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Young differential equations with power type nonlinearities

Jorge A. León[a], David Nualart[b], Samy Tindel[c,∗]

[a] Depto. de Control Automático, CINVESTAV-IPN, Apartado Postal 14-740, 07000 México, D.F, Mexico
[b] Department of Mathematics, University of Kansas, 405 Snow Hall, Lawrence, KS, USA
[c] Department of Mathematics, Purdue University, 150 N. University Street, W. Lafayette, IN 47907, USA

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Abstract

In this note we give several methods to construct nontrivial solutions to the equation $dy_t = \sigma(y_t)\,dx_t$, where x is a γ-Hölder $\mathbb{R}^d$-valued signal with $\gamma \in (1/2, 1)$ and σ is a function behaving like a power function $|\xi|^\kappa$, with $\kappa \in (0, 1)$. In this situation, classical Young integration techniques allow to get existence and uniqueness results whenever $\gamma(\kappa + 1) > 1$, while we focus on cases where $\gamma(\kappa + 1) \le 1$. Our analysis then relies on Zähle's extension (Zähle, 1998) of Young's integral allowing to cover the situation at hand.
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1. Introduction

Let $T > 0$ be a fixed arbitrary horizon, and consider a noisy function $x : [0, T] \to \mathbb{R}^d$ in the Hölder space $\mathcal{C}^\gamma([0, T];\ \mathbb{R}^d)$, with $\gamma > 1/2$. Let $\sigma^1, \dots, \sigma^d$ be some vector fields on $\mathbb{R}^m$, a be an initial data in $\mathbb{R}^m$ and consider the following integral equation

$$y_t = a + \sum_{j=1}^{d} \int_0^t \sigma^j(y_u)\,dx_u^j, \qquad t \in [0, T]. \tag{1}$$

∗ Corresponding author.
E-mail addresses: jleon@ctrl.cinvestav.mx (J.A. León), nualart@ku.edu (D. Nualart), stindel@purdue.edu (S. Tindel).

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When $\sigma^1, \ldots, \sigma^d$ are smooth enough, Eq. (1) can be solved thanks to fractional calculus [8,12,16] or Young integration techniques. Extensions of these methods, thanks to the rough paths theory (see e.g. [4,9]), also allow to handle cases of signals with regularity lower than 1/2.

In the current paper, we are concerned with a different, though very natural problem: can we define and solve Eq. (1) for coefficients which are only Hölder continuous? Stated in such a generality the question is still open, but we consider here the special case of a coefficient σ behaving like a power function.

This problem has quite a long story, and a full answer in the case of a 1-dimensional equation driven by a standard Brownian motion is given in [6,15]. The basic idea on which Watanabe–Yamada's contribution relies, is the following a priori estimate. Consider Eq. (1) driven by a Brownian motion B, with a non-linearity $\sigma(\xi) = |\xi|^\kappa$ where $\kappa > 1/2$. Namely, let y be a solution to

$$y_t = a + \int_0^t |y_u|^\kappa \, dB_u, \quad t \in [0, T], \tag{2}$$

where the differential with respect to B is understood in the Itô sense. Then obviously the main problem in order to estimate y is its behavior close to 0, since elsewhere $\xi \mapsto |\xi|^\kappa$ is a Lipschitz function. For $n \geq 1$ we thus consider an approximation φ_n of the function $\xi \mapsto |\xi|$ such that $\varphi_n \in C_b^2(\mathbb{R})$, $\varphi_n \geq 0$ and $\|\varphi_n^{(2)}\|_\infty \leq n$. Then applying Itô's formula to Eq. (2) we get

$$\mathbf{E}\left[\varphi_n(y_t)\right] = \varphi_n(a) + \frac{1}{2}\int_0^t \mathbf{E}\left[\varphi_n^{(2)}(y_u)\,|y_u|^{2\kappa}\right] du. \tag{3}$$

The right hand side of Eq. (3) is then controlled by noticing that, whenever $|y_u| \leq 1/n$, we have $|\varphi_n^{(2)}(y_u)|\,|y_u|^{2\kappa} \leq n^{-(2\kappa-1)}$. This quantity converges to 0 as $n \to \infty$, which is the key step in order to control $\mathbf{E}[\varphi_n(y_t)]$ in [15].

The method described above in order to handle the Brownian case is short and elegant, but fails to give a true intuition of the phenomenon allowing to solve Eq. (1) with a power type coefficient. This intuition has been highlighted in [10,11], though in the much more technical context of the stochastic heat equation. In order to understand the main idea, let us go back to Eq. (1) understood in the Young sense. Then two cases can be thought of (we restrict our considerations to 1-dimensional paths in the remainder of the introduction for notational sake):

(i) One expects y to be an element of $\mathcal{C}^\gamma$, since the equation is driven by $x \in \mathcal{C}^\gamma$. This means that $\sigma(y)$ should lie in $\mathcal{C}^{\kappa\gamma}$. When κ satisfies $\kappa\gamma + \gamma > 1$, each integral $\int_0^t \sigma(y_u)\,dx_u$ can thus be defined as a usual Young integral, and Eq. (1) is solved thanks to classical methods as in [4,8,12,16].

(ii) Let us now consider the case $\kappa\gamma + \gamma \leq 1$. If one wishes to define the integral $\int_0^t \sigma(y_u)\,dx_u$ properly when y_u is close to 0, the heuristic argument is as follows: when y_u is small the equation is basically noiseless, so that $\sigma(y)$ should be considered as a $\mathcal{C}^\kappa$-Hölder function instead of a $\mathcal{C}^{\kappa\gamma}$-Hölder function. This means that the expected condition on κ in order to solve Eq. (1) is just $\kappa + \gamma > 1$.

As mentioned above, this strategy has been successfully implemented in [10,11] in a Brownian SPDE context. It heavily relies on the regularity gain when y hits 0. In our case, we will follow two directions which are somehow different in their nature: (i) We will see that if y does not hit 0 too sharply, this condition being quantified in an integral way, then the integrals $\int_0^t \sigma(y_u)\,dx_u$ still have a good chance to be defined even if $\kappa\gamma + \gamma < 1$. One can then construct a solution of

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(1) in this landmark. (ii) Another approach consists in quantifying the regularity gain enforced by Eq. (1) when the solution y approaches 0. In this way, one can get some uniform a priori Hölder bounds on y and invoke some compactness arguments.

To be more specific, we shall proceed as follows:

(1) We start with a general lemma on Young integration. Namely (see Proposition 2.4 for a precise statement), we consider η such that $(\kappa+\eta)\gamma > 1-\gamma$. We also consider a path $y \in \mathcal{C}^\gamma$ and a function σ behaving like a power function $|\xi|^\kappa$. By adding the assumption $|y|^{-1} \in L^q([0,\tau])$ with $q = \frac{\eta}{\gamma(\kappa+\eta)}$, we prove that $\int_0^t \sigma(y_u)\,dx_u$ is well defined as a Young-type integral and gives rise to a γ-Hölder function. Notice that we have carried out this part of our program with fractional integration techniques because the calculations are easily expressed in this setting. We can however link the integral we obtain with Riemann sums, as will be shown in Theorem 2.6.

(2) With this integration result in hand, we consider the 1-dimensional version of Eq. (1) and perform a Lamperti-type transformation $y_t = \phi^{-1}(x_t)$, where $\phi(\xi) = \int_0^\xi [\sigma(s)]^{-1} ds$. Then we prove that y is a solution to our equation of interest by identifying the Young integral $\int_0^t \sigma(y_u)\,dx_u$ for $y_t = \phi^{-1}(x_t)$. Our result is valid for any κ such that $\gamma(1+\kappa) < 1$, and we refer to Theorem 3.7 for a precise statement.

(3) In case of a multidimensional setting, our global strategy is different. Namely, we will base our consideration on the fact that when y_u is close to 0, its regularity is higher than expected (as mentioned above). Specifically, our basic a priori estimate for (1) states that whenever a solution y satisfies $|y_u| \le 2^{-k}$ for u lying in an interval I, then we also have $|y_t - y_s|$ of order $2^{-\kappa k}|t-s|^\gamma$ for $s,t \in I$. Our regularity gain is thus expressed by the coefficient $2^{-\kappa k}$ above. This gain is sufficient to get to the existence of a γ-Hölder continuous solution to Eq. (1) in the d-dimensional case. We will then construct a solution which vanishes as soon as it hits the origin (see Theorem 4.15).

Summarizing the considerations above, we are able to get existence theorems for Eq. (1) with power type nonlinearities in a wide range of cases. The situation would obviously be clearer if we could get the corresponding pathwise uniqueness results, like in the aforementioned Refs. [6,10,11,15]. However, these articles handle the case of Itô type equations, for which uniqueness is expected. In our Stratonovich–Young case uniqueness of the solution is ruled out, since both the nontrivial solution we shall construct and the solution $y \equiv 0$ solve Eq. (1) when $a = 0$. We shall go back to this issue below.

Our paper is structured as follows: the Young's integral related to our power type coefficient is studied in Section 2. Section 3 deals with its application to the existence of solutions to Eq. (1) in dimension 1. The other approach, based on the a priori regularity gain of the solution when it hits 0, is developed in Section 4. Finally, in Section 5 we discuss the application of these results to the case of stochastic differential equations driven by a fractional Brownian motion.

Notations. Throughout the article, we use the following conventions: for 2 quantities a and b, we write $a \lesssim b$ if there exists a universal constant c (which might depend on the parameters of the model, such as, $\gamma, \kappa, \eta, \alpha, T, \dots$) such that $a \le c\,b$. If f is a vector-valued function defined on an interval $[0,T]$ and $s,t \in [0,T]$, δf_{st} denotes the increment $f_t - f_s$.

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2. An extension of Young's integral

This section is devoted to an extension of Young's integral using fractional calculus techniques, which will be suitable to handle Eq. (1) with Hölder-type and singular nonlinearities. We shall first recall some general elements of fractional calculus.

2.1. Elements of fractional calculus

We restrict this introduction to real-valued functions for notational sake. Consider $0 \le a < b \le T$ and an $L^1([0,T])$-function f. For $t \in [a,b]$ and $\alpha \in (0,1)$ the fractional integrals of f are defined as

$$I^{\alpha}_{a+} f_t = \frac{1}{\Gamma(\alpha)} \int_a^t (t-r)^{\alpha-1} f_r \, dr, \quad \text{and} \quad I^{\alpha}_{b-} f_t = \frac{1}{\Gamma(\alpha)} \int_t^b (r-t)^{\alpha-1} f_r \, dr.$$

For any $p \ge 1$, we denote by $I^{\alpha}_{a+}(L^p)$ the image of $L^p([a,b])$ by I^{α}_{a+}, and similarly for $I^{\alpha}_{b-}(L^p)$.

The inverse of the operators I^{α}_{a+} and I^{α}_{b-} are called fractional derivatives, and are defined as follows. For $f \in I^{\alpha}_{a+}(L^p)$ and $t \in [a,b]$ we set

$$D^{\alpha}_{a+} f_t = L^p - \lim_{\varepsilon \downarrow 0} \frac{1}{\Gamma(1-\alpha)} \left(\frac{f_t}{(t-a)^{\alpha}} + \alpha \int_a^{t-\varepsilon} \frac{f_t - f_r}{(t-r)^{1+\alpha}} \, dr \right), \tag{4}$$

where we use the convention $f_r = 0$ on $[a,b]^c$. In the same way, for $f \in I^{\alpha}_{b-}(L^p)$ and $t \in [a,b]$, we set

$$D^{\alpha}_{b-} f_t = L^p - \lim_{\varepsilon \downarrow 0} \frac{1}{\Gamma(1-\alpha)} \left(\frac{f_t}{(b-t)^{\alpha}} + \alpha \int_{t+\varepsilon}^b \frac{f_t - f_r}{(r-t)^{1+\alpha}} \, dr \right). \tag{5}$$

By [14, Remark 13.2] we have that, for $p > 1$, $f \in I^{\alpha}_{a+}(L^p)$ (resp. $f \in I^{\alpha}_{b-}(L^p)$) if and only if $f \in L^p([a,b])$ and the limit in the right-hand side of (4) (resp. (5)) exists. In this case $f = I^{\alpha}_{a+}(D^{\alpha}_{a+} f)$ (resp. $f = I^{\alpha}_{b-}(D^{\alpha}_{b-} f)$). It is not difficult to see that, as a consequence of the proof of [14, Theorem 13.2], the fact that $f \in L^p([a,b])$, $\frac{f(\cdot)}{(\cdot - a)^{\alpha}}$ and $\int_a^{\cdot} \frac{f(\cdot) - f_r}{(\cdot - r)^{1+\alpha}} dr$ (resp. $\frac{f(\cdot)}{(b-\cdot)^{\alpha}}$ and $\int_{\cdot}^b \frac{f(\cdot) - f_r}{(r-\cdot)^{1+\alpha}} dr$) belong to $L^p([a,b])$ implies that $f \in I^{\alpha}_{a+}(L^p)$ (resp. $f \in I^{\alpha}_{b-}(L^p)$) and

$$D^{\alpha}_{a+} f_t = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f_t}{(t-a)^{\alpha}} + \alpha \int_a^t \frac{f_t - f_r}{(t-r)^{1+\alpha}} \, dr \right) \tag{6}$$

(resp.

$$D^{\alpha}_{b-} f_t = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f_t}{(b-t)^{\alpha}} + \alpha \int_t^b \frac{f_t - f_r}{(r-t)^{1+\alpha}} \, dr \right) \Bigg).$$

Notice that $\mathcal{C}^{\alpha+\varepsilon}([a,b]) \subset I^{\alpha}_{a+}(L^p)$, with $\varepsilon > 0$. In the same manner, we have $\mathcal{C}^{\alpha+\varepsilon}([a,b]) \subset I^{\alpha}_{b-}(L^p)$.

