs}}^{-1} = \mu_{r,k}^m \circ \pi_{S_{rs}}^{-1}, \quad (3.17)$$

$$\lim_{s \rightarrow \infty} \mu_{r,k,s,rs}^m = \mu_{r,k,s}^m. \quad (3.18)$$

Note that two of these are the convergence of the *conditional* measures. In comparing with the weak convergence of $\{\mu_{r,k}^{n,m}\}$ in Lemma 3.1, it is noted that the convergence of the conditional measures is much more delicate. It involves a variety of strong convergence of the conditioned variable s .

In each step, we prove the bounds of the densities being uniform in n, s ((4.9) and (5.1)) and the related quantities as well as the convergence of measures as above. The uniformity of the bounds is the crucial point of the proof. We emphasize that we can carry out the proof because we treat the cut-off measures $\{\mu_{r,k}^{n,m}\}$ defined by (3.1). This cut-off is done by the set $H_{r,k}$. Therefore, the assumption (H.3) plays a significant role in the proof of Theorem 2.1.

The first step consists of nine lemmas and a proposition. Recall the expressions (3.3) and (3.8). We have already proved the uniform bound of $\int_{S_r^m} e^{-\tilde{\mathcal{H}}_r^n(x)} \Lambda_r(dx)$ in Lemma 3.3, and will prove that for $\tilde{\sigma}_{r,k,s,rs}^{n,m}$ in Lemma 4.2. We then prove weak convergence $\lim_{n \rightarrow \infty} \mu_{r,k}^{n,m} \circ \pi_{S_{rs}}^{-1} = \mu_{r,k}^m \circ \pi_{S_{rs}}^{-1}$ and L^1 convergence of their densities (Lemma 4.3, Proposition 4.4). In this schema, we will have to prove the convergence of both $\tilde{\sigma}_{r,k,s,rs}^{n,m}(x)$ and $e^{-\tilde{\mathcal{H}}_r^n(x)}$. Since the convergence of

$e^{-\tilde{\mathcal{H}}_r^n(x)}$ has been done by Lemmas 3.2 and 3.3, we will concentrate on that for $\tilde{\sigma}_{r,k,s,rs}^{n,m}(x)$. Proposition 4.4 is the main result of this section. We will prove Proposition 4.4 by using Lemma 4.5 and consecutive five lemmas (Lemmas 4.6–4.10).

The second step consists of two lemmas. In Lemma 5.1, we prove the absolute continuity of the measures $\mu_{r,k,s,rs}^m$ and the uniform bound (5.1) of their densities $\sigma_{r,k,s,rs}^m(x)$. Finally, in Lemma 5.2, we prove the convergence of $\sigma_{r,k,s,rs}^m(x)$ as $s \rightarrow \infty$ using martingale convergence theorems to complete the proof of the quasi-Gibbs property.

4. Proof of the first step

In Lemma 4.2, we will give both sides bounds of $\tilde{\sigma}_{r,k,s,rs}^{n,m}(x)$. For this purpose, we control the sum of the interactions in (2.16) and (2.17). We begin by setting

$$d_{S_{rs}}^n(s, t) = \min \left\{ \sum_{i=1}^n |s_i - t_i| \right\} \quad \text{for } s, t \in S_{rs}^n, \quad (4.1)$$

where the minimum is taken over the labeling such that $\pi_{S_{rs}}(s) = \sum_{i=1}^n \delta_{s_i}$ and $\pi_{S_{rs}}(t) = \sum_{i=1}^n \delta_{t_i}$. Let

$$B_r^q = \{x; |x - S_r| < 1/q\} \setminus S_r. \quad (4.2)$$

That is, B_r^q is the intersection of S_r^c and the $1/q$ -neighborhood of S_r . Let $A_{rs,l}^{n,q}$ be the subset of $S_{rs}^n \cap H_{s,l}$ with no particles in B_r^q . Namely,

$$A_{rs,l}^{n,q} = \{s \in S_{rs}^n \cap H_{s,l}; s(B_r^q) = 0\}. \quad (4.3)$$

Lemma 4.1. (1) Set $c_8(k) = mk \cdot \text{diam}(S_r)$. Then, for each $k \in \mathbb{N}$,

$$\sup_{r \leq s < t \in \mathbb{N}} \sup_{x, x' \in S_{rs}^m} \sup_{s \in H_{r,k}^n} |\tilde{\psi}_{r,st}^n(x, s) - \tilde{\psi}_{r,st}^n(x', s)| \leq c_8. \quad (4.4)$$

(2) Let $S_{rs}^n = \{x \in S; x(S_{rs}) = n\}$ and let $H_{s,l}^n$ be as in (2.18). Then, for each $n, l \in \mathbb{N}$,

$$\sup_{r \leq s < t \in \mathbb{N}} \sup_{y, y' \in S_{rs}^n} \sup_{\substack{s \in H_{s,l}^n \\ y \neq y'}} \left\{ \frac{|\tilde{\psi}_{rs,st}^n(y, s) - \tilde{\psi}_{rs,st}^n(y', s)|}{d_{S_{rs}}^n(y, y')} \right\} \leq l. \quad (4.5)$$

(3) Let $c_9 = c_9(m, n, q, r, s, l)$ be the constant defined by

$$c_9 = \sup_{n \in \mathbb{N}} \sup_{x \in S_r^m} \sup_{\substack{y, y' \in A_{rs,l}^{n,q} \\ y \neq y'}} \left\{ \frac{|\tilde{\psi}_{r,rs}^n(x, y) - \tilde{\psi}_{r,rs}^n(x, y')|}{d_{S_{rs}}^n(y, y')} \right\}. \quad (4.6)$$

Then we have $c_9 < \infty$.

Proof. (4.4) follows from (2.16), (2.17), and (2.18) immediately.

We next prove (4.5). Let $\{y_i\}_i^n$ and $\{y'_i\}_i^n$ be labels such that $\pi_{S_{rs}}(y) = \sum_{i=1}^n \delta_{y_i}$ and that $\pi_{S_{rs}}(y') = \sum_{i=1}^n \delta_{y'_i}$. Then we have

$$\begin{aligned}
\frac{|\tilde{\Psi}_{rs,st}^n(\mathbf{y}, \mathbf{s}) - \tilde{\Psi}_{rs,st}^n(\mathbf{y}', \mathbf{s})|}{\sum_{i=1}^n |y_i - y'_i|} &= \frac{\left| \sum_{i=1}^n \{ \tilde{\Psi}_{rs,st}^n(y_i, \mathbf{s}) - \tilde{\Psi}_{rs,st}^n(y'_i, \mathbf{s}) \} \right|}{\sum_{i=1}^n |y_i - y'_i|} \\
&\leq \frac{\sum_{i=1}^n |\tilde{\Psi}_{rs,st}^n(y_i, \mathbf{s}) - \tilde{\Psi}_{rs,st}^n(y'_i, \mathbf{s})|}{\sum_{i=1}^n |y_i - y'_i|} \\
&\leq \max_{i=1, \dots, n} \left\{ \frac{|\tilde{\Psi}_{rs,st}^n(y_i, \mathbf{s}) - \tilde{\Psi}_{rs,st}^n(y'_i, \mathbf{s})|}{|y_i - y'_i|} \right\} \\
&\leq L.
\end{aligned} \tag{4.7}$$

Here we used the inequality $\{\sum_i^n a_i\}/\{\sum_i^n b_i\} \leq \max\{a_m/b_m; m = 1, \dots, n\}$ valid for $a_i \geq 0$ and $b_j > 0$ in the third line. We also used (2.18) and $S_{rs} \subset S_s$ in the last line. Taking the maximum of the labels on the left-hand side of (4.7), we obtain (4.5).

The proof of (4.6) is similar to (4.5). Indeed, in the same fashion as above, we deduce that

$$\begin{aligned}
\frac{|\tilde{\Psi}_{r,st}^n(\mathbf{x}, \mathbf{y}) - \tilde{\Psi}_{r,st}^n(\mathbf{x}, \mathbf{y}')|}{\sum_{j=1}^n |y_j - y'_j|} &\leq \max_{j=1, \dots, n} \left\{ \frac{|\tilde{\Psi}_{r,st}^n(\mathbf{x}, y_j) - \tilde{\Psi}_{r,st}^n(\mathbf{x}, y'_j)|}{|y_j - y'_j|} \right\} \\
&= \max_{j=1, \dots, n} \left\{ \frac{|\Psi_{r,st}^n(\mathbf{x}, y_j) - \Psi_{r,st}^n(\mathbf{x}, y'_j)|}{|y_j - y'_j|} \right\}.
\end{aligned} \tag{4.8}$$

Here the second line follows from (2.17).

