Accepted Manuscript

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PII: S0304-4149(17)30086-8
DOI: http://dx.doi.org/10.1016/j.spa.2017.03.013
Reference: SPA 3103

To appear in: *Stochastic Processes and their Applications*

Received date: 23 February 2016
Revised date: 9 December 2016
Accepted date: 14 March 2017

Please cite this article as: S. Lou, C. Ouyang, Local times of stochastic differential equations driven by fractional Brownian motions, *Stochastic Processes and their Applications* (2017), http://dx.doi.org/10.1016/j.spa.2017.03.013

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Local times of stochastic differential equations driven by fractional Brownian motions

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December 9, 2016

Abstract

In this paper, we study the existence and (Hölder) regularity of local times of stochastic differential equations driven by fractional Brownian motions. In particular, we show that in one dimension and in the rough case $H < 1/2$, the Hölder exponent (in t) of the local time is $1 - H$, where H is the Hurst parameter of the driving fractional Brownian motion.

Keywords: local time, fractional Brownian motion, stochastic differential equation

1 Introduction

In this paper, we consider the following stochastic differential equation (SDE)

$$X_t = x + \int_0^t V_0(X_s)ds + \sum_{i=1}^m \int_0^t V_i(X_s)dB_s^i, \quad t \in [0,T], \tag{1.1}$$

where $x \in \mathbb{R}^d$, $V_0, V_1, \cdots, V_d$ are C^∞-bounded vector fields on $\mathbb{R}^d$ and $\{B_t\}_{0\le t\le T}$ is an m-dimensional fractional Brownian motion. Throughout this paper, we assume that in (1.1), the fractional Brownian motion has Hurst parameter $H \in (1/4, 1)$ and the vector fields V_i satisfy the uniform elliptic condition. When $H \in (1/2, 1)$, the above equation is understood in Young's sense. When $H \in (1/4, 1/2)$, stochastic integrals in equation (1.1) are interpreted as rough path integrals (see, e.g., [8, 10]) which extends the Young's integral. Existence and uniqueness of solutions to the above equation can be found, for example, in [13]. In particular, when $H = \frac{1}{2}$, this notion of solution coincides with the

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solution of the corresponding Stratonovitch stochastic differential equation. It is also clear now (cf. [1, 6, 7, 11]) that under Hörmander's condition the law of the solution X_t has a smooth density with respect to the Lebesgue measure on $\mathbb{R}^d$.

We are interested in the existence and regularity of local times of the solution X to equation (1.1). For a d-dimensional fractional Brownian motion B itself, its local time has been studied intensively under the framework of Gaussian random fields and is now well-understood (see, e.g., [4], [9] and [5]). The challenge to investigate local times of X is that it is not a Gaussian process in general. Many tools developed for Gaussian random fields can not be directly applied. For example, the Fourier transform of the law of X is not easy to analyze in such a non-Gaussian setting.

Our approach relies on a sharp estimate of the joint density of finite distributions of X (Theorem 2.4 below). Essentially, this density estimate plays a similar role to the "local nondeterminism" condition that is often used in the content of Gaussian random fields. The main result of our investigation is summarized as follows.

Fix any small positive number a. Let $L(t,x)$ be the local time (occupation density) of X up to time t and $L^a(t,x)$ the occupation density of X over the time interval $[a,t]$. Define the pathwise Hölder exponent of $L^a(\cdot,x)$ by

$$\alpha(t)=\sup\left\{\alpha>0,\limsup_{\delta\to 0}\sup_{x\in\mathbb{R}^d}\frac{L^a(t+\delta,x)-L^a(t,x)}{\delta^\alpha}=0\right\}. \tag{1.2}$$

Theorem 1.1. *Let X be the solution to equation (1.1).*

(1) Almost surely, when $dH<1$, the local time $L(t,x)$ of X exists for any fixed t; when $dH>1$, the local time does not exist for any t.

(2) Assume $d=1$ and $1/4<H<1/2$. There exists a version of $L^a(t,x)$ that is jointly continuous in (t,x). Moreover, for any $\beta<1-H$, $L^a(t,x)$ is β-Hölder continuous in t, uniformly in x. And its pathwise Hölder exponent is given by

$$\alpha(t)=1-H,\quad \text{a.s.}\quad \textit{for all } t\in[a,T].$$

Finally, let us briefly explain why we have to impose the technical assumption $d=1$ and $1/2<H<1/4$ for the Hölder regularity of $L^a(t,x)$. In order to establish the continuity of $L^a(t,x)$ in the space variable, the natural approach is to provide an upper bound for

$$\mathbb{E}|L^a(t,x)-L^a(t,y)|^n.$$

Our observation is that the above moment can be bounded from above by quantities of the form

$$|x-y|^n\int dx_1,...,dx_n\int_{[a,T]^n}|\partial_x^n p_{t_1,...,t_n}(x_1,...,x_n)|dt_1,...,dt_n,$$

where $p_{t_1,...,t_n}(x_1,...,x_n)$ is the joint density of $(X_{t_1},...,X_{t_n})$. In general, one needs $n>d$ in order to conclude the continuity of $L^a(t,x)$ in x. However, as one can see from Theorem

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2.4, large n and H tend to blow up the time integral above. Consequently, we have to take small n and H in order for our approach to succeed. Indeed the computation in Lemma 4.2 shows that we have to restrict our discussion under the assumption $n \leq 2$ and $(1+d)H < 1$, i.e., $d = 1$ and $H < 1/2$.

The rest of the paper is organized as follows. In Section 2, we introduce some basic tools for analyzing SDEs driven by fractional Brownian motions. In particular, we establish the key estimate for the joint density of $(X_{t_1}, ..., X_{t_n})$, which enables us to establish both the existence of local time in Section 2 and the regularity of the local time in Section 3.

2 SDEs driven by fractional Brownian motions

In this section, we present some tools for analyzing SDEs driven by fractional Brownian motions which will be needed for the remainder of the paper.

Let $B = \{B_t = (B_t^1, \ldots, B_t^m),\ t \in [0,T]\}$ be a m-dimensional fractional Brownian motion with Hurst parameter $H \in (0,1)$. That is, B is a centered Gaussian process whose covariance structure is induced by

$$R(t,s) := \mathbb{E} B_s^i B_t^j = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right)\delta_{ij}, \quad i,j = 1, \ldots, m. \tag{2.1}$$

It can be shown, by a standard application of Kolmogorov's criterion, that B admits a continuous version whose paths are γ-Hölder continuous for any $\gamma < H$.

2.1 Malliavin calculus

We introduce the basic framework of Malliavin calculus in this subsection. The reader is invited to read the corresponding chapters in [14] for further details. Let $\mathcal{E}$ be the space of $\mathbb{R}^m$-valued step functions on $[0,1]$, and $\mathcal{H}$ the closure of $\mathcal{E}$ for the scalar product:

$$\langle (\mathbf{1}_{[0,t_1]}, \cdots, \mathbf{1}_{[0,t_m]}), (\mathbf{1}_{[0,s_1]}, \cdots, \mathbf{1}_{[0,s_m]}) \rangle_{\mathcal{H}} = \sum_{i=1}^{m} R(t_i, s_i).$$

$\mathcal{H}$ is the reproducing kernel Hilbert space for B.

Some isometry arguments allow to define the Wiener integral $B(h) = \int_0^1 \langle h_s, dB_s \rangle$ for any element $h \in \mathcal{H}$, with the additional property $\mathbb{E}[B(h_1)B(h_2)] = \langle h_1, h_2 \rangle_{\mathcal{H}}$ for any $h_1, h_2 \in \mathcal{H}$. An $\mathcal{F}$-measurable real valued random variable F is said to be cylindrical if it can be written, for a given $n \geq 1$, as

$$F = f\left(B(\phi^1), \ldots, B(\phi^n)\right),$$

where $\phi^i \in \mathcal{H}$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a C^∞ bounded function with bounded derivatives. The set of cylindrical random variables is denoted by $\mathcal{S}$.

