

Bridge representation and modal-path approximation

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Abstract

The article shows a bridge representation for the joint density of a system of stochastic processes consisting of a Brownian motion with drift coupled with a correlated fractional Brownian motion with drift. As a result, a small time approximation of the joint density is readily obtained by substituting the conditional expectation under the bridge measure by a single path: the modal-path from the initial point to the terminal point.

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1. Introduction

Stochastic modeling with long range dependence processes has nowadays become ubiquitous. Applications of such processes range from models for traffic, telecommunication, geophysics to finance. In this regard, among other continuous time processes, fractional Brownian motion is probably the most frequently used base model for long range dependence due to its Gaussianity and close relationship with the classical Brownian motion.

In the field of quantitative finance, stochastic differential equations driven by fractional Brownian motions with different Hurst parameters are considered in option pricing theory in

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order to capture certain stylized facts observed in the market. For the model to be free of arbitrage opportunity, the underlying asset itself has to be driven by a Brownian motion, which is discussed in Cheridito [4] and Rogers [13] for instance. On the other hand, there is empirical evidence showing that the volatility of logarithmic returns of the underlying asset exhibits long range dependence (see for example Bollerslev and Mikkelsen [3] and Granger and Hyung [7] for S&P500 index and Tschernig [15] for foreign exchange rate). Thus, the price dynamic of the underlying asset is naturally modeled by a stochastic system driven by a mixture of Brownian and fractional Brownian motions. However, as the probability density of such a model is concerned, to our knowledge, little is known in determining tractable analytic expressions or asymptotic expansions for the joint density, which is in part due to the lack of analytic tools from PDE theory.

In this paper, we consider the stochastic system consisting of a Brownian motion with drift coupled with a correlated fractional Brownian motion with Hurst parameter $H \in (0, 1)$ with drift. Modulo a Gaussian prefactor, we aim to derive a bridge representation for the joint density (see Theorem 2.2), which accordingly yields a small time asymptotic of the heat kernel type to the lowest order as shown in Theorem 3.3. The technique used to derive these results is a natural extension of the one dimensional case considered in Rogers [12] and Wang and Gatheral [16].

To obtain the bridge representation for the joint density, we follow the line of thought as in [12] and [16] which we briefly summarize in the following. A general nondegenerate diffusion is transformed into a Brownian motion with drift by applying the Lamperti transformation. Girsanov's theorem is then applied to define a new equivalent measure so that the resulting process is driftless under the new measure. Finally, modulo a Gaussian density, the bridge representation for the transition density is obtained by conditioning on the terminal point of Brownian motion (see for example Theorem 2 in [16]). With this bridge representation, a small time asymptotic expansion of the transition density is readily obtained by expanding the Brownian bridge expectation around a deterministic path, the most-likely-path (see [16] for more details). We remark that the trick of applying Lamperti transformation to unitize the diffusion coefficient in one dimensional case is generally not applicable in higher dimensions due to geometric obstructions.

After addressing some technical conditions, we are allowed to apply Girsanov's theorem to de-drift the coupled Brownian and fractional Brownian motions under a new measure. However, the integrands (see (A.11) and (A.12)) required in defining the Radon–Nikodym derivative for the new measure are more involved due to the appearance of the defining kernel of the fractional Brownian motion.

Modal-path approximation of the joint density is thus obtained by evaluating the bridge representation along a single deterministic path, the modal-path connecting the initial point and the terminal point. The rationale is as follows. In small time, the densities of the corresponding bridges are peaked around their modes (hence the name modal-path). On the other hand, under the new measure the two processes considered are jointly Gaussian, and hence, the modes are simply given by their expectations. For Brownian bridge ($H = \frac{1}{2}$), the modal-path is the straight line connecting the initial and terminal points of the bridge. While for fractional Brownian bridge ($H \neq \frac{1}{2}$), we use the form of Volterra bridge as in Baudoin and Coutin [1] to determine the modal-path, which is not a straight line. As the Hurst parameter H approaches one half, the modal-path gets closer to the straight line connecting the initial and terminal points. Additionally, as H approaches zero, the modal-path travels very quickly to the midpoint, stays around the midpoint till almost to the end, then travels very quickly to the terminal point. This in a sense creates a jump-like behavior (see Remark 3.1 for more details). In this article, for technical

reasons, the proof for modal-path approximation in small time works for $H < \frac{3}{4}$ (see Remark 3.2 for more details).

It is worth mentioning that recent papers by Baudoin and Ouyang [2], Inahama [10,9] and Yamada [17] study a heat kernel type expansion for the joint density of solution to SDEs driven by fractional Brownian motions in small time. The driving fractional Brownian motions in these papers are all assumed to have the same Hurst parameter $H > 1/2$, except for Inahama [10] where the case $H \in (1/3, 1/2]$ is studied. Thus, it is conceivable that in logarithmic scale the lowest order in the expansion of the probability density is of t^{2H} as $t \rightarrow 0^+$.

On the other hand, as closed-form expression is concerned, Zeng, Chen, and Yang [18] derived the density of a one dimensional Ornstein–Uhlenbeck process driven by fractional Brownian motion in closed form representation by solving a Fokker–Planck type of equation satisfied by the density function. The density in this case is unsurprisingly Gaussian (see (3.6) in [18]).

The rest of the paper is organized as follows. The main result of bridge representation is proved in Section 2. Section 3 gives the modal-path approximation of the joint density and an error analysis of the approximation. Finally, the paper concludes with specific examples of the modal-path approximations. For reader's convenience, we review basics on fractional Brownian motion, fractional differentiation, and fractional integration in the Appendix.

2. Model specification and change of probability measures

Throughout the text, $B = \{B_t, t \in [0, \infty)\}$ and $W = \{W_t, t \in [0, \infty)\}$ denote two independent standard Brownian motions defined on the complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \in [0, \infty)})$ satisfying the usual conditions. $B^H = \{B_t^H, t \in [0, \infty)\}$ is a fractional Brownian motion with Hurst parameter $H \in (0, 1)$ generated by B . In this paper we understand B^H as the Volterra–Gaussian process given by

$$B_t^H = \int_0^t K_H(t, s) dB_s,$$

where K_H is given by (A.3) in the Appendix.

Let $T > 0$ be a fixed number. We shall make use of the following notations. Let $C([0, T])$ denote the space of continuous functions defined on $[0, T]$, and $C^\lambda([0, T])$ denote the space of Hölder continuous functions on $[0, T]$ of order $\lambda \in (0, 1)$. The supremum norm and C^λ norm are defined respectively as

$$\|f\|_{T, \infty} = \sup_{0 \leq t \leq T} |f(t)| \quad \text{if } f \in C([0, T]),$$

and

$$\|f\|_{T, \lambda} = \|f\|_{T, \infty} + \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{|t - s|^\lambda}, \quad \text{if } f \in C^\lambda([0, T]).$$

2.1. The model

Consider the two dimensional stochastic system

$$\begin{cases} X_t = x_0 + \rho B_t + \sqrt{1 - \rho^2} W_t + \int_0^t h_1(s, X_s, Y_s) ds, \\ Y_t = y_0 + B_t^H + \int_0^t h_2(s, X_s, Y_s) ds, \end{cases} \quad (2.1)$$

where $(X_0, Y_0) = (x_0, y_0)$ is the initial point, $\rho \in (-1, 1)$, and the two functions $h_1, h_2 : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are deterministic. By construction, X_t is a Brownian motion with drift h_1 and Y_t is a fractional Brownian motion of Hurst parameter H with drift h_2 .

The following assumptions on h_1 and h_2 guarantee the existence and uniqueness of the solution to (2.1).

Assumption 1.

- (a) The functions h_1 and h_2 are Lipschitz in x, y uniformly for t . That is, there exists a constant $L > 0$ such that

$$|h_i(t, x_1, y_1) - h_i(t, x_2, y_2)| \leq L(|x_1 - x_2| + |y_1 - y_2|), \quad i = 1, 2, \quad (2.2)$$

for all $t \in [0, T]$ and $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$.

- (b) (i) If $H > \frac{1}{2}$, there exist two constants $L > 0$ and $\gamma \in (H - \frac{1}{2}, \frac{1}{2})$ such that the function h_1 satisfies

$$|h_1(t, 0, 0)| \leq L, \quad \forall t \in [0, T],$$

and the function h_2 satisfies

$$|h_2(t, x, y) - h_2(s, x, y)| \leq L|t - s|^\gamma, \quad \forall s, t \in [0, T], \quad \forall (x, y) \in \mathbb{R}^2, \quad (2.3)$$

i.e., h_2 is Hölder continuous in t of order γ uniformly for x and y .

- (ii) If $H \leq \frac{1}{2}$, there exists a constant $L > 0$ such that

$$|h_i(t, 0, 0)| \leq L, \quad \forall s, t \in [0, T], \quad i = 1, 2.$$

Remark 2.1. The conditions in Assumption 1 imply that the functions h_1 and h_2 satisfy the following linear growth condition: there exists a constant $K > 0$ such that

$$|h_i(t, x, y)| \leq K(1 + |x| + |y|), \quad i = 1, 2, \quad (2.4)$$

for all $t \in [0, T]$ and $(x, y) \in \mathbb{R}^2$.

Since we consider small time asymptotic in this work, we always assume that $T \leq 1$. We use the conventions of

$$\|\cdot\|_\infty = \|\cdot\|_{1,\infty} \quad \text{and} \quad \|\cdot\|_\lambda = \|\cdot\|_{1,\lambda}.$$

The following theorem establishes the existence and uniqueness of the solution to (2.1) and the regularity of the solution trajectories under Assumption 1.

Theorem 2.1. *Let the conditions in Assumption 1 be satisfied. Then, there exists a positive constant δ such that the system (2.1) has a unique solution (X, Y) when $T < \delta$. Moreover, the trajectories of X and Y satisfy $X \in C^{\frac{1}{2}-\epsilon}([0, T])$ and $Y \in C^{H-\epsilon}([0, T])$ almost surely for every $0 < \epsilon < \min\{\frac{1}{2}, H\}$.*

Proof. We use the contraction mapping theorem to prove the existence and uniqueness of the solution. Let $(x^i, y^i), i = 1, 2$, be two stochastic processes taking values in $C([0, T])$. Define

$$\begin{cases} X_t^i = x_0 + \rho B_t + \sqrt{1 - \rho^2} W_t + \int_0^t h_1(s, x_s^i, y_s^i) ds, \\ Y_t^i = y_0 + B_t^H + \int_0^t h_2(s, x_s^i, y_s^i) ds, \end{cases}$$

for each $i = 1, 2$. [Assumption 1](#) implies that

$$\|X^1 - X^2\|_{T,\infty} + \|Y^1 - Y^2\|_{T,\infty} \leq 2LT (\|x^1 - x^2\|_{T,\infty} + \|y^1 - y^2\|_{T,\infty}).$$

We choose $\delta = \frac{1}{2L}$, and hence, by the contraction mapping theorem we obtain the existence and uniqueness of the solution in $C([0, T])$ for any $T < \delta$.