Note that Ψ^n converges to Ψ compactly and uniformly in $C^1(S \times S \setminus \{x = y\})$ by the assumption (2.15). Recall that $|x_k - y_j| \geq 1/q$ by (4.3). Then we deduce the claim $c_9 < \infty$ from (4.8). \square

Lemma 4.2. Let $c_{10} = c_4(n_0)e^{\{\sup_{n \in \mathbb{N}} |\mathbf{m}_r^n|\} 2mb_r + 3c_8}$. Then, for $\mu_{r,k}^{n,m}$ -a.e. \mathbf{s} , it holds that, for all $\mathbf{x} \in \mathbf{S}_r^m$, $r < s \in \mathbb{N}$, and $n_0 \leq n \in \mathbb{N}$

$$c_{10}^{-1} \leq \tilde{\sigma}_{r,k,\mathbf{s},rs}^{n,m}(\mathbf{x}) \leq c_{10}. \tag{4.9}$$

Proof. Since the diameter of S_r is b_r and the number of particles in S_r is m , we see that

$$\left| \sum_{x'_i \in S_r} x'_i - \sum_{x_i \in S_r} x_i \right| \leq 2mb_r \quad \text{for all } \mathbf{x}, \mathbf{x}' \in \mathbf{S}_r^m.$$

Hence we deduce from this, (3.10) and (4.4) that

$$\begin{aligned}
\frac{\tilde{\sigma}_{r,k,\mathbf{s},rs}^{n,m}(\mathbf{x})}{\tilde{\sigma}_{r,k,\mathbf{s},rs}^{n,m}(\mathbf{x}')} &= \frac{e^{-\mathbf{m}_r^n \cdot \sum_{x_i \in S_r} x_i - \tilde{\Psi}_{r,rs}^n(\mathbf{x}, \mathbf{s})} \tilde{\tau}_{r,rs}^n(\mathbf{x}, \mathbf{s})}{e^{-\mathbf{m}_r^n \cdot \sum_{x'_i \in S_r} x'_i - \tilde{\Psi}_{r,rs}^n(\mathbf{x}', \mathbf{s})} \tilde{\tau}_{r,rs}^n(\mathbf{x}', \mathbf{s})} \\
&\leq e^{|\mathbf{m}_r^n| 2mb_r + c_8} \frac{\tilde{\tau}_{r,rs}^n(\mathbf{x}, \mathbf{s})}{\tilde{\tau}_{r,rs}^n(\mathbf{x}', \mathbf{s})}.
\end{aligned} \tag{4.10}$$

We set $\Xi = \Xi_r^m$ and $\hat{\Xi} = \hat{\Xi}_r^m$ by

$$\begin{aligned}\Xi &= \{(n, s, x, x'); n \in \mathbb{N}, r < s \in \mathbb{N}, x, x' \in \mathbf{S}_r^m\}, \\ \hat{\Xi} &= \{(n, s, t, x, x'); n \in \mathbb{N}, r < s < t \in \mathbb{N}, x, x' \in \mathbf{S}_r^m\}.\end{aligned}$$

Then, by (3.12), we have for $\mu_{r,k}^{n,m}$ -a.e. \mathbf{s}

$$\begin{aligned}& \sup_{\Xi} \left\{ \frac{\tilde{\tau}_{r,rs}^n(x, \mathbf{s})}{\tilde{\tau}_{r,rs}^n(x', \mathbf{s})} \right\} \\ &= \sup_{\Xi} \left\{ \frac{\int_{\mathbf{S}} 1_{H_{r,k}}(\pi_{S_{rs}}(\mathbf{s}) + z) e^{-\tilde{\Psi}_{r,s\infty}^n(x,z) - \tilde{\Psi}_{rs,s\infty}^n(\mathbf{s},z)} \mu_{r,k}^{n,m} \circ \pi_{S_{s\infty}}^{-1}(dz)}{\int_{\mathbf{S}} 1_{H_{r,k}}(\pi_{S_{rs}}(\mathbf{s}) + z) e^{-\tilde{\Psi}_{r,s\infty}^n(x',z) - \tilde{\Psi}_{rs,s\infty}^n(\mathbf{s},z)} \mu_{r,k}^{n,m} \circ \pi_{S_{s\infty}}^{-1}(dz)} \right\} \\ &= \sup_{\hat{\Xi}} \left\{ \frac{\int_{\mathbf{S}} 1_{H_{r,k}}(\pi_{S_{rs}}(\mathbf{s}) + z) e^{-\tilde{\Psi}_{r,st}^n(x,z) - \tilde{\Psi}_{rs,st}^n(\mathbf{s},z)} \mu_{r,k}^{n,m} \circ \pi_{S_{s\infty}}^{-1}(dz)}{\int_{\mathbf{S}} 1_{H_{r,k}}(\pi_{S_{rs}}(\mathbf{s}) + z) e^{-\tilde{\Psi}_{r,st}^n(x',z) - \tilde{\Psi}_{rs,st}^n(\mathbf{s},z)} \mu_{r,k}^{n,m} \circ \pi_{S_{s\infty}}^{-1}(dz)} \right\} \\ &\leq e^{2c_8}.\end{aligned}\tag{4.11}$$

Here we used $\mu^n(\mathbf{S}(S) = N_n) = 1$ in the third line, and (4.4) in the last line.

Let $c_{11} = \sup_{n \in \mathbb{N}} |m_n| 2mb_r + 3c_8$. Then we deduce $c_{11} < \infty$ from (2.20). Hence (4.10) and (4.11) yield that

$$\sup_{\Xi} \frac{\tilde{\sigma}_{r,k,s,rs}^{n,m}(x)}{\tilde{\sigma}_{r,k,s,rs}^{n,m}(x')} \leq e^{c_{11}} \quad \text{for } \mu_{r,k}^{n,m}\text{-a.e. } \mathbf{s}.\tag{4.12}$$

For $\mu_{r,k}^{n,m}$ -a.e. \mathbf{s} , we deduce from (4.12) that for all $r < s \in \mathbb{N}$ and $n \in \mathbb{N}$

$$e^{-c_{11}} \tilde{\sigma}_{r,k,s,rs}^{n,m}(x') \leq \tilde{\sigma}_{r,k,s,rs}^{n,m}(x) \leq e^{c_{11}} \tilde{\sigma}_{r,k,s,rs}^{n,m}(x') \quad \text{for all } x, x' \in \mathbf{S}_r^m.\tag{4.13}$$

Multiply (4.13) by $1_{\mathbf{S}_r^m}(x') e^{-\tilde{\mathcal{H}}_r^n(x')}$ and integrate w.r.t. $\Lambda_r(dx')$. Note that by (3.4), we have $\int_{\mathbf{S}_r^m} \tilde{\sigma}_{r,k,s,rs}^{n,m}(x') e^{-\tilde{\mathcal{H}}_r^n(x')} \Lambda_r(dx') = 1$. Then we deduce that for $\mu_{r,k}^{n,m}$ -a.e. \mathbf{s} ,

$$e^{-c_{11}} \leq \tilde{\sigma}_{r,k,s,rs}^{n,m}(x) \int_{\mathbf{S}_r^m} e^{-\tilde{\mathcal{H}}_r^n(x')} \Lambda_r(dx') \leq e^{c_{11}} \quad \text{for all } x \in \mathbf{S}_r^m.$$

This combined with Lemma 3.3 yields (4.9). \square

Lemma 4.3. $\mu_{r,k}^{n,m} \circ \pi_{S_{rs}}^{-1}$ converges weakly to $\mu_{r,k}^m \circ \pi_{S_{rs}}^{-1}$ as $n \rightarrow \infty$.

Proof. Let E be the discontinuity points of $\pi_{S_{rs}}$, namely

$$E = \left\{ \mathbf{s} \in \mathbf{S}; \lim_{n \rightarrow \infty} \pi_{S_{rs}}(\mathbf{s}_n) \neq \pi_{S_{rs}}(\mathbf{s}) \text{ for some } \{\mathbf{s}_n\} \text{ such that } \lim_{n \rightarrow \infty} \mathbf{s}_n = \mathbf{s} \right\}.$$

Then by (H.1), we deduce that $\mu_{r,k}^m(E) \leq \mu(E) = 0$. Since $\mu_{r,k}^{n,m}$ converges weakly to $\mu_{r,k}^m$ by Lemma 3.1 and the discontinuity points of $\pi_{S_{rs}}^{-1}$ are $\mu_{r,k}^m$ -measure zero, we obtain Lemma 4.3. \square

Let $\mathcal{H}_{rs} = \mathcal{H}_{S_{rs}}^{\Phi, \Psi}$ and $\tilde{\mathcal{H}}_{rs}^n$ such that

$$\tilde{\mathcal{H}}_{rs}^n(x) = \sum_{x_i \in S_{rs}} \{ \Phi^n(x_i) - m_\infty^n \cdot x_i \} + \sum_{x_i, x_j \in S_{rs}, i < j} \Psi^n(x_i, x_j). \quad (4.14)$$

By (2.12) and (2.13), we see that $\mu_{r,k}^{n,m} \circ \pi_{S_{rs}}^{-1}$ and $\mu_{r,k}^m \circ \pi_{S_{rs}}^{-1}$ are absolutely continuous w.r.t. $e^{-\tilde{\mathcal{H}}_{rs}^n} \Lambda_{rs}$ and $e^{-\mathcal{H}_{rs}} \Lambda_{rs}$, respectively. Hence, we denote by Δ^n and Δ their Radon–Nikodym densities, respectively. Namely,

$$\Delta^n(s) = \frac{\mu_{r,k}^{n,m} \circ \pi_{S_{rs}}^{-1}(ds)}{e^{-\tilde{\mathcal{H}}_{rs}^n} \Lambda_{rs}(ds)}, \quad \Delta(s) = \frac{\mu_{r,k}^m \circ \pi_{S_{rs}}^{-1}(ds)}{e^{-\mathcal{H}_{rs}} \Lambda_{rs}(ds)}. \quad (4.15)$$

These density functions are $\sigma[\pi_{S_{rs}}]$ -measurable and defined for Λ_{rs} -a.s. s . We naturally regard these as functions on S defined for Λ -a.s. s by taking $\Delta^n(s) = \Delta^n(\pi_{S_{rs}}(s))$ and $\Delta(s) = \Delta(\pi_{S_{rs}}(s))$. Similar convention will be used for $\Delta_l^{n,n}$ introduced in (4.20).