The Malliavin derivative is defined as follows: for $F \in \mathcal{S}$, the derivative of F is the $\mathbb{R}^m$ valued stochastic process $(\mathbf{D}_t F)_{0 \leq t \leq 1}$ given by

$$\mathbf{D}_t F = \sum_{i=1}^{n} \phi^i(t) \frac{\partial f}{\partial x_i}\left(B(\phi^1), \ldots, B(\phi^n)\right).$$

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More generally, we can introduce iterated derivatives by $\mathbf{D}^k_{t_1,\dots,t_k}F = \mathbf{D}_{t_1}\dots\mathbf{D}_{t_k}F$. For any $p \geq 1$, we denote by $\mathbb{D}^{k,p}$ the closure of the class of cylindrical random variables with respect to the norm

$$\|F\|_{k,p} = \left(\mathbb{E}\left(F^p\right) + \sum_{j=1}^{k}\mathbb{E}\left(\left\|\mathbf{D}^j F\right\|^p_{\mathcal{H}^{\otimes j}}\right)\right)^{\frac{1}{p}},$$

and

$$\mathbb{D}^{\infty} = \bigcap_{p\geq 1}\bigcap_{k\geq 1}\mathbb{D}^{k,p}.$$

Let $F = (F^1,\dots,F^n)$ be a random vector whose components are in $\mathbb{D}^{\infty}$. Define the Malliavin matrix of F by

$$\gamma_F = (\langle \mathbf{D}F^i, \mathbf{D}F^j\rangle_{\mathcal{H}})_{1\leq i,j\leq n}.$$

Then F is called *non-degenerate* if γ_F is invertible *a.s.* and

$$(\det\gamma_F)^{-1} \in \cap_{p\geq 1}L^p(\Omega).$$

It is a classical result that the law of a non-degenerate random vector admits a smooth density with respect to the Lebesgue measure on $\mathbb{R}^n$.

It is well-known that for a fractional Brownian motion B, there is an underlying Wiener process W such that

$$B_t = \int_0^t K_H(t,s)dW_s,$$

where $K(t,s)$ is a deterministic kernel whose expression is explicit. Based on the above representation, one can consider fractional Brownian motions and hence functionals of fractional Brownian motions as functionals of the underlying Wiener process W. This observation allows us to perform Malliavin calculus with respect to the Wiener process W. We shall perform Malliavin calculus with respect to both B and W. In order to distinguish them, the Malliavin derivatives (and corresponding Sobolev spaces, respectively) with respect to W will be denoted by D (and by $D^{k,p}$, respectively). The relation between the two operators $\mathbf{D}$ and D is given by the following (see e.g. [14, Proposition 5.2.1]).

Proposition 2.1. *Let $D^{1,2}$ be the Malliavin-Sobolev space corresponding to the Wiener process W. Then $\mathbb{D}^{1,2} = D^{1,2}$, and for any $F \in D^{1,2}$ we have $\mathrm{D}F = K^*\mathbf{D}F$,where K^* is the isometry between $L^2([0,T])$ and $\mathcal{H}$.*

It is known that B and the Wiener process W generate the same filtration which we denote by $\{\mathcal{F}_t;\, t\in[0,1]\}$. Set $L^2_t \equiv L^2([t,1])$ and $\mathbb{E}_t = \mathbb{E}(\cdot\,|\mathcal{F}_t)$. For a random variable F and $t\in[0,1]$, define, for $k\geq 0$ and $p>0$, the conditional Sobolev norm

$$\|F\|_{k,p;t} = \left(\mathbb{E}_t\left[F^p\right] + \sum_{j=1}^{k}\mathbb{E}_t\left[\left\|D^j F\right\|^p_{(L^2_t)^{\otimes j}}\right]\right)^{\frac{1}{p}}.$$

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By convention, when $k = 0$ we always write $\|F\|_{p;t} = \|F\|_{0,p;t}$. The conditional Malliavin matrix of F is given by

$$\Gamma_{F,t} = \left(\langle DF^i, DF^j \rangle_{L^2_t} \right)_{1 \le i,\, j \le n}. \tag{2.2}$$

The following is a conditional version of Proposition 2.1.4 of [14].

Proposition 2.2. *Fix $k \ge 1$. Let $F = (F^1, ..., F^n)$ be a random vector and G a random variable. Assume both F and G are smooth in the Malliavin sense and $(\det \Gamma_{F,s})^{-1}$ has finite moments of all orders. Then for any multi-index $\alpha \in \{1, \dots, n\}^k, k \ge 1$, there exists an element $H^s_\alpha(F, G) \in \cap_{p \ge 1} \cap_{m \ge 0} D^{m,p}$ such that*

$$\mathbb{E}_s \left[(\partial_\alpha \varphi)(F)\, G \right] = \mathbb{E}_s \left[\varphi(F)\, H^s_\alpha(F, G) \right], \qquad \varphi \in \mathcal{C}^\infty_p(\mathbb{R}^n),$$

where $H^s_\alpha(F, G)$ is recursively defined by

$$H^s_{(i)}(F, G) = \sum_{j=1}^n \delta_s \left(G \left(\Gamma^{-1}_{F,s} \right)_{ij} DF^j \right), \quad H^s_\alpha(F, G) = H^s_{(\alpha_k)}(F, H^s_{(\alpha_1, \dots, \alpha_{k-1})}(F, G)).$$

Here δ_s denotes the Skorohod integral with respect to the Wiener process W on the interval $[s, 1]$. (See [14, Section 1.3.2] for a detailed account of the definition of δ_s.) Furthermore, the following norm estimate holds true:

$$\|H^s_\alpha(F, G)\|_{p;s} \le c_{p,q} \|\Gamma^{-1}_{F,s} DF\|^k_{k, 2^{k-1} r; s} \|G\|^k_{k,q;s},$$

where $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

2.2 SDEs driven by fractional Brownian motions and density estimate

Consider the following stochastic differential equation driven by a fractional Brownian motion with Hurst parameter $H > 1/4$,

$$X_t = x_0 + \sum_{i=1}^m \int_0^t V_i(X_s) dB^i_s + \int_0^t V_0(X_s) ds, \qquad t \in [0, T]. \tag{2.3}$$

Here $V_i, i = 0, 1, ..., d$ are C^∞-bounded vector fields on $\mathbb{R}^d$ which form a uniform elliptic system. Note that under the uniform ellipticity condition, we have $m \ge d$.

Recall that D is the Malliavin derivative operator with respect to the underlying Wiener process W and $\Gamma_{F,t}$ is defined in (2.2) for a random variable F. The following estimate is a restatement of Proposition 5.9 in [2].

Lemma 2.3. *Let $a \in (0, T)$, and consider $H \in (1/4, 1)$. Then there exists a constant $C > 0$ depending on a such that for $a \le s \le t \le T$ the following holds:*

$$\begin{aligned} \|\Gamma^{-1}_{X_t - X_s, s}\|^d_{d, 2^{d+2}; s} &\le \frac{C}{(t-s)^{2dH}} \left(\mathbb{E}_s (1 + G) \right)^{\frac{d}{2^{d+2}}} \\ \|D(X_t - X_s)\|^d_{d, 2^{d+2}; s} &\le C (t-s)^{dH} \left(\mathbb{E}_s (1 + G) \right)^{\frac{d}{2^{d+2}}}, \end{aligned}$$

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where G is a random variable smooth in the Malliavin sense and has finite moments to any order.

The above lemma allows us to estimate the joint density of $(X_{t_1},...,X_{t_n})$, or equivalently the joint density of $(X_{t_1}, X_{t_2}-X_{t_1},...,X_{t_n}-X_{t_{n-1}})$.

Theorem 2.4. *Fix $a \in (0,T)$ and $\gamma < H$. Let $\tilde{p}_{t_1,...,t_n}(\xi_1,...,\xi_n)$ be the joint density of the random vector $(X_{t_1}, X_{t_2}-X_{t_1},...,X_{t_n}-X_{t_{n-1}})$, $a \le t_1 < \cdots < t_n \le T$. For any non-negative integers $k_1,...,k_n$, there exists a constant $C_1, C_2 > 0$ (depending on a) such that*

$$\begin{aligned}&\partial_{\xi_1}^{k_1}\dots\partial_{\xi_n}^{k_n}\tilde{p}_{t_1,...,t_n}(\xi_1,...,\xi_n)\\ &\le C_1\frac{1}{(t_2-t_1)^{(d+k_2)H}}e^{-\frac{|\xi_2|^{2\gamma}}{2|t_2-t_1|^{2\gamma^2}}}\cdots\frac{1}{(t_n-t_{n-1})^{(d+k_n)H}}e^{-\frac{|\xi_n|^{2\gamma}}{C_2|t_n-t_{n-1}|^{2\gamma^2}}}.\end{aligned}$$

Remark 2.5. In the above upper bound, the term $t_1^{-(d+k_1)H}$ has been absorbed in the constant C_1, given the fact that $t_1 \ge a > 0$.