From [\(2.1\)](#) and [\(2.4\)](#), we get

$$|X_t| \leq |x_0| + \rho \|B\|_\infty + \sqrt{1 - \rho^2} \|W\|_\infty + K \int_0^t (1 + |X_s| + |Y_s|) ds,$$

and

$$|Y_t| \leq |y_0| + \|B^H\|_\infty + K \int_0^t (1 + |X_s| + |Y_s|) ds.$$

Thus, Gronwall's inequality implies that

$$\begin{aligned} & \|X\|_{T,\infty} + \|Y\|_{T,\infty} \\ & \leq \left(|x_0| + |y_0| + \rho \|B\|_\infty + \sqrt{1 - \rho^2} \|W\|_\infty + \|B^H\|_\infty + 2KT \right) e^{2KT}. \end{aligned} \quad (2.5)$$

From [\(2.1\)](#), [\(2.4\)](#) and [\(2.5\)](#), we observe that

$$\begin{aligned} |X_t - X_s| & \leq \rho |B_t - B_s| + \sqrt{1 - \rho^2} |W_t - W_s| + \left| \int_s^t h_1(r, X_r, Y_r) dr \right| \\ & \leq \rho |B_t - B_s| + \sqrt{1 - \rho^2} |W_t - W_s| + C (1 + \|B\|_\infty + \|W\|_\infty + \|B^H\|_\infty) |t - s|, \end{aligned}$$

where C denotes a generic constant depending on $|x_0|$, $|y_0|$, ρ , L and K . Note that C may vary from line to line and in the proofs of some later lemmas will also depend on H .

Then, for any $\epsilon \in (0, \frac{1}{2})$, the following estimate can be obtained

$$\|X\|_{T, \frac{1}{2}-\epsilon} \leq \rho \|B\|_{\frac{1}{2}-\epsilon} + \sqrt{1 - \rho^2} \|W\|_{\frac{1}{2}-\epsilon} + C (1 + \|B\|_\infty + \|W\|_\infty + \|B^H\|_\infty). \quad (2.6)$$

Similarly, for any $\epsilon \in (0, H)$, the following estimate holds

$$\|Y\|_{T, H-\epsilon} \leq \|B^H\|_{H-\epsilon} + C (1 + \|B\|_\infty + \|W\|_\infty + \|B^H\|_\infty). \quad (2.7)$$

The proof is completed. \square

2.2. Change of measures

Next, we discuss a change of measures, where under the new measure X , Y become standard Brownian and fractional Brownian motions, respectively. Heuristically, the new measure $\tilde{\mathbb{P}}$ would be defined by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp \left\{ - \int_0^T \tilde{h}_1(t) dW_t - \frac{1}{2} \int_0^T \tilde{h}_1^2(t) dt - \int_0^T \tilde{h}_2(t) dB_t - \frac{1}{2} \int_0^T \tilde{h}_2^2(t) dt \right\}, \quad (2.8)$$

where \tilde{h}_1 and \tilde{h}_2 are defined in [\(A.11\)–\(A.12\)](#). The well-definedness of \tilde{h}_1 and \tilde{h}_2 is established in [Lemma A.2](#) of the [Appendix](#).

The following lemma asserts that the two processes \tilde{h}_1 and \tilde{h}_2 satisfy Novikov's condition.

Lemma 2.1. *There exists a small $t_0 \leq T$ such that the adapted processes \tilde{h}_1 and \tilde{h}_2 satisfy Novikov's condition in $[0, t_0]$. That is,*

$$\mathbb{E} \left[\exp \left\{ \frac{1}{2} \int_0^{t_0} |\tilde{h}_1(t)|^2 dt + \frac{1}{2} \int_0^{t_0} |\tilde{h}_2(t)|^2 dt \right\} \right] < \infty. \quad (2.9)$$

Proof. Case $H \leq \frac{1}{2}$: From (A.9), (2.5) and the linear growth property (2.4) of h_2 , we get

$$\begin{aligned} |\tilde{h}_2(t)| &= c_H^{-1} t^{H-\frac{1}{2}} \left| \int_0^t (t-s)^{-\frac{1}{2}-H} s^{\frac{1}{2}-H} h_2(s, X_s, Y_s) ds \right| \\ &\leq C (1 + \|B\|_\infty + \|W\|_\infty + \|B^H\|_\infty) t^{H-\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}-H} s^{\frac{1}{2}-H} ds \\ &= CB \left(\frac{1}{2} - H, \frac{3}{2} - H \right) (1 + \|B\|_\infty + \|W\|_\infty + \|B^H\|_\infty) t^{\frac{1}{2}-H} \\ &\leq C (1 + \|B\|_\infty + \|W\|_\infty + \|B^H\|_\infty), \end{aligned} \quad (2.10)$$

where $B(\cdot, \cdot)$ is the Beta function.

From (A.12), (2.10), the linear growth condition (2.4) on h_1 and (2.5), we obtain

$$|\tilde{h}_1(t)| \leq C (1 + \|B\|_\infty + \|W\|_\infty + \|B^H\|_\infty).$$

Thus, we obtain

$$\int_0^t (|\tilde{h}_1(s)|^2 + |\tilde{h}_2(s)|^2) ds \leq C (1 + \|B\|_\infty^2 + \|W\|_\infty^2 + \|B^H\|_\infty^2) t. \quad (2.11)$$

Therefore, the estimate (2.11) and Fernique's theorem (see [6]) imply (2.9) for a small enough t_0 .

Case $H > \frac{1}{2}$: From (A.7), we can see that

$$\tilde{h}_2(t) = \frac{c_H^{-1}}{\Gamma(\frac{3}{2} - H)} (a(t) + b(t)), \quad (2.12)$$

where

$$a(t) = t^{\frac{1}{2}-H} h_2(t, X_t, Y_t) + \left(H - \frac{1}{2} \right) \int_0^t \frac{h_2(t, X_t, Y_t) - h_2(s, X_s, Y_s)}{(t-s)^{H+\frac{1}{2}}} ds,$$

and

$$b(t) = \left(H - \frac{1}{2} \right) t^{H-\frac{1}{2}} \int_0^t \frac{(t^{\frac{1}{2}-H} - s^{\frac{1}{2}-H}) h_2(s, X_s, Y_s)}{(t-s)^{H+\frac{1}{2}}} ds.$$

For any small positive number $\epsilon < 1 - H < \frac{1}{2}$, from the conditions (2.2), (2.3) and (2.4) on h_2 , (2.5), (2.6) and (2.7) it follows that

$$\begin{aligned} |a(t)| &\leq C (1 + \|B\|_\infty + \|W\|_\infty + \|B^H\|_\infty) t^{\frac{1}{2}-H} + C \int_0^t (t-s)^{\gamma-H-\frac{1}{2}} ds \\ &\quad + C \left(1 + \|B\|_\infty + \|W\|_\infty + \|B^H\|_\infty + \|B\|_{\frac{1}{2}-\epsilon} + \|W\|_{\frac{1}{2}-\epsilon} + \|B^H\|_{H-\epsilon} \right) \\ &\quad \times \int_0^t (t-s)^{-H-\epsilon} ds \\ &\leq C (1 + \|B\|_\infty + \|W\|_\infty + \|B^H\|_\infty) t^{\frac{1}{2}-H} + Ct^{\gamma-H+\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
& + C \left(1 + \|B\|_\infty + \|W\|_\infty + \|B^H\|_\infty + \|B\|_{\frac{1}{2}-\epsilon} \right. \\
& \left. + \|W\|_{\frac{1}{2}-\epsilon} + \|B^H\|_{H-\epsilon} \right) t^{1-H-\epsilon}.
\end{aligned} \tag{2.13}$$

For the term $b(t)$, we shall apply the fact that the integral $\int_0^1 \frac{u^{\frac{1}{2}-H}-1}{(1-u)^{H+\frac{1}{2}}} du = \alpha_H$ is a finite number depending only on H . Using the linear growth condition (2.4) on h_2 , (2.5) and the change of variables $u = \frac{s}{t}$, we obtain

$$\begin{aligned}
|b(t)| & \leq C \left(1 + \|B\|_\infty + \|W\|_\infty + \|B^H\|_\infty \right) t^{\frac{1}{2}-H} \int_0^1 \frac{u^{\frac{1}{2}-H}-1}{(1-u)^{H+\frac{1}{2}}} du \\
& = C\alpha_H \left(1 + \|B\|_\infty + \|W\|_\infty + \|B^H\|_\infty \right) t^{\frac{1}{2}-H}.
\end{aligned} \tag{2.14}$$

From (2.12)–(2.14), we can show that

$$\begin{aligned}
|\tilde{h}_2(t)| & \leq C \left(1 + \|B\|_\infty + \|W\|_\infty + \|B^H\|_\infty \right) t^{\frac{1}{2}-H} \\
& + C \left(1 + \|B\|_\infty + \|W\|_\infty + \|B^H\|_\infty + \|B\|_{\frac{1}{2}-\epsilon} + \|W\|_{\frac{1}{2}-\epsilon} + \|B^H\|_{H-\epsilon} \right).
\end{aligned} \tag{2.15}$$

From (A.12), (2.15), the linear growth condition (2.4) on h_1 and (2.5), we obtain

$$\begin{aligned}
|\tilde{h}_1(t)| & \leq C \left(1 + \|B\|_\infty + \|W\|_\infty + \|B^H\|_\infty \right) t^{\frac{1}{2}-H} \\
& + C \left(1 + \|B\|_\infty + \|W\|_\infty + \|B^H\|_\infty + \|B\|_{\frac{1}{2}-\epsilon} + \|W\|_{\frac{1}{2}-\epsilon} + \|B^H\|_{H-\epsilon} \right).
\end{aligned} \tag{2.16}$$

Then, it is easy to see that

$$\begin{aligned}
& \int_0^t (|\tilde{h}_1(s)|^2 + |\tilde{h}_2(s)|^2) ds \\
& \leq C \left(1 + \|B\|_\infty^2 + \|W\|_\infty^2 + \|B^H\|_\infty^2 \right) t^{2-2H} \\
& + C \left(1 + \|B\|_\infty^2 + \|W\|_\infty^2 + \|B^H\|_\infty^2 + \|B\|_{\frac{1}{2}-\epsilon}^2 + \|W\|_{\frac{1}{2}-\epsilon}^2 + \|B^H\|_{H-\epsilon}^2 \right) t.
\end{aligned} \tag{2.17}$$

Therefore, from the estimate (2.17) and Fernique's theorem (see [6]), we conclude that there exists a small enough t_0 such that (2.9) holds. \square

We let $T \leq t_0$ and restrict ourselves to the small interval $[0, T]$ hereafter for convenience. By Lemma 2.1, the equivalent probability measure $\tilde{\mathbb{P}}$ via the Radon–Nikodym derivative by (2.8) is well-defined.