The following is the main result of this section.

Proposition 4.4. $\Delta^n e^{-\tilde{\mathcal{H}}_{rs}^n}$ converges to $\Delta e^{-\mathcal{H}_{rs}}$ in $L^1(S, \Lambda)$ as $n \rightarrow \infty$.

We devote the rest of this section to the proof of Proposition 4.4. This proof is rather long, and we will complete it after preparing a sequence of lemmas.

Lemma 4.5. Assume that $\{\Delta^n e^{-\tilde{\mathcal{H}}_{rs}^n}\}_{n \in \mathbb{N}}$ are relative compact in $L^1(S, \Lambda)$. Then Proposition 4.4 holds.

Proof. If $\{\Delta^n e^{-\tilde{\mathcal{H}}_{rs}^n}\}_{n \in \mathbb{N}}$ are relatively compact in $L^1(S, \Lambda)$, then their limit points are unique and equal to $\Delta e^{-\mathcal{H}_{rs}}$ by Lemma 4.3. \square

To prove the relative compactness as above, we use various kinds of cut-off procedures. Recall that $S_{rs}^n = \{x \in S; x(S_{rs}) = n\}$. We set $\Delta^{n,n} = \Delta^n 1_{S_{rs}^n}$. Then we have

$$\Delta^n = \sum_{n=0}^{\infty} \Delta^{n,n}. \quad (4.16)$$

We begin by considering a cut-off of $\Delta^n e^{-\tilde{\mathcal{H}}_{rs}^n}$ according to (4.16).

Lemma 4.6. For each $\epsilon > 0$, there exists an n_0 such that

$$\sup_{n \in \mathbb{N}} \left\| \left\{ \sum_{n=n_0}^{\infty} \Delta^{n,n} \right\} e^{-\tilde{\mathcal{H}}_{rs}^n} \right\|_{L^1(S, \Lambda)} < \epsilon. \quad (4.17)$$

Proof. By Lemma 4.3, we see that the sequence $\{\mu_{r,k}^{n,m} \circ \pi_{S_{rs}}^{-1}\}$ is tight. Hence we deduce that for each $\epsilon > 0$ there exists an n_0 such that

$$\sup_{n \in \mathbb{N}} \mu_{r,k}^{n,m} \left(\sum_{n=n_0}^{\infty} S_{rs}^n \right) < \epsilon, \quad (4.18)$$

which is equivalent to (4.17). \square

According to (4.17), the relative compactness of $\{\Delta^n e^{-\tilde{\mathcal{H}}_{rs}^n}\}_{n \in \mathbb{N}}$ in $L^1(\mathbf{S}, \Lambda)$ follows from that of $\{\Delta^{n,n} e^{-\tilde{\mathcal{H}}_{rs}^n}\}_{n \in \mathbb{N}}$ for each $n \in \mathbb{N}$. Hence, we fix $n \in \mathbb{N}$ in the rest of this section.

Let $H_{s,l}$ and $H_{s,l}^n$ be as in (2.18). We consider new sequences of cut-off measures $\{\mu_l^{n,n}\}_{l \in \mathbb{N}}$ such that

$$\mu_l^{n,n} = \mu_{r,k}^{n,m}(\cdot \cap S_{rs}^n \cap H_{s,l}). \quad (4.19)$$

Then, since $\mu_{r,k}^{n,m}((H_{s,l}^n)^c) = 0$ and $H_{s,l}^n \subset H_{s,l}$, we have $\mu_l^{n,n} = \mu_{r,k}^{n,m}(\cdot \cap S_{rs}^n \cap H_{s,l}^n)$.

Let $\Delta_l^{n,n}$ be the Radon–Nikodym density of $\mu_l^{n,n} \circ \pi_{S_{rs}}^{-1}$ w.r.t. $e^{-\tilde{\mathcal{H}}_{rs}^n} \Lambda_{rs}$. Then by definition

$$\Delta_l^{n,n}(\mathbf{s}) = \frac{\mu_l^{n,n} \circ \pi_{S_{rs}}^{-1}(d\mathbf{s})}{e^{-\tilde{\mathcal{H}}_{rs}^n} \Lambda_{rs}(d\mathbf{s})}. \quad (4.20)$$

We naturally regard $\Delta_l^{n,n}(\mathbf{s})$ as a $\sigma[\pi_{S_{rs}}]$ -measurable function defined on \mathbf{S} .

Lemma 4.7. *Let $\Delta_l^{n,n}$ be as in (4.20). Then, for each $n \in \mathbb{N}$, we have*

$$\lim_{l \rightarrow \infty} \limsup_{n \in \mathbb{N}} \|\Delta^{n,n} e^{-\tilde{\mathcal{H}}_{rs}^n} - \Delta_l^{n,n} e^{-\tilde{\mathcal{H}}_{rs}^n}\|_{L^1(\mathbf{S}, \Lambda)} = 0. \quad (4.21)$$

Proof. Since $\mu_l^{n,n} \leq \mu_{r,k}^{n,m}$ by (4.19), we see that $\Delta_l^{n,n} e^{-\tilde{\mathcal{H}}_{rs}^n} \leq \Delta^{n,n} e^{-\tilde{\mathcal{H}}_{rs}^n}$. This together with (4.15) and (4.20) yields

$$\|\Delta^{n,n} e^{-\tilde{\mathcal{H}}_{rs}^n} - \Delta_l^{n,n} e^{-\tilde{\mathcal{H}}_{rs}^n}\|_{L^1(\mathbf{S}, \Lambda)} \leq \mu_{r,k}^{n,m}(H_{s,l}^c). \quad (4.22)$$

From $\mu_{r,k}^{n,m} \leq \mu^n$ and (2.19) we deduce that

$$\lim_{l \rightarrow \infty} \limsup_{n \in \mathbb{N}} \mu_{r,k}^{n,m}(H_{s,l}^c) \leq \lim_{l \rightarrow \infty} \limsup_{n \in \mathbb{N}} \mu^n(H_{s,l}^c) = 0. \quad (4.23)$$

Combining (4.22) and (4.23) yields (4.21). \square

According to Lemma 4.7, the only task that remains is to prove the relative compactness of $\{\Delta_l^{n,n} e^{-\tilde{\mathcal{H}}_{rs}^n}\}_{n \in \mathbb{N}}$ in $L^1(\mathbf{S}, \Lambda)$ for all sufficiently large $l \in \mathbb{N}$. Hence, we fix such an $l \in \mathbb{N}$ in the rest of this section. Let $A_{rs,l}^{n,q}$ be as in (4.3).

Lemma 4.8. *For each $\epsilon > 0$, there exists a $q_0 \in \mathbb{N}$ such that, for all $q \geq q_0$,*

$$\sup_{n \in \mathbb{N}} \|\Delta_l^{n,n} e^{-\tilde{\mathcal{H}}_{rs}^n} - \Delta_l^{n,n} e^{-\tilde{\mathcal{H}}_{rs}^n} 1_{A_{rs,l}^{n,q}}\|_{L^1(\mathbf{S}, \Lambda)} \leq \epsilon. \quad (4.24)$$

Proof. By the definitions of $A_{rs,l}^{n,q}$ and B_r^q , and from the property of 1-correlation function we deduce that

$$\|\Delta_l^{n,n} e^{-\tilde{\mathcal{H}}_{rs}^n} - \Delta_l^{n,n} e^{-\tilde{\mathcal{H}}_{rs}^n} 1_{A_{rs,l}^{n,q}}\|_{L^1(\mathbf{S}, \Lambda)} \leq \mu_{r,k}^{n,m}((A_{rs,l}^{n,q})^c) \leq \int_{B_r^q} \rho_n^1(x) dx.$$

We deduce from (2.12) and (2.13) that, for each $\epsilon > 0$, there exists a $q_0 \in \mathbb{N}$ such that

$$\sup_{n \in \mathbb{N}} \int_{B_r^q} \rho_n^1(x) dx \leq \epsilon \quad \text{for all } q \geq q_0.$$

Combining these two inequalities, we obtain (4.24). \square

We will prove that $\{\Delta_l^{n,n} e^{-\tilde{\mathcal{H}}_{rs}^n} 1_{A_{rs,l}^{n,q}}\}_{n \in \mathbb{N}}$ are relatively compact in $L^1(\mathbb{S}, \Lambda)$ for each $n \in \mathbb{N}$ and for all sufficiently large $l, q \in \mathbb{N}$. For this, we will prove both the relative compactness of $\{e^{-\tilde{\mathcal{H}}_{rs}^n} 1_{A_{rs,l}^{n,q}}\}_{n \in \mathbb{N}}$ in $L^1(\mathbb{S}, \Lambda)$, and that of $\{\Delta_l^{n,n}\}_{n \in \mathbb{N}}$ in $C_b(A_{rs,l}^{n,q})$ with uniform norm $\|\cdot\|_{C_b(A_{rs,l}^{n,q})}$, where $\|f\|_{C_b(A_{rs,l}^{n,q})} = \sup\{|f(y)|; y \in A_{rs,l}^{n,q}\}$.