Proof. The proof is similar to the one for Theorem 3.5 in [12]. We only prove the case of $n=3$. The general case is almost identical. The positive constants c_i, $1 \le i \le 4$, may change from line to line.

First observe that $\tilde{p}_{t_1,t_2,t_3}$ can be expressed as

$$\begin{aligned}&\partial_{\xi_1}^{k_1}\partial_{\xi_2}^{k_2}\partial_{\xi_3}^{k_3}\tilde{p}_{t_1,t_2,t_3}(\xi_1,\xi_2,\xi_3)\\ =&\partial_{\xi_1}^{k_1}\partial_{\xi_2}^{k_2}\partial_{\xi_3}^{k_3}\mathbb{E}\left[\delta_{\xi_1}(X_{t_1})\,\delta_{\xi_2}(X_{t_2}-X_{t_1})\,\delta_{\xi_3}(X_{t_3-t_2})\right], \quad \text{for} \quad \xi_1,\xi_2,\xi_3\in\mathbb{R}^d,\\ =&\partial_{\xi_1}^{k_1}\partial_{\xi_2}^{k_2}\mathbb{E}\left[\delta_{\xi_1}(X_{t_1})\,\delta_{\xi_2}(X_{t_2}-X_{t_1})\,\partial_{\xi_3}^{k_3}\mathbb{E}_{t_2}\left[\delta_{\xi_3}(X_{t_3}-X_{t_2})\right]\right].\end{aligned}$$

Due to Proposition 2.2, $M_{t_2t_3} = \partial_{\xi_3}^{k_3}\mathbb{E}_{t_2}\left[\delta_{\xi_2}(X_{t_3}-X_{t_2})\right]$ can be bounded from above as follows:

$$|M_{t_2t_3}| \le c_1\|\Gamma^{-1}_{X_{t_3}-X_{t_2},t_2}\|^{d+k_3}_{d+k_3,2^{d+k_3+2},t_2}\,\|D(X_{t_3}-X_{t_2})\|^{d+k_3}_{d+k_3,2^{d+k_3+2};t_2}\left(\mathbb{E}_{t_2}[\mathbf{1}_{(X_{t_3}-X_{t_2}>\xi_3)}]\right)^{1/2},$$

where c_1 is some positive constant. Therefore, Lemma 2.3 yields that for some constant $c_2>0$,

$$\begin{aligned}&\partial_{\xi_1}^{k_1}\partial_{\xi_2}^{k_2}\partial_{\xi_3}^{k_3}p_{t_1,t_2,t_3}(\xi_1,\xi_2,\xi_3) \qquad (2.4)\\ \le&\frac{c_2}{(t_3-t_2)^{(d+k_3)H}}\partial_{\xi_1}^{k_1}\partial_{\xi_2}^{k_2}\mathbb{E}\left[\delta_{\xi_1}(X_{t_1})\,\delta_{\xi_2}(X_{t_2}-X_{t_1})\left(\mathbb{E}_{t_2}(1+G)^2\right)^{\frac{d+k_3}{2^{d+k_3+1}}}\left(\mathbb{E}_{t_2}[\mathbf{1}_{(X_{t_3}-X_{t_2}>\xi_3)}]\right)^{1/2}\right]\end{aligned}$$

To proceed, denote by $\mathbf{B}$ the lift of B as a rough path. Set

$$\mathcal{N}_{\gamma,q}(\mathbf{B}) = \int_0^T\int_0^T\frac{d(\mathbf{B}_v,\mathbf{B}_u)^q}{|v-u|^{\gamma q}}\,dudv,$$

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where $\gamma < H$ and q a nonnegative integer, and

$$d(\mathbf{B}_v, \mathbf{B}_u) = \sum_{k=1}^{[1/H]} \left\| \int_{v \le s_1 \le \ldots \le s_k \le u} dB^{\otimes k} \right\|^{1/k}.$$

$\mathcal{N}_{\gamma,q}(\mathbf{B})$ is the Besov norm of $\mathbf{B}$ (as a function in t). The reason we need to consider these Besov norms $\mathcal{N}_{\gamma,q}(\mathbf{B})$ is that they are smooth in the Malliavin sense.

Fix any $\varepsilon > 0$ satisfying $\gamma + \varepsilon < H$. It has been shown in [12, Theorem 3.5] that for some $c > 0$,

$$|X_t - X_s| \le c\,|t-s|^{\gamma}(1 + \mathcal{N}_{\gamma+\epsilon,2q}(\mathbf{B})^{1/2q})^{1/\gamma},$$

and there exists a constant $\lambda_0 > 0$ such that

$$\mathbb{E} \exp\left\{ \lambda_0 \mathcal{N}_{\gamma+\epsilon,2q}(\mathbf{B})^{1/q} \right\} < \infty.$$

That is, $\mathcal{N}_{\gamma+\epsilon,2q}(\mathbf{B})^{1/2q}$ has Gaussian tail. Thus, for $\lambda < \lambda_0$ we can find a constant $c_3 > 0$,

$$\mathbb{E}_{t_2}\left[\mathbf{1}_{(X_{t_3} - X_{t_2} > \xi_3)} \right] \le c_3 \exp\left\{ -\lambda \frac{|\xi_3|^{2\gamma}}{|t_3 - t_2|^{2\gamma^2}} \right\} \mathbb{E}_{t_2} \exp\left\{ \lambda (1 + \mathcal{N}_{\gamma+\epsilon,2q}(\mathbf{B}))^{2/2q} \right\}.$$

Plugging this inequality into (2.4), we end up with:

$$\begin{aligned} &\partial_{\xi_1}^{k_1} \partial_{\xi_2}^{k_2} \partial_{\xi_3}^{k_3} \tilde{p}_{t_1,t_2,t_3}(\xi_1,\xi_2,\xi_3) \qquad (2.5) \\ \le &\frac{c_3}{(t_3 - t_2)^{(d+k_3)H}} e^{-\lambda \frac{|\xi_3|^{2\gamma}}{c_4 |t-s|^{2\gamma^2}}}\, \partial_{\xi_1}^{k_1} \partial_{\xi_2}^{k_2} \mathbb{E}\left[\delta_{\xi_1}(X_{t_1})\, \delta_{\xi_2}(X_{t_2} - X_{t_1})\, \Psi \right] \end{aligned}$$

where Ψ is a random variable that is smooth in the Malliavin calculus sense. [Ψ only depends on G and $\mathcal{N}_{\gamma+\epsilon,2q}(\mathbf{B})^{1/2q}$.]