Let $g, f \in L^1([0,T])$ be two functions such that, for some $\alpha \in (0,1)$, $f \in I^{\alpha}_{a+}(L^1)$ and $g^{b-} \in I^{1-\alpha}_{b-}(L^1)$, where $g^{b-}_r = g_r - g_{b-}$. In this case we say that f is integrable with respect to g if and only if $(D^{\alpha}_{a+} f) D^{1-\alpha}_{b-} g^{b-}_r \in L^1([a,b])$. In this case we define the integral $\int_a^b f \, dg$ in the following way

$$\int_a^b f_r \, dg_r := \int_a^b (D^{\alpha}_{a+} f_r) D^{1-\alpha}_{b-} g^{b-}_r \, dr. \tag{7}$$

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If $f \in \mathcal{C}^{\alpha}([a,b])$ and $g \in \mathcal{C}^{\lambda}([a,b])$ (i.e., f and g are α-Hölder and λ-Hölder continuous, respectively) with $\alpha+\lambda>1$, then it can be checked that $\int_a^b f_r\,dg_r$ is well-defined, and that it coincides with Young's integral defined as a limit of Riemann sums (see [16, Theorem 4.2.1]). We shall analyze below this integral under hypotheses suited to our purposes (see Proposition 2.4), that is, to solve Eq. (1).

2.2. *The fractional integral*

We assume in this section that x is a γ-Hölder continuous and real valued signal. In this section, as in the remaining of this paper, we assume that $\gamma \in (1/2, 1)$. Consider the following additional assumption on the coefficient $\sigma : \mathbb{R}^m \to \mathbb{R}^m$.

Hypothesis 2.1. The function $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ satisfies $\sigma(0)=0$ and

$$|\sigma(\xi_2)-\sigma(\xi_1)| \lesssim \left| |\xi_2|^{\kappa} - |\xi_1|^{\kappa} \right|, \quad \xi_1, \xi_2 \in \mathbb{R}^m, \tag{8}$$

for some $\kappa \in (0,1)$ such that $\gamma(\kappa+1)<1$.

Remark 2.2. In order to understand the implications of Hypothesis 2.1, note that if σ fulfills condition (8) and if we consider $\xi_1, \xi_2 \in \mathbb{R}^m$ such that $|\xi_1| = |\xi_2|$, then we obviously have $\sigma(\xi_2) = \sigma(\xi_1)$. Thus (8) implies that σ is a radial function, that is, $\sigma(\xi) = \rho(|\xi|)$, where $\rho : [0,\infty) \to \mathbb{R}^m$. On the other hand, it is not difficult to see that a radial function $\sigma(\xi) = \rho(|\xi|)$ such that $\rho \in \mathcal{C}^1((0,\infty))$, $\rho(0)=0$ and $|\rho^{(1)}(y)| \lesssim y^{\kappa-1}$, $y>0$, satisfies inequality (8).

For a function σ satisfying Hypothesis 2.1, we define

$$\mathcal{N}_{\kappa,\sigma} := \sup \left\{ \frac{|\sigma(\xi_2)-\sigma(\xi_1)|}{||\xi_2|^{\kappa} - |\xi_1|^{\kappa}|} : \xi_2, \xi_1 \in \mathbb{R}^m, |\xi_1| \neq |\xi_2| \right\}. \tag{9}$$

We now label the following auxiliary result for further use.

Lemma 2.3. *Assume σ satisfies Hypothesis 2.1. Then we have*

$$|\sigma(\xi_2)-\sigma(\xi_1)| \le \frac{\kappa}{\kappa+\eta} \mathcal{N}_{\kappa,\sigma} \left(|\xi_2|^{-\eta} + |\xi_1|^{-\eta} \right) |\xi_2 - \xi_1|^{\kappa+\eta},$$

for any $0 \le \eta \le 1-\kappa$ and $\xi_1, \xi_2 \in \mathbb{R}^m \setminus \{0\}$.

Proof. The case $\eta = 0$ or $\eta = 1-\kappa$ is obvious, so we assume $0<\eta<1-\kappa$. Without loss of generality, we can assume that $|\xi_1| \le |\xi_2|$. According to (8), we can write

$$\begin{aligned} |\sigma(\xi_2)-\sigma(\xi_1)| &\le \mathcal{N}_{\kappa,\sigma} \left(|\xi_2|^{\kappa} - |\xi_1|^{\kappa} \right) \\ &= \kappa \mathcal{N}_{\kappa,\sigma} \int_{|\xi_1|}^{|\xi_2|} z^{\kappa-1} dz \le \kappa \mathcal{N}_{\kappa,\sigma} |\xi_1|^{-\eta} \int_{|\xi_1|}^{|\xi_2|} z^{\kappa+\eta-1} dz \\ &\le \kappa \mathcal{N}_{\kappa,\sigma} |\xi_1|^{-\eta} \int_{|\xi_1|}^{|\xi_2|} (z - |\xi_1|)^{\kappa+\eta-1}\, dz, \end{aligned}$$

which yields our claim. □

We are now ready to provide a result on the integral defined in (7). To do this, for any $\lambda \in (0,1)$ and $\eta > 0$, we introduce the space

$$\mathcal{C}^{\lambda}_{\eta}([0,T];\mathbb{R}^m) = \{ y \in \mathcal{C}^{\lambda}([0,T];\mathbb{R}^m) : |y|^{-1} \in L^{\eta}([0,T];\mathbb{R}) \} \tag{10}$$

and use the convention

$$\|f\|_\lambda := \sup_{0\le s<t\le T} \frac{|f(t)-f(s)|}{(t-s)^\lambda},$$

for any $f \in \mathcal{C}^\lambda([0,T];\mathbb{R}^m)$.

Proposition 2.4. *Assume that σ satisfies Hypothesis 2.1 and $\frac{1-\gamma(1+\kappa)}{\gamma} < \eta < 1-\kappa$. Then the following results hold true:*

(i) *If $y \in \mathcal{C}^\gamma_\eta([0,T];\mathbb{R}^m)$, then, for any $t \in [0,T]$, the integral*

$$[\Lambda(y)]_t := \int_0^t \sigma(y_s)\,dx_s,$$

is well defined in the sense of relation (7).

(ii) *Consider $y \in \mathcal{C}^\gamma_{\frac{\eta}{\gamma(\kappa+\eta)}}([0,T];\mathbb{R}^m)$. Then $\Lambda(y)$ belongs to the space $\mathcal{C}^\gamma([0,T];\mathbb{R}^m)$, and the following bound holds true:*

$$\|\Lambda(y)\|_\gamma \lesssim \|x\|_\gamma \left(\|\sigma(y)\|_\infty + \mathcal{N}_{\kappa,\sigma}\|y\|_\gamma^{\kappa+\eta} \left(\int_0^T |y_s|^{-\frac{\eta}{\gamma(\kappa+\eta)}}\,ds \right)^{\gamma(\kappa+\eta)} \right), \tag{11}$$

where $\mathcal{N}_{\kappa,\sigma}$ has been introduced in (9).

Remark 2.5. Taking into account that the function $\eta \to \frac{\eta}{\eta+\kappa}$ is strictly increasing we deduce that $\eta > \frac{1}{\gamma} - 1 - \kappa$ if and only if $\frac{\eta}{\gamma(\kappa+\eta)} > \frac{1-\gamma-\kappa\gamma}{\gamma(1-\gamma)}$. Therefore, the integrability condition for $|y|^{-1}$ in statement (ii) is stronger than that in statement (i).

Proof of Proposition 2.4. Let α be such that $1-\gamma < \alpha < \gamma(\kappa+\eta)$, which implies $\alpha\gamma^{-1} - \kappa < \eta < 1-\kappa$. Let $0 \le t_1 < t_2 \le T$. Recall that the integral $\int_{t_1}^{t_2} [\sigma(y)]_s\,dx_s$ is defined by formula (7). To show that this integral exists and to establish suitable estimates, we first analyze the fractional derivative of x

$$\begin{aligned} \left| D^{1-\alpha}_{t_2-} x^{t_2-}_s \right| &= \frac{1}{\Gamma(\alpha)} \left| \frac{x_s - x_{t_2}}{(t_2-s)^{1-\alpha}} + (1-\alpha)\int_s^{t_2} \frac{x_s - x_r}{(r-s)^{2-\alpha}} dr \right| \\ &\lesssim \|x\|_\gamma (t_2-s)^{\alpha+\gamma-1} + \|x\|_\gamma \int_s^{t_2} (r-s)^{\alpha+\gamma-2} dr \\ &\lesssim \|x\|_\gamma (t_2-s)^{\alpha+\gamma-1}, \end{aligned} \tag{12}$$

where we have used the fact that $\alpha+\gamma > 1$ for the last step. Hence, we can write

$$\int_{t_1}^{t_2} \left| [D^\alpha_{t_1+}\sigma(y)]_s\, D^{1-\alpha}_{t_2-} x^{t_2-}_s \right| ds \lesssim \|x\|_\gamma \left(J^1_{t_1t_2} + J^2_{t_1t_2} \right),$$

with

$$J^1_{t_1t_2} = \|\sigma(y)\|_\infty \int_{t_1}^{t_2} (s-t_1)^{-\alpha}(t_2-s)^{\alpha+\gamma-1} ds$$

and

$$J^2_{t_1t_2} = \int_{t_1}^{t_2} \left(\int_{t_1}^s \frac{|\sigma(y_s)-\sigma(y_u)|}{(s-u)^{\alpha+1}} du \right) (t_2-s)^{\alpha+\gamma-1} ds.$$

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It is now readily checked that

$$J^1_{t_1t_2} \lesssim \|\sigma(y)\|_\infty (t_2-t_1)^\gamma. \tag{13}$$

For the term $J^2_{t_1t_2}$, invoking Lemma 2.3 and some elementary algebraic manipulations, we get

$$\begin{aligned}
J^2_{t_1t_2} &\lesssim \mathcal{N}_{\kappa,\sigma}\int_{t_1}^{t_2}(t_2-s)^{\alpha+\gamma-1}\int_{t_1}^{s}\left(|y_s|^{-\eta}+|y_u|^{-\eta}\right)\frac{|y_s-y_u|^{\kappa+\eta}}{(s-u)^{\alpha+1}}duds\\
&\lesssim \mathcal{N}_{\kappa,\sigma}\|y\|_\gamma^{\kappa+\eta}\int_{t_1}^{t_2}(t_2-s)^{\alpha+\gamma-1}\int_{t_1}^{s}\left(|y_s|^{-\eta}+|y_u|^{-\eta}\right)(s-u)^{\gamma(\kappa+\eta)-\alpha-1}duds\\
&\lesssim \mathcal{N}_{\kappa,\sigma}\|y\|_\gamma^{\kappa+\eta}\Bigg(\int_{t_1}^{t_2}(t_2-s)^{\alpha+\gamma-1}|y_s|^{-\eta}\int_{t_1}^{s}(s-u)^{\gamma(\kappa+\eta)-\alpha-1}duds\\
&\quad+\int_{t_1}^{t_2}|y_u|^{-\eta}\int_{u}^{t_2}(t_2-s)^{\alpha+\gamma-1}(s-u)^{\gamma(\kappa+\eta)-\alpha-1}dsdu\Bigg).
\end{aligned}\tag{14}$$

Notice that $\eta>\alpha\gamma^{-1}-\kappa$ implies that $\gamma(\kappa+\eta)-\alpha>0$. This implies that the integral $\int_{t_1}^{t_2}[\sigma(y)]_s\,dx_s$ is well defined, provided $|y|^{-1}\in L^\eta([0,T];\mathbb{R})$.