Lemma 2.2. *Under the probability measure $\tilde{\mathbb{P}}$, the processes $\tilde{W} = \{\tilde{W}_t = W_t + \int_0^t \tilde{h}_1(s) ds, t \in [0, T]\}$ and $\tilde{B} = \{\tilde{B}_t = B_t + \int_0^t \tilde{h}_2(s) ds, t \in [0, T]\}$ become two independent Brownian motions, and the process $\tilde{B}^H = \{\tilde{B}_t^H = B_t^H + \int_0^t K_H(t, s) \tilde{h}_2(s) ds, t \in [0, T]\}$ becomes a fractional Brownian motion. Henceforth, under the $\tilde{\mathbb{P}}$ -measure W and B become two Brownian motions with drift respectively.*

2.3. Bridge representation of the joint density under $\tilde{\mathbb{P}}$

The purpose of this section is to show the bridge representation (2.21) of the joint density of (X_T, Y_T) .

The following lemma is required in determining the covariance matrix of (X_t, Y_t) under $\tilde{\mathbb{P}}$ defined in (2.8).

Lemma 2.3. *The integral $\int_0^t K_H(t, u)du$ is given explicitly as*

$$\int_0^t K_H(t, u)du = \kappa_H t^{H+\frac{1}{2}}, \quad (2.18)$$

where the constant κ_H is given by

$$\kappa_H = c_H \frac{B\left(\frac{3}{2} - H, H + \frac{1}{2}\right)}{H + \frac{1}{2}}, \quad (2.19)$$

where c_H is the constant appearing in (A.3).

Proof. Recall the function $K_H(t, s)$ in (A.3). By the change of variables $s = tr$, one can calculate

$$\begin{aligned} \int_0^t \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} ds &= t^{H+\frac{1}{2}} \int_0^1 r^{\frac{1}{2}-H} (1-r)^{H-\frac{1}{2}} dr \\ &= t^{H+\frac{1}{2}} B\left(\frac{3}{2} - H, H + \frac{1}{2}\right). \end{aligned}$$

By changing the order of the integrals and applying the change of variables $s = ur$, we have

$$\begin{aligned} \int_0^t s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du ds &= \int_0^t \int_0^u s^{\frac{1}{2}-H} u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} ds du \\ &= \int_0^t u^{H-\frac{1}{2}} \int_0^1 r^{\frac{1}{2}-H} (1-r)^{H+\frac{1}{2}} dr du = \frac{t^{H+\frac{1}{2}} B\left(\frac{3}{2} - H, H + \frac{1}{2}\right)}{H + \frac{1}{2}}. \end{aligned}$$

Thus, by integrating $K_H(t, s)$ with respect to s over the interval $[0, t]$, we obtain (2.18). \square

Remark 2.2. Note that $\int_0^t K_H(t, u)du = \mathbb{E}[B_t B_t^H] \leq \sqrt{\mathbb{E}[|B_t|^2]} \sqrt{\mathbb{E}[|B_t^H|^2]} = t^{H+\frac{1}{2}}$. Together with (2.18), it implies that $\kappa_H \leq 1$ and the equality holds when $H = \frac{1}{2}$. Hence, we have

$$0 < 1 - \rho^2 \kappa_H^2 \leq 1, \quad (2.20)$$

since $\rho \in (-1, 1)$.

Next theorem is one of our main results in this paper which gives a bridge representation for the joint density of (X_T, Y_T) under the probability measure \mathbb{P} .

Theorem 2.2 (Bridge Representation of the Joint Density). *Let X and Y respectively be Brownian and fractional Brownian motions with drift and initial condition (x_0, y_0) satisfying (2.1). The law of (X_T, Y_T) under the probability \mathbb{P} is absolutely continuous with respect to the Lebesgue measure and its joint density $p_T(x, y|x_0, y_0)$ has the following bridge representation*

$$\begin{aligned} p_T(x, y|x_0, y_0) \\ = \phi(x - x_0, y - y_0) \tilde{\mathbb{E}}_{x,y} \left[e^{\int_0^T \tilde{h}_1(t) d\tilde{W}_t - \frac{1}{2} \int_0^T \tilde{h}_1^2(t) dt + \int_0^T \tilde{h}_2(t) d\tilde{B}_t - \frac{1}{2} \int_0^T \tilde{h}_2^2(t) dt} \right], \end{aligned} \quad (2.21)$$

where $\tilde{\mathbb{E}}_{x,y}[\cdot] = \tilde{\mathbb{E}}[\cdot | X_T = x, Y_T = y]$ denotes the conditional expectation with respect to the probability measure $\tilde{\mathbb{P}}$ given the terminal point $(X_T, Y_T) = (x, y)$, ϕ is the bivariate Gaussian density

$$\phi(\xi, \eta) = \frac{1}{2\pi T^{H+\frac{1}{2}} \sqrt{1 - \rho^2 \kappa_H^2}} \times \exp \left\{ -\frac{1}{2(1 - \rho^2 \kappa_H^2)} \left[\left(\frac{\xi}{\sqrt{T}} \right)^2 - 2\rho \kappa_H \left(\frac{\xi}{\sqrt{T}} \right) \left(\frac{\eta}{T^H} \right) + \left(\frac{\eta}{T^H} \right)^2 \right] \right\}, \quad (2.22)$$

and the processes \tilde{h}_1 and \tilde{h}_2 are determined by (A.11) and (A.12).

To start with the bridge representation, we first determine the dynamics of (X_t, Y_t) under the $\tilde{\mathbb{P}}$ -measure. Recall from (A.11) and (A.12) that we can rewrite the processes X and Y as

$$X_t = x_0 + \rho \tilde{B}_t + \sqrt{1 - \rho^2} \tilde{W}_t, \quad (2.23)$$

$$Y_t = y_0 + \tilde{B}_t^H. \quad (2.24)$$

Thus, (X, Y) is a Gaussian process under $\tilde{\mathbb{P}}$.

Now we are in position to complete the proof of the bridge representation for the joint density p_T .

Proof of Theorem 2.2. Since X and Y are Brownian and fractional Brownian motions under $\tilde{\mathbb{P}}$ respectively, we have

$$\tilde{\mathbb{E}}[X_t] = x_0, \quad \tilde{\mathbb{E}}[Y_t] = y_0, \quad \forall t \in [0, T],$$

and, for $s, t \in [0, T]$,

$$\text{Cov}(X_t, X_s) = \tilde{\mathbb{E}}[(X_s - x_0)(X_t - x_0)] = s \wedge t,$$

and

$$\text{Cov}(Y_t, Y_s) = \tilde{\mathbb{E}}[(Y_s - y_0)(Y_t - y_0)] = R_H(s, t),$$

where R_H is the autocovariance function for the fractional Brownian motion as given in (A.2). The covariance between X_t and Y_s is determined by applying the Itô isometry as

$$\begin{aligned} \text{Cov}(X_t, Y_s) &= \tilde{\mathbb{E}}[(X_t - x_0)(Y_s - y_0)] = \tilde{\mathbb{E}} \left[\left(\rho \tilde{B}_t + \sqrt{1 - \rho^2} \tilde{W}_t \right) \int_0^s K_H(s, u) d\tilde{B}_u \right] \\ &= \rho \int_0^{s \wedge t} K_H(s, u) du, \quad \forall s, t \in [0, T]. \end{aligned} \quad (2.25)$$

Thus, the joint density \tilde{p}_T of the bivariate Gaussian variable (X_T, Y_T) under $\tilde{\mathbb{P}}$ is given by

$$\tilde{p}_T(x, y | x_0, y_0) = \frac{1}{2\pi \sqrt{|\Sigma(T)|}} e^{-\frac{1}{2} \mathbf{x}' \Sigma(T)^{-1} \mathbf{x}} = \phi(x - x_0, y - y_0),$$

where $\Sigma(T)$ denotes the covariance matrix of (X_T, Y_T) given by

$$\Sigma(T) = \begin{bmatrix} \text{Cov}(X_T, X_T) & \text{Cov}(X_T, Y_T) \\ \text{Cov}(X_T, Y_T) & \text{Cov}(Y_T, Y_T) \end{bmatrix} = \begin{bmatrix} T & \rho_H T^{H+\frac{1}{2}} \\ \rho_H T^{H+\frac{1}{2}} & T^{2H} \end{bmatrix}. \quad (2.26)$$

Note that (2.20) implies that the matrix $\Sigma(T)$ is invertible.

From (2.8) we have, for any bounded and continuous function f defined on \mathbb{R}^2 ,

$$\begin{aligned} & \int p_T(x, y|x_0, y_0) f(x, y) dx dy \\ &= \mathbb{E}[f(X_T, Y_T)] = \tilde{\mathbb{E}} \left[f(X_T, Y_T) \frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} \right] \\ &= \tilde{\mathbb{E}} \left[f(X_T, Y_T) e^{\int_0^T \tilde{h}_1(t) d\tilde{W}_t - \frac{1}{2} \int_0^T \tilde{h}_1^2(t) dt + \int_0^T \tilde{h}_2(t) d\tilde{B}_t - \frac{1}{2} \int_0^T \tilde{h}_2^2(t) dt} \right] \\ &= \int \tilde{p}_T(x, y|x_0, y_0) f(x, y) \tilde{\mathbb{E}}_{x,y} \left[e^{\int_0^T \tilde{h}_1(t) d\tilde{W}_t - \frac{1}{2} \int_0^T \tilde{h}_1^2(t) dt + \int_0^T \tilde{h}_2(t) d\tilde{B}_t - \frac{1}{2} \int_0^T \tilde{h}_2^2(t) dt} \right] dx dy. \end{aligned}$$

Finally, since f is arbitrary, we obtain the bridge representation (2.21). \square

3. Modal-path approximation

The bridge representation (2.21) of the joint density given in Theorem 2.2 albeit succinct is hard to calculate in practice; owing to the complexity in defining the processes \tilde{h}_1, \tilde{h}_2 and the involvement of the stochastic integrals with respect to Brownian motions under the new measure $\tilde{\mathbb{P}}$. In this section, we approximate the following conditional expectation in the bridge representation in Theorem 2.2

$$\tilde{\mathbb{E}}_{x,y} \left[e^{\int_0^T \tilde{h}_1(t) d\tilde{W}_t - \frac{1}{2} \int_0^T \tilde{h}_1^2(t) dt + \int_0^T \tilde{h}_2(t) d\tilde{B}_t - \frac{1}{2} \int_0^T \tilde{h}_2^2(t) dt} \right] \quad (3.1)$$

by evaluating the integrand along the modal-path (thus the term “modal-path approximation”). We also provide an error estimate of the modal-path approximation. The idea is to replace the processes X and Y in the bridge representation by their expectations under the new measure $\tilde{\mathbb{P}}$. Since X and Y are Brownian and fractional Brownian bridges respectively under the $\tilde{\mathbb{P}}$ -measure, the expectations consist of the mode of the joint density of X_t and Y_t that are easily obtained as in Remark 3.1. We summarize the result in Theorem 3.3.