Now we can continue our computation by writing

$$\partial_{\xi_1}^{k_1} \partial_{\xi_2}^{k_2} \mathbb{E}\left[\delta_{\xi_1}(X_{t_1})\, \delta_{\xi_2}(X_{t_2} - X_{t_1})\, \Psi \right] = \partial_{\xi_1}^{k_1} \mathbb{E}\left[\delta_{\xi_1}(X_{t_1})\, \partial_{\xi_2}^{k_2} \mathbb{E}_{t_1}[\delta_{\xi_2}(X_{t_2} - X_{t_1})\, \Psi] \right],$$

and repeating the above argument to establish an upper bound for $M_{t_2 t_1} = \partial_{\xi_2}^{k_2} \mathbb{E}_{t_1}[\delta_{\xi_2}(X_{t_2} - X_{t_1})\, \Psi]$. Observe that the existence of a smooth random variable Ψ does not affect the upper bound estimate when applying integration by parts. [But we may need to choose λ small enough to ensure sufficient integrability for Ψ and its Malliavin derivatives.] Hence,

$$\begin{aligned} &\partial_{\xi_1}^{k_1} \partial_{\xi_2}^{k_2} \partial_{\xi_3}^{k_3} \tilde{p}_{t_1,t_2,t_3}(\xi_1,\xi_2,\xi_3) \qquad (2.6) \\ \le &c_3 \frac{1}{(t_3 - t_2)^{(d+k_3)H}} e^{-\lambda \frac{|\xi_3|^{2\gamma}}{c_4 |t_3 - t_2|^{2\gamma^2}}}\, \frac{1}{(t_2 - t_1)^{(d+k_2)H}} e^{-\lambda \frac{|\xi_2|^{2\gamma}}{c_4 |t_2 - t_1|^{2\gamma^2}}}\, \partial_{\xi_1}^{k_1} \mathbb{E}\left[\delta_{\xi_1}(X_{t_1})\, \Phi \right] \end{aligned}$$

where Φ is a random variable that is smooth in the Malliavin calculus sense. [Φ still only depends on G and $\mathcal{N}_{\gamma+\epsilon,2q}(\mathbf{B})^{1/2q}$.] We can now integrate (2.6) safely by parts in order to regularize the term $\delta_{\xi_1}(X_s)$, which finishes the proof.

□

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The following technical result will be needed later. Let $p_{t_1,...,t_n}(x_1,...,x_n)$ be the joint density of $(X_{t_1},...,X_{t_n}), a \le t_1 \le,...,\le t_n \le T$, and set

$$v(x_1,...,x_n) = \int_{[a,T]^n} p_{t_1,...,t_n}(x_1,...,x_n)dt_1,...,dt_n. \tag{2.7}$$

Corollary 2.6. *If $(1+d)H < 1$, then v is uniformly continuous in $(x_1,...,x_n)$.*

Proof. Let $\bar{x} = (x_1,...,x_n)$ and $\bar{y} = (y_1,...,y_n)$.

$$\begin{aligned} |v(\bar{x}) - v(\bar{y})| \le & \int_{[a,T]^n} |p_{t_1,...,t_n}(\bar{x}) - p_{t_1,...,t_n}(\bar{y})|dt_1,...,dt_n \\ \le & |\bar{x} - \bar{y}| \int_{[a,T]^n} |\partial_{\bar{x}} p_{t_1,...,t_n}(\bar{x}^*)|dt_1,...,dt_n. \end{aligned} \tag{2.8}$$

Observe that

$$p_{t_1,...,t_n}(x_1,...,x_n) = \tilde{p}_{t_1,...,t_n}(x_1, x_2 - x_1, x_3 - x_2,..., x_n - x_{n-1}).$$

It follows immediately from Theorem 2.4 that

$$\begin{aligned} |\partial_{\bar{x}} p_{t_1,...,t_n}(\bar{x})| \le & c_1 \sum_{k=2}^{n} \frac{1}{|t_2 - t_2|^{dH}} \cdots \frac{1}{|t_k - t_{k-1}|^{(1+d)H}} \cdots \frac{1}{|t_n - t_{n-1}|^{dH}} \\ \le & c_2 \frac{1}{|t_2 - t_1|^{(1+d)H}} \cdots \frac{1}{|t_n - t_{n-1}|^{(1+d)H}}. \end{aligned} \tag{2.9}$$

By combining (2.8) and (2.9), and since there is no singularity in the integral (2.8) in t under our assumption $(1+d)H < 1$, the proof is complete. □

3 Existence of local time

For any time interval $I \subset [0,T]$, define

$$L(I,A) := \lambda(\{s \in I : X_s \in A\}), \qquad A \subset \mathbb{R}^d, \tag{3.1}$$

where λ is the Lebesgue measure on $\mathbb{R}^+$. It is clear that $L(I,\cdot)$ is the occupation measure on $\mathbb{R}^d$ of X during time interval I. If $L(I,\cdot)$ is absolutely continuous with respect to the Lebesgue measure λ_d on $\mathbb{R}^d$, we denote the Radon-Nikodym derivative by

$$L(I,x) = \frac{dL(I,\cdot)}{d\lambda_d},$$

and call $L(I,x)$ the occupation density of X at x during time interval I. Thus it holds for all measurable $A \subset \mathbb{R}^d$ that

$$L(I,A) = \int_{x \in A} L(I,x)dx.$$

Whenever there is no confusion, we always write $L(t,A) = L([0,t],A)$ and $L(t,x) = L([0,t],x)$ for the occupation measure and occupation density, respectively.

The following theorem is a classical result on the existence of occupation density (local time) for a stochastic process X ([9, Theorem 21.12]).

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Theorem 3.1. *For any interval $I \subset \mathbb{R}^+$, suppose*

$$\liminf_{\epsilon \downarrow 0} \frac{1}{\epsilon^d} \int_I \mathbb{P}\{(|X_s - X_u|) \leq \epsilon\} ds < \infty \quad \text{for a.e. } u \in I,$$

then the occupation density $L(I, x)$ exists almost surely.

On account of the above theorem, we are ready to state our result on the existence of local time for the solution to equation (2.3).

Theorem 3.2. *Let X be the solution to equation (2.3). Assume that $dH < 1$, then for each fixed $t \in [0, T]$, X has local time $L(t, x)$ almost surely.*

Proof. Fix a $t \in [0, T]$ and $a \in (0, t]$. It follows from Theorem 2.4 that the probability density $\tilde{p}_{s,u}(z)$ of $X_u - X_s$ satisfies that for some $c_1, c_2 > 0$,

$$\tilde{p}_{s,u}(z) \leq \frac{c_1}{(u-s)^{dH}} \exp\left\{-\frac{|z|^{2\gamma}}{c_2(u-s)^{2\gamma^2}}\right\}, \text{ for all } a \leq s < u \leq T,$$

where $\gamma < H$. It follows immediately that under our assumption $dH < 1$, we have for the solution X to equation (2.3)

$$\liminf_{\epsilon \downarrow 0} \frac{1}{\epsilon^d} \int_a^t \mathbb{P}\{(|X_s - X_u|) \leq \epsilon\} ds < \infty \quad \text{for all } u \in [a, t].$$

Thus, by Theorem 3.1, there exists a null set $N_a \subset \Omega$ such that the occupation density $L([a, t], x)(\omega)$ exists for all $\omega \notin N_a$. Since the occupation density exists for all $\omega \notin N_a$, we can work on the following modification (in x) of it (for all $\omega \notin N_a$), which is still denoted by $L([a, t], x)$,

$$L([a, t], x) = \liminf_{\epsilon \downarrow 0} \frac{L([a, t], B(x, \epsilon))}{C_d \epsilon^d} = \liminf_{\epsilon \downarrow 0} \frac{1}{C_d \epsilon^d} \int_a^t \mathbf{1}_{B(x,\epsilon)}(X_s) ds.$$

Here, C_d is such that $C_d \epsilon^d$ gives the volume of the ball $B(x, \epsilon)$ in $\mathbb{R}^d$. The reason to work on this version of $L([a, t], x)$ is that it is monotone in a.

Observe that we have for all $\omega \notin N_a$,

$$L([a, t], A) = \int_{x \in A} L([a, t], x) dx, \qquad \text{for all Broel } A \subset \mathbb{R}^d.$$

Letting $a \downarrow 0$ (along a sequence a_n) in the above and by monotone convergence theorem, we have for all $\omega \notin \cup_{n=1}^{\infty} N_{a_n}$

$$L([0, t], A) = \lim_{a_n \downarrow 0} L([a_n, t], A) = \int_{x \in A} \lim_{a_n \downarrow 0} L([a_n, t], x) dx.$$

Hence, almost surely, the occupation density $L([0, t], x)$ of the occupation measure $L([0, t], A)$ exists (which is $\lim_{a_n \downarrow 0} L([a_n, t], x)$). □

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Theorem 3.3. *Let X be the same as in Theorem 3.2. If $dH > 1$, X does not have a local time $L(t,x)$ for any $t \in [0,T]$ almost surely.*

Proof. It has been shown in [12] that, with probability one, the Hausdorff dimension of the sample paths of X over the time interval $[0,T]$ is $\min\{d, 1/H\} = 1/H < d$, when $dH > 1$. Hence the sample paths of X has Lebesgue measure zero almost surely, by the definition of Hausdorff dimension. On the other hand, recall the definition of the occupation measure of X in (3.1). Almost surely, it is a positive and finite measure supported in the sample path of X. Therefore it can not be absolutely continuous with respect to the Lebesgue measure λ_d. The proof is thus completed. □

4 Regularity of local time

Unless otherwise stated, it is assumed that $(1+d)H < 1$ throughout this section. As a remark, in the case $d = 1$, any $H < 1/2$ satisfies this condition.