Applying Hölder's inequality with $p^{-1}=\gamma(\kappa+\eta)$ and $q^{-1}=1-p^{-1}$, and assuming $|y|^{-1}\in L^{\eta/(\gamma(\kappa+\eta))}([0,T];\mathbb{R})$, yields

$$\begin{aligned}
J^2_{t_1t_2} &\lesssim \mathcal{N}_{\kappa,\sigma}\|y\|_\gamma^{\kappa+\eta}\left(\int_{t_1}^{t_2}|y_u|^{-p\eta}du\right)^{1/p}\\
&\quad\times\left[\left(\int_{t_1}^{t_2}(t_2-s)^{q(\alpha+\gamma-1)}(s-t_1)^{q(\gamma(\kappa+\eta)-\alpha)}ds\right)^{1/q}\right.\\
&\quad\left.+\left(\int_{t_1}^{t_2}(t_2-u)^{q\gamma(\kappa+\eta+1)-q}du\right)^{1/q}\right].
\end{aligned}$$

Now a simple analysis of the exponents in the above relation implies

$$J^2_{t_1t_2}\lesssim \mathcal{N}_{\kappa,\sigma}\|y\|_\gamma^{\kappa+\eta}\left(\int_0^T|y_s|^{-\frac{\eta}{\gamma(\kappa+\eta)}}ds\right)^{\gamma(\kappa+\eta)}(t_2-t_1)^\gamma. \tag{15}$$

Finally, the estimates (11) follow from (13) and (15). The proof is now complete. □

2.3. *The integral via Riemann sums*

The next goal is to see that the integral $\Lambda(y)$ given in Proposition 2.4 can be approximated by Riemann sums. Towards this end, for any $n\geq 2$, we consider a uniform partition $\Pi_n=\{a=t_1<t_2<\cdots<t_n=b\}$ of the interval $[a,b]\subset[0,T]$, such that $|\Pi_n|:=\frac{b-a}{n-1}=t_{j+1}-t_j$ for all $j\in\{1,2,\ldots,n-1\}$. For y as in Proposition 2.4(i), we define the following approximation based on Π_n

$$z^n_s=\sum_{i=2}^{n}\frac{1}{|\Pi_n|}\left(\int_{t_{i-1}}^{t_i}\sigma(y_r)dr\right)\mathbf{1}_{(t_{i-1},t_i]}(s),\qquad s\in[a,b]. \tag{16}$$

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We observe that, owing to [16, Corollary 2.3], we have

$$\int_a^b z_s^n dx_s = \sum_{i=2}^n \frac{1}{|\Pi_n|} \left(\int_{t_{i-1}}^{t_i} \sigma(y_s) ds \right) \delta x_{t_{i-1} t_i},$$

where the left hand side is understood as in relation (7) and where we recall that $\delta x_{uv} := x_v - x_u$. The convergence of $\int_a^b z_s^n\, dx_s$ is given in the following theorem, which is the main result of this subsection. Here we use the Definitions (10) and (16).

Theorem 2.6. *Suppose that σ satisfies Hypothesis 2.1. Let η be such that $\frac{1-\gamma(1+\kappa)}{\gamma} < \eta < 1-\kappa$. Consider $y \in \mathcal{C}_\eta^\gamma([0,T];\mathbb{R}^m)$. Then for all $0 \le a < b \le T$ we have*

$$\lim_{n\to\infty} \int_a^b z_s^n dx_s = \int_a^b \sigma(y_s) dx_s,$$

where z^n is defined in (16).

In order to prove this theorem, we first go through a series of auxiliary results.

Lemma 2.7. *Let σ satisfy Hypothesis 2.1, $y \in \mathcal{C}^\gamma([0,T];\mathbb{R}^m)$ and consider $[a,b] \subset [0,T]$. Then for all $s \in [a,b]$ we have*

$$\left|\sigma(y_s) - z_s^n\right| \le \mathcal{N}_{\kappa,\sigma} \|y\|_\gamma^\kappa \, |\Pi_n|^{\kappa\gamma} .$$

Proof. For $s \in (a,b]$, the definition of z^n gives

$$\begin{aligned}
\left|\sigma(y_s) - z_s^n\right| &= \sum_{i=2}^n \left| \sigma(y_s) - \frac{1}{|\Pi_n|} \int_{t_{i-1}}^{t_i} \sigma(y_r) dr \right| \mathbf{1}_{(t_{i-1},t_i]}(s) \\
&\le \sum_{i=2}^n \frac{1}{|\Pi_n|} \left(\int_{t_{i-1}}^{t_i} |\sigma(y_s) - \sigma(y_r)| dr \right) \mathbf{1}_{(t_{i-1},t_i]}(s) \\
&\le \mathcal{N}_{\kappa,\sigma} \sum_{i=2}^n \frac{1}{|\Pi_n|} \left(\int_{t_{i-1}}^{t_i} \left| |y_s|^\kappa - |y_r|^\kappa \right| dr \right) \mathbf{1}_{(t_{i-1},t_i]}(s).
\end{aligned}$$

Since y is γ-Hölder continuous, we thus have

$$\begin{aligned}
\left|\sigma(y_s) - z_s^n\right| &\le \mathcal{N}_{\kappa,\sigma} \|y\|_\gamma^\kappa \sum_{i=2}^n \frac{1}{|\Pi_n|} \left(\int_{t_{i-1}}^{t_i} |s-r|^{\kappa\gamma} dr \right) \mathbf{1}_{(t_{i-1},t_i]}(s) \\
&\le \mathcal{N}_{\kappa,\sigma} \|y\|_\gamma^\kappa \sum_{i=2}^n |\Pi_n|^{\kappa\gamma} \mathbf{1}_{(t_{i-1},t_i]}(s),
\end{aligned}$$

which completes the proof. $\square$

We now estimate the Hölder regularity of our approximation z^n.

Lemma 2.8. *Let σ and y be functions verifying the assumptions of Theorem 2.6. Then, for $a < u < s \le b$, we have*

$$\left|z_s^n - z_u^n\right| \lesssim \|y\|_\gamma^{\kappa+\eta} \left(\Phi_{u,s}^n + \Psi_{u,s}^n \right),$$

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where

$$\Phi^n_{u,s} = |\Pi_n|^{\gamma(\kappa+\eta)-1} \sum_{2\le j<i\le n} \left(\int_{t_{i-1}}^{t_i} |y_r|^{-\eta} dr + \int_{t_{j-1}}^{t_j} |y_r|^{-\eta} dr \right) \mathbf{1}_{(t_{j-1},t_j]}(u) \mathbf{1}_{(t_{i-1},t_i]}(s)$$

and

$$\Psi^n_{u,s} = \frac{(s-u)^{\gamma(\kappa+\eta)}}{|\Pi_n|} \sum_{2\le j<i\le n} \left(\int_{t_{i-1}}^{t_i} |y_r|^{-\eta} dr + \int_{t_{j-1}}^{t_j} |y_r|^{-\eta} dr \right) \mathbf{1}_{(t_{j-1},t_j]}(u) \mathbf{1}_{(t_{i-1},t_i]}(s).$$

Proof. Assume $s \in (t_{i-1}, t_i]$. If u lies into $(t_{i-1}, t_i]$ too, then $|z^n_s - z^n_u| = 0$ by definition of z^n. We now assume that $u \in (t_{j-1}, t_j]$ with $j \in \{2, \dots, i-1\}$. Then it is readily checked that

$$\begin{aligned} z^n_s - z^n_u &= \frac{1}{|\Pi_n|} \left(\int_{t_{i-1}}^{t_i} \sigma(y_r)\, dr - \int_{t_{j-1}}^{t_j} \sigma(y_r)\, dr \right) \\ &= \frac{1}{|\Pi_n|} \int_{t_{j-1}}^{t_j} \big(\sigma(y_{r+t_{i-1}-t_{j-1}}) - \sigma(y_r)\big)\, dr. \end{aligned}$$

Therefore, thanks to Lemma 2.3 we obtain

$$\begin{aligned} \left|z^n_s - z^n_u\right| &\lesssim \mathcal{N}_{\kappa,\sigma} \frac{1}{|\Pi_n|} \int_{t_{j-1}}^{t_j} \left(|y_{r+t_{i-1}-t_{j-1}}|^{-\eta} + |y_r|^{-\eta}\right) \left|y_{r+t_{i-1}-t_{j-1}} - y_r\right|^{\kappa+\eta} dr \\ &\lesssim \mathcal{N}_{\kappa,\sigma} \frac{\|y\|_\gamma^{\kappa+\eta} |t_{i-1} - t_{j-1}|^{\gamma(\kappa+\eta)}}{|\Pi_n|} \int_{t_{j-1}}^{t_j} \left(|y_{r+t_{i-1}-t_{j-1}}|^{-\eta} + |y_r|^{-\eta}\right) dr, \end{aligned}$$

from which we derive

$$\begin{aligned} \left|z^n_s - z^n_u\right| \lesssim\ & \frac{\|y\|_\gamma^{\kappa+\eta}(s-u+|\Pi_n|)^{\gamma(\kappa+\eta)}}{|\Pi_n|} \\ & \times \sum_{2\le j<i\le n} \left(\int_{t_{i-1}}^{t_i} |y_r|^{-\eta} dr + \int_{t_{j-1}}^{t_j} |y_r|^{-\eta} dr \right) \mathbf{1}_{(t_{j-1},t_j]}(u) \mathbf{1}_{(t_{i-1},t_i]}(s). \end{aligned}$$

Our claim is now easily deduced. □

The next result will help to handle some of the terms appearing in Lemma 2.8.

Lemma 2.9. *Let the assumptions of Theorem 2.6 prevail, and consider the path $\Phi^n : [a,b]^2 \to \mathbb{R}_+$ introduced in Lemma 2.8. We also introduce the following measure on $[a,b]^2$*

$$\mu(du, ds) = (s-u)^{-\alpha-1}(b-s)^{\alpha+\gamma-1} \mathbf{1}_{\{u<s\}}\, du ds, \tag{17}$$

where α is such that $1-\gamma < \alpha < \gamma(\kappa+\eta)$. Then Φ^n converges to zero in $L^1([a,b]^2, \mu)$, as $n \to \infty$.

Proof. We can write

$$\begin{aligned} \|\Phi^n\|_{L^1([a,b]^2,\mu)} \lesssim\ & |\Pi_n|^{-1+\gamma(\kappa+\eta)} \sum_{2\le j<i\le n} \left(\int_{t_{i-1}}^{t_i} |y_r|^{-\eta} dr + \int_{t_{j-1}}^{t_j} |y_r|^{-\eta} dr \right) \\ & \times \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s-u)^{-1-\alpha} du ds \lesssim I_1^n + I_2^n, \end{aligned} \tag{18}$$

where

$$I_1^n = |\Pi_n|^{-1+\gamma(\kappa+\eta)} \sum_{i=3}^n \left(\int_{t_{i-1}}^{t_i} |y_r|^{-\eta} dr + \int_{t_{i-2}}^{t_{i-1}} |y_r|^{-\eta} dr \right) \int_{t_{i-1}}^{t_i} \int_{t_{i-2}}^{t_{i-1}} (s-u)^{-1-\alpha} du ds$$

and

$$I_2^n = |\Pi_n|^{-1+\gamma(\kappa+\eta)} \sum_{i=4}^n \sum_{j=2}^{i-2} \left(\int_{t_{i-1}}^{t_i} |y_r|^{-\eta} dr + \int_{t_{j-1}}^{t_j} |y_r|^{-\eta} dr \right) \times \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s-u)^{-1-\alpha} du ds.$$

We now bound the terms I_1^n and I_2^n separately.

It is easily seen from the expression of I_1^n that

$$I_1^n \lesssim |\Pi_n|^{\gamma(\kappa+\eta)-\alpha} \sum_{i=2}^n \int_{t_{i-1}}^{t_i} |y_r|^{-\eta} dr = |\Pi_n|^{\gamma(\kappa+\eta)-\alpha} \int_a^b |y_r|^{-\eta} dr.$$

Hence, due to the fact that $\gamma(\kappa+\eta)-\alpha>0$, we obtain $\lim_{n\to\infty} I_1^n = 0$.