3.1. The conditional expectation of (X_t, Y_t) given its terminal point under the $\tilde{\mathbb{P}}$ -measure

For notational simplicity, we denote $x_t^{x,y} = \tilde{\mathbb{E}}_{x,y}[X_t]$ and $y_t^{x,y} = \tilde{\mathbb{E}}_{x,y}[Y_t]$, and we use the following notations hereafter:

$$\bar{\rho} = \sqrt{1 - \rho^2}, \quad \rho_H = \rho \kappa_H, \quad \bar{\rho}_H = \sqrt{1 - \rho_H^2}.$$

By straightforward application of Lemma A.4 in the Appendix, we obtain the following expression for the modal-path

$$\begin{bmatrix} x_t^{x,y} \\ y_t^{x,y} \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \Sigma(t; T) \Sigma(T)^{-1} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}, \quad (3.2)$$

where the matrix $\Sigma(t; T)$ is given by

$$\Sigma(t; T) = \begin{bmatrix} \text{Cov}(X_t, X_T) & \text{Cov}(X_t, Y_T) \\ \text{Cov}(Y_t, X_T) & \text{Cov}(Y_t, Y_T) \end{bmatrix} = \begin{bmatrix} t & \rho \int_0^t K_H(T, u) du \\ \rho_H t^{H+\frac{1}{2}} & R_H(t, T) \end{bmatrix} \quad (3.3)$$

and $\Sigma(T)$ is given by (2.26).

Hence, by straightforward calculations, we obtain the explicit expression for the modal path as

$$x_t^{x,y} = \tilde{\mathbb{E}}[X_t^{x,y}] = x_0 + m_{11}(t; T)(x - x_0) + m_{12}(t; T)(y - y_0), \quad (3.4)$$

$$y_t^{x,y} = \tilde{\mathbb{E}}[Y_t^{x,y}] = y_0 + m_{21}(t; T)(x - x_0) + m_{22}(t; T)(y - y_0), \quad (3.5)$$

where

$$m_{11}(t; T) = \frac{1}{\bar{\rho}_H^2} \left(\frac{t}{T} - \frac{\rho \rho_H}{T^{H+\frac{1}{2}}} \int_0^t K_H(T, s) ds \right), \quad (3.6)$$

$$m_{12}(t; T) = \frac{1}{\bar{\rho}_H^2} \left(-\rho_H \frac{t}{T^{H+\frac{1}{2}}} + \frac{\rho}{T^{2H}} \int_0^t K_H(T, s) ds \right), \quad (3.7)$$

$$m_{21}(t; T) = \frac{\rho_H}{\bar{\rho}_H^2} \left(\frac{t^{H+\frac{1}{2}}}{T} - \frac{R_H(t, T)}{T^{H+\frac{1}{2}}} \right), \quad (3.8)$$

$$m_{22}(t; T) = \frac{1}{\bar{\rho}_H^2} \left(-\rho_H^2 \left\{ \frac{t}{T} \right\}^{H+\frac{1}{2}} + \frac{R_H(t, T)}{T^{2H}} \right). \quad (3.9)$$

See Figs. 1 and 2 for plots of modal-paths in various cases. We remark that, if $H = \frac{1}{2}$, then, (3.4) and (3.5) reduce to

$$x_t^{x,y} = x_0 + \frac{t}{T}(x - x_0),$$

$$y_t^{x,y} = y_0 + \frac{t}{T}(y - y_0),$$

which is simply the straight line connecting (x_0, y_0) and (x, y) as expected even though $X_t^{x,y}$ and $Y_t^{x,y}$ are correlated. On the other hand, if $H \neq \frac{1}{2}$ but $\rho = 0$, then $\rho_H = 0$ and $\bar{\rho}_H = 1$. It follows that

$$x_t^{x,y} = x_0 + \frac{t}{T}(x - x_0),$$

$$y_t^{x,y} = y_0 + \frac{R_H(t, T)}{T^{2H}}(y - y_0).$$

In either case, there are no interactions between $x_t^{x,y}$ and $y_t^{x,y}$.

Remark 3.1. We present plots of modal-paths with various Hurst parameters H and correlation coefficients ρ in Figs. 1 and 2. Terminal time is set as $T = 1$, initial and terminal points are chosen as $(x_0, y_0) = (0, 0)$ and $(x_T, y_T) = (1, 1)$ respectively. As one can see in the plots, when the driving Brownian motions are positively correlated ($\rho \geq 0$), the smaller the Hurst parameter, the curvier the modal-path. When H is close to zero, we observe a jump-like behavior in the modal-path for all the ρ 's. When $H \approx \frac{1}{2}$, the modal-paths all look like straight lines independent of the values of ρ . The negatively correlated case ($\rho < 0$) behaves much more differently than the positive cases when H is away from one half.

3.2. Small time asymptotic of the joint density

Define for $i = 1, 2$

$$\tilde{h}_i(s) = h_i(s, x_s^{x,y}, y_s^{x,y})$$

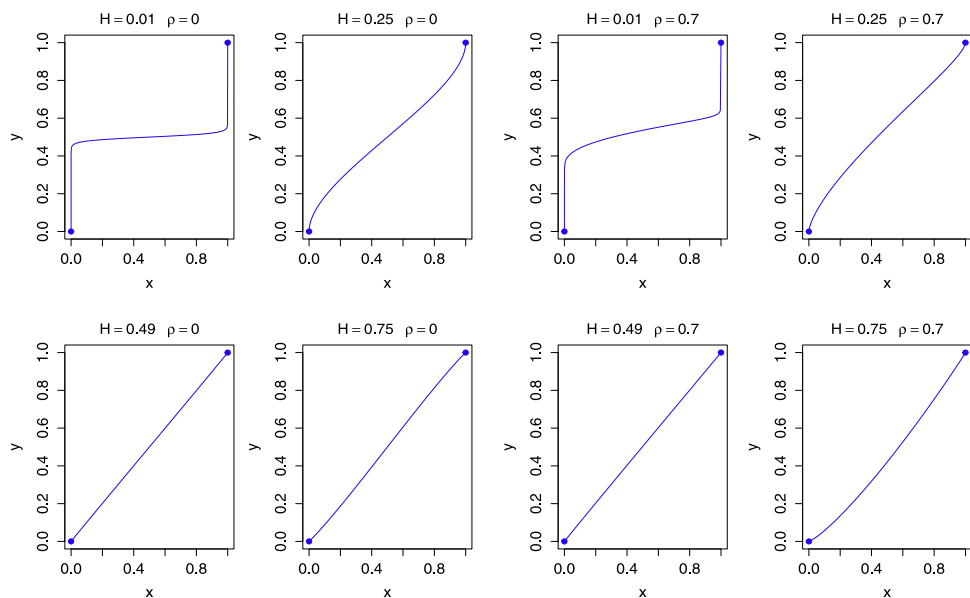


Fig. 1. The plots of modal-paths from $(x_0, y_0) = (0, 0)$ to $(x, y) = (1, 1)$ within the time interval $[0, 1]$ with $\rho = 0, 0.7$ and Hurst parameters $H = 0.01, 0.25, 0.49,$ and 0.75 .

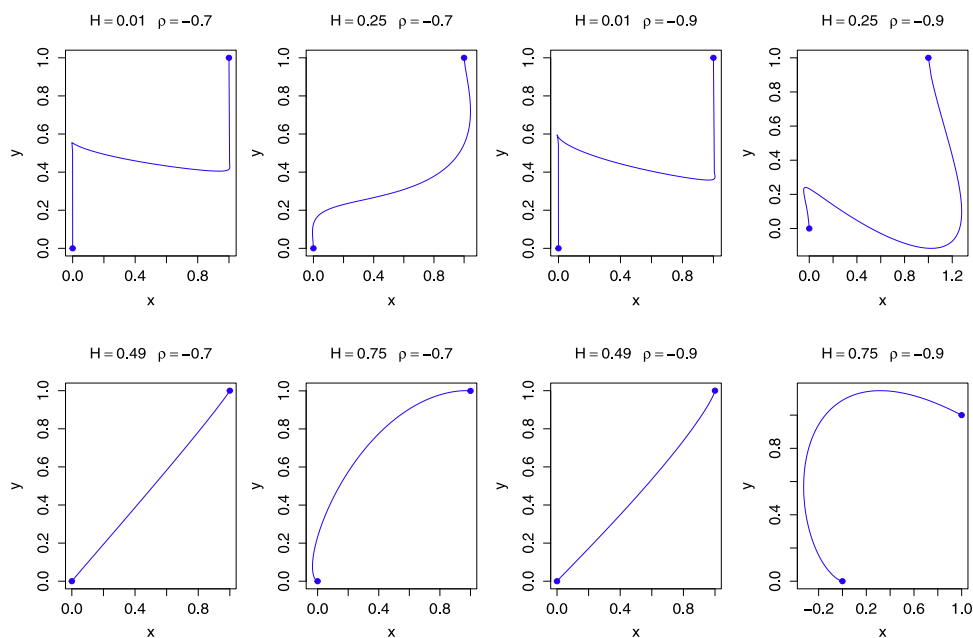


Fig. 2. The plots of modal-paths from $(x_0, y_0) = (0, 0)$ to $(x, y) = (1, 1)$ within the time interval $[0, 1]$ with $\rho = -0.7, -0.9$ and Hurst parameters $H = 0.01, 0.25, 0.49,$ and 0.75 .

and the \hat{h}_i 's by

$$\rho \hat{h}_2(t) + \sqrt{1 - \rho^2} \hat{h}_1(t) = \bar{h}_1(t), \quad (3.10)$$

$$\hat{h}_2(t) = \mathcal{K}_H^{-1} \left(\int_0^t \bar{h}_2(s) ds \right) (t). \quad (3.11)$$

In other words, $\bar{h}_i(s)$ represents the value of h_i evaluated along the modal path and the \hat{h}_i 's are solution to the system of equations similar to (A.11) and (A.12) except that the X_s and Y_s are substituted by the modal path. The well-definedness of \hat{h}_1 and \hat{h}_2 is proved in Lemma A.3 in the Appendix.