Fix a small positive number $a \in (0,T]$, and let $L^a(t,A)$ and $L^a(t,x)$ be the occupation measure and occupation density respectively on the time interval $[a,t]$. That is

$$L^a(t,A) = L([a,t],A), \quad \text{and} \quad L^a(t,x) = L([a,t],x).$$

In order to use Kolmogorov type continuity theorems to conclude various continuities of $L^a(t,x)$ in t and x, we seek to provide upper bounds for quantities in the form of $\mathbb{E}|L^a(t,x) - L^a(t,y)|^n$. For this purpose, it is necessary to specify, for each fixed t and x, a version of $L^a(t,x)$ more carefully. Let

$$L_0^a(t,x) = \liminf_{\epsilon \downarrow 0} \frac{1}{C_d \epsilon^d} \int_a^t \mathbf{1}_{B(x,\epsilon)}(X_s) ds, \tag{4.1}$$

which, by our discussion before, exists as a finite limit for a.e. x almost surely. Set

$$L_\epsilon^a(t,x) = \frac{1}{C_d \epsilon^d} \int_a^t \mathbf{1}_{B(x,\epsilon)}(X_s) ds \tag{4.2}$$

Recall the content of Corollary 2.6, standard argument shows that the uniform continuity of v implies that $L_\epsilon^a(t,x)$ is Cauchy in $L^n(\mathbb{P})$, uniformly in x. [See the argument in p42-p43 in [9].] It follows that there is a subsequence $\{\epsilon_n\}_{n\geq 1}$ such that $L_{\epsilon_n}^a(t,x)$ converges uniformly in $L^n(\mathbb{P})$ and converges almost surely for each fixed x. Hence the limit exists in (4.1) through the subsequence ϵ_n both uniformly in $L^n(\mathbb{P})$ and almost surely for each fixed x, and there is no problem then in arguing as if ϵ_n is the full sequence ϵ in the limit in (4.1). This limit, which we denote by $L^a(t,x)$, is the version we work with throughout our discussion below.

Denote by $I_R = [-R,R] \subset \mathbb{R}$. The following theorem is a restatement of Theorem 3.1 of [3].

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Theorem 4.1. *Let $Y(t,x), (t,x) \in [0,T] \times I_R$ be a stochastic process of two parameters. Suppose there are positive constants n, C and c, d such that*

$$\mathbb{E}|Y(t,x+k) - Y(t,x)|^n \le C|k|^{1+c}, \quad \text{for } x, x+k \in I_R, t = 0, T; \tag{4.3}$$

$$\mathbb{E}|Y(t+h,x) - Y(t,x)|^n \le C|h|^{1+d}, \quad \text{for all } t, t+h \in [0,T], \text{ and } x \in I_R; \tag{4.4}$$

$$\mathbb{E}|Y(t+h,x+k) - Y(t,x+k) - Y(t+h,x) + Y(t,x)|^n \le C|k|^{1+c}|h|^{1+d}, \quad \text{for all } t, t+h \in [0,T] \text{ and } x, x+k \in I_R. \tag{4.5}$$

Then there exists a version of the process Y jointly continuous in (t,x). Moreover, for every $\gamma < d/r$ there exist random variables η and Δ which are almost surely positive and finite such that

$$|Y(t+h,x) - Y(t,x)| \le \Delta |h|^\gamma, \text{ for all } t, t+h \in [0,T], x \in I_R, \text{ and } |h| < \eta.$$

Our primary goal for this section is to find a version of the local time $L^a(t,x)$ with jointly Hölder continuity in (t,x). Furthermore, on account of Theorem 4.1, we show that the Hölder regularity of the local time in the time variable is uniform in the space variable. The next three lemmas establish the three inequalities in the condition of Theoerm 4.1. We start off with the Hölder continuity in the space variable.

Lemma 4.2. *Assume $d = 1$ and $1/4 < H < 1/2$. For any $t \in [a,T]$, and α satisfying $1 < \alpha < \frac{1}{H} - 1$, there exists a constant $C_1 > 0$ (depending on a and T) such that for any $x, y \in \mathbb{R}$,*

$$\mathbb{E}\left|L^a(t,x) - L^a(t,y)\right|^2 \le C_1 |x-y|^\alpha. \tag{4.6}$$

Proof. By our choice of $L^a(t,x)$ in the paragraph after (4.1), and by Fatou's lemma,

$$\begin{aligned}\mathbb{E}\left|L^a(t,x) - L^a(t,y)\right|^2 &= \mathbb{E}\lim_{\epsilon \downarrow 0} \left| \frac{L^a(t,B(x,\epsilon)) - L^a(t,B(y,\epsilon))}{C_d \epsilon} \right|^2 \\ &\le \liminf_{\epsilon \downarrow 0} \frac{1}{C_d^2 \epsilon^2} \mathbb{E}\left|L^a(t,B(x,\epsilon)) - L^a(t,B(y,\epsilon))\right|^2.\end{aligned}$$

Let $p_{u,s}(z_1,z_2)$ be the joint density of (X_u, X_s). By change of variables, we have

$$\begin{aligned}&\mathbb{E}\left|L^a(t,B(x,\epsilon)) - L^a(t,B(y,\epsilon))\right|^2 \\ =&2\mathbb{E}\int_{s=a}^t \int_{u=a}^s \left(\mathbf{1}_{B(x,\epsilon)}(X_u) - \mathbf{1}_{B(y,\epsilon)}(X_u)\right)\left(\mathbf{1}_{B(x,\epsilon)}(X_s) - \mathbf{1}_{B(y,\epsilon)}(X_s)\right) ds du \\ =&2\int_{s=a}^t \int_{u=a}^s \int_{\mathbb{R}^2} \left(\mathbf{1}_{B(x,\epsilon)}(z_1) - \mathbf{1}_{B(y,\epsilon)}(z_1)\right)\left(\mathbf{1}_{B(x,\epsilon)}(z_2) - \mathbf{1}_{B(y,\epsilon)}(z_2)\right) p_{u,s}(z_1,z_2) dz_1 dz_2 ds du \\ =&2\int_{s=a}^t \int_{u=a}^s \int\int_{z_1,z_2 \in B(x,\epsilon)} \big[p(z_1,z_2) - p(z_1-(x-y),z_2) \\ &\qquad + p(z_1-(x-y), z_2-(x-y)) - p(z_1, z_2-(x-y))\big] dz_2 dz_1 du ds.\end{aligned}$$

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For notation convenience, we introduce

$$F_{x,y}(z_1,z_2) = p(z_1,z_2) - p(z_1-(x-y),z_2) + p(z_1-(x-y),z_2-(x-y)) - p(z_1,z_2-(x-y)).$$

By Theorem 2.4 and Taylor expansion of $F_{x,y}(z_1,z_2)$ to the first and second order (with respect to $x-y$), there exist a constant c_1 such that

$$F_{x,y}(z_1,z_2) \le |x-y|\frac{c_1}{|s-u|^{2H}}, \tag{4.7}$$

and

$$F_{x,y}(z_1,z_2) \le |x-y|^2\frac{c_1}{|s-u|^{3H}}. \tag{4.8}$$

Write

$$\begin{aligned}
&\mathbb{E}\,|L^a(t,B(x,\epsilon)) - L^a(t,B(y,\epsilon))|^2 = 2\int_{s=a}^{t}\int_{u=a}^{s}\int\!\!\int_{z_1,z_2\in B(x,\epsilon)} F_{x,y}(z_1,z_2)dz_2dz_1duds \\
=&2\int\!\!\int_{|s-u|^H>|x-y|}\int\!\!\int_{z_1,z_2\in B(x,\epsilon)} F_{x,y}(z_1,z_2)dz_2dz_1duds \\
&+2\int\!\!\int_{|s-u|^H\le|x-y|}\int\!\!\int_{z_1,z_2\in B(x,\epsilon)} F_{x,y}(z_1,z_2)dz_2dz_1duds.
\end{aligned} \tag{4.9}$$