As far as I_2^n is concerned, a simple scaling argument entails

$$I_2^n \lesssim |\Pi_n|^{\gamma(\kappa+\eta)-\alpha} \sum_{i=4}^n \sum_{j=2}^{i-2} \left(\int_{t_{i-1}}^{t_i} |y_r|^{-\eta} dr + \int_{t_{j-1}}^{t_j} |y_r|^{-\eta} dr \right) \times \int_{i-1}^{i} \int_{j-1}^{j} (s-u)^{-1-\alpha} du ds,$$

and roughly bounding the term $s-u$ by $i-j-1$ in the integral above, we get

$$I_2^n \lesssim |\Pi_n|^{\gamma(\kappa+\eta)-\alpha} \sum_{i=4}^n \sum_{j=2}^{i-2} \left(\int_{t_{i-1}}^{t_i} |y_r|^{-\eta} dr + \int_{t_{j-1}}^{t_j} |y_r|^{-\eta} dr \right) (i-j-1)^{-1-\alpha}$$
$$\lesssim |\Pi_n|^{\gamma(\kappa+\eta)-\alpha} \sum_{i=2}^n \int_{t_{i-1}}^{t_i} |y_r|^{-\eta} dr \sum_{k=1}^{n-1} k^{-1-\alpha} \lesssim |\Pi_n|^{\gamma(\kappa+\eta)-\alpha} \int_a^b |y_r|^{-\eta} dr.$$

We thus get $\lim_{n\to\infty} I_2^n = 0$, again according to the fact that $\gamma(\kappa+\eta)-\alpha>0$.

Finally, taking into account $\lim_{n\to\infty} I_1^n = 0$, $\lim_{n\to\infty} I_2^n = 0$ and relation (18), our claim is now proved. □

Still having in mind a bound on the terms of Lemma 2.8, we state the following intermediate result.

Lemma 2.10. *Assume the hypotheses of Lemma 2.9 hold true and let Ψ^n be as in Lemma 2.8. Then as $n\to\infty$, Ψ^n converges in $L^1([a,b]^2,\mu)$ to the function Ψ defined as follows*

$$\Psi_{u,s} = \left(|y_s|^{-\eta} + |y_u|^{-\eta}\right)(s-u)^{\gamma(\kappa+\eta)} \mathbf{1}_{\{u<s\}}.$$

Proof. The result is an immediate consequence of the fact that $|y|^{-\eta} \in L^1([a,b])$, together with the conditions $\alpha+\gamma-1>0$ and $\gamma(\kappa+\eta)>\alpha$. □

We are now ready to give the proof of Theorem 2.6.

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Proof of Theorem 2.6. Let α be such that $1-\gamma<\alpha<\gamma(\kappa+\eta)$. Owing to (12) we can write

$$\left|\int_a^b \left(z_s^n-\sigma(y_s)\right) dx_s\right| = \left|\int_a^b \left[D_{a+}^{\alpha}\left(\sigma(y)-z^n\right)\right]_s D_{b-}^{1-\alpha}x_s^{b-}ds\right| \lesssim \|x\|_\gamma \left(L_1^n+L_2^n\right),$$

where

$$L_1^n=\int_a^b \frac{\left|\sigma(y_s)-z_s^n\right|}{(s-a)^{\alpha}}(b-s)^{\alpha+\gamma-1}ds$$

and

$$L_2^n=\int_a^b \left(\int_a^s \frac{\left|\sigma(y_s)-z_s^n-\left(\sigma(y_u)-z_u^n\right)\right|}{(s-u)^{\alpha+1}}du\right)(b-s)^{\alpha+\gamma-1}ds.$$

Moreover, notice that invoking Lemma 2.7 we can deduce that $L_1^n \lesssim \mathcal{N}_{\kappa,\sigma}\|y\|_\gamma^{\kappa}|\Pi_n|^{\kappa\gamma}$. Therefore L_1^n goes to zero as $n\to\infty$. Thus, in order to finish the proof we only need to see that L_2^n converges to zero as $n\to\infty$.

In order to study the limit of L_2^n, first notice that thanks to Lemma 2.7 we can write

$$\left|\sigma(y_s)-z_s^n-\left(\sigma(y_u)-z_u^n\right)\right| \le \left|\sigma(y_s)-z_s^n\right|+\left|\sigma(y_u)-z_u^n\right| \lesssim \mathcal{N}_{\kappa,\sigma}\|y\|_\gamma^{\kappa}|\Pi_n|^{\kappa\gamma}, \quad (19)$$

which implies that the integrand in L_2^n converges to zero as n tends to infinity, for each u and s such that $a\le u<s\le b$. On the other hand, we can also bound the rectangular increment $\sigma(y_s)-z_s^n-(\sigma(z_u)-z_u^n)$ as follows

$$\left|\sigma(y_s)-z_s^n-\left(\sigma(y_u)-z_u^n\right)\right| \le |\sigma(y_s)-\sigma(y_u)|+\left|z_s^n-z_u^n\right|. \quad (20)$$

Lemma 2.3 plus the fact that $y\in\mathcal{C}_\eta^\gamma$ imply that

$$|\sigma(y_s)-\sigma(y_u)| \lesssim \left(|y_s|^{-\eta}+|y_u|^{-\eta}\right)|y_s-y_u|^{\kappa+\eta} \lesssim \left(|y_s|^{-\eta}+|y_u|^{-\eta}\right)(s-u)^{(\kappa+\eta)\gamma}.$$

Since $(\kappa+\eta)\gamma>\alpha$, we get that the term $|\sigma(y_s)-\sigma(y_u)|$ is integrable in $[a,b]^2$ with respect to the measure $\mu(du,ds)=(s-u)^{-\alpha-1}(b-s)^{\alpha+\gamma-1}\mathbf{1}_{\{u<s\}}duds$ introduced in Eq. (17). Moreover, the term $\left|z_s^n-z_u^n\right|$ is bounded, up to a constant, by $\Phi_{u,s}^n+\Psi_{u,s}^n$ (see Lemma 2.8). Applying the dominated convergence theorem as stated in [13, Theorem 11.4.18], together with Lemmas 2.9 and 2.10, we deduce that L_2^n tends to 0 as n tends to infinity, which finishes the proof. □

3. One-dimensional differential equations

The purpose of this section is to obtain existence results for the system (1) in dimension 1, that is for the following equation:

$$y_t=\int_0^t \sigma(y_s)dx_s, \quad t\ge 0. \quad (21)$$

Here, recall that we assume $x\in\mathcal{C}^\gamma([0,T];\mathbb{R})$, with $\gamma\in(1/2,1)$. We now give a general condition on the coefficient σ in (21), which will prevail for the remainder of this section.

Hypothesis 3.1. We suppose that $\sigma:\mathbb{R}\to\mathbb{R}_+$ satisfies Hypothesis 2.1, and moreover

(i) σ is a continuous function such that $\sigma(\xi)>0$ for $\xi\neq 0$.
(ii) $1/\sigma$ is integrable on compact neighborhoods of zero.

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Remark 3.2. Notice that a basic example of a function σ satisfying Hypothesis 3.1 is any power coefficient of the form $\sigma(\xi) = C|\xi|^\kappa$, where $\kappa < 1$ is such that $\gamma < \frac{1}{1+\kappa}$. Another example is given by $\sigma(\xi) = |\xi|^\kappa + \sin(|\xi|^\kappa)$.

With Hypothesis 3.1 in mind, we shall solve Eq. (21) thanks to an approximation procedure. We first state the following lemma, whose elementary proof is left to the reader.

Lemma 3.3. *Let σ be a function satisfying Hypothesis 3.1 and $n \in \mathbb{N}$. We consider the sequence $\{\tilde{\xi}_n, n \geq 1\}$, where $\tilde{\xi}_1 \in [0,1]$ is the first time such that $\sigma(\tilde{\xi}_1) = \max_{0\leq\xi\leq 1}\sigma(\xi)$ and $\tilde{\xi}_{n+1}$ is the first time such that $\sigma(\tilde{\xi}_{n+1}) = \max_{0\leq\xi\leq\frac{\tilde{\xi}_n}{2}}\sigma(\xi)$. Let us also define the following function on $\mathbb{R}$:*

$$\sigma_n(\xi) = \begin{cases} \sigma(\xi), & |\xi| > \tilde{\xi}_n, \\ \sigma(\tilde{\xi}_n), & |\xi| \leq \tilde{\xi}_n. \end{cases}$$

Then σ_n satisfies (8), with $\mathcal{N}_{\kappa,\sigma_n} \leq \mathcal{N}_{\kappa,\sigma}$, where $\mathcal{N}_{\kappa,\sigma}$ is given in (9).

We shall construct a solution to Eq. (21) by means of a Lamperti type transformation for σ, which has been used, among another applications, to study the existence of a unique solution for ordinary differential equations (see, for instance, the proof of Theorem 5.1 in Hartman [5]). This transform is classically defined in the following way.

Lemma 3.4. *Let σ be a function fulfilling Hypothesis 3.1 and σ_n be defined as in Lemma 3.3. For those two functions and $\xi \in \mathbb{R}$, we set*

$$\phi(\xi) = \int_0^\xi \frac{ds}{\sigma(s)} \quad \text{and} \quad \phi_n(\xi) = \int_0^\xi \frac{ds}{\sigma_n(s)}. \tag{22}$$

Then ϕ and ϕ_n are both invertible and, for any $\xi \in \mathbb{R}$, we have $|\phi^{-1}(\xi)| \leq |\phi_n^{-1}(\xi)|$, where ϕ^{-1}, ϕ_n^{-1} stand for the respective inverse of ϕ and ϕ_n.

Proof. The result is an immediate consequence of the inequalities $\phi_n \leq \phi$ on $\mathbb{R}_+$ and $\phi \leq \phi_n$ on $\mathbb{R}_-$, which follow from our definition (22). □

The next result states the uniform (in n) Lipschitz regularity of ϕ_n^{-1}.

Lemma 3.5. *Let $M > 0$. Then, there is a constant $c_M > 0$ such that*

$$|\phi_n^{-1}(\xi_1) - \phi_n^{-1}(\xi_2)| \leq c_M|\xi_1 - \xi_2|,$$

for all ξ_1 and ξ_2 such that $|\xi_1|, |\xi_2| \leq M$ and for all $n \in \mathbb{N}$.

Proof. Suppose $|\xi| \leq M$. By (22) and Lemma 3.4, we get

$$\left|\frac{d\phi_n^{-1}(\xi)}{d\xi}\right| = \left|\sigma_n(\phi_n^{-1}(\xi))\right| \leq \mathcal{N}_{\kappa,\sigma}\left|\phi_n^{-1}(\xi)\right| \leq \mathcal{N}_{\kappa,\sigma}\left|\phi_n^{-1}(M)\right|.$$

In addition, observe that $\lim_{n\to\infty}\phi_n^{-1}(M) = \phi^{-1}(M)$, which means in particular that the sequence $\{\phi_n^{-1}(M), n \geq 1\}$ is bounded. Thus a direct application of the mean value theorem finishes the proof. □

We now proceed to the approximation of Eq. (21).

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Proposition 3.6. *Suppose that $\gamma \in (1/2, 1)$, $x \in \mathcal{C}^{\gamma}([0,T])$ and $x_0 = 0$. Also suppose that Hypothesis 3.1 holds. Let $n \in \mathbb{N}$ and $y_t^n = \phi_n^{-1}(x_t)$. Then y^n solves the following equation*

$$y_t^n = \int_0^t \sigma_n(y_s^n)\, dx_s, \quad \text{for all } t \geq 0,$$

where the integral with respect to x is understood in Young's sense.

Proof. We first observe that the function ϕ_n^{-1} is locally Lipschitz due to Lemma 3.5. The function σ_n is also locally Lipschitz according to Lemma 2.3. Therefore, $\sigma_n(\phi_n^{-1}(x_s))$ is locally γ-Hölder continuous. Thus, invoking the usual change of variable in Young's integral (see e.g. [16, Theorem 4.3.1]) and recalling that $\gamma > 1/2$, we obtain

$$y_t^n = \int_0^t \sigma_n(\phi_n^{-1}(x_s))dx_s = \int_0^t \sigma_n(y_s^n)dx_s, \quad t \geq 0,$$

and the proof is complete. □

We now turn to the main result of this section which states the convergence of y^n to a solution to Eq. (21). We recall that $\gamma > 1/2$ again.

Theorem 3.7. *Assume that σ satisfies Hypothesis 3.1. Consider η such that $\frac{1-\gamma(1+\kappa)}{\gamma} < \eta < 1-\kappa$. Let ϕ be the function given by (22), and suppose that $x \in \mathcal{C}^{\gamma}([0,T])$ is such that $|\phi^{-1}(x)|^{-\eta} \in L^1([0,T])$ and $x_0 = 0$. Then the function $y = \phi^{-1}(x)$ is a solution of the equation*

$$y_t = \int_0^t \sigma(y_s)dx_s, \quad t \geq 0,$$

where the integral $\int_0^t \sigma(y_s)dx_s$ is understood as in Proposition 2.4.