As the time interval is very small in our small time asymptotic, we approximate the two processes \tilde{h}_1, \tilde{h}_2 by their respective modal-path approximations \hat{h}_1, \hat{h}_2 , resulting in the conditional expectation

$$\tilde{\mathbb{E}}_{x,y} \left[e^{\int_0^T \hat{h}_1(t) d\tilde{W}_t - \frac{1}{2} \int_0^T \hat{h}_1^2(t) dt + \int_0^T \hat{h}_2(t) d\tilde{B}_t - \frac{1}{2} \int_0^T \hat{h}_2^2(t) dt} \right]. \quad (3.12)$$

Now we can evaluate the conditional expectation (3.12) explicitly since the two random variables

$$G_1 := \int_0^T \hat{h}_1(t) d\tilde{W}_t, \quad \text{and} \quad G_2 := \int_0^T \hat{h}_2(t) d\tilde{B}_t$$

are jointly Gaussian. Denote the integral of h by $\langle h \rangle = \int_0^T h(t) dt$ for any $h \in L^1([0, T])$.

Lemma 3.1. *The logarithm of (3.12) has the explicit expression*

$$\begin{aligned} \omega(T) &= \log \tilde{\mathbb{E}}_{x,y} \left[e^{\int_0^T \hat{h}_1(t) d\tilde{W}_t - \frac{1}{2} \int_0^T \hat{h}_1^2(t) dt + \int_0^T \hat{h}_2(t) d\tilde{B}_t - \frac{1}{2} \int_0^T \hat{h}_2^2(t) dt} \right] \\ &= \omega_1(T) - \frac{1}{\bar{\rho}_H^2} \left\{ \left(\frac{\bar{\rho} \langle \hat{h}_1 \rangle}{\sqrt{T}} + \frac{\rho \langle \hat{h}_2 \rangle}{\sqrt{T}} - \frac{\rho_H \langle \bar{h}_2 \rangle}{T^H} \right)^2 + \left(\frac{\bar{\rho}_H \langle \bar{h}_2 \rangle}{T^H} \right)^2 \right\}, \end{aligned} \quad (3.13)$$

where

$$\begin{aligned} \omega_1(T) &= \frac{1}{\bar{\rho}_H^2} \left\{ \left(\frac{\bar{\rho} \langle \hat{h}_1 \rangle}{\sqrt{T}} + \frac{\rho \langle \hat{h}_2 \rangle}{\sqrt{T}} - \frac{\rho_H \langle \bar{h}_2 \rangle}{T^H} \right) \left(\frac{x - x_0}{\sqrt{T}} \right) \right. \\ &\quad \left. - \rho_H \left(\frac{\bar{\rho} \langle \hat{h}_1 \rangle}{\sqrt{T}} + \frac{\rho \langle \hat{h}_2 \rangle}{\sqrt{T}} - \frac{\rho_H \langle \bar{h}_2 \rangle}{T^H} \right) \left(\frac{y - y_0}{T^H} \right) + \bar{\rho}_H^2 \frac{\langle \bar{h}_2 \rangle}{T^H} \left(\frac{y - y_0}{T^H} \right) \right\}. \end{aligned}$$

Furthermore, since the last two terms in the brackets on the right-hand side of (3.13) are of higher order compared to the others, we have, as $T \rightarrow 0$,

$$\omega(T) = \omega_1(T) + O(T^\alpha), \quad (3.14)$$

where

$$\alpha = \begin{cases} 3 - 4H & \text{if } H \in (\frac{1}{2}, 1), \\ 2H & \text{if } H \in (0, \frac{1}{2}]. \end{cases} \quad (3.15)$$

Remark 3.2. The explicit expression (3.15) holds true for all $H \in (0, 1)$. Since we are interested in positive powers for T as $T \rightarrow 0$, we only consider $H < \frac{3}{4}$ in our modal-path approximation.

For this reason, in the next small time asymptotic result about the joint density, we add the requirement of $H < \frac{3}{4}$.

The following is the main theorem of the present paper.

Theorem 3.3 (Small Time Asymptotic of the Joint Density). Assume that $H < \frac{3}{4}$. The joint probability density $p_T(x, y|x_0, y_0)$ given by (2.21) in Theorem 2.2 has the asymptotic expansion as $T \rightarrow 0$

$$p_T(x, y|x_0, y_0) = \phi(x - x_0, y - y_0)e^{\omega_1(T)}(1 + o(T^\beta)), \quad (3.16)$$

where $\beta \in (0, \alpha)$ (α is defined in (3.15)), ϕ is the Gaussian density given in (2.22) and $\omega_1(T)$ is defined in (3.14).

3.3. Proof of Lemma 3.1

We prove Lemma 3.1 in this subsection.

Proof. Consider the Gaussian random vector $\mathbf{Z} := (G_1, G_2, X_T, Y_T)'$. Note that \mathbf{Z} has expectation $(0, 0, x_0, y_0)'$ and covariance matrix Σ_Z

$$\Sigma_Z = \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{D}' & \Sigma(T) \end{bmatrix},$$

where

$$\mathbf{C} = \begin{bmatrix} \langle \hat{h}_1^2 \rangle & 0 \\ 0 & \langle \hat{h}_2^2 \rangle \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \bar{\rho}\langle \hat{h}_1 \rangle & 0 \\ \rho\langle \hat{h}_2 \rangle & \langle \hat{h}_2 K_H(T, \cdot) \rangle = \langle \bar{h}_2 \rangle \end{bmatrix},$$

and $\Sigma(T)$ is defined in (2.26). Let $\mathbf{1} = (1, 1)'$ denote the 2×1 column vector with both components being equal to 1.

By applying Lemma A.4 in the Appendix, we decompose the Gaussian vector $(G_1, G_2)'$ as

$$\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \mathbf{D}\Sigma(T)^{-1} \begin{bmatrix} X_T - x_0 \\ Y_T - y_0 \end{bmatrix} + \begin{bmatrix} V_1 \\ V_2 \end{bmatrix},$$

where $(V_1, V_2)'$ is a Gaussian vector independent of $(X_T, Y_T)'$ with zero expectation and covariance matrix given by $\mathbf{C} - \mathbf{D}\Sigma(T)^{-1}\mathbf{D}'$. Moreover, by straightforward computations, one can show that the matrix $\mathbf{D}\Sigma(T)^{-1}\mathbf{D}'$ has the following explicit expression

$$\begin{aligned} & \mathbf{D}\Sigma(T)^{-1}\mathbf{D}' \\ &= \frac{1}{\bar{\rho}_H^2} \begin{bmatrix} \left(\frac{\bar{\rho}\langle \hat{h}_1 \rangle}{\sqrt{T}} \right)^2 & \frac{\bar{\rho}\langle \hat{h}_1 \rangle}{\sqrt{T}} \frac{\rho\langle \hat{h}_2 \rangle}{\sqrt{T}} - \frac{\bar{\rho}\langle \hat{h}_1 \rangle}{\sqrt{T}} \frac{\rho_H\langle \bar{h}_2 \rangle}{T^H} \\ \frac{\bar{\rho}\langle \hat{h}_1 \rangle}{\sqrt{T}} \frac{\rho\langle \hat{h}_2 \rangle}{\sqrt{T}} - \frac{\bar{\rho}\langle \hat{h}_1 \rangle}{\sqrt{T}} \frac{\rho_H\langle \bar{h}_2 \rangle}{T^H} & \left(\frac{\rho\langle \hat{h}_2 \rangle}{\sqrt{T}} \right)^2 - 2 \frac{\rho\langle \hat{h}_2 \rangle}{\sqrt{T}} \frac{\rho_H\langle \bar{h}_2 \rangle}{T^H} + \left(\frac{\langle \bar{h}_2 \rangle}{T^H} \right)^2 \end{bmatrix}. \end{aligned}$$

Thus, the above decomposition implies that (3.12) equals

$$\begin{aligned} & e^{-\frac{1}{2} \int_0^T \hat{h}_1^2(t) dt - \frac{1}{2} \int_0^T \hat{h}_2^2(t) dt} \tilde{\mathbb{E}}_{x,y} [e^{G_1 + G_2}] \\ &= e^{-\frac{1}{2}(\langle \hat{h}_1^2 \rangle + \langle \hat{h}_2^2 \rangle)} \times \exp \left\{ \mathbf{1}' \mathbf{D}\Sigma(T)^{-1} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} \right\} \times \tilde{\mathbb{E}} [e^{V_1 + V_2}]. \end{aligned} \quad (3.17)$$

Note that

$$\begin{aligned} & \mathbf{1}' \mathbf{D} \Sigma(T)^{-1} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} \\ &= \frac{1}{\bar{\rho}_H^2} \left\{ \left(\frac{\bar{\rho} \langle \hat{h}_1 \rangle}{\sqrt{T}} + \frac{\rho \langle \hat{h}_2 \rangle}{\sqrt{T}} - \frac{\rho_H \langle \bar{h}_2 \rangle}{T^H} \right) \left(\frac{x - x_0}{\sqrt{T}} \right) \right. \\ & \quad \left. - \rho_H \left(\frac{\bar{\rho} \langle \hat{h}_1 \rangle}{\sqrt{T}} + \frac{\rho \langle \hat{h}_2 \rangle}{\sqrt{T}} - \frac{\rho_H \langle \bar{h}_2 \rangle}{T^H} \right) \left(\frac{y - y_0}{T^H} \right) + \bar{\rho}_H^2 \frac{\langle \bar{h}_2 \rangle}{T^H} \left(\frac{y - y_0}{T^H} \right) \right\}, \end{aligned} \quad (3.18)$$

and

$$\begin{aligned} & \mathbb{E} \left[e^{V_1 + V_2} \right] = e^{\frac{1}{2} \text{var}(V_1 + V_2)} = e^{\mathbf{1}'(\mathbf{C} - \mathbf{D} \Sigma(T)^{-1} \mathbf{D}') \mathbf{1}} \\ &= e^{\frac{1}{2} (\langle \hat{h}_1^2 \rangle + \langle \hat{h}_2^2 \rangle)} e^{-\mathbf{1}'(\mathbf{D} \Sigma(T)^{-1} \mathbf{D}') \mathbf{1}} \\ &= e^{\frac{1}{2} (\langle \hat{h}_1^2 \rangle + \langle \hat{h}_2^2 \rangle)} \times \exp \left\{ -\frac{1}{\bar{\rho}_H^2} \left(\left(\frac{\bar{\rho} \langle \hat{h}_1 \rangle}{\sqrt{T}} + \frac{\rho \langle \hat{h}_2 \rangle}{\sqrt{T}} - \frac{\rho_H \langle \bar{h}_2 \rangle}{T^H} \right)^2 - \left(\frac{\bar{\rho}_H \langle \bar{h}_2 \rangle}{T^H} \right)^2 \right) \right\}. \end{aligned} \quad (3.19)$$

Therefore, by combining (3.17)–(3.19) we obtain (3.13).