We take care of the two terms on the right hand side of (4.9) separately. Let $\delta>0$ be a fixed constant satisfying $d+1+\delta<1/H$. For the first term, it holds

$$\begin{aligned}
&\int\!\!\int_{|s-u|^H>|x-y|}\int\!\!\int_{z_1,z_2\in B(x,\epsilon)} F_{x,y}(z_1,z_2)dz_2dz_1duds \\
\le&\int\!\!\int_{|s-u|^H>|x-y|}\int\!\!\int_{z_1,z_2\in B(x,\epsilon)} |x-y|^2\frac{c_2}{|s-u|^{3H}}\,dz_1dz_2duds \qquad [\text{by (4.8)}] \\
\le&\ c_2\,\epsilon^2|x-y|^{1+\delta}\int\!\!\int_{|s-u|^H>|x-y|}\frac{1}{|s-u|^{(2+\delta)H}}\,duds \le c_2\,\epsilon^2|x-y|^{1+\delta}
\end{aligned} \tag{4.10}$$

Now for the second term on the right hand side of (4.9), due to (4.7) it holds

$$\begin{aligned}
&\int\!\!\int_{|s-u|^H\le|x-y|}\int\!\!\int_{z_1,z_2\in B(x,\epsilon)} F_{x,y}(z_1,z_2)dz_2dz_1duds \\
\le&\int\!\!\int_{|s-u|^H\le|x-y|}\int\!\!\int_{z_1,z_2\in B(x,\epsilon)} |x-y|\frac{c_3}{|s-u|^{(d+1)H}}\,dz_1dz_2duds \qquad [\text{by (4.7)}] \\
\le&\ c_3\,\epsilon^2|x-y|\int\!\!\int_{|s-u|^H\le|x-y|}\frac{1}{|s-u|^{2H}}\,duds \\
\le&\ c_3\,\epsilon^2|x-y|\cdot|x-y|^{(1-2H)/H} = c_3\,\epsilon^2|x-y|^{\frac{1}{H}-1},
\end{aligned} \tag{4.11}$$

where $\frac{1}{H}-1>1+\delta$. Therefore, the proof is complete by choosing $\alpha=1+\delta$ in the statement of the theorem. □

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It follows immediately by Kolmogorov continuity theorem that, when $d = 1$ and $H < 1/2$, there exists a modification of $L^a(t,x)$ that is α-Hölder continuous in x for any $\alpha < \frac{1}{2H} - 1$.

In the following discussion, we seek to establish the Hölder regularity of $L^a(t,x)$ in the time variable. We start with two technical lemmas which correspond to (4.4) and (4.5) in Theorem 4.1

Lemma 4.3. *Assume $dH < 1$. For all $x \in \mathbb{R}^d$, and even $n \in \mathbb{N}$, there exists some positive constant C_2 (depending on a and n) such that for all $x \in \mathbb{R}^d$, all $a < s < t \le T$,*

$$\mathbb{E}\,|L^a(t,x) - L^a(s,x)|^n \le C_2 |t-s|^{(1-dH)(n-1)+1}. \tag{4.12}$$

Proof. As before, we only need to find an upper bound for the following:

$$\begin{aligned}
&\mathbb{E}\,|L^a(t,B(x,\epsilon)) - L^a(s,B(x,\epsilon))|^n \\
&= n!\mathbb{E}\int_{u_n=s}^{t}\int_{u_{n-1}=s}^{u_n}\cdots\int_{u_1=s}^{u_2} \mathbf{1}_{B(x,\epsilon)}(X_{u_1})\cdots\mathbf{1}_{B(x,\epsilon)}(X_{u_n})du_1\cdots du_n \\
&= n!\mathbb{E}\int_{u_n=s}^{t}\int_{u_{n-1}=s}^{u_n}\cdots\int_{u_1=s}^{u_2}\int_{z_1,\cdots z_n\in B(x,\epsilon)} p_{u_1,\cdots,u_n}(z_1,\cdots,z_n)dz_1\cdots dz_n du_1\cdots du_n.
\end{aligned}$$

In the above, $p_{u_1,\ldots,u_n}(z_1,\cdots,z_n)$ is the joint density of $(X_{u_1},...,X_{u_n})$. By Theorem 2.4, clearly we have that for some $c_1 > 0$ (which may change from line to line),

$$p_{u_1,\ldots,u_n}(z_1,\cdots,z_n) \le c_1 \frac{1}{(u_n-u_{n-1})^{dH}}\cdots\frac{1}{(u_2-u_1)^{dH}},$$

Hence the right hand side above is upper bounded by

$$\begin{aligned}
&c_1 n!\epsilon^{nd}\int_{u_n=s}^{t}\int_{u_{n-1}=s}^{u_n}\cdots\int_{u_1=s}^{u_2}\frac{1}{(u_n-u_{n-1})^{dH}}\cdots\frac{1}{(u_2-u_1)^{dH}}\,du_1\cdots du_n \\
\le& c_1 n!\epsilon^{nd}\int_{u_n=s}^{t}\int_{u_{n-1}=s}^{u_n}\cdots\int_{u_2=s}^{u_3}\frac{1}{(u_n-u_{n-1})^{dH}}\cdots\frac{1}{(u_3-u_2)^{dH}}(u_2-s)^{1-dH}\,du_2\cdots du_n \\
\le& c_1 n!\epsilon^{nd}\int_{u_n=s}^{t}\int_{u_{n-1}=s}^{u_n}\cdots\int_{u_2=s}^{u_3}\frac{1}{(u_n-u_{n-1})^{dH}}\cdots\frac{1}{(u_3-u_2)^{dH}}(u_3-s)^{1-dH}\,du_2\cdots du_n \\
\le& c_1 n!\epsilon^{nd}\int_{u_n=s}^{t}\int_{u_{n-1}=s}^{u_n}\cdots\int_{u_2=s}^{u_3}\frac{1}{(u_n-u_{n-1})^{dH}}\cdots\frac{1}{(u_4-u_3)^{dH}}(u_3-s)^{2(1-dH)}\,du_3\cdots du_n \\
\le\cdots\le& c_1 n!\epsilon^{nd}\int_{u_n=s}^{t}(u_n-s)^{(1-dH)(n-1)}du_n \le c_1 n!\epsilon^{nd}(t-s)^{(1-dH)(n-1)+1}
\end{aligned} \tag{4.13}$$

The proof is complete. □

Again on account of Kolmogorov continuity theorem, there exists a modification of $L^a(t,x)$ being β-Hölder continuous in t for any $\beta < 1 - dH$. The next lemma is needed for the uniform Hölder continuity of the local time in the time variable (uniform in x).

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Lemma 4.4. *Assume $d = 1$ and $H \in (1/4, 1/2)$. For any $\delta \in (0, \frac{1}{H} - 2)$ and any even integer $n \in \mathbb{N}$, there exists some $C_3 > 0$ such that*

$$\mathbb{E}\left[L^a([s,t],x) - L^a([s,t],y)\right]^n \le C_3|x-y|^{1+\delta} \cdot |t-s|^{1+(1-H)(n-3)}, \tag{4.14}$$

for all $a \le s < t \le T$ and $x, y \in \mathbb{R}$.

Proof. Denote by

$$G_{x,y;\epsilon}(z) = \mathbf{1}_{B(x,\epsilon)}(z) - \mathbf{1}_{B(y,\epsilon)}(z).$$

We divide the proof into three steps.