Remark 3.8. Note that y is a non-trivial solution (i.e., it is not identically zero) and that $z \equiv 0$ is also a solution of Eq. (21). So, in general, this equation may have several solutions.

Proof of Theorem 3.7. Let y^n be as in Proposition 3.6. For each $\xi \in \mathbb{R}$ we have $\phi_n^{-1}(\xi) \to \phi^{-1}(\xi)$ as n tends to infinity. Hence, y^n converges point-wise to y as n tends to infinity. Therefore, thanks to Proposition 3.6, we are reduced to show that for all $t \geq 0$

$$I(t) := \lim_{n\to\infty} \int_0^t \left[\sigma_n(y_s^n) - \sigma(y_s)\right] dx_s = 0.$$

Otherwise stated, according to Proposition 2.4, we have to check that, for $t \geq 0$,

$$\lim_{n\to\infty} \int_0^t \left[D_{0+}^{\alpha}\left(\sigma(y) - \sigma_n(y^n)\right)\right]_s D_{t-}^{1-\alpha} x_s^{t-}\, ds = 0, \tag{23}$$

where α is such that $1-\gamma < \alpha < \gamma(\kappa+\eta)$. In order to prove relation (23), we first invoke definition (6) and relation (12). For $s \in [0,T]$, this gives

$$\left|\left[D_{0+}^{\alpha}\left(\sigma(y) - \sigma_n(y^n)\right)\right]_s D_{t-}^{1-\alpha} x_s^{t-}\right| \lesssim \|x\|_{\gamma} \left(I_{1,n}(s) + \int_0^s I_{2,n}(s,r)dr\right), \tag{24}$$

where

$$I_{1,n}(s) = \frac{\left|\sigma(\phi^{-1}(x_s)) - \sigma_n(\phi_n^{-1}(x_s))\right|}{s^{\alpha}}$$

and

$$I_{2,n}(s,r) = \frac{\left|\sigma(\phi^{-1}(x_s)) - \sigma_n(\phi_n^{-1}(x_s)) - \left(\sigma(\phi^{-1}(x_r)) - \sigma_n(\phi_n^{-1}(x_r))\right)\right|}{(s-r)^{1+\alpha}}.$$

Going back to our aim (23), we are reduced to prove that

$$\lim_{n\to\infty} \int_0^t I_{1,n}(s)\, ds = 0, \quad \text{and} \quad \lim_{n\to\infty} \int_0^t \int_0^s I_{2,n}(s,r)\, dr ds = 0. \tag{25}$$

Moreover, thanks to the very definition of σ_n, we have that for all $0 \le r < s \le t$, $I_{1,n}(s) \to 0$ and $I_{2,n}(s,r) \to 0$, as $n \to \infty$. Our claim (25) is thus ensured if we can bound $I_{1,n}(s)$ and $I_{2,n}(s,r)$ properly.

Let us start with a bound on the term $I_{1,n}(s)$. As in the proof of Lemma 3.5 we can show that $I_{1,n}(s)$ is bounded by a constant times $s^{-\alpha}$ for all $n \in \mathbb{N}$. This is enough to apply the dominated convergence theorem.

In order to bound the term $I_{2,n}$, we apply Lemmas 2.3, 3.4 and 3.5, and the fact that σ_n satisfies (8) with $\mathcal{N}_{\kappa,\sigma_n} \le \mathcal{N}_{\kappa,\sigma}$ (see Lemma 3.3) to establish

$$\begin{aligned} I_{2,n}(s,r) &\le (s-r)^{-\alpha-1}\left(|\sigma(\phi^{-1}(x_s)) - \sigma(\phi^{-1}(x_r))| + |\sigma_n(\phi_n^{-1}(x_s)) - \sigma_n(\phi_n^{-1}(x_r))|\right) \\ &\lesssim (s-r)^{\gamma(\kappa+\eta)-\alpha-1}\Big(|\phi^{-1}(x_s)|^{-\eta} \\ &\qquad + |\phi^{-1}(x_r)|^{-\eta} + |\phi_n^{-1}(x_s)|^{-\eta} + |\phi_n^{-1}(x_r)|^{-\eta}\Big) \\ &\le (s-r)^{\gamma(\kappa+\eta)-\alpha-1}\left(|\phi^{-1}(x_s)|^{-\eta} + |\phi^{-1}(x_r)|^{-\eta}\right). \end{aligned}$$

We can thus conclude by the dominated convergence theorem, thanks to the fact that $\gamma(\kappa+\eta) - \alpha > 0$. We get the second claim in (25), which completes the proof of our theorem. □

Remark 3.9. A small variant of our calculations also allows to construct a solution to the initial value problem

$$y_t = a + \int_0^t \sigma(y_s) dx_s, \quad t \ge 0, \tag{26}$$

for a general $a \in \mathbb{R}$. Indeed, along the same lines as for Theorem 3.7, one can prove that $y_t = \phi^{-1}(x_t + \phi(a))$ is a solution of (26) if $|\phi^{-1}(x_t + \phi(a))|^{-\eta} \in L^1([0,T])$.

4. Multidimensional differential equations

We now turn to the multidimensional setting of Eq. (1). As mentioned in the introduction, our considerations will rely on regularity gain estimates for the solution when it approaches 0, similarly to [10,11]. Before we deal with these regularity estimates, we will first introduce some new notation.

4.1. Setting

In the remainder of the article, we assume that $x \in \mathcal{C}^\gamma([0,T];\mathbb{R}^d)$ and that each component σ^j, $j = 1,\ldots,d$ in the coefficients of Eq. (1), satisfies Hypothesis 2.1. As in the previous section, we need an additional hypothesis that says that σ^j behaves as a power function.

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Hypothesis 4.1. We suppose that for each $j = 1, \ldots, d, \sigma^j : \mathbb{R}^m \to \mathbb{R}^m$ satisfies Hypothesis 2.1 (with the same κ as in the previous sections), and moreover:

(i) For any $\xi \in \mathbb{R}^m$ we have $|\sigma^j(\xi)| \gtrsim |\xi|^\kappa$.
(ii) σ^j is differentiable with $\nabla\sigma^j$ locally Hölder continuous of order larger than $\frac{1}{\gamma} - 1$ in the set $\{|\xi| \neq 0\}$.

Fix $a \in \mathbb{R}^m$, $a \neq 0$, and we consider equation

$$y_t = a + \sum_{j=1}^{d} \int_0^t \sigma^j(y_u)\, dx_u^j, \quad t \in [0, T]. \tag{27}$$

Using an approximation of σ^j similar to Lemma 3.3 and applying known results on existence and uniqueness of solutions to equations driven by Hölder continuous functions (see e.g. [4]), it is easy to show the following result.

Proposition 4.2. *Suppose that Hypothesis 4.1(ii) holds, and let T be a given strictly positive time horizon. Then, there exist a continuous function y defined on $[0, T]$ and an instant $\tau \leq T$, such that one of the following two possibilities holds:*

(A) *$\tau = T$, y is nonzero on $[0, T]$, $y \in \mathcal{C}^\gamma([0, T]; \mathbb{R}^m)$ and y solves Eq. (27) on $[0, T]$, where the integrals $\int \sigma^j(y_u)\, dx_u^j$ are understood in the usual Young sense.*
(B) *We have $\tau < T$. Then for any $t < \tau$, the path y sits in $\mathcal{C}^\gamma([0, t]; \mathbb{R}^m)$ and y solves Eq. (27) on $[0, t]$. Furthermore, $y_s \neq 0$ on $[0, \tau)$, $\lim_{t\to\tau} y_t = 0$ and $y_t = 0$ on the interval $[\tau, T]$.*

Notice that our option (A) above leads to classical solutions of Eq. (27). In the rest of this section, we will assume (B), that is the function y given by Proposition 4.2 vanishes in the interval $[\tau, T]$. We remark that the integral in case (B) is not the one defined in Proposition 2.4, which requires suitable integrability conditions on σ^j. Our aim is thus to prove the following two facts:

- The path y is globally γ-Hölder continuous on $[0, T]$.
- The integrals $\int \sigma^j(y_u)\, dx_u^j$ can be understood as limits of Riemann sums, and y solves Eq. (27) on $[0, T]$.

Observe that in order to achieve this aim, we will need some additional hypotheses on x. We shall also assume $\gamma + \kappa > 1$, which is a natural condition in our context (as explained in the introduction).

Remark 4.3. As mentioned in the introduction, we implement here the regularity gain strategy inspired by the Brownian SPDE case (cf [10,11]). An outline of this strategy is the following:

(i) Our basic regularity gain result is Proposition 4.8. It states that if a solution y satisfies $|y_u| \leq 2^{-k}$ for u lying in an interval I, then we also have $|y_t - y_s|$ of order $2^{-\kappa k}|t - s|^\gamma$ for $s, t \in I$.
(ii) Proposition 4.8 enables to get a lower bound on the amount of time that y spends in intervals of the form $[a2^{-k}, b2^{-k}]$. We can get a matching upper bound by adding a roughness assumption on x. This roughness assumption amounts to assert that the main contributions in the increments of a solution y are always of the form $y_t - y_s \approx \sigma(y_s)[x_t - x_s]$. Our considerations in this direction are summarized in Section 4.3.

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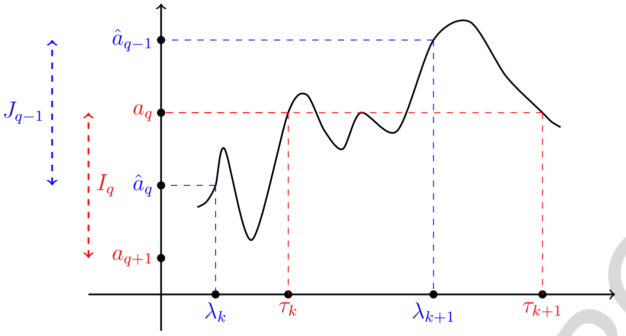

Fig. 1. An example of path with stopping times.

(iii) With the sharp bound of step (ii) in hand, a careful analysis of the increments of y enables to obtain the desired global bound in γ-Hölder spaces. This is the contents of Proposition 4.13.

Remark 4.4. As the reader might see in the sequel, the amount of efforts devoted to prove that y is globally γ-Hölder continuous is arguably very large. However, the stability of $\mathcal{C}^\gamma$ by the map $x \mapsto y$ is of fundamental importance in the analysis of differential systems like (27). In addition, we believe that some of the techniques developed here might also be useful to analyze rough PDEs with power type coefficients.

Now we split the interval $[0, \tau)$ as follows. We first define $a_q = 2^{-q}$ and we introduce a decomposition of the space $\mathbb{R}_+$, which is the state space for $|y|$, into the following sets:

$$I_{-1} = [1, \infty), \quad \text{and} \quad I_q = [a_{q+1}, a_q), \quad q \geq 0.$$

We also need to define the intervals:

$$J_{-1} = [3/4, \infty), \quad \text{and} \quad J_q = \left[\frac{a_{q+2} + a_{q+1}}{2}, \frac{a_{q+1} + a_q}{2}\right) =: \left[\hat{a}_{q+1}, \hat{a}_q\right), \quad q \geq 0.$$

Notice that $\hat{a}_q = \frac{3}{2^{q+2}}$. We now construct a partition of $[0, \tau)$ as follows. Assume that $|a| \in I_{q_0}$, and set $\lambda_0 = 0$ and

$$\tau_0 = \inf\left\{t \geq 0 : |y_t| \notin I_{q_0}\right\}.$$

In this case $y_{\tau_0} \in J_{\hat{q}_0}$ with $\hat{q}_0 \in \{q_0, q_0 - 1\}$. We then set:

$$\lambda_1 = \inf\left\{t \geq \tau_0 : |y_t| \notin J_{\hat{q}_0}\right\}.$$

In this way we recursively construct a sequence of stopping times $\lambda_0 < \tau_0 < \cdots < \lambda_k < \tau_k$ such that

$$|y_t| \in \left[\frac{b_1}{2^{q_k}}, \frac{b_2}{2^{q_k}}\right], \quad \text{for } t \in [\lambda_k, \tau_k] \cup [\tau_k, \lambda_{k+1}], \tag{28}$$

where $b_1 = \frac{3}{8}$, $b_2 = \frac{3}{4}$ and $q_{k+1} = q_k + \ell$, with $\ell \in \{-1, 0, 1\}$, assuming that $q_k \geq 1$. Notice that if $q_k = 0$ or $q_k = 1$, then the upper bound b_2 may be infinity. This construction is depicted in Fig. 1.