Furthermore, in the case $H \in (\frac{1}{2}, \frac{3}{4})$, by straightforward calculation based on (A.33), (A.37) and (A.38), we obtain the following estimates

$$\begin{aligned} \frac{\langle \bar{h}_2 \rangle}{T^H} &\leq C(1 + |x - x_0| + |y - y_0|)T^{1-H} + C|y - y_0|T^{\frac{3-4H}{2}} \\ &\leq C(1 + |x - x_0| + |y - y_0|)T^{\frac{3-4H}{2}}, \end{aligned} \quad (3.20)$$

and, for $i = 1, 2$,

$$\begin{aligned} \frac{\langle \hat{h}_i \rangle}{\sqrt{T}} &\leq C(1 + |x - x_0| + |y - y_0|)T^{1-H} + C|y - y_0|T^{\frac{3-4H}{2}} + CT^{1+\gamma-H} \\ &\quad + C|x - x_0|T^{1-H} + C|y - y_0|(T^{\frac{3-4H}{2}} + T^{1-H}) \\ &\leq C(1 + |x - x_0| + |y - y_0|)T^{\frac{3-4H}{2}}. \end{aligned} \quad (3.21)$$

Then, (3.13), (3.20) and (3.21) imply (3.14) for the case $H \in (\frac{1}{2}, \frac{3}{4})$.

In the case $H \in (0, \frac{1}{2}]$, it follows from (A.47)–(A.49) the inequalities

$$\begin{aligned} \frac{\langle \bar{h}_2 \rangle}{T^H} &\leq C(1 + |x - x_0| + |y - y_0|)T^{1-H} + C|x - x_0|T^{\frac{1}{2}} \\ &\leq C(1 + |x - x_0| + |y - y_0|)T^H, \end{aligned} \quad (3.22)$$

and, for $i = 1, 2$,

$$\begin{aligned} \frac{\langle \hat{h}_i \rangle}{\sqrt{T}} &\leq C(1 + |x - x_0| + |y - y_0|)T^{\frac{1}{2}} + C|x - x_0|T^H \\ &\leq C(1 + |x - x_0| + |y - y_0|)T^H. \end{aligned} \quad (3.23)$$

Therefore, by (3.13), (3.22) and (3.23) we obtain (3.14) for the case $H \in (0, \frac{1}{2}]$. \square

3.4. Proof of Theorem 3.3

This subsection is devoted to the proof of Theorem 3.3.

Proof. For any fixed $(x, y) \in \mathbb{R}^2$, we denote

$$\xi_T(x, y) = e^{\int_0^T \hat{h}_1(t) d\tilde{W}_t - \frac{1}{2} \int_0^T \hat{h}_1^2(t) dt + \int_0^T \hat{h}_2(t) d\tilde{B}_t - \frac{1}{2} \int_0^T \hat{h}_2^2(t) dt},$$

$$\xi_T = e^{\int_0^T \tilde{h}_1(t) d\tilde{W}_t - \frac{1}{2} \int_0^T \tilde{h}_1^2(t) dt + \int_0^T \tilde{h}_2(t) d\tilde{B}_t - \frac{1}{2} \int_0^T \tilde{h}_2^2(t) dt},$$

and

$$\begin{aligned} \eta_T(x, y) &= \int_0^T (\tilde{h}_1(t) - \hat{h}_1(t)) d\tilde{W}_t - \frac{1}{2} \int_0^T (\tilde{h}_1(t)^2 - \hat{h}_1(t)^2) dt \\ &\quad + \int_0^T (\tilde{h}_2(t) - \hat{h}_2(t)) d\tilde{B}_t - \frac{1}{2} \int_0^T (\tilde{h}_2(t)^2 - \hat{h}_2(t)^2) dt. \end{aligned} \quad (3.24)$$

This theorem claims that $\mathbb{E}_{x,y}[\xi_T(x, y)]$ is the limit of $\mathbb{E}_{x,y}[\xi_T]$ as $T \rightarrow 0$. From [Theorem 2.2](#) and the calculation of $\omega(T)$ in [Lemma 3.1](#) we only need to show that, for any bounded and continuous function f defined on \mathbb{R}^2 , the following limit holds

$$\begin{aligned} &\lim_{T \rightarrow 0} \int_{\mathbb{R}^2} \tilde{p}_T(x, y | x_0, y_0) f(x, y) \left(\tilde{\mathbb{E}}_{x,y}[\xi_T] - \tilde{\mathbb{E}}_{x,y}[\xi_T(x, y)] \right) dx dy \\ &= \lim_{T \rightarrow 0} \tilde{\mathbb{E}}[f(X_T, Y_T)(\xi_T - \xi_T(x, y))] = 0. \end{aligned} \quad (3.25)$$

Notice that

$$\lim_{T \rightarrow 0} \left| \tilde{\mathbb{E}}[f(X_T, Y_T)(\xi_T - \xi_T(x, y))] \right| \leq \|f\|_{\infty} \lim_{T \rightarrow 0} \tilde{\mathbb{E}}[|\xi_T - \xi_T(x, y)|]. \quad (3.26)$$

Thus, in order to show (3.25), it suffices to show

$$\lim_{T \rightarrow 0} \tilde{\mathbb{E}}[|\xi_T - \xi_T(x, y)|] = 0. \quad (3.27)$$

Using the inequality $|e^u - e^v| \leq \frac{e^u + e^v}{2} |u - v|$ for any $u, v \in \mathbb{R}$, we have the following bound

$$\begin{aligned} |\xi_T - \xi_T(x, y)| &\leq \frac{1}{2} (\xi_T + \xi_T(x, y)) |\eta_T(x, y)| \\ &= \frac{1}{2} \tilde{D}_T(x, y) + \frac{1}{2} \hat{D}_T(x, y), \end{aligned} \quad (3.28)$$

where

$$\tilde{D}_T(x, y) = \xi_T |\eta_T(x, y)| = e^{\int_0^T \tilde{h}_1(t) d\tilde{W}_t - \frac{1}{2} \int_0^T \tilde{h}_1^2(t) dt + \int_0^T \tilde{h}_2(t) d\tilde{B}_t - \frac{1}{2} \int_0^T \tilde{h}_2^2(t) dt} |\eta_T(x, y)|,$$

and

$$\hat{D}_T(x, y) = \xi_T(x, y) |\eta_T(x, y)| = e^{\int_0^T \hat{h}_1(t) d\tilde{W}_t - \frac{1}{2} \int_0^T \hat{h}_1^2(t) dt + \int_0^T \hat{h}_2(t) d\tilde{B}_t - \frac{1}{2} \int_0^T \hat{h}_2^2(t) dt} |\eta_T(x, y)|.$$

First, for any $r > 0$, we will estimate $\tilde{\mathbb{E}}[|\eta_T(x, y)|^r]$. From (3.24), the Burkholder–Davis–Gundy inequality and the Cauchy–Schwarz inequality, it implies that

$$\begin{aligned} &\tilde{\mathbb{E}}[|\eta_T(x, y)|^r] \\ &\leq \tilde{\mathbb{E}} \left[\left(\int_0^T |\tilde{h}_1(t) - \hat{h}_1(t)|^2 dt \right)^{\frac{r}{2}} \right] + \tilde{\mathbb{E}} \left[\left(\int_0^T |\tilde{h}_2(t) - \hat{h}_2(t)|^2 dt \right)^{\frac{r}{2}} \right] \\ &\quad + \tilde{\mathbb{E}} \left[\left(\int_0^T (|\tilde{h}_1(t)|^2 + |\hat{h}_1(t)|^2) dt \right)^{\frac{r}{2}} \left(\int_0^T (|\tilde{h}_1(t) - \hat{h}_1(t)|^2) dt \right)^{\frac{r}{2}} \right] \\ &\quad + \tilde{\mathbb{E}} \left[\left(\int_0^T (|\tilde{h}_2(t)|^2 + |\hat{h}_2(t)|^2) dt \right)^{\frac{r}{2}} \left(\int_0^T (|\tilde{h}_2(t) - \hat{h}_2(t)|^2) dt \right)^{\frac{r}{2}} \right]. \end{aligned} \quad (3.29)$$

From Lemma A.5 in the Appendix, we can bound the right-hand side of (3.29) in the case $H \leq \frac{1}{2}$ and the case $H \in (\frac{1}{2}, \frac{3}{4})$.

When $H \leq \frac{1}{2}$, the right-hand side of (3.29) can be bounded by

$$\begin{aligned} & C(1 + \tilde{\mathbb{E}} \left[\|\tilde{B}\|_\infty^r + \|\tilde{W}\|_\infty^r + \|\tilde{B}^H\|_\infty^r \right] + |x - x_0|^r + |y - y_0|^r) T^{Hr} \\ & + C(1 + \tilde{\mathbb{E}} \left[\|\tilde{B}\|_\infty^{2r} + \|\tilde{W}\|_\infty^{2r} + \|\tilde{B}^H\|_\infty^{2r} \right] + |x - x_0|^{2r} + |y - y_0|^{2r}) T^{2Hr} \\ & \rightarrow 0, \text{ as } T \rightarrow 0. \end{aligned} \quad (3.30)$$

When $H \in (\frac{1}{2}, \frac{3}{4})$, the right-hand side of (3.29) can be bounded by

$$\begin{aligned} & C(1 + \tilde{\mathbb{E}} \left[\|\tilde{B}\|_{\frac{1}{2}-\epsilon}^r + \|\tilde{W}\|_{\frac{1}{2}-\epsilon}^r + \|\tilde{B}^H\|_{\frac{1}{2}-\epsilon}^r + \|\tilde{B}\|_\infty^r + \|\tilde{W}\|_\infty^r + \|\tilde{B}^H\|_\infty^r \right]) T^{(1-H)r} \\ & + C(|x - x_0|^r + |y - y_0|^r) T^{(1-H)r} + C|y - y_0|^r T^{\frac{(3-4H)r}{2}} \\ & + C(1 + \tilde{\mathbb{E}} \left[\|\tilde{B}\|_{\frac{1}{2}-\epsilon}^{2r} + \|\tilde{W}\|_{\frac{1}{2}-\epsilon}^{2r} + \|\tilde{B}^H\|_{\frac{1}{2}-\epsilon}^{2r} + \|\tilde{B}\|_\infty^{2r} + \|\tilde{W}\|_\infty^{2r} + \|\tilde{B}^H\|_\infty^{2r} \right]) T^{(2-2H)r} \\ & + C(|x - x_0|^{2r} + |y - y_0|^{2r}) T^{(2-2H)r} + C|y - y_0|^{2r} T^{(3-4H)r} \\ & \rightarrow 0, \text{ as } T \rightarrow 0. \end{aligned} \quad (3.31)$$

Therefore, by (3.29)–(3.31) we can show

$$\lim_{T \rightarrow 0} \tilde{\mathbb{E}} \left[|\eta_T(x, y)|^r \right] = 0, \quad (3.32)$$

for any $r > 0$.