Step 1: For any $\epsilon > 0$, the following holds:

$$\begin{aligned}
&\mathbb{E}\left[L^a\left([s,t], B(x,\epsilon)\right) - L^a\left([s,t], B(y,\epsilon)\right)\right]^n \\
=&n!\mathbb{E}\int_{s\le u_1\le\cdots\le u_n\le t} G_{x,y;\epsilon}(X_{u_1})\cdots G_{x,y;\epsilon}(X_{u_n})\,du_1\cdots du_n \\
=&n!\int_{s\le u_1\le\cdots\le u_n\le t} du_1\cdots du_n\,\mathbb{E}\Big[\mathbf{1}_{B(x,\epsilon)}(X_{u_1})\mathbf{1}_{B(x,\epsilon)}(X_{u_2}) - \mathbf{1}_{B(y,\epsilon)}(X_{u_1})\mathbf{1}_{B(y,\epsilon)}(X_{u_2}) \\
&\quad - \mathbf{1}_{B(y,\epsilon)}(X_{u_1})\mathbf{1}_{B(x,\epsilon)}(X_{u_2}) + \mathbf{1}_{B(y,\epsilon)}(X_{u_1})\mathbf{1}_{B(y,\epsilon)}(X_{u_2})\Big] G_{x,y;\epsilon}(X_{u_3})\cdots G_{x,y;\epsilon}(X_{u_n}).
\end{aligned}$$

We first estimate the probability expectation in the above. Note

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{B(x,\epsilon)}(X_{u_1})\mathbf{1}_{B(x,\epsilon)}(X_{u_2}) - \mathbf{1}_{B(y,\epsilon)}(X_{u_1})\mathbf{1}_{B(y,\epsilon)}(X_{u_2}) \\
&\quad - \mathbf{1}_{B(y,\epsilon)}(X_{u_1})\mathbf{1}_{B(x,\epsilon)}(X_{u_2}) + \mathbf{1}_{B(y,\epsilon)}(X_{u_1})\mathbf{1}_{B(y,\epsilon)}(X_{u_2})\Big] G_{x,y;\epsilon}(X_{u_3})\cdots G_{x,y;\epsilon}(X_{u_n}) \\
=&\int_{\mathbb{R}^n}\Big[\mathbf{1}_{B(x,\epsilon)}(z_1)\mathbf{1}_{B(x,\epsilon)}(z_2) - \mathbf{1}_{B(y,\epsilon)}(z_1)\mathbf{1}_{B(y,\epsilon)}(z_2) - \mathbf{1}_{B(y,\epsilon)}(z_1)\mathbf{1}_{B(x,\epsilon)}(z_2) \\
&\quad + \mathbf{1}_{B(y,\epsilon)}(z_1)\mathbf{1}_{B(y,\epsilon)}(z_2)\Big] G_{x,y;\epsilon}(X_{u_3})\cdots G_{x,y;\epsilon}(X_{u_n}) p_{u_1,\cdots,u_n}(z_1,\cdots,z_n)dz_1\cdots dz_n,
\end{aligned} \tag{4.15}$$

where $p_{u_1,\cdots u_n}(z_1,\cdots z_n)$ is the joint density for $(X_{u_1},...,X_{u_n})$. For notation convenience, we denote by $\bar{z} = (z_3,...,z_n)$, and write $p(z_1,z_2,\bar{z}) = p_{u_1,\ldots,u_n}(z_1,...,z_n)$. By doing change of variable three times in (4.15) to make all the terms in the square brackets equal to $\mathbf{1}_{B(x,\epsilon)}(z_1)\mathbf{1}_{B(x,\epsilon)}(z_2)$, we can write the last display of (4.15) as

$$\int_{\mathbb{R}^n} \mathbf{1}_{B(x,\epsilon)}(z_1)\mathbf{1}_{B(x,\epsilon)}(z_2)G_{x,y;\epsilon}(z_3)\cdots G_{x,y;\epsilon}(z_n)F_{x,y}(z_1,z_2,\bar{z})dz_1dz_2d\bar{z},$$

where

$$\begin{aligned}
&F_{x,y}(z_1,z_2,\bar{z}) \\
=&p(z_1,z_2,\bar{z}) - p(z_1,z_2-(x-y),\bar{z}) - p(z_1-(x-y),z_2,\bar{z}) + p(z_1-(x-y),z_2-(x-y),\bar{z}).
\end{aligned}$$

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Hence we have for any even integer n,

$$\begin{aligned}
&\mathbb{E}\left[L^a\left([s,t],B(x,\epsilon)\right)-L^a\left([s,t],B(y,\epsilon)\right)\right]^n\\
\leq\, & n!\int_{s\leq u_1\leq\cdots\leq u_n\leq t} du_1\cdots du_n\\
&\int_{\mathbb{R}^n}\mathbf{1}_{B(x,\epsilon)}(z_1)\mathbf{1}_{B(x,\epsilon)}\left(z_2\right)|G_{x,y;\epsilon}(z_3)\cdots G_{x,y;\epsilon}(z_n)|\cdot|F_{x,y}(z_1,z_2,\bar{z})|dz_1dz_2d\bar{z}. \quad (4.16)
\end{aligned}$$

Setp 2: Applying Taylor's expansion to the first and second order, and by the estimates of derivatives of the joint density $p(z_1,z_2,\bar{z})$ (Theorem 2.4), it can be seen immediately that there exist constants c_i such that the following upper bounds hold.

$$|F_{x,y}(z_1,z_2,\bar{z})|\leq c_1|x-y|\left(\frac{1}{(u_3-u_2)^H}+\frac{1}{(u_2-u_1)^H}\right)\frac{1}{(u_n-u_{n-1})^H}\cdots\frac{1}{(u_2-u_1)^H}; \quad (4.17)$$

and

$$|F_{x,y}(z_1,z_2,\bar{z})|\leq c_2|x-y|^2\left(\frac{1}{(u_3-u_2)^{2H}}+\frac{1}{(u_2-u_1)^{2H}}\right)\frac{1}{(u_n-u_{n-1})^H}\cdots\frac{1}{(u_2-u_1)^H}. \quad (4.18)$$

In order to get (4.18) as above, we have used the elementary inequality

$$(u_3-u_2)(u_2-u_1)\geq\min\{(u_3-u_2)^2,(u_2-u_1)^2\}$$

for the cross term.

Setp 3: We split the right hand-side of (4.16) into J_1+J_2 where

$$\begin{aligned}
J_1=\, & n!\int_{\{s\leq u_1<\cdots<u_n\leq t\}\cap\{u_2-u_1\geq u_3-u_2\}} du_1\cdots du_n\\
&\int_{\mathbb{R}^n}\mathbf{1}_{B(x,\epsilon)}(z_1)\mathbf{1}_{B(x,\epsilon)}\left(z_2\right)|G_{x,y;\epsilon}(z_3)\cdots G_{x,y;\epsilon}(z_n)||F_{x,y}(z_1,z_2,\bar{z})|dz_1dz_2d\bar{z}, \quad (4.19)
\end{aligned}$$

and J_2 is defined similarly for $\{u_2-u_1<u_3-u_2\}$.

In the following, we only show how to bound J_1, as J_2 can be treated in a similar manner. Below, the positive constants c_i $(1\leq i\leq 5)$ may change from line to line. From our bound of $F_{x,y}(z_1,z_2,\bar{z})$, we clearly have

$$\begin{aligned}
&J_1\\
\leq\, & c_1\epsilon^n\int_{\left\{\substack{s\leq u_1<\cdots<u_n\leq t\\ u_2-u_1\geq u_3-u_2}\right\}}\left(\frac{|x-y|}{(u_3-u_2)^H}\wedge\frac{|x-y|^2}{(u_3-u_2)^{2H}}\right)\frac{1}{(u_n-u_{n-1})^H}\cdots\frac{1}{(u_2-u_1)^H}du_1\cdots du_n\\
\leq\, & c_1\epsilon^n\int_{\{s\leq u_1<\cdots<u_n\leq t\}}\left(\frac{|x-y|}{(u_3-u_2)^H}\wedge\frac{|x-y|^2}{(u_3-u_2)^{2H}}\right)\frac{1}{(u_n-u_{n-1})^H}\cdots\frac{1}{(u_2-u_1)^H}du_1\cdots du_n\\
\leq\, & c_1\epsilon^n(t-s)^{1-H}\int_{\{s\leq u_3<\cdots<u_n\leq t\}}du_3\cdots du_n\,\frac{1}{(u_n-u_{n-1})^H}\cdots\frac{1}{(u_4-u_3)^H}\\
&\int_{u_2=s}^{u_3}\left(\frac{|x-y|}{(u_3-u_2)^H}\wedge\frac{|x-y|^2}{(u_3-u_2)^{2H}}\right)\frac{1}{|u_3-u_2|^H}du_2. \quad (4.20)
\end{aligned}$$