Finally, let us justify a simplification in notations which will prevail until the end of this Section.

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Remark 4.5. Notice that, owing to our Hypothesis 2.1, our problem relies heavily on radial variables in $\mathbb{R}^m$. Therefore, in order to alleviate vectorial notations, we will carry out the computations below for $m = d = 1$. This allows us in particular to drop the exponents j in our formulae. The reader will easily generalize our considerations to higher dimensions.

4.2. Regularity estimates

Let us start with a decomposition lemma for the solution to the regularized Eq. (27). We recall a convention which will prevail until the end of the paper: for a function f defined on $[0, T]$, we set $\delta f_{st} = f_t - f_s$.

Lemma 4.6. *Let $0 \le s < t < \tau$. For $l \ge 0$ we consider the dyadic partition Π_{st}^l of $[s, t]$ defined by $t_i^l = s + 2^{-l} i(t - s)$ for $l \ge 0$ and $i = 0, \ldots, 2^l$. Then one can write:*

$$\delta y_{st} = \sigma(y_s)\,\delta x_{st} + \sum_{l=1}^{\infty} K_{st}^l, \tag{29}$$

where

$$K_{st}^l = \sum_{i=0}^{2^l-1} [\delta\sigma(y)]_{t_{2i}^{l+1} t_{2i+1}^{l+1}}\ \delta x_{t_{2i+1}^{l+1} t_{2i+2}^{l+1}}.$$

Proof. Since $s, t \in [0, \tau)$, the integral $\int_s^t \sigma(y_u)\,dx_u$ is a usual Young integral, which is thus limit of Riemann sums along dyadic partitions. Let us write J_{st}^l for those Riemann sums, and notice that

$$J_{st}^l = \sum_{i=0}^{2^l-1} \sigma\left(y_{t_i^l}\right) \delta x_{t_i^l t_{i+1}^l} \tag{30}$$

$$= \sum_{i=0}^{2^l-1} \sigma\left(y_{t_{2i}^{l+1}}\right) \left[\delta x_{t_{2i}^{l+1} t_{2i+1}^{l+1}} + \delta x_{t_{2i+1}^{l+1} t_{2i+2}^{l+1}}\right]. \tag{31}$$

Then, we know from usual Young integration that J_{st}^l converges, as $l \to \infty$, to $\int_s^t \sigma(y_u)\,dx_u$. Therefore, we can write

$$\int_s^t \sigma(y_u)\,dx_u = \sigma(y_s)\,\delta x_{st} + \sum_{l=0}^{\infty} \left(J_{st}^{l+1} - J_{st}^l\right).$$

Resorting to expression (30) for J_{st}^{l+1} and to expression (31) for J_{st}^l above, some elementary algebraic manipulations reveal that $J_{st}^{l+1} - J_{st}^l = K_{st}^l$, which ends the proof. □

Let us state an additional (harmless) hypothesis on our noise x, which will be crucial in order to get sharp regularity estimates.

Hypothesis 4.7. There exists $\varepsilon_1 > 0$ such that for $\gamma_1 = \gamma + \varepsilon_1$, we have $\|x\|_{\gamma_1} < \infty$ and $\gamma_1 + \gamma\kappa < 1$.

We are now ready to give the basis of the strategy alluded to above, based on a regularity gain when y is close to 0.

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Proposition 4.8. *Assume σ satisfies Hypothesis 4.1 and x is such that 4.7 is fulfilled. Then the following bounds hold true:*

(i) *There exist constants $c_{0,x}$ and $c_{1,x}$ such that for $s,t \in [\lambda_k, \lambda_{k+1})$ satisfying*

$$|t-s| \le c_{0,x}\, 2^{-\alpha q_k}, \quad \text{with } \alpha := \frac{1-\kappa}{\gamma}, \tag{32}$$

we have the following bound:

$$|\delta y_{st}| \le c_{1,x}\, 2^{-q_k \kappa} |t-s|^{\gamma}. \tag{33}$$

(ii) *With Hypothesis 4.7 in mind, we get a refined decomposition for δy_{st}. Namely, if s,t are two instants in $[\lambda_k, \lambda_{k+1})$ such that (32) holds true, we have the following relation for δy_{st}:*

$$\delta y_{st} = \sigma(y_s)\, \delta x_{st} + r_{st}, \quad \text{with } |r_{st}| \le c_{2,x}\, 2^{-\kappa_{\varepsilon_1} q_k} |t-s|^{\gamma}, \tag{34}$$

where we have set $\kappa_{\varepsilon_1} = \kappa + \varepsilon_1 \alpha$.

Proof. For $k \ge 1$ and $\nu > 0$ we set

$$\|y\|_{\gamma,k,\nu} = \sup\left\{ \frac{|\delta y_{uv}|}{|v-u|^{\gamma}} : u, v \in [\lambda_k, \lambda_{k+1}),\ |v-u| \le \frac{c_0}{2^{\nu}} \right\},$$

where the constants c_0 and ν will be tuned on later.

Step 1: *Proof of* (33). Pick $s,t \in [\lambda_k, \lambda_{k+1})$ such that $|s-t| \le c_0 2^{-\nu}$. Recall that we consider the dyadic partitions of $[s,t]$, with $t_i^l = s + 2^{-l} i(t-s)$ for $l \ge 1$ and $i = 0, \dots, 2^l$. Start from decomposition (29). Then, since both $|y_s|$ and $|y_t|$ lie into $[b_1 2^{-q_k}, b_2 2^{-q_k}]$ and σ verifies Hypothesis 2.1, we obviously have

$$|\sigma(y_s)\, \delta x_{st}| \le c_1 \|x\|_{\gamma} |t-s|^{\gamma} 2^{-q_k \kappa}, \tag{35}$$

where $c_1 = \mathcal{N}_{\kappa,\sigma} b_2^{\kappa}$.

In the remainder of this proof, we denote $t_{2i}^{l+1}, t_{2i+1}^{l+1}$ by t_{2i}, t_{2i+1}, respectively, to simplify the notation. We now bound the quantity $[\delta\sigma(y)]_{t_{2i}t_{2i+1}} \delta x_{t_{2i+1}t_{2i+2}}$ popping up in (29). Thanks to Lemma 2.3, for any $\eta \le 1-\kappa$ we have

$$\left|[\delta\sigma(y)]_{t_{2i}t_{2i+1}}\right| \le \mathcal{N}_{\kappa,\sigma} \left(|y_{t_{2i}}|^{-\eta} + |y_{t_{2i+1}}|^{-\eta} \right) \left|\delta y_{t_{2i}t_{2i+1}}\right|^{\kappa+\eta}.$$

Thus, since $y_{t_{2i}}, y_{t_{2i+1}} \in [b_1 2^{-q_k}, b_2 2^{-q_k}]$ we get

$$\left|[\delta\sigma(y)]_{t_{2i}t_{2i+1}}\right| \left|\delta x_{t_{2i+1}t_{2i+2}}\right| \le \mathcal{N}_{\kappa,\sigma} 2 b_1^{-\eta} \|x\|_{\gamma} \|y\|_{\gamma,k,\nu}^{\kappa+\eta} 2^{q_k \eta} \left|\frac{t-s}{2^l}\right|^{(1+\kappa+\eta)\gamma}. \tag{36}$$

We choose η above such that $\gamma(1+\kappa+\eta) = 2\gamma$. It is readily checked that such a η verifies

$$\eta = 1-\kappa.$$

Furthermore, with this value of η in hand, relation (36) becomes

$$\left|[\delta\sigma(y)]_{t_{2i}t_{2i+1}}\right| \left|\delta x_{t_{2i+1}t_{2i+2}}\right| \le \mathcal{N}_{\kappa,\sigma} 2 b_1^{\kappa-1} \|x\|_{\gamma} \|y\|_{\gamma,k,\nu} 2^{q_k(1-\kappa)} \left|\frac{t-s}{2^l}\right|^{2\gamma}. \tag{37}$$

Plugging this inequality into the terms K_{st}^l of (29) we end up with

$$\sum_{l=1}^{\infty} |K_{st}^l| \le c_{3,x} \|y\|_{\gamma,k,\nu} 2^{q_k(1-\kappa)} |t-s|^{2\gamma}, \tag{38}$$

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where we have set $c_{3,x} = \frac{\mathcal{N}_{\kappa,\sigma} 2 b_1^{\kappa-1}}{2^{2\gamma}-1} \|x\|_\gamma$. Reporting (35) and (38) into (29), this yields

$$|\delta y_{st}| \le c_1 \|x\|_\gamma |t-s|^\gamma 2^{-q_k \kappa} + A_{st}^2, \quad \text{with } A_{st}^2 = c_{3,x} \|y\|_{\gamma,k,\nu}\, 2^{q_k(1-\kappa)}\, |t-s|^{2\gamma}. \tag{39}$$

We should now bound the term A_{st}^2 as a γ-Hölder increment. Indeed, recalling that we assume $|t-s| \le c_0 2^{-\nu}$, we get

$$A_{st}^2 \le c_{3,x} c_0^\gamma\, 2^{q_k(1-\kappa)-\nu\gamma} \|y\|_{\gamma,k,\nu} |t-s|^\gamma. \tag{40}$$

We now choose c_0 and ν so that $c_{3,x} c_0^\gamma\, 2^{q_k(1-\kappa)-\nu\gamma} \le \frac{1}{2}$. It is readily checked that this is achieved for c_0 small enough and $\nu = \alpha q_k := \gamma^{-1}(1-\kappa) q_k$ given by (32). With those values of c_0 and ν in hand, relation (39) becomes

$$\|y\|_{\gamma,k,\nu} \le c_1 \|x\|_\gamma 2^{-q_k \kappa} + \frac{1}{2} \|y\|_{\gamma,k,\nu},$$

from which (33) is easily deduced, with $c_{1,x} = 2 c_1 \|x\|_\gamma$.

Step 2: *Proof of* (34). Go back to relation (37) and invoke Hypothesis 4.7 in order to get

$$\left|[\delta\sigma(y)]_{t_{2i} t_{2i+1}}\right| \left|\delta x_{t_{2i+1} t_{2i+2}}\right| \le \mathcal{N}_{\kappa,\sigma} 2 b_1^{\kappa-1} \|x\|_{\gamma_1} \|y\|_{\gamma,k,\nu}\, 2^{q_k(1-\kappa)} \left|\frac{t-s}{2^l}\right|^{2\gamma+\varepsilon_1}.$$

Moreover, according to (29), the term r_{st} in (34) is given by $\sum_{l=1}^\infty K_{st}^l$. Proceeding as for relations (38) and (39), we obtain that

$$|r_{st}| \le \sum_{l=1}^\infty \left|K_{st}^l\right| \le A_{st}^2 = \tilde{c}_{3,x} \|y\|_{\gamma,k,\nu}\, 2^{q_k(1-\kappa)}\, |t-s|^{2\gamma+\varepsilon_1}, \tag{41}$$

where $\tilde{c}_{3,x} = \frac{\mathcal{N}_{\kappa,\sigma} 2 b_1^{\kappa-1}}{2^{2\gamma+\varepsilon_1}-1} \|x\|_{\gamma_1}$.

We now plug the a priori bound (33) on $\|y\|_{\gamma,k,\nu}$ we have just obtained, and read the regularity of A^2 in γ-Hölder norm. Similarly to (40), we can recast (41) as:

$$A_{st}^2 \le \tilde{c}_{3,x} c_0^{\gamma+\varepsilon_1} 2^{q_k(1-\kappa)-\nu(\gamma+\varepsilon_1)}\, c_{1,x}\, 2^{-q_k \kappa} |t-s|^\gamma.$$

Let us recall that $\nu = \alpha q_k$. Therefore we obtain:

$$A_{st}^2 \le \tilde{c}_{3,x} c_0^{\gamma+\varepsilon_1} c_{1,x} 2^{-q_k(\kappa+\alpha\varepsilon_1)\gamma} |t-s|^\gamma.$$

Taking into account the fact that $\kappa_{\varepsilon_1} = \kappa + \alpha\varepsilon_1$, this finishes the proof of (34). □

In the sequel we shall need some regularity estimates for y on time scales slightly larger than $2^{-\alpha q_k}$ with $\alpha = \gamma^{-1}(1-\kappa)$. This is the contents of the following property.