We begin by proving the first claim.

Lemma 4.9. $\{e^{-\tilde{\mathcal{H}}_{rs}^n} 1_{A_{rs,l}^{n,q}}\}_{n \in \mathbb{N}}$ converge to $e^{-\mathcal{H}_{rs}} 1_{A_{rs,l}^{n,q}}$ in $L^1(\mathbb{S}, \Lambda)$, and

$$\inf_{n \in \mathbb{N}} \|e^{-\tilde{\mathcal{H}}_{rs}^n} 1_{A_{rs,l}^{n,q}}\|_{L^1(\mathbb{S}, \Lambda)} > 0 \quad \text{for all sufficiently large } l, q \in \mathbb{N}. \quad (4.25)$$

Proof. From (2.14) and (2.15) together with (4.14), we deduce that

$$\lim_{n \rightarrow \infty} e^{-\tilde{\mathcal{H}}_{rs}^n(x)} 1_{A_{rs,l}^{n,q}}(x) = e^{-\mathcal{H}_{rs}(x)} 1_{A_{rs,l}^{n,q}}(x) \quad \text{for } \Lambda\text{-a.e. } x, \quad (4.26)$$

$$\sup_{n \in \mathbb{N}} \sup_{x \in \mathbb{S}} \{e^{-\tilde{\mathcal{H}}_{rs}^n(x)} 1_{A_{rs,l}^{n,q}}(x)\} < \infty. \quad (4.27)$$

From (4.26) and (4.27), combined with the Lebesgue convergence theorem, we deduce the first claim. In turn, we deduce that

$$\lim_{n \rightarrow \infty} \|e^{-\tilde{\mathcal{H}}_{rs}^n} 1_{A_{rs,l}^{n,q}}\|_{L^1(\mathbb{S}, \Lambda)} = \|e^{-\mathcal{H}_{rs}} 1_{A_{rs,l}^{n,q}}\|_{L^1(\mathbb{S}, \Lambda)}. \quad (4.28)$$

From (2.14) and (2.15), we have, for all sufficiently large $l, q \in \mathbb{N}$,

$$\|e^{-\tilde{\mathcal{H}}_{rs}^n} 1_{A_{rs,l}^{n,q}}\|_{L^1(\mathbb{S}, \Lambda)} > 0 \quad (\forall n \in \mathbb{N}), \quad \|e^{-\mathcal{H}_{rs}} 1_{A_{rs,l}^{n,q}}\|_{L^1(\mathbb{S}, \Lambda)} > 0. \quad (4.29)$$

Combining (4.28) and (4.29) yields (4.25). \square

We next prove the second claim.

Lemma 4.10. $\{\Delta_l^{n,n}\}_{n \in \mathbb{N}}$ are relatively compact in $C_b(A_{rs,l}^{n,q})$ with uniform norm.

Proof. From the definition of $\Delta_l^{n,n}$ (see (3.1), (4.19) and (4.20)), we see that

$$\|\Delta_l^{n,n} e^{-\tilde{\mathcal{H}}_{rs}^n} 1_{A_{rs,l}^{n,q}}\|_{L^1(\mathbb{S}, \Lambda)} = \mu_l^{n,n} \circ \pi_{S_{rs}}^{-1}(A_{rs,l}^{n,q}) \leq 1. \quad (4.30)$$

Note that $\pi_{S_{rs}^c} = \pi_{S_r} + \pi_{S_{s\infty}}$. Hence we write $\pi_{S_{rs}^c}(\mathbf{s}) = \mathbf{x} + \mathbf{z}$, where $\mathbf{x} \in \pi_{S_r}(\mathbb{S})$ and $\mathbf{z} \in \pi_{S_{s\infty}}(\mathbb{S})$. With this notation, $\Delta_l^{n,n}(\mathbf{y})$ can be written as

$$\Delta_l^{n,n}(\mathbf{y}) = c_{12} \int_{\mathbb{S}} 1_{H_{r,k}^n \cap H_{s,l}^n}(\mathbf{x} + \pi_{S_{rs}}(\mathbf{y}) + \mathbf{z}) e^{-\tilde{\Psi}_{r,rs}^n(\mathbf{x}, \mathbf{y}) - \tilde{\Psi}_{rs,s\infty}^n(\mathbf{y}, \mathbf{z})} \mu_l^{n,n} \circ \pi_{S_{rs}^c}^{-1}(d\mathbf{x} d\mathbf{z})$$

with positive constant c_{12} . Let $c_{13} = \sup\{e^{(c_9+l)d_{S_{rs}}^n(\mathbf{y}, \mathbf{y}')} ; \mathbf{y}, \mathbf{y}' \in A_{rs,l}^{n,q}\}$. Then applying (4.5) and (4.6) to $\tilde{\Psi}_{r,rs}^n(\mathbf{x}, \mathbf{y})$ and $\tilde{\Psi}_{rs,s\infty}^n(\mathbf{y}, \mathbf{z})$ respectively, we deduce from Lemma 4.1 that

$$\sup_{n \in \mathbb{N}} \sup_{\mathbf{y}, \mathbf{y}' \in A_{rs,l}^{n,q}} \left\{ \frac{\Delta_l^{n,n}(\mathbf{y})}{\Delta_l^{n,n}(\mathbf{y}')} \right\} \leq \sup_{\mathbf{y}, \mathbf{y}' \in A_{rs,l}^{n,q}} e^{(c_9+l)d_{S_{rs}}^n(\mathbf{y}, \mathbf{y}')} = c_{13} < \infty. \quad (4.31)$$

Hence from (4.30) and (4.31), we see that

$$\begin{aligned} \|\Delta_l^{n,n}\|_{C_b(\mathbf{A}_{rs,l}^{n,q})} \cdot \|e^{-\tilde{\mathcal{H}}_{rs}^n} 1_{\mathbf{A}_{rs,l}^{n,q}}\|_{L^1(\mathbf{S}, \Lambda)} &= \left\| \left(\frac{\|\Delta_l^{n,n}\|_{C_b(\mathbf{A}_{rs,l}^{n,q})}}{\Delta_l^{n,n}} \right) \Delta_l^{n,n} e^{-\tilde{\mathcal{H}}_{rs}^n} 1_{\mathbf{A}_{rs,l}^{n,q}} \right\|_{L^1(\mathbf{S}, \Lambda)} \\ &\leq c_{13} \|\Delta_l^{n,n} e^{-\tilde{\mathcal{H}}_{rs}^n} 1_{\mathbf{A}_{rs,l}^{n,q}}\|_{L^1(\mathbf{S}, \Lambda)} \quad \text{by (4.31)} \\ &\leq c_{13} \quad \text{by (4.30).} \end{aligned} \quad (4.32)$$

Combining (4.25) and (4.32) yields

$$\sup_{n \in \mathbb{N}} \|\Delta_l^{n,n}\|_{C_b(\mathbf{A}_{rs,l}^{n,q})} \leq \frac{c_{13}}{\inf_{n \in \mathbb{N}} \|e^{-\tilde{\mathcal{H}}_{rs}^n} 1_{\mathbf{A}_{rs,l}^{n,q}}\|_{L^1(\mathbf{S}, \Lambda)}} < \infty. \quad (4.33)$$

Taking the logarithm of (4.31) and interchanging the role of y and y' , we see that, for all $y, y' \in \mathbf{A}_{rs,l}^{n,q}$,

$$\sup_{n \in \mathbb{N}} \{|\log \Delta_l^{n,n}(y) - \log \Delta_l^{n,n}(y')|\} \leq (c_9 + l) d_{\mathbf{S}_{rs}^n}(y, y'). \quad (4.34)$$

Then we deduce from the inequality

$$|x - y| \leq \max\{x, y\} |\log x - \log y| \quad \text{for } x, y > 0$$

and (4.34) that, for all $y, y' \in \mathbf{A}_{rs,l}^{n,q}$,

$$\begin{aligned} \sup_{n \in \mathbb{N}} \{|\Delta_l^{n,n}(y) - \Delta_l^{n,n}(y')|\} &\leq \sup_{n \in \mathbb{N}} \{\|\Delta_l^{n,n}\|_{C_b(\mathbf{A}_{rs,l}^{n,q})} |\log \Delta_l^{n,n}(y) - \log \Delta_l^{n,n}(y')|\} \\ &\leq c_{14}(c_9 + l) d_{\mathbf{S}_{rs}^n}(y, y'), \end{aligned} \quad (4.35)$$

where we set $c_{14} = \sup_{n \in \mathbb{N}} \|\Delta_l^{n,n}\|_{C_b(\mathbf{A}_{rs,l}^{n,q})}$. Since $c_{14} < \infty$ by (4.33), we deduce from (4.35) that $\{\Delta_l^{n,n}\}_{n \in \mathbb{N}}$ are equi-continuous in $C_b(\mathbf{A}_{rs,l}^{n,q})$ for each $q \in \mathbb{N}$.