Now, we fix $p > 0$ and $q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$. First, let us show $\lim_{T \rightarrow 0} \tilde{\mathbb{E}} \left[\hat{D}_T(x, y) \right] = 0$. Using Hölder's inequality for conditional expectation, by (3.32), Part (b) for the case $H \in (\frac{1}{2}, \frac{3}{4})$ and Part (f) for the case $H \leq \frac{1}{2}$ in Lemma A.5 we obtain

$$\begin{aligned} & \lim_{T \rightarrow 0} \tilde{\mathbb{E}} \left[\hat{D}_T(x, y) \right] \\ & \leq \lim_{T \rightarrow 0} \left(\tilde{\mathbb{E}} \left[e^{\int_0^T p \tilde{h}_1(t) d\tilde{W}_t - \frac{p^2}{2} \int_0^T \tilde{h}_1^2(t) dt + \int_0^T p \tilde{h}_2(t) d\tilde{B}_t - \frac{p^2}{2} \int_0^T \tilde{h}_2^2(t) dt} \right] \right)^{\frac{1}{p}} \left(\tilde{\mathbb{E}} \left[|\eta_T(x, y)|^q \right] \right)^{\frac{1}{q}} \\ & = \left(\lim_{T \rightarrow 0} e^{\frac{p(p-1)}{2} \int_0^T (\tilde{h}_1^2(t) + \tilde{h}_2^2(t)) dt} \right) \left(\lim_{T \rightarrow 0} \left(\tilde{\mathbb{E}} \left[|\eta_T(x, y)|^q \right] \right)^{\frac{1}{q}} \right) = 1 \cdot 0 = 0. \end{aligned} \quad (3.33)$$

Next, we will show $\lim_{T \rightarrow 0} \tilde{\mathbb{E}} \left[\tilde{D}_T(x, y) \right] = 0$. Using the techniques in the proof of Lemma 2.1 and the estimates in Lemma A.5, we can find some $0 < t_1 < 1$ such that

$$\tilde{\mathbb{E}} \left[e^{\frac{p^2}{2} \int_0^{t_1} \tilde{h}_1^2(t) dt + \frac{p^2}{2} \int_0^{t_1} \tilde{h}_1^2(t) dt} \right] < \infty,$$

and hence

$$\tilde{\mathbb{E}} \left[e^{\int_0^T p \tilde{h}_1(t) d\tilde{W}_t - \frac{p^2}{2} \int_0^T \tilde{h}_1^2(t) dt + \int_0^T p \tilde{h}_2(t) d\tilde{B}_t - \frac{p^2}{2} \int_0^T \tilde{h}_2^2(t) dt} \right] = 1. \quad (3.34)$$

Without loss of generality, we assume $T < t_1$. Applying Hölder's inequality, the Cauchy–Schwarz inequality, Part (a) for the case $H \in (\frac{1}{2}, \frac{3}{4})$ and Part (e) for the case $H \leq \frac{1}{2}$ in

Lemma A.5, (3.32), (3.34) and the dominated convergence theorem, one can obtain

$$\begin{aligned}
 & \lim_{T \rightarrow 0} \tilde{\mathbb{E}} \left[\tilde{D}_T(x, y) \right] \\
 & \leq \lim_{T \rightarrow 0} \left[\left(\tilde{\mathbb{E}} \left[e^{\int_0^T p \tilde{h}_1(t) d\tilde{W}_t - \frac{p^2}{2} \int_0^T \tilde{h}_1^2(t) dt + \int_0^T p \tilde{h}_2(t) d\tilde{B}_t - \frac{p^2}{2} \int_0^T \tilde{h}_2^2(t) dt} \right] \right)^{\frac{1}{p}} \right. \\
 & \quad \times \left. \left(\tilde{\mathbb{E}} \left[e^{(p-1)q \int_0^T \tilde{h}_1^2(t) dt + (p-1)q \int_0^T \tilde{h}_2^2(t) dt} \right] \right)^{\frac{1}{2q}} \left(\tilde{\mathbb{E}} \left[|\eta_T(x, y)|^{2q} \right] \right)^{\frac{1}{2q}} \right] \\
 & = \lim_{T \rightarrow 0} \left(\tilde{\mathbb{E}} \left[e^{(p-1)q \int_0^T \tilde{h}_1^2(t) dt + (p-1)q \int_0^T \tilde{h}_2^2(t) dt} \right] \right)^{\frac{1}{2q}} \lim_{T \rightarrow 0} \left(\tilde{\mathbb{E}} \left[|\eta_T(x, y)|^{2q} \right] \right)^{\frac{1}{2q}} \\
 & \leq 1 \cdot 0 = 0.
 \end{aligned} \tag{3.35}$$

Therefore, (3.28), (3.33) and (3.35) imply (3.27), which completes the proof. \square

3.5. Examples

We illustrate the $\omega(T)$ in the modal-path approximation (3.16) more explicitly by considering the following particular examples. Example 2 shows the recovery of the classical heat kernel expansion when $H = \frac{1}{2}$.

Example 1. If the drift terms h_1 and h_2 are both independent of x and y , we have $\tilde{h}_i = \hat{h}_i$, for $i = 1, 2$. One can easily verify that the representation

$$p_T(x, y | x_0, y_0) = \phi(x - x_0, y - y_0) e^{\omega(T)}$$

as in (3.16) is exact.

Example 2 (Recovery of Classical Heat Kernel Expansion up to Zeroth Order). Let $H = \frac{1}{2}$. Note that in this case $\rho_H = \rho$, $\hat{h}_2 = \bar{h}_2$. Thus, $\bar{\rho} \hat{h}_1 = \bar{h}_1 - \rho \bar{h}_2$. The function ω simplifies to

$$\begin{aligned}
 \omega(T) = & \frac{1}{\bar{\rho}^2} \left\{ \langle \bar{h}_1 \rangle \left(\frac{x - x_0}{T} \right) - \rho \langle \bar{h}_2 \rangle \left(\frac{x - x_0}{T} \right) - \rho \langle \bar{h}_1 \rangle \left(\frac{y - y_0}{T} \right) + \langle \bar{h}_2 \rangle \left(\frac{y - y_0}{T} \right) \right\} \\
 & + O(T).
 \end{aligned}$$

Notice that the last expression is exactly the work done by the vector field $h_1 \partial_1 + h_2 \partial_2$ along the geodesic connecting (x_0, y_0) to (x, y) . In this case, the geodesic is simply the straight line connecting (x_0, y_0) and (x, y) . Hence, the small time approximation of $p_T(x, y | x_0, y_0)$ reads

$$\phi(x - x_0, y - y_0) e^{\frac{1}{\bar{\rho}^2} \left\{ \langle \bar{h}_1 \rangle \left(\frac{x - x_0}{T} \right) - \rho \langle \bar{h}_2 \rangle \left(\frac{x - x_0}{T} \right) - \rho \langle \bar{h}_1 \rangle \left(\frac{y - y_0}{T} \right) + \langle \bar{h}_2 \rangle \left(\frac{y - y_0}{T} \right) \right\}} \{1 + O(T)\} \tag{3.36}$$

as $T \rightarrow 0$. It recovers the classical heat kernel expansion to zeroth order in the two dimensional Euclidean case.

Remark 3.3 (Classical Heat Kernel Expansion). Let $q(T, x_T, y_T | t, x_t, y_t)$ be the transition density of a two dimensional diffusion process from (x_t, y_t) at time t to (x_T, y_T) at time T . Then as $t \rightarrow T$, q has the following heat kernel expansion up to zeroth order

$$q_T(T, x_T, y_T | t, x_t, y_t) = \frac{1}{2\pi(T-t)} e^{-\frac{d^2}{2(T-t)}} e^{\int_{\gamma} \langle V_{\gamma}(s), \dot{\gamma}(s) \rangle ds} \{1 + O(T-t)\}, \tag{3.37}$$

where d denotes the geodesic distance between (x_t, y_t) and (x_T, y_T) associated with the Riemann metric determined by the diffusion matrix of the underlying process, assumed uniformly elliptic. $\int_{\gamma} \langle V, \dot{\gamma} \rangle ds$ represents the work done by the vector field V , given by the drift of the underlying process, along the geodesic γ connecting (x_t, y_t) to (x_T, y_T) . See for instance Hsu [8] (Theorem 5.1.1) for more detailed discussions on heat kernel expansion. In the case of flat geometry (Euclidean), the geodesic is simply a straight line connecting the initial and terminal points and the geodesic distance is the Euclidean distance. When $H = \frac{1}{2}$, we showed in Example 2 that $\omega(T)$ recovers (3.37) in the Euclidean case.

Example 3 (Uncorrelated Case, i.e., $\rho = 0$). In this case, since $\rho_H = \rho\kappa_H = 0$, $\bar{\rho} = \bar{\rho}_H = 1$, and notice that $\hat{h}_1 = \bar{h}_1$, ω reduces to

$$\omega(T) = -\left(\frac{\langle \bar{h}_1 \rangle}{\sqrt{T}}\right)^2 - \left(\frac{\langle \bar{h}_2 \rangle}{T^H}\right)^2 + \langle \bar{h}_1 \rangle \left(\frac{x - x_0}{T}\right) + \langle \bar{h}_2 \rangle \left(\frac{y - y_0}{T^H}\right).$$

Since there are no interactions between $x_t^{x,y}$ and $y_t^{x,y}$, we have

$$\omega(T) = \langle \bar{h}_1 \rangle \left(\frac{x - x_0}{T}\right) + \langle \bar{h}_2 \rangle \left(\frac{y - y_0}{T^H}\right) + O(T^\alpha),$$

where

$$\alpha = \begin{cases} 2 - 2H & \text{if } H \in (\frac{1}{2}, 1), \\ 1 & \text{if } H \in (0, \frac{1}{2}). \end{cases}$$

Thus, the small time approximation of $p_T(x, y|x_0, y_0)$ in this case reads

$$\phi(x - x_0, y - y_0) e^{\langle \bar{h}_1 \rangle \left(\frac{x - x_0}{T}\right) + \langle \bar{h}_2 \rangle \left(\frac{y - y_0}{T^H}\right)} (1 + o(T^\beta)),$$

as $T \rightarrow 0$, for any $\beta \in (0, \alpha)$, which can be regarded as a generalization of the heat kernel expansion up to zeroth order (see (3.36) in Example 2). Note also that in this case, we obtain the small time approximation of the joint density $p_T(x, y|x_0, y_0)$ for all $H \in (0, 1)$.