Corollary 4.9. *Under the same hypotheses as in Proposition 4.8, consider $\varepsilon_2 > 0$ such that*

$$\varepsilon_2 < \min\left(\gamma^{-1}(1-\kappa), \kappa(1-\gamma)^{-1}, \frac{\kappa + \gamma^{-1}(1-\kappa)\varepsilon_1}{1+\varepsilon_1}\right).$$

Then there exists a constant $c_{4,x} = 2^{1-\gamma} c_{0,x}$ such that for $s,t \in [\lambda_k, \lambda_{k+1})$ satisfying $|t-s| \le c_{4,x} 2^{-(\alpha-\varepsilon_2) q_k}$ with $\alpha = \gamma^{-1}(1-\kappa)$ we have

$$|\delta y_{st}| \le c_{5,x} 2^{-q_k \kappa_{\varepsilon_2}^-} |t-s|^\gamma, \quad \textit{with } \kappa_{\varepsilon_2}^- = \kappa - (1-\gamma)\varepsilon_2. \tag{42}$$

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Moreover, under the same conditions on s, t, decomposition (34) still holds true, with

$$|r_{st}| \le c_{6,x} 2^{-q_k \kappa_{\varepsilon_1,\varepsilon_2}} |t-s|^{\gamma}, \quad \textit{where } \kappa_{\varepsilon_1,\varepsilon_2} = \kappa + \alpha\varepsilon_1 - \varepsilon_2 - \varepsilon_1\varepsilon_2. \tag{43}$$

Proof. We take up the notation introduced for the proof of Proposition 4.8, and we split again our computations in 2 steps.

Step 1: *Proof of* (42). Start from inequality (33), which is valid for $|t-s| \le c_{0,x} 2^{-\alpha q_k}$. Now let $m \in \mathbb{N}$ and consider $s, t \in [\lambda_k, \lambda_{k+1})$ such that $c_{0,x}(m-1)2^{-\alpha q_k} < |t-s| \le c_{0,x} m 2^{-\alpha q_k}$. We partition the interval $[s, t]$ by setting $t_j = s + c_{0,x} j 2^{-\alpha q_k}$ for $j = 0, \ldots, m-1$ and $t_m = t$. Then we simply write

$$|\delta y_{st}| \le \sum_{j=0}^{m-1} |\delta y_{t_j t_{j+1}}| \le c_{1,x} 2^{-q_k \kappa} \sum_{j=0}^{m-1} \left(t_{j+1} - t_j\right)^{\gamma} \le c_{1,x} 2^{-q_k \kappa} m^{1-\gamma} |t-s|^{\gamma},$$

where the last inequality stems from the fact that $t_{j+1} - t_j \le (t-s)/m$. Now the upper bound (42) is easily deduced by applying the above inequality to $m = [2^{\varepsilon_2 q_k}] + 1$.

Step 2: *Proof of* (43). Once (42) is proven, we go again through the estimation of K_{st}^{l}. Replacing $\|y\|_{\gamma,k,\nu}$ by $c_{5,x} 2^{-q_k \kappa_{\bar{\varepsilon}_2}}$ in (41), we end up with

$$|r_{st}| \le c_{6,x}\, 2^{-q_k \kappa_{\bar{\varepsilon}_2}} 2^{q_k(1-\kappa)} 2^{-q_k(\alpha-\varepsilon_2)(\gamma+\varepsilon_1)} |t-s|^{\gamma} = c_{6,x}\, 2^{-q_k \kappa_{\varepsilon_1,\varepsilon_2}} |t-s|^{\gamma},$$

which is our claim (43). □

4.3. *Estimates for stopping times*

Thanks to the regularity estimates of the previous section, we get a bound on the difference $\lambda_{k+1} - \lambda_k$ which roughly states that a solution to Eq. (27), cannot go too sharply to 0.

Proposition 4.10. *The sequence of stopping times $\{\lambda_k,\ k \ge 1\}$ defined by (28) satisfies*

$$\lambda_{k+1} - \lambda_k \ge c_{7,x}\, 2^{-\alpha q_k}, \tag{44}$$

where $\alpha = (1-\kappa)/\gamma$.

Proof. We shall prove that $\tau_k - \lambda_k$ satisfies a lower bound of the form

$$\tau_k - \lambda_k \ge c_{7,x}\, 2^{-\alpha q_k}. \tag{45}$$

Along the same lines we can prove a similar bound for $\lambda_{k+1} - \tau_k$, and this will prove our claim (44).

Inequality (45) is obtained in the following way. We observe that in order to get out of the interval $[\lambda_k, \tau_k)$, an increment of size $2^{-(q_k+1)}$ must occur. Indeed, at λ_k the solution is at the middle point of I_{q_k} and the length of this interval is of order 2^{-q_k}. However, relation (33) asserts that if $|\delta y_{st}| \ge 2^{-(q_k+1)}$ and $|t-s| \le c_{0,x} 2^{-\alpha q_k}$, then we must have

$$c_{1,x} \frac{|t-s|^{\gamma}}{2^{\kappa q_k}} \ge \frac{1}{2^{q_k+1}}, \tag{46}$$

which implies

$$|t-s| \ge \left(2c_{1,x}\right)^{-\frac{1}{\gamma}} 2^{-\frac{(1-\kappa)q_k}{\gamma}} = \left(2c_{1,x}\right)^{-\frac{1}{\gamma}} 2^{-\alpha q_k}.$$

This finishes our proof. □

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In order to sharpen Proposition 4.10, we introduce a roughness hypothesis on x, borrowed from [1]. As we shall see, this assumption is satisfied when x is a fractional Brownian motion.

Hypothesis 4.11. We assume that for $\hat{\varepsilon}$ arbitrarily small there exists a constant $c > 0$ such that for every s in $[0, T]$, every ϵ in $(0, T/2]$, and every u in $\mathbb{R}^d$ with $|u| = 1$, there exists t in $[0, T]$ such that $\epsilon/2 < |t - s| < \epsilon$ and

$$|\langle u, \delta x_{st}\rangle| > c\,\epsilon^{\gamma+\hat{\varepsilon}}.$$

The largest such constant c is called the modulus of $(\gamma+\hat{\varepsilon})$-Hölder roughness of x, and is denoted by $L_{\gamma,\hat{\varepsilon}}(x)$.

Under this hypothesis, we are also able to upper bound the difference $\lambda_{k+1} - \lambda_k$ in a useful way. To this aim, recall that option (B) in Proposition 4.2 is assumed below. It yields the relation $\lim_{k\to\infty} q_k = \infty$. Also remember that $\{\lambda_k,\ k \ge 1\}$ is given in (28), and that $\alpha = (1-\kappa)/\gamma$.

Proposition 4.12. *For all $\varepsilon_2 < \frac{\alpha\varepsilon_1}{1+\gamma+\varepsilon_1} \wedge \frac{\kappa}{1-\gamma}$ and q_k large enough, the sequence of stopping times $\{\lambda_k,\ k \ge 1\}$ satisfies*

$$\lambda_{k+1} - \lambda_k \le c_{x,\varepsilon_2} 2^{-q_k(\alpha-\varepsilon_2)}. \tag{47}$$

Furthermore, inequality (42) can be extended as follows: there exists a constant c_x such that for $s, t \in [\lambda_k, \lambda_{k+1})$ we have

$$|\delta y_{st}| \le c_x 2^{-\kappa_{\varepsilon_2}^{-} q_k} |t-s|^{\gamma}. \tag{48}$$

Proof. If (47) does not hold, this implies that there exists $\varepsilon_2 < \frac{\alpha\varepsilon_1}{1+\gamma+\varepsilon_1} \wedge \frac{\kappa}{1-\gamma}$ satisfying the condition of Corollary 4.9 so that for any constant C the inequality

$$\lambda_{k+1} - \lambda_k \ge C 2^{-q_k(\alpha-\varepsilon_2)} \tag{49}$$

holds for infinitely many values of k. This implies that

$$\lambda_{k+1} - \lambda_k \ge C\, 2^{-q_k(1-\kappa)/(\gamma+\hat{\varepsilon})}, \tag{50}$$

if we choose $\hat{\varepsilon}$ small enough so that $(1-\kappa)/(\gamma+\hat{\varepsilon}) \ge \alpha - \varepsilon_2$. We wish to exhibit a contradiction, namely that one can find $s, t \in [\lambda_k, \lambda_{k+1}]$ such that $|\delta y_{st}| > |J_{q_k}|$, where $|J_{q_k}|$ denotes the size of J_{q_k}.

In order to lower bound $|\delta y_{st}|$, let us first invoke Hypothesis 4.11. Since our computations are performed in the one-dimensional case for notational sake, we can in fact recast Hypothesis 4.11 as follows. Choose

$$\varepsilon := \frac{c_1\, 2^{-\frac{q_k(1-\kappa)}{\gamma+\hat{\varepsilon}}}}{\left[L_{\gamma,\hat{\varepsilon}}(x)\right]^{\frac{1}{\gamma+\hat{\varepsilon}}}} \le C\, 2^{-\frac{q_k(1-\kappa)}{\gamma+\hat{\varepsilon}}},$$

which can be achieved by taking the constant C large enough, for a given constant c_1. Then there exist $s, t \in [\lambda_k, \lambda_{k+1}]$ satisfying

$$\frac{\varepsilon}{2} \le |t-s| \le \varepsilon, \quad \text{and} \quad |\delta x_{st}| \ge c_1^{\gamma+\hat{\varepsilon}}\, 2^{-q_k(1-\kappa)}. \tag{51}$$

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Notice that c_1 can be made arbitrarily large, by playing with k and $\hat{\varepsilon}$. In addition, we can use the fact that $|\sigma(y_s)| \geq c2^{-q_k\kappa}$ whenever $s \in [\lambda_k, \lambda_{k+1}]$. Indeed, this follows from Hypothesis 4.1 and the fact that $y_s \geq b_1 2^{-q_k} \geq 2^{-q_k-2}$. This entails, for s, t as in (51)

$$|\sigma(y_s)\delta x_{st}| \geq c c_1^{\gamma+\hat{\varepsilon}} 2^{-q_k}.$$

If (49) holds true, we can now choose c_1 so that $c c_1^{\gamma+\hat{\varepsilon}} \geq 6$. This yields

$$|\sigma(y_s)\delta x_{st}| \geq 6 \cdot 2^{-q_k} = 2|J_{q_k}|.$$

In particular the size of this increment is larger than twice the size of J_{q_k}.

We now assume again that we have chosen $\hat{\varepsilon}$ small enough so that $(1-\kappa)/(\gamma+\hat{\varepsilon}) \geq \alpha - \varepsilon_2$. Then the upper bound on $|t-s|$ in (51) also implies $|t-s| \leq c_{8,x} 2^{-q_k(\alpha-\varepsilon_2)}$. For the two instants s, t exhibited in relation (51), we resort to decomposition (29) together with the bound (43). This yields

$$|\delta y_{st}| \gtrsim A^1_{st} - A^2_{st}, \quad \text{with } A^1_{st} = 6 \cdot 2^{-q_k}, \quad A^2_{st} \leq c_{6,x} 2^{-q_k\kappa_{\varepsilon_1,\varepsilon_2}} |t-s|^{\gamma} \leq c_{9,x} 2^{-q_k\mu_{\varepsilon_2}},$$

where we recall that $\kappa_{\varepsilon_1,\varepsilon_2} = \kappa + \alpha\varepsilon_1 - \varepsilon_2 - \varepsilon_1\varepsilon_2$ and where we obtain

$$\mu_{\varepsilon_2} = \kappa_{\varepsilon_1,\varepsilon_2} + (\alpha - \varepsilon_2)\gamma = 1 + \alpha\varepsilon_1 - (1+\gamma+\varepsilon_1)\varepsilon_2.$$

Our aim is now to prove that A^2_{st} can be made negligible with respect to 2^{-q_k} when q_k is large enough. This is achieved whenever $\mu_{\varepsilon_2} > 1$, and this condition can be met by picking ε_1 large enough and ε_2 small enough. Summarizing our considerations, we have thus shown that A^1_{st} is larger than twice $|J_{q_k}| = 3 \cdot 2^{-q_k}$ and that A^2_{st} is negligible with respect to A^1_{st} as q_k gets large. This proves our claim (47). □

4.4. Hölder continuity

We shall use the following notation, valid for $\lambda \in (0,1)$, a time horizon $t \in [0,T]$ and a function from $[0,t]$ to $\mathbb{R}^m$:

$$\|f\|_{\lambda,t} := \sup_{0 \leq s < u \leq t} \frac{|\delta f_{st}|}{|u-s|^{\lambda}}, \quad \text{where } \delta f_{st} = f_t - f_s. \tag{52}$$

Then, we have the following result, which is our first main objective in this section.