From (4.33) and (4.35), we deduce that $\{\Delta_l^{n,n}\}_{n \in \mathbb{N}}$ are equi-continuous and uniformly bounded in $C_b(\mathbf{A}_{rs,l}^{n,q})$ for each $q \in \mathbb{N}$. Hence, applying the Ascoli–Arzelá theorem to $\{\Delta_l^{n,n}\}$ completes the proof of Lemma 4.10. \square

We are now in a position to complete the proof of Proposition 4.4.

Proof of Proposition 4.4. From Lemmas 4.9 and 4.10, we deduce that $\{e^{-\tilde{\mathcal{H}}_{rs}^n} 1_{\mathbf{A}_{rs,l}^{n,q}}\}_{n \in \mathbb{N}}$ are convergent sequences in $L^1(\mathbf{S}, \Lambda)$ and that $\{\Delta_l^{n,n}\}_{n \in \mathbb{N}}$ are relatively compact in $C_b(\mathbf{A}_{rs,l}^{n,q})$ for each $n \in \mathbb{N}$ and for all sufficiently large $l, q \in \mathbb{N}$. Then we conclude that

$$\{\Delta_l^{n,n} e^{-\tilde{\mathcal{H}}_{rs}^n} 1_{\mathbf{A}_{rs,l}^{n,q}}\}_{n \in \mathbb{N}}$$

are relatively compact in $L^1(\mathbf{S}, \Lambda)$ for such $n, l, q \in \mathbb{N}$. Combining this with Lemmas 4.6–4.8, we see that $\{\Delta_l^{n,n} e^{-\tilde{\mathcal{H}}_{rs}^n}\}_{n \in \mathbb{N}}$ are relatively compact in $L^1(\mathbf{S}, \Lambda)$. Hence by Lemma 4.5, we complete the proof of Proposition 4.4. \square

5. Proof of the second step

We devote this section to the proof of the second step.

Let $\mu_{r,k}^m = \mu(\cdot \cap S_{r,k}^m)$ be as in Definition 2.1. Let $\mu_{r,k,s,rs}^m$ be the regular conditional probability defined by

$$\mu_{r,k,s,rs}^m = \mu_{r,k}^m(\pi_{S_r}(\mathbf{s}) \in d\mathbf{x} | \pi_{S_{rs}}(\mathbf{s})).$$

We begin by proving uniform upper and lower bounds of Radon–Nikodym densities of $\mu_{r,k,s,rs}^m$ w.r.t. $e^{-\mathcal{H}_r(\mathbf{x})} \Lambda_r(d\mathbf{x})$.

Lemma 5.1. (1) For $\mu_{r,k}^m$ -a.e. \mathbf{s} , the regular conditional probability $\mu_{r,k,s,rs}^m$ is absolutely continuous w.r.t. $e^{-\mathcal{H}_r(\mathbf{x})} \Lambda_r(d\mathbf{x})$.

(2) Let $\sigma_{r,k,s,rs}^m$ be the Radon–Nikodym densities of $\mu_{r,k,s,rs}^m$ w.r.t. $e^{-\mathcal{H}_r(\mathbf{x})} \Lambda_r(d\mathbf{x})$. Then, for each $r, s, m, k \in \mathbb{N}$ such that $r < s$ and $\mu_{r,k}^m$ -a.e. \mathbf{s} ,

$$c_{10}^{-1} \leq \sigma_{r,k,s,rs}^m(\mathbf{x}) \leq c_{10} \quad \text{for } \mu_{r,k,s,rs}^m\text{-a.e. } \mathbf{x}. \quad (5.1)$$

Here c_{10} is the positive constant given in Lemma 4.2.

Proof. We first prove the claim (1). Similar to the case of Lemma 4.3, we see that $\mu_{r,k}^{n,m} \circ (\pi_{S_r}, \pi_{S_{rs}})^{-1}$ converge weakly to $\mu_{r,k}^m \circ (\pi_{S_r}, \pi_{S_{rs}})^{-1}$ as $n \rightarrow \infty$. Hence, for $f, g \in C_b(S)$, we have

$$\int_S f(\pi_{S_r}(\mathbf{s}))g(\pi_{S_{rs}}(\mathbf{s}))d\mu_{r,k}^{n,m} = \lim_{n \rightarrow \infty} \int_S f(\pi_{S_r}(\mathbf{s}))g(\pi_{S_{rs}}(\mathbf{s}))d\mu_{r,k}^{n,m}. \quad (5.2)$$

By Lemma 4.2 and the diagonal argument, there exist subsequences of $\{\tilde{\sigma}_{r,k,s,rs}^{n,m}\}_{n \in \mathbb{N}}$, denoted by the same symbol, with a limit $\sigma_{r,k,s,rs}^m$ such that, for all $k, m, r < s \in \mathbb{N}$,

$$\lim_{n \rightarrow \infty} \tilde{\sigma}_{r,k,s,rs}^{n,m}(\pi_{S_r}(\mathbf{s})) = \sigma_{r,k,s,rs}^m(\pi_{S_r}(\mathbf{s})) \quad * \text{-weakly in } L^\infty(S, \Lambda). \quad (5.3)$$

Here $\sigma_{r,k,s,rs}^m$ is a function such that $\sigma_{r,k,s,rs}^m(\mathbf{x}) = \sigma_{r,k,\pi_{S_{rs}}(\mathbf{s}),rs}^m(\pi_{S_r}(\mathbf{x}))$. Let

$$F^n(\mathbf{s}) = f(\pi_{S_r}(\mathbf{s}))g(\pi_{S_{rs}}(\mathbf{s}))\Delta^n(\mathbf{s})e^{-\tilde{\mathcal{H}}_r^n(\mathbf{s})}, \quad (5.4)$$

$$F(\mathbf{s}) = f(\pi_{S_r}(\mathbf{s}))g(\pi_{S_{rs}}(\mathbf{s}))\Delta(\mathbf{s})e^{-\mathcal{H}_r(\mathbf{s})}. \quad (5.5)$$

Then by Proposition 4.4, we see that F^n converge to F in $L^1(S, \Lambda)$. This combined with (5.3) implies

$$\lim_{n \rightarrow \infty} \int_S F^n(\mathbf{s})\tilde{\sigma}_{r,k,s,rs}^{n,m}(\mathbf{s})d\Lambda = \int_S F(\mathbf{s})\sigma_{r,k,s,rs}^m(\mathbf{s})d\Lambda. \quad (5.6)$$

By (5.2), (5.6) and $\Delta(\mathbf{y})e^{-\mathcal{H}_{rs}(\mathbf{y})} \Lambda_{rs}(d\mathbf{y}) = \mu_{r,k}^m \circ \pi_{S_{rs}}^{-1}(d\mathbf{y})$, we obtain

$$\int_S f(\mathbf{x})g(\mathbf{y})d\mu_{r,k}^m = \int_S f(\mathbf{x})g(\mathbf{y})\sigma_{r,k,s,rs}^m(\mathbf{x})e^{-\mathcal{H}_r(\mathbf{x})} \Lambda_r(d\mathbf{x})\mu_{r,k}^m \circ \pi_{S_{rs}}^{-1}(d\mathbf{y}),$$

where $\mathbf{x} = \pi_{S_r}(\mathbf{s})$ and $\mathbf{y} = \pi_{S_{rs}}(\mathbf{s})$. Hence, we obtain (1) with density $\sigma_{r,k,s,rs}^m$.