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For the integral with respect to u_2 in the last display of (4.20), we have for any $\delta < \frac{1}{H} - 2$,

$$\begin{aligned}
&\int_{u_2=s}^{u_3} \left(\frac{|x-y|}{(u_3-u_2)^H} \wedge \frac{|x-y|^2}{(u_3-u_2)^{2H}} \right) \frac{1}{|u_3-u_2|^H} du_2 \\
&= \int_{(u_3-u_2)^H < |x-y|} \left(\frac{|x-y|}{(u_3-u_2)^H} \wedge \frac{|x-y|^2}{(u_3-u_2)^{2H}} \right) \frac{1}{|u_3-u_2|^H} du_2 \\
&\qquad + \int_{(u_3-u_2)^H \geq |x-y|} \left(\frac{|x-y|}{(u_3-u_2)^H} \wedge \frac{|x-y|^2}{(u_3-u_2)^{2H}} \right) \frac{1}{|u_3-u_2|^H} du_2 \\
&\leq \int_{(u_3-u_2)^H < |x-y|} \frac{|x-y|}{(u_3-u_2)^{2H}} du_2 + \int_{(u_3-u_2)^H \geq |x-y|} \frac{|x-y|^2}{(u_3-u_2)^{3H}} du_2 \\
&\leq |x-y| \cdot |x-y|^{\frac{1}{H}[1-2H]} + \int_{(u_3-u_2)^H \geq |x-y|} \frac{|x-y|^{(1+\delta)}}{(u_3-u_2)^{(2+\delta)H}} du_2 \\
&\leq \left(|x-y|^{\frac{1}{H}-1} + |x-y|^{1+\delta} \right) \leq |x-y|^{1+\delta}. \qquad (4.21)
\end{aligned}$$

To complete the computation for (4.20), by following the same line of computation as the proof of Lemma 4.3, we obtain

$$\int_{\{s \leq u_3 < \cdots u_n \leq t\}} \frac{1}{(u_n - u_{n-1})^{dH}} \cdots \frac{1}{(u_4 - u_3)^{dH}} du_3 \cdots du_n \leq c_2 (t-s)^{(1-dH)(n-3)+1}. \quad (4.22)$$

Combining (4.21) and (4.22), we have shown that for any $\delta < \frac{1}{H} - 2$, there exists some constant $c_3 > 0$

$$J_1 \leq c_3 \epsilon^n |x-y|^{1+\delta} (t-s)^{(1-dH)(n-3)+1}. \quad (4.23)$$

Following same line of argument as above, it holds for some $c_4 > 0$ that

$$J_2 \leq c_4 \epsilon^n |x-y|^{1+\delta} (t-s)^{(1-dH)(n-3)+1}. \quad (4.24)$$

Therefore we have established the following desired upper bound for (4.16):

$$\mathbb{E}\left[L^a\left([s,t], B(x,\epsilon)\right) - L^a\left([s,t], B(y,\epsilon)\right) \right]^n \leq c_5 \epsilon^n |x-y|^{1+\delta} (t-s)^{(1-H)(n-3)+1}. \quad (4.25)$$

The proof is thus complete. □

Remark 4.5. Clearly, Lemma 4.2 is a special case of Lemma 4.4. The reason that we keep Lemma 4.2 is because the proof of this simpler result provides a better idea of the more complex proof of Lemma 4.4. We also remark that by taking $s = a$ in Lemma 4.4, we have for any $\delta \in (0, \frac{1}{H} - 2)$ and even $n \in \mathbb{N}$, there exists a positive constant C_4 depending on T and a such that

$$\mathbb{E}\left[L^a(t,x) - L^a(t,y) \right]^n \leq C_4 |x-y|^{1+\delta}. \quad (4.26)$$

Now that all the three conditions in Theorem 4.1 have been established in the three lemmas above, the following theorem results from Theorem 4.1 naturally.

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Theorem 4.6. *Assume $d = 1$ and $H < 1/2$. For every $\beta < 1 - H$, there exists a version of the local time $L^a(t, x)$ jointly continuous in $(t, x) \in [a, T] \times \mathbb{R}$. Moreover, there exist random variables η and Δ which are almost surely positive and finite such that*

$$|L^a(t + h, x) - L^a(t, x)| \leq \Delta |h|^\beta, \text{ for all } x, t, h \text{ satisfying } t, t + h \in [a, T], \text{ and } |h| < \eta.$$

Proof. Fix $R > 0$. Apply Theorem 4.1 to $Y(t, x) = L([a, a + t], x)$ and by Lemma 4.3, Lemma 4.4 and (4.26), we obtain for each $m \in \mathbb{N}$ a joint continuous version $Y_m(t, x)$ of $Y(t, x)$ on $[0, T] \times I_{mR}$ which is β-Hölder continuous in time uniformly in $x \in I_{mR}$, for all $\beta < 1 - H$.

Observe that for $m' > m$, $Y_{m'}(t, x)$ is a continuous extension of $Y_m(t, x)$ from $[0, T] \times I_{mR}$ to $[0, T] \times I_{m'R}$. Therefore, we can define a continuous version of Y on $[0, T] \times \mathbb{R}$ by

$$Y(t, x) = Y_m(t, x), \quad \text{for } (t, x) \in [0, T] \times I_{mR}.$$

The proof is complete by observing that $Y(t, x) = L([a, a + t], x)$ has compact support in $x \in \mathbb{R}$ (for each fixed $\omega \in \Omega$). □

Recall the definition of pathwise Hölder exponent of $L^a(t, x)$ in (1.2). We are now ready to state the main result of this section.

Theorem 4.7. *Assume $d = 1$ and $1/4 < H < 1/2$. Let X be the solution to equation (2.3). The pathwise Hölder exponent of $L^a(t, x)$ is given by*

$$\alpha(t) = 1 - H, \quad \text{a.s.} \quad \textit{for all } t \in [a, T].$$

Proof. It follows immediately from Theorem 4.6 that $\alpha_L \geq 1 - H$. We show $\alpha_L \leq 1 - H$ in what follows. Fix any $t > 0$. Since the local time vanishes outside the range of X, it holds

$$\begin{aligned} \delta &= \int_{\mathbb{R}^d} (L(t + \delta, x) - L(t, x))\, dx \\ &\leq \sup_{x \in \mathbb{R}^d} (L(t + \delta, x) - L(t, x)) \sup_{s,u \in [t,t+\delta]} |X_u - X_s| \\ &\leq 2 \sup_{x \in \mathbb{R}^d} (L(t + \delta, x) - L(t, x)) \sup_{s \in [t,t+\delta]} |X_t - X_s|. \end{aligned} \tag{4.27}$$

Now in order to show $\alpha_L \leq 1 - H$, it suffices to observe for any $\alpha > 0$, by (4.27), it holds that

$$\frac{\sup_{x \in \mathbb{R}^d} (L(t + \delta, x) - L(t, x))}{\delta^\alpha} \geq \frac{\delta^{1-\alpha}}{2 \sup_{s \in [t,t+\delta]} |X_t - X_s|}.$$

On the other hand, by the construction of rough integrals, we have, for any $\gamma < H$, a finite random variable G_γ such that for all $s, t \in [0, T]$,

$$|X_t - X_s| \leq G_\gamma |t - s|^\gamma.$$

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Hence for all $\gamma < H$,

$$\frac{\sup_{x\in\mathbb{R}^d}\left(L(t+\delta,x)-L(t,x)\right)}{\delta^{\alpha}} \geq \frac{\delta^{1-\alpha}}{G_\gamma \delta^{\gamma}} = \frac{\delta^{1-\alpha-\gamma}}{G_\gamma}.$$

For any $\alpha > 1-H$, we can always find a $\gamma < H$ such that the right hand side above tends to infinity as $\delta \downarrow 0$, which implies $\alpha(t) \leq 1-H$. The proof is complete. □

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