Proposition 4.13. *Suppose that σ satisfies Hypothesis 4.1 and that our noise x satisfies Hypotheses 4.7 and 4.11. We also assume that $\gamma + \kappa > 1$. Then, the function y given in Proposition 4.2 belongs to $\mathcal{C}^{\gamma}([0,T];\mathbb{R}^m)$.*

Proof. Remember that we are assuming that y satisfies condition (B) in Proposition 4.2. Consider first $s = \lambda_k$ and $t = \lambda_l$ with $k < l$. We start by decomposing the increments $|\delta y_{st}|$ as follows

$$|\delta y_{st}| \leq \sum_{j=k}^{l-1} \left|\delta y_{\lambda_j\lambda_{j+1}}\right|.$$

Then owing to Proposition 4.12 we have $\lambda_{k+1} - \lambda_k \leq c_{x,\varepsilon_2} 2^{-q_k(\alpha-\varepsilon_2)}$ for k large enough. We can thus apply Corollary 4.9, which yields

$$|\delta y_{st}| \leq \sum_{j=k}^{l-1} \left|\delta y_{\lambda_j\lambda_{j+1}}\right| \leq c_{5,x} \sum_{j=k}^{l-1} 2^{-q_j\kappa^-_{\varepsilon_2}} |\lambda_{j+1} - \lambda_j|^{\gamma}. \tag{53}$$

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Furthermore, inequality (44) entails:

$$2^{-\frac{q_j(1-\kappa)}{\gamma}} \lesssim c_{7,x}^{-1}\left(\lambda_{j+1}-\lambda_j\right) \quad\Longrightarrow\quad 2^{-q_j\kappa_{\varepsilon_2}^{-}} \le (c_{7,x})^{-\frac{\gamma\kappa_{\varepsilon_2}^{-}}{1-\kappa}}\left(\lambda_{j+1}-\lambda_j\right)^{\frac{\gamma\kappa_{\varepsilon_2}^{-}}{1-\kappa}}.$$

Plugging this information into (53) and setting $c_{10,x} = c_{5,x}(c_{7,x})^{-\frac{\gamma\kappa_{\varepsilon_2}^{-}}{1-\kappa}}$, we end up with:

$$|\delta y_{st}| \le c_{10,x}\sum_{j=k}^{l-1}|\lambda_{j+1}-\lambda_j|^{\mu_{\varepsilon_2}}, \quad \text{with } \mu_{\varepsilon_2} = \gamma\left(1+\frac{\kappa_{\varepsilon_2}^{-}}{1-\kappa}\right).$$

We now wish the exponent μ_{ε_2} to be of the form $\mu_{\varepsilon_2} = 1+\varepsilon_3$ with $\varepsilon_3 > 0$. Since $\kappa_{\varepsilon_2}^{-}$ is arbitrarily close to κ, it is readily checked that this can be achieved as long as $\gamma+\kappa > 1$. Recalling that $s = \lambda_k$ and $t = \lambda_l$, one can thus recast the previous inequality as

$$|\delta y_{st}| \le c_{10,x}\sum_{j=k}^{l-1}\left|\lambda_{j+1}-\lambda_j\right|^{1+\varepsilon_3} \le c_{10,x}|\lambda_l-\lambda_k|^{1+\varepsilon_3} \le c_{10,x}\,\tau^{1+\varepsilon_3-\gamma}|t-s|^{\gamma},$$

which is consistent with our claim.

The general case $s < \lambda_k \le \lambda_l < t$ is treated by decomposing δy_{st} as

$$\delta y_{st} = \delta y_{s\lambda_k} + \delta y_{\lambda_k\lambda_l} + \delta y_{\lambda_l t}.$$

Then resort to (48) in order to bound $\delta y_{s\lambda_k}$ and $\delta y_{\lambda_l t}$. $\square$

The next proposition says that if (B) holds, the function y can be obtained as the limit of a suitable sequence of Riemann sums.

Proposition 4.14. *Let y be the function given in Proposition 4.2. For all $0 \le s < t \le T$, let Π_{st} be the set of partitions of $[s,t]$, denoted generically by $\pi = \{s = t_0 < \cdots < t_m = t\}$. For $\varepsilon > 0$ arbitrarily small, define*

$$\Pi_{st}^{\varepsilon} = \left\{\pi \in \Pi_{st};\ \text{there exists } j^* \text{ such that } t_{j^*} < \tau \le t_{j^*+1} \text{ and } \eta \le |\tau - t_{j^*}| \le 2\eta\right\},$$

where $\eta = c_x\varepsilon^{1/\gamma}$ for a strictly positive constant c_x and τ is introduced in Proposition 4.2. Then under the conditions of Proposition 4.13, one can find $\pi \in \Pi_{st}^{\varepsilon}$ such that:

$$\left|\int_s^t \sigma(y_u)\,dx_u - \sum_{t_j\in\pi}\sigma(y_{t_j})\,\delta x_{t_j t_{j+1}}\right| \le \varepsilon. \tag{54}$$

Proof. Consider a partition π lying in Π_{st}^{ε}, and set $S_\pi = \sum_{t_i\in\pi}\sigma(y_{t_i})\,\delta x_{t_i t_{i+1}}$. Since $y_u = 0$ for $u \ge \tau$, it is worth noting that we also have

$$S_\pi = S_{\pi^*} + \sigma(y_{t_{j^*}})\,\delta x_{t_{j^*}t_{j^*+1}}, \quad \text{where } S_{\pi^*} \equiv \sum_{j<j^*}\sigma(y_{t_j})\,\delta x_{t_j t_{j+1}}.$$

Then we can write

$$|\delta y_{st} - S_\pi| \le \left|\delta y_{st_{j^*}} - S_{\pi^*}\right| + \left|\delta y_{t_{j^*}\tau}\right| + \left|\sigma(y_{t_{j^*}})\,\delta x_{t_{j^*}t_{j^*+1}}\right| := I_1 + I_2 + I_3.$$

We now bound separately the 3 terms on the right hand side above. For the term I_2 we have

$$I_2 \le \|y\|_\gamma|\tau - t_{j^*}|^{\gamma} \le c_x(2\eta)^{\gamma}.$$

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We can obviously choose a constant c_x such that if $\eta = c_x \varepsilon^{1/\gamma}$, then $I_2 \le \frac{\varepsilon}{3}$. Thanks to the same kind of elementary considerations, we can also make the term I_3 smaller than $\frac{\varepsilon}{3}$. In order to bound I_1, we invoke the fact that $|\tau - t_{j^*}| \ge \eta$ and we set

$$Q_\eta = \inf\{|y_s| : s < \tau - \eta\}.$$

Observe that $Q_\eta > 0$. In addition, by Hypothesis 4.1(ii), σ is differentiable and locally Hölder continuous of order $\frac{1}{\gamma} - 1$ on $[Q_\eta, \infty)$. By usual convergence of Riemann sums for Young integrals, we thus have

$$\lim_{\pi \in \Pi_{st_{j^*}}, |\pi| \to 0} I_1 = \lim_{\pi \in \Pi_{st_{j^*}}, |\pi| \to 0} |\delta y_{st} - S_\pi| = 0.$$

Therefore we can choose $|\pi|$ so that $I_1 \le \frac{\varepsilon}{3}$. Putting together our upper bounds on I_1, I_2 and I_3, the proof of (54) is now finished. □

Finally we can summarize the considerations of this section into the following theorem.

Theorem 4.15. *Consider Eq. (27), and let T be a given strictly positive time horizon. We suppose that Hypothesis 4.1 holds for the coefficient σ, and that Hypotheses 4.7 and 4.11 are satisfied for our noise x. Then, there exist a continuous function y defined on $[0, T]$ and an instant $\tau \le T$, such that one of the following two possibilities holds:*

(A) *$\tau = T$, y is nonzero on $[0, T]$, $y \in \mathcal{C}^\gamma([0, T]; \mathbb{R}^m)$ and y solves Eq. (27) on $[0, T]$, where the integrals $\int \sigma^j(y_u)\, dx_u^j$ are understood in the usual Young sense.*

(B) *We have $\tau < T$. Then for any $t < \tau$, the path y sits in $\mathcal{C}^\gamma([0, T]; \mathbb{R}^m)$ and y solves Eq. (27) on $[0, T]$, where the integrals $\int \sigma^j(y_u)\, dx_u^j$ are understood as in Proposition 4.14. Furthermore, $y_s \ne 0$ on $[0, \tau)$, $\lim_{t \to \tau} y_t = 0$ and $y_t = 0$ on the interval $[\tau, T]$.*

5. Application to fractional Brownian motion

Let $B^H = \{B_t^H,\ t \in [0, T]\}$ be a standard d-dimensional fractional Brownian motion with the Hurst parameter $H \in (\frac{1}{2}, 1)$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$, that is, the components of B^H are independent centered Gaussian processes with covariance

$$\mathbf{E}(B_t^{H,i} B_s^{H,i}) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t - s|^{2H}\right),$$

for any $s, t \in [0, T]$. It is clear that $\mathbf{E}|B_t^H - B_s^H|^2 = d|t - s|^{2H}$, and, as a consequence, the trajectories of B^H are γ-Hölder continuous for any $\gamma < H$. Consider the m-dimensional stochastic differential equation

$$X_t = x_0 + \sum_{j=1}^d \int_0^t \sigma^j(X_s) dB_s^{H,j}, \quad 0 \le t \le T, \tag{55}$$

where $x_0 \in \mathbb{R}^m$. If σ is Hölder continuous of order $\kappa > \frac{1}{H} - 1$, then, there exists a solution X which has Hölder continuous trajectories of order γ, for any $\gamma < H$. This was proved by Lyons in [8] using the Young's integral and p-variation estimates. An extension of this result where there is a measurable drift with linear growth was given by Duncan and Nualart in [2]. Under this weak assumption of σ we cannot expect the uniqueness of a solution, which requires σ to be differentiable with partial derivatives Hölder continuous of order larger than $\frac{1}{H} - 1$ (see [8,12]).

The results proved in the previous sections allow us to construct examples of existence of solutions for Eq. (55), when σ is Hölder continuous of order κ and $\kappa \le \frac{1}{H} - 1$.

Example 5.1. Suppose that $m = d = 1$, $x_0 = 0$ and $\sigma(\xi) = C|\xi|^\kappa$, with $\kappa \le \frac{1}{H} - 1$. Then, the process

$$X_t = \phi^{-1}(B_t^H),$$

where $\phi(\xi) = \int_0^\xi \frac{dx}{\sigma(x)}$ satisfies Eq. (55), where the integral is a path-wise integral defined in Proposition 2.4. Indeed, it suffices to show that the assumptions of Theorem 3.7 hold. Taking into account that ϕ^{-1} satisfies

$$\mathrm{sgn}(\phi^{-1}(\xi))|(\phi^{-1}(\xi))|^{1-\kappa} = C(1-\kappa)\xi,$$

for any $\xi \in \mathbb{R}$, we get $|(\phi^{-1}(\xi))| = [C(1-\kappa)]^{\frac{1}{1-\kappa}} |\xi|^{\frac{1}{1-\kappa}}$. Therefore, for any $\eta < 1 - \kappa$, Q4

$$\mathbf{E} \int_0^T |\phi^{-1}(B_s^H)|^{-\eta} ds = [C(1-\kappa)]^{-\frac{\eta}{1-\kappa}} \mathbf{E} \int_0^T |B_s^H|^{-\frac{\eta}{1-\kappa}} ds < \infty.$$

This implies $\int_0^T |\phi^{-1}(B_s^H)|^{-\eta} ds < \infty$ almost surely, and we can apply Theorem 3.7.

Example 5.2. Consider Eq. (55) in the multidimensional case, with $x_0 \neq 0$. Suppose that each component σ^j satisfies Hypothesis 4.1 with $\kappa \le \frac{1}{H} - 1$ and observe that B^H satisfies Hypotheses 4.7 and 4.11. Then, we can apply Propositions 4.2 and 4.13, and conclude that there exist a stochastic process X such that, if Q5

$$\tau = \inf\{t > 0 : \; X_t = 0\} \wedge T,$$

then,

$$X_t = \left(x_0 + \sum_{j=1}^d \int_0^t \sigma^j(X_s)\, dB_s^{H,j} \right) \mathbf{1}_{[0,\tau)}(t),$$

where for $t < \tau$, the stochastic integral is understood as a path-wise Young integral. Moreover, the process X satisfies $X \in \mathcal{C}^\gamma([0,T]; \mathbb{R}^m)$ for any $\gamma < H$. Moreover, Proposition 4.2 implies that $X_t \equiv 0$, for $t \ge \tau$.

Uncited references

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[3] and [7].

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