By (4.9) and (5.3), we see that $\sigma_{r,k,s,rs}^m$ satisfies (5.1), which implies (2). \square

Lemma 5.2. Let $\mu_{r,k,s}^m(dx)$ be as in (2.7). Let $\sigma_{r,k,s,rs}^m$ be as in Lemma 5.1. Then the following limit exists. For $\mu_{r,k}^m$ -a.s. \mathbf{s} ,

$$\sigma_{r,k,s}^m(\mathbf{x}) := \lim_{s \rightarrow \infty} \sigma_{r,k,s,rs}^m(\mathbf{x}) \quad \text{for } \mu_{r,k,s}^m\text{-a.s. } \mathbf{x}. \quad (5.7)$$

Moreover, $\sigma_{r,k,s}^m$ satisfies for $\mu_{r,k}^m$ -a.e. \mathbf{s}

$$c_{10}^{-1} \leq \sigma_{r,k,s}^m(\mathbf{x}) \leq c_{10} \quad \text{for } \mu_{r,k,s}^m\text{-a.e. } \mathbf{x} \quad (5.8)$$

$$\sigma_{r,k,s}^m(\mathbf{x}) e^{-\mathcal{H}_r(\mathbf{x})} \Lambda_r(d\mathbf{x}) = \mu_{r,k,s}^m(d\mathbf{x}). \quad (5.9)$$

Proof. The proof of this lemma is exactly the same as Lemma 5.5 in [18]. However, we give the proof here for the reader's convenience.

Define $M_s : \mathbf{S} \rightarrow \mathbb{R}$ by $M_s(\mathbf{s}) = \sigma_{r,k,s,rs}^m(\mathbf{x})$, where $\mathbf{x} = \pi_{S_r}(\mathbf{s})$. Recall that $\sigma_{r,k,s,rs}^m$ is the Radon–Nikodym density of $\mu_{r,k,s,rs}^m$ w.r.t. $e^{-\mathcal{H}_r(\mathbf{x})} \Lambda_r(d\mathbf{x})$ and that $\mu_{r,k,s,rs}^m = \mu_{r,k,\pi_{S_{rs}}(\mathbf{s}),rs}^m$ by construction. Hence,

$$M_s(\mathbf{s}) e^{-\mathcal{H}_r(\mathbf{x})} \Lambda_r(d\mathbf{x}) = \mu_{r,k,\pi_{S_{rs}}(\mathbf{s}),rs}^m(d\mathbf{x}). \quad (5.10)$$

Let $\mathcal{F}_s = \sigma[\pi_{S_r}, \pi_{S_{rs}}]$, where $r < s \leq \infty$. Then by (5.10), we see that $\{M_s\}_{s \in [r, \infty)}$ is an (\mathcal{F}_s) -martingale, which implies $M_\infty(\mathbf{s}) := \lim_{s \rightarrow \infty} M_s(\mathbf{s})$ exists for $\mu_{r,k}^m$ -a.e. \mathbf{s} . Since

$$M_s(\mathbf{s}) = \sigma_{r,k,\pi_{S_{rs}}(\mathbf{s}),rs}^m(\mathbf{x}), \quad \text{where } \mathbf{x} = \pi_{S_r}(\mathbf{s}),$$

we write $M_\infty(\mathbf{s}) = \sigma_{r,k,s}^m(\mathbf{x})$. By construction, $\sigma_{r,k,s}^m(\mathbf{x}) = \sigma_{r,k,\pi_{S_{r\infty}}(\mathbf{s})}^m(\mathbf{x}) = \sigma_{r,k,\pi_{S_{\infty}^c}(\mathbf{s})}^m(\mathbf{x})$ and, for $\mu_{r,k}^m$ -a.s. \mathbf{s} , we can regard $\sigma_{r,k,s}^m(\mathbf{x})$ as a $\sigma[\pi_{S_r}]$ -measurable function in \mathbf{x} . Hence, through the disintegration (2.8), we obtain (5.7).

We immediately obtain (5.8) from (5.1) and (5.7).

We see that $\{M_s\}_{s \in [r, \infty)}$ is uniformly integrable by (5.1). Hence, by (5.7), we see that $M_s(\mathbf{s})$ converges to $M_\infty(\mathbf{s}) = \sigma_{r,k,s}^m(\mathbf{x})$ strongly in $L^1(\mathbf{S}_r^m, \mu_{r,k,s}^m)$, which combined with (5.10) and the definition $M_s(\mathbf{s}) = \sigma_{r,k,s,rs}^m(\mathbf{x})$ yields (5.9). \square

Proof of Theorem 2.1. By Lemma 3.1, we see that $\{\mu_{r,k}^m\}$ satisfies (2.5). Moreover, by (5.8) and (5.9), we deduce that $\mu_{r,k,s}^m$ satisfies (2.6), which completes the proof of Theorem 2.1. \square

6. A sufficient condition of (2.19) in (H.3)

In this section, we give a sufficient condition of (2.19) in (H.3) when $d = 1, 2$ and Ψ^n satisfy (2.21). So $\Psi^n(x, y) := \Psi(x, y) = -\beta \log |x - y|$ are logarithmic functions by assumption. If $d = 2$, we regard \mathbb{R}^2 as \mathbb{C} by the natural correspondence: $\mathbb{R}^2 \ni (x, y) \mapsto x + \sqrt{-1}y \in \mathbb{C}$. To unify both the cases we regard \mathbb{R} as a subset of \mathbb{C} in an obvious manner. We denote by $\Re[\cdot]$ and $\Im[\cdot]$ the real and imaginary parts of \cdot , respectively. We remark that $z/|z|^2 = 1/\bar{z} \in \mathbb{C}$.

We consider the Taylor expansion of $\Psi(x, y)$.

Lemma 6.1. Assume (2.21). Let $x, y \in \mathbb{C}$ such that $|x| < |y|$. Then

$$\Psi(x, y) - \Psi(0, y) = \beta \sum_{\ell=1}^{\infty} \frac{1}{\ell} \Re \left[\left(\frac{x}{y} \right)^\ell \right]. \quad (6.1)$$

Here $\Re[\cdot]$ denotes the real part of $\cdot \in \mathbb{C}$.

Proof. Let $r = |x|/|y|$ and $\theta = \angle(x, y)$. Then we see that

$$\begin{aligned}\Psi(x, y) - \Psi(0, y) &= -\frac{\beta}{2} \log \left| \frac{x}{|y|} - \frac{y}{|y|} \right|^2 \\ &= -\frac{\beta}{2} \log (1 + r^2 - 2r \cos \theta) \\ &= -\frac{\beta}{2} \{\log(1 - re^{i\theta}) + \log(1 - re^{-i\theta})\}.\end{aligned}$$

Hence, (6.1) follows from the Taylor expansion. \square

Let $S_{rs} = S_s \setminus S_r = \{y \in S; b_r \leq |y| < b_s\}$ be as before, where S_r and b_r are given by (2.3). We set $\Psi_{rs}(x, y) = \sum_{y_i \in S_{rs}} \Psi(x, y_i)$, where $y = \sum_i \delta_{y_i}$. By (6.1), we easily see that

$$\begin{aligned}\Psi_{rs}(x, y) - \Psi_{rs}(w, y) &= \beta \sum_{\ell=1}^{\infty} \frac{1}{\ell} \sum_{y_i \in S_{rs}} \Re \left[\frac{x^\ell - w^\ell}{y_i^\ell} \right] \\ &= \beta \sum_{\ell=1}^{\infty} \frac{1}{\ell} \Re \left[(x^\ell - w^\ell) \cdot \sum_{y_i \in S_{rs}} \frac{1}{y_i^\ell} \right].\end{aligned}\quad (6.2)$$

Recall the notation $x \cdot m_s^n$ in (2.14). If $d = 2$, then $m_s^n = (m_{s,1}^n, m_{s,2}^n) \in \mathbb{R}^2$ by definition, and so $x \cdot m_s^n = x_1 m_{s,1}^n + x_2 m_{s,2}^n$. Since we interpret x as complex numbers, we set $x \cdot m_s^n = \Re[x] m_{s,1}^n + \Im[x] m_{s,2}^n = \Re[x \tilde{m}_s^n]$. Since $x = x_1 + \sqrt{-1}x_2$, we then have

$$\begin{aligned}x \cdot (m_r^n - m_s^n) &= x_1(m_{r,1}^n - m_{s,1}^n) + x_2(m_{r,2}^n - m_{s,2}^n) \\ &= \Re[x(\tilde{m}_r^n - \tilde{m}_s^n)].\end{aligned}$$

Here, in the second line, we regard x , m_r^n , and m_s^n as complex numbers.

Lemma 6.2. *Let*

$$\tilde{\Psi}_{rs}^n(x, y) = \Psi_{rs}(x, y) + \Re[x(\tilde{m}_r^n - \tilde{m}_s^n)].\quad (6.3)$$

Then the following holds with finite constants c_{15} and c_{16} .

$$\begin{aligned}\sup_{x, w \in S_r, x \neq w} \frac{|\tilde{\Psi}_{rs}^n(x, y) - \tilde{\Psi}_{rs}^n(w, y)|}{|x - w|} &\leq |F_{rs}^n(y)| + c_{15} \sum_{\ell=2}^{\ell_0-1} \left| \sum_{y_i \in S_{rs}} \frac{1}{y_i^\ell} \right| \\ &\quad + c_{16} \sum_{y_i \in S_{rs}} \frac{b_r^{\ell_0}}{|y_i|^{\ell_0} - b_r^{\ell_0}}.\end{aligned}\quad (6.4)$$

Here $y = \sum_j \delta_{y_j}$ as before and

$$F_{rs}^n(y) = \Re \left[\beta \left(\sum_{y_i \in S_{rs}} \frac{1}{y_i} \right) + (\tilde{m}_r^n - \tilde{m}_s^n) \right],\quad (6.5)$$

$$c_{15} = |\beta| \cdot \max_{1 \leq \ell < \ell_0} \sup_{x, w \in S_r, x \neq w} \frac{|x^\ell - w^\ell|}{\ell |x - w|},\quad (6.6)$$

$$c_{16} = |\beta| \ell_0 \cdot \sup_{\ell_0 \leq \ell} \sup_{x, w \in S_r, x \neq w} \frac{|x^\ell - w^\ell|}{b_r^\ell \ell |x - w|}. \quad (6.7)$$

Proof. We first check the finiteness of c_{15} and c_{16} . Indeed, $c_{15} < \infty$ is clear. Note that, $|x|/b_r < 1$ on S_r . Thus, the Lipschitz norm of the function x^ℓ/b_r^ℓ on S_r is uniformly bounded in ℓ , which implies $c_{16} < \infty$.

Since $|\Re[ab]| \leq |a||b|$, we deduce from (6.2) that for $x, w \in S_r$ with $x \neq w$

$$\frac{|\tilde{\Psi}_{rs}^n(x, y) - \tilde{\Psi}_{rs}^n(w, y)|}{|x - w|} \leq |F_{rs}^n(y)| + |\beta| \left\{ \sum_{\ell=2}^{\infty} \frac{|x^\ell - w^\ell|}{\ell |x - w|} \right\} \left| \sum_{y_i \in S_{rs}} \frac{1}{y_i^\ell} \right|. \quad (6.8)$$

We easily see that

$$\begin{aligned} & |\beta| \left\{ \sum_{\ell=2}^{\infty} \frac{|x^\ell - w^\ell|}{\ell |x - w|} \right\} \left| \sum_{y_i \in S_{rs}} \frac{1}{y_i^\ell} \right| \\ &= |\beta| \left\{ \sum_{\ell=2}^{\ell_0-1} \frac{|x^\ell - w^\ell|}{\ell |x - w|} \right\} \left| \sum_{y_i \in S_{rs}} \frac{1}{y_i^\ell} \right| + |\beta| \left\{ \sum_{\ell=\ell_0}^{\infty} \frac{|x^\ell - w^\ell|}{\ell |x - w|} \right\} \left| \sum_{y_i \in S_{rs}} \frac{1}{y_i^\ell} \right| \\ &\leq c_{15} \sum_{\ell=2}^{\ell_0-1} \left| \sum_{y_i \in S_{rs}} \frac{1}{y_i^\ell} \right| + \frac{c_{16}}{\ell_0} \sum_{\ell=\ell_0}^{\infty} \sum_{y_i \in S_{rs}} \frac{b_r^\ell}{|y_i|^\ell} \\ &= c_{15} \sum_{\ell=2}^{\ell_0-1} \left| \sum_{y_i \in S_{rs}} \frac{1}{y_i^\ell} \right| + c_{16} \sum_{y_i \in S_{rs}} \frac{b_r^{\ell_0}}{|y_i|^{\ell_0} - b_r^{\ell_0}}. \end{aligned} \quad (6.9)$$

Here, in the last line, we used the formula

$$\sum_{\ell=\ell_0}^{\infty} \frac{a^\ell}{b^\ell} = \frac{a^{\ell_0}}{b^{\ell_0}} \frac{b}{b-a} \leq \ell_0 \frac{a^{\ell_0}}{b^{\ell_0} - a^{\ell_0}} \quad \text{for } 0 < a \leq b.$$

If $a = b$, then we interpret $\sum_{\ell=\ell_0}^{\infty} a^\ell/b^\ell = \infty$.

Combining (6.8) and (6.9) completes the proof of Lemma 6.2. \square

Taking (6.3)–(6.5) into account, we set for $r, k, \ell \in \mathbb{N}$

$$\mathcal{U}_{r,1,k} = \sum_{n=1}^{\infty} \left\{ y \in \mathcal{S}; y(S) = N_n, \sup_{r < s \in \mathbb{N}} |F_{rs}^n(y)| \leq k \right\}, \quad (6.10)$$

$$\mathcal{U}_{r,\ell,k} = \left\{ y \in \mathcal{S}; \sup_{r < s \in \mathbb{N}} \left| \sum_{y_i \in S_{rs}} \frac{1}{y_i^\ell} \right| \leq k \right\} \quad \text{if } 2 \leq \ell, \quad (6.11)$$

$$\bar{\mathcal{U}}_{r,\ell,k} = \left\{ y \in \mathcal{S}; \left\{ \sum_{y_i \in S_{r\infty}} \frac{1}{|y_i|^\ell - b_r^\ell} \right\} \leq k \right\}. \quad (6.12)$$

We introduce the new condition (H.5).

(H.5) For each $r \in \mathbb{N}$, there exists an ℓ_0 such that $2 \leq \ell_0 \in \mathbb{N}$ and that

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu^n(\bar{\mathcal{U}}_{r,\ell_0,k}^c) = 0, \quad (6.13)$$

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu^n(\mathcal{U}_{r,\ell,k}^c) = 0 \quad \text{for all } 1 \leq \ell < \ell_0. \quad (6.14)$$

We now state the main theorem of this section.

Theorem 6.3. Assume (2.21) and $S = \mathbb{C}$. Then (H.5) implies (2.19).

Remark 6.1. If $d = 1$, then

$$|F_{rs}^n(\mathbf{y})| = |v_{1,rs}(\mathbf{y})|. \quad (6.15)$$

Hence Theorem 6.3 is also valid for the proof of Theorem 2.2. In fact, we see that

$$\mathcal{U}_{r,1,k} = \sum_{n=1}^{\infty} \left\{ \mathbf{y} \in \mathcal{S}; \mathbf{y}(S) = N_n, \sup_{r < s \in \mathbb{N}} |v_{1,rs}(\mathbf{y})| \leq k \right\}. \quad (6.16)$$

Proof. Set $c_{17} = c_{16}b_r^{\ell_0}$. Then from (6.4) we deduce that

$$\begin{aligned} & \sup_{r < s \in \mathbb{N}} \sup_{x, w \in S_r, x \neq w} \frac{|\tilde{\Psi}_{rs}^n(x, \mathbf{y}) - \tilde{\Psi}_{rs}^n(w, \mathbf{y})|}{|x - w|} \\ & \leq \left\{ \sup_{r < s \in \mathbb{N}} |F_{rs}^n(\mathbf{y})| \right\} + c_{15} \sum_{\ell=2}^{\ell_0-1} \left\{ \sup_{r < s \in \mathbb{N}} \left| \sum_{y_i \in S_{rs}} \frac{1}{y_i^\ell} \right| \right\} + c_{17} \left\{ \sum_{y_i \in S_{r\infty}} \frac{1}{|y_i|^{\ell_0} - b_r^{\ell_0}} \right\}. \end{aligned} \quad (6.17)$$

Combining this with (2.18) and (6.10)–(6.12), we deduce that

$$\mathcal{H}_{r,k} \supset \left\{ \bigcap_{\ell=1}^{\ell_0-1} \mathcal{U}_{r,\ell,k/(\ell_0 c_{15})} \right\} \cap \bar{\mathcal{U}}_{r,\ell_0,k/(\ell_0 c_{17})}.$$

Hence, we obtain

$$\mu^n(\mathcal{H}_{r,k}^c) \leq \left\{ \sum_{\ell=1}^{\ell_0-1} \mu^n(\mathcal{U}_{r,\ell,k/(\ell_0 c_{15})}^c) \right\} + \mu^n(\bar{\mathcal{U}}_{r,\ell_0,k/(\ell_0 c_{17})}^c). \quad (6.18)$$

This together with (H.5) implies (2.19), which completes the proof. \square

7. Proof of Theorem 2.2

In this section, we complete the proof of Theorem 2.2. For this we check the conditions of (H.5). We begin with (6.13), the first condition of (H.5).

Lemma 7.1. Assume $S = \mathbb{C}$ and (H.2). Then (6.13) follows from (2.24).

Proof. Let b_r be as in (2.3). We divide the set $S_{r\infty} = \{b_r \leq |x| < \infty\}$ in (6.12) into two parts $S_{r(r+1)} = \{b_r \leq |x| < b_{r+1}\}$ and $S_{(r+1)\infty} = \{b_{r+1} \leq |x| < \infty\}$. Let

$$\begin{aligned} \mathcal{V}_{1,k} &= \left\{ \mathbf{x} \in \mathcal{S}; \left\{ \sum_{x_i \in S_{r(r+1)}} \frac{1}{|x_i|^{\ell_0} - b_r^{\ell_0}} \right\} \leq \frac{k}{2} \right\} \\ \mathcal{V}_{2,k} &= \left\{ \mathbf{x} \in \mathcal{S}; \left\{ \sum_{x_i \in S_{(r+1)\infty}} \frac{1}{|x_i|^{\ell_0} - b_r^{\ell_0}} \right\} \leq \frac{k}{2} \right\}, \quad \text{where } \mathbf{x} = \sum_i \delta_{x_i}. \end{aligned}$$

Then clearly $\bar{U}_{r,\ell_0,k} \supset V_{1,k} \cap V_{2,k}$. To estimate $V_{1,k}$, we observe that for $x = \sum_i \delta_{x_i}$

$$\sum_{x_i \in S_{r(r+1)}} \frac{1}{|x_i|^{\ell_0} - b_r^{\ell_0}} \leq \left\{ \sup_{x_i \in S_{r(r+1)}} \frac{1}{|x_i|^{\ell_0} - b_r^{\ell_0}} \right\} \cdot x(S_{r(r+1)}).$$

Here $x(S_{r(r+1)})$ is the number of points x_i in $S_{r(r+1)}$. Considering this, we set

$$V_{3,k} = \left\{ x \in \mathbb{S}; \sup_{x_i \in S_{r(r+1)}} \frac{1}{|x_i|^{\ell_0} - b_r^{\ell_0}} \leq \sqrt{k/2} \right\},$$

$$V_{4,k} = \{x \in \mathbb{S}; x(S_{r(r+1)}) \leq \sqrt{k/2}\}.$$

Then we have $V_{1,k} \supset V_{3,k} \cap V_{4,k}$. We therefore obtain $\bar{U}_{r,\ell_0,k} \supset V_{2,k} \cap V_{3,k} \cap V_{4,k}$ by combining these two inclusions. Hence, we deduce (6.13) from

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu^n(V_{l,k}^c) = 0 \quad \text{for all } l = 2, 3, 4. \quad (7.1)$$

We will check (7.1) for each $l = 2, 3, 4$.

As for (7.1) with $l = 2$, according to the Chebyshev inequality, we have

$$\begin{aligned} \mu^n(V_{2,k}^c) &\leq \frac{2}{k} E^{\mu^n} \left[\sum_{x_i \in S_{(r+1)\infty}} \frac{1}{|x_i|^{\ell_0} - b_r^{\ell_0}} \right] \\ &= \frac{2}{k} \int_{S_{(r+1)\infty}} \left\{ \frac{1}{|x|^{\ell_0} - b_r^{\ell_0}} \right\} \rho_n^1(x) dx \\ &= \frac{2}{k} \int_{S_{(r+1)\infty}} \left\{ \frac{|x|^{\ell_0}}{|x|^{\ell_0} - b_r^{\ell_0}} \frac{1}{|x|^{\ell_0}} \right\} \rho_n^1(x) dx \\ &\leq \frac{2}{k} \left\{ \frac{b_{r+1}^{\ell_0}}{b_{r+1}^{\ell_0} - b_r^{\ell_0}} \right\} \cdot \int_{S_{(r+1)\infty}} \left\{ \frac{1}{|x|^{\ell_0}} \right\} \rho_n^1(x) dx. \end{aligned} \quad (7.2)$$

Here we used the fact that $t^{\ell_0}/\{t^{\ell_0} - b_r^{\ell_0}\}$ is decreasing in $t \in (b_r, \infty)$, which implies

$$\sup_{x \in S_{(r+1)\infty}} \frac{|x|^{\ell_0}}{|x|^{\ell_0} - b_r^{\ell_0}} \leq \frac{b_{r+1}^{\ell_0}}{b_{r+1}^{\ell_0} - b_r^{\ell_0}}.$$

By (2.24) and (7.2), we obtain (7.1) with $l = 2$.

We next consider (7.1) with $l = 3$. Let

$$U_k = \{x \in S_{r(r+1)}; b_r^{\ell_0} \leq |x|^{\ell_0} < b_r^{\ell_0} + \sqrt{2/k}\}.$$

It is not difficult to see that U_k is non-increasing and $\lim_{k \rightarrow \infty} U_k = \emptyset$. We note that

$$\begin{aligned} V_{3,k}^c &= \left\{ x \in \mathbb{S}; \inf_{x_i \in S_{r(r+1)}} \{|x_i|^{\ell_0} - b_r^{\ell_0}\} < \sqrt{2/k} \right\} \\ &= \{x \in \mathbb{S}; 1 \leq x(U_k)\}. \end{aligned} \quad (7.3)$$

Here we use the convention such that $\inf \emptyset = \infty$; that is, we interpret $x \notin V_{3,k}^c$ when $x(S_{r(r+1)}) = 0$. Let $c_{18} = \sup\{\rho_n^1(x); n \in \mathbb{N}, x \in S_{r(r+1)}\}$. Then by (2.13), we have $C_{18} < \infty$.

From the second equality in (7.3) and the Chebyshev inequality, we obtain

$$\mu^n(\mathbf{V}_{3,k}^c) \leq E^{\mu^n}[\mathbf{x}(U_k)] = \int_{U_k} \rho_n^1(x) dx \leq c_{18} \int_{U_k} dx. \quad (7.4)$$

Hence, we deduce (7.1) with $l = 3$ from (7.4) and $\lim_{k \rightarrow \infty} U_k = \emptyset$.

We finally consider (7.1) with $l = 4$. From the Chebyshev inequality, we obtain

$$\mu^n(\mathbf{V}_{4,k}^c) \leq \sqrt{\frac{2}{k}} E^{\mu^n}[\mathbf{x}(S_{r(r+1)})] = \sqrt{\frac{2}{k}} \int_{S_{r(r+1)}} \rho_n^1(x) dx \leq \sqrt{\frac{2}{k}} c_{18} \int_{S_{r(r+1)}} dx.$$

This immediately yields (7.1) with $l = 4$. \square

We proceed with (6.14), the second condition of (H.5).

Lemma 7.2. *Let the same assumptions as Lemma 7.1 hold. Then (6.14) follows from (2.25).*

Proof. By (2.25), we can and do choose $\{b_r\}$ and $c_{19} > 0$ in such a way that

$$\sup_{n \in \mathbb{N}} \|\mathbf{v}_{\ell, b_r \infty}\|_{L^1(\mathbb{S}, \mu^n)} \leq c_{19} 3^{-r} \quad \text{for all } r \in \mathbb{N}, 1 \leq \ell < \ell_0. \quad (7.5)$$

We note that $\mathbf{v}_{\ell, b_r b_s}(x) = \mathbf{v}_{\ell, b_r \infty}(x) - \mathbf{v}_{\ell, b_s \infty}(x)$. Then by (6.11), we see that

$$\begin{aligned} \mu^n(\{\mathbf{U}_{r, \ell, k}\}^c) &= \mu^n\left(\sup_{r < s \in \mathbb{N}} |\mathbf{v}_{\ell, b_r \infty} - \mathbf{v}_{\ell, b_s \infty}| > k\right) \\ &\leq \mu^n\left(|\mathbf{v}_{\ell, b_r \infty}| > \frac{k}{2}\right) + \mu^n\left(\sup_{r < s \in \mathbb{N}} |\mathbf{v}_{\ell, b_s \infty}| > \frac{k}{2}\right) \\ &\leq \mu^n\left(|\mathbf{v}_{\ell, b_r \infty}| > \frac{k}{2}\right) + \sum_{s=r+1}^{\infty} \mu^n\left(|\mathbf{v}_{\ell, b_s \infty}| > \frac{k}{2}\right) \\ &\leq \frac{2}{k} \cdot \left\{ \sum_{s=r}^{\infty} \|\mathbf{v}_{\ell, b_s \infty}\|_{L^1(\mathbb{S}, \mu^n)} \right\}. \end{aligned} \quad (7.6)$$

Here we used the Chebyshev inequality in the last line.

From (7.6), we deduce that

$$\sup_{n \in \mathbb{N}} \mu^n(\{\mathbf{U}_{r, \ell, k}\}^c) \leq \frac{2}{k} \cdot \left\{ \sum_{s=r}^{\infty} \sup_{n \in \mathbb{N}} \|\mathbf{v}_{\ell, b_s \infty}\|_{L^1(\mathbb{S}, \mu^n)} \right\}. \quad (7.7)$$

Then from (7.5) and (7.7), we see that

$$\sup_{n \in \mathbb{N}} \mu^n(\{\mathbf{U}_{r, \ell, k}\}^c) \leq \frac{2}{k} \cdot \frac{c_{19} 3^{-r}}{1 - 3^{-1}}. \quad (7.8)$$

Taking $k \rightarrow \infty$ in (7.8), we obtain (6.14). \square

Proof of Theorem 2.2. From Lemmas 7.1 and 7.2, we deduce that the assumption (H.5) in Theorem 6.3 holds. Hence from Theorem 6.3, we obtain (2.19) in (H.3). Therefore Theorem 2.2 follows from Theorem 2.1. \square

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