Example 4. Consider the case where both h_1 and h_2 are linear functions of x and y , say,

$$h_i(t, x, y) = \alpha_i(t)x + \beta_i(t)y + \gamma_i(t), \quad i = 1, 2.$$

We impose the following conditions:

(a1) If $H > \frac{1}{2}$, we assume that $\alpha_2(0) = \beta_2(0) = 0$ and there exist two constants $L > 0$ and $\gamma \in (H - \frac{1}{2}, \frac{1}{2})$ such that

$$|\alpha_1(t)| + |\beta_1(t)| + |\gamma_1(t)| \leq L, \quad \forall t \in [0, T],$$

and

$$|\alpha_2(t) - \alpha_2(s)| + |\beta_2(t) - \beta_2(s)| + |\gamma_2(t) - \gamma_2(s)| \leq L|t - s|^\gamma, \quad \forall s, t \in [0, T].$$

(a2) If $H \leq \frac{1}{2}$, we assume that there exists a constant $L > 0$ such that

$$|\alpha_i(t)| + |\beta_i(t)| + |\gamma_i(t)| \leq L, \quad \forall t \in [0, T], \quad i = 1, 2.$$

Note that the above conditions ensure (2.2) and (2.4), and hence Theorem 2.1 stays true and (2.5), (2.6) and (2.7) still hold. Moreover, though (2.3) cannot be guaranteed in this example, we

have the following estimate, for any small enough ϵ ,

$$\begin{aligned}
 & |h_2(t, X_t, Y_t) - h_2(s, X_s, Y_s)| \\
 & \leq |\gamma_2(t) - \gamma_2(s)| + |\alpha_2(t) - \alpha_2(s)| |X_t| + |\alpha_2(s)| |X_t - X_s| \\
 & \quad + |\beta_2(t) - \beta_2(s)| |Y_t| + |\beta_2(s)| |Y_t - Y_s| \\
 & \leq C(1 + \|B\|_\infty + \|W\|_\infty + \|B^H\|_\infty) |t - s|^\gamma \\
 & \quad + C(1 + \|B\|_\infty + \|W\|_\infty + \|B^H\|_\infty + \|B\|_{\frac{1}{2}-\epsilon} + \|W\|_{\frac{1}{2}-\epsilon}) |t - s|^{\frac{1}{2}-\epsilon} \\
 & \quad + C(1 + \|B\|_\infty + \|W\|_\infty + \|B^H\|_\infty + \|B^H\|_{H-\epsilon}) |t - s|^{H-\epsilon}.
 \end{aligned} \tag{3.38}$$

Hence, by Lemma A.2, the \tilde{h}_i 's are well defined. Using the above estimate and modifying the proof slightly in Lemma 2.1, it follows that Novikov's condition in Lemma 2.1 and the change of measures in Lemma 2.2 sustain, thereby all the main results in this paper hold for this linear system.

More importantly, the restriction $H < \frac{3}{4}$ in Theorem 3.3 can be removed in this linear case. In fact, because of the Hölder continuity of the coefficients α_2 and β_2 in the case $H > \frac{1}{2}$, we can improve the estimates in parts (b) and (c) in Lemma A.5 to

$$\int_0^T (|\hat{h}_1(t)|^2 + |\hat{h}_2(t)|^2) dt \leq C(1 + |x - x_0|^2 + |y - y_0|^2) T^{2-2H},$$

and

$$\begin{aligned}
 & \int_0^T (|\tilde{h}_1(t) - \hat{h}_1(t)|^2 + |\tilde{h}_2(t) - \hat{h}_2(t)|^2) dt \\
 & \leq C(1 + \|\tilde{B}\|_{\frac{1}{2}-\epsilon}^2 + \|\tilde{W}\|_{\frac{1}{2}-\epsilon}^2 + \|\tilde{B}^H\|_{\frac{1}{2}-\epsilon}^2 + \|\tilde{B}\|_\infty^2 + \|\tilde{W}\|_\infty^2 + \|\tilde{B}^H\|_\infty^2) T^{2-2H} \\
 & \quad + C(|x - x_0|^2 + |y - y_0|^2) T^{2-2H}.
 \end{aligned}$$

Therefore, in this linear case Theorem 3.3 holds without the restriction $H < \frac{3}{4}$. Furthermore, based on the above estimates, we can deduce $\omega(T)$ to be

$$\begin{aligned}
 \omega(T) = & \frac{1}{\bar{\rho}_H^2} \left\{ \left(\frac{\bar{\rho}\langle\hat{h}_1\rangle}{\sqrt{T}} + \frac{\rho\langle\hat{h}_2\rangle}{\sqrt{T}} - \frac{\rho_H\langle\bar{h}_2\rangle}{T^H} \right) \left(\frac{x - x_0}{\sqrt{T}} \right) \right. \\
 & \left. - \rho_H \left(\frac{\bar{\rho}\langle\hat{h}_1\rangle}{\sqrt{T}} + \frac{\rho\langle\hat{h}_2\rangle}{\sqrt{T}} - \frac{\rho_H\langle\bar{h}_2\rangle}{T^H} \right) \left(\frac{y - y_0}{T^H} \right) + \bar{\rho}_H^2 \frac{\langle\bar{h}_2\rangle}{T^H} \left(\frac{y - y_0}{T^H} \right) \right\} \\
 & + O(T^\alpha),
 \end{aligned}$$

where

$$\alpha = \begin{cases} 2 - 2H & \text{if } H \in (\frac{1}{2}, 1), \\ 2H & \text{if } H \in (0, \frac{1}{2}]. \end{cases}$$

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Appendix

In this appendix, after reviewing basic but essential background technicalities for dealing with the fractional Brownian motion, several lemmas that are used in the main part will be established.

A.1. Representation of fractional Brownian motion on an interval

Let $a, b \in \mathbb{R}$ with $a < b$. Let $f \in L^1([a, b])$ and $\alpha > 0$. The left-sided fractional Riemann–Liouville integrals of f of order α are defined for almost all $t \in (a, b)$ by

$$I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds,$$

where $\Gamma(\alpha) = \int_0^{\infty} r^{\alpha-1} e^{-r} dr$ is the Euler gamma function. Let $I_{a+}^{\alpha}(L^p([a, b]))$ be the image of $L^p([a, b])$ by the operator I_{a+}^{α} .

Fractional integration admits the following composition formulas:

$$I_{a+}^{\alpha} I_{a+}^{\beta} f = I_{a+}^{\alpha+\beta} f, \quad (\text{A.1})$$

for any $f \in L^1([a, b])$.

If $f \in I_{a+}^{\alpha}(L^p([a, b]))$ and $0 < \alpha < 1$ then the left-side Weyl derivative is defined as

$$D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(t-a)^{\alpha}} + \alpha \int_a^t \frac{f(t) - f(s)}{(t-s)^{\alpha+1}} ds \right) 1_{(a,b)}(t)$$

for almost all $t \in (a, b)$ (the convergence of the integrals at the singularity $s = t$ holds point-wise for almost all $t \in (a, b)$ if $p = 1$ and moreover in L^p -sense if $1 < p < \infty$).

From Theorems 3.5 and 3.6 in [14], we have:

(i) If $\alpha < \frac{1}{p}$ and $q = \frac{p}{1-\alpha p}$ then

$$I_{a+}^{\alpha}(L^p([a, b])) \subset L^q([a, b]).$$

(ii) If $\beta > \alpha$, then

$$C^{\beta}([a, b]) \subset I_{a+}^{\alpha}(L^p([a, b])), \quad \forall p > 1.$$

In the following sections, let $T > 0$ be a fixed number.

Definition 1. A centered Gaussian process $B^H = \{B_t^H; t \in [0, T]\}$ is called fractional Brownian motion (fBm for short) with Hurst parameter $H \in (0, 1)$ if it has the covariance function

$$R_H(s, t) = \mathbb{E}(B_s^H B_t^H) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}), \quad (\text{A.2})$$

for all $s, t \in [0, T]$.

For $H = \frac{1}{2}$, the process $B^{\frac{1}{2}}$ is a standard Brownian motion. For $H \neq \frac{1}{2}$, the fBm B^H is not a semimartingale. It follows from (A.2) that

$$\mathbb{E}(|B_t^H - B_s^H|^2) = |t-s|^{2H}.$$

Furthermore, by Kolmogorov's continuity criterion, B^H is Hölder continuous of order β for all $\beta < H$.

The fractional Brownian motion B^H has the following integral representation (see [5] and [11])

$$B_t^H = \int_0^T K_H(t, s) dB_s,$$

where $B = \{B_t, t \in [0, T]\}$ is a standard Brownian motion and

$$K_H(t, s) = c_H \left[\left(\frac{t}{s} \right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2} \right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right] \times \mathbf{1}_{[0,t]}(s). \quad (\text{A.3})$$

$$\text{with } c_H = \left[\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)\Gamma(H+\frac{1}{2})} \right]^{1/2}.$$

For any $H \in (0, 1)$, consider the integral transform

$$(\mathcal{K}_H f)(t) = \int_0^T K_H(t, s) f(s) ds. \quad (\text{A.4})$$

Then, we have the following important fact (see Theorem 2.1 in [5] and (10.22) in [14]).

Lemma A.1. *The operator \mathcal{K}_H is an isomorphism from $L^2([0, T])$ onto $I_{0+}^{H+\frac{1}{2}}(L^2([0, T]))$ and it can be expressed in terms of fractional integrals as follows*

$$\mathcal{K}_H f = c_H I_{0+}^1 u^{H-\frac{1}{2}} I_{0+}^{H-\frac{1}{2}} u^{\frac{1}{2}-H} f, \quad \text{if } H > \frac{1}{2}, \quad (\text{A.5})$$

$$\mathcal{K}_H f = c_H I_{0+}^{2H} u^{\frac{1}{2}-H} I_{0+}^{\frac{1}{2}-H} u^{H-\frac{1}{2}} f, \quad \text{if } H \leq \frac{1}{2}. \quad (\text{A.6})$$

From (A.5) and (A.6), the inverse operator \mathcal{K}_H^{-1} is given by

$$\mathcal{K}_H^{-1} h = c_H^{-1} s^{H-\frac{1}{2}} D_{0+}^{H-\frac{1}{2}} s^{\frac{1}{2}-H} h', \quad \text{if } H > \frac{1}{2}, \quad (\text{A.7})$$

$$\mathcal{K}_H^{-1} h = c_H^{-1} s^{\frac{1}{2}-H} D_{0+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} D_{0+}^{2H} h, \quad \text{if } H \leq \frac{1}{2}, \quad (\text{A.8})$$

for all $h \in I_{0+}^{H+\frac{1}{2}}(L^2([0, T]))$, where h' is the derivative of h if h is absolutely continuous. For the case $H \leq \frac{1}{2}$, if h is absolutely continuous, we can apply (10.6) in [14] to get

$$\mathcal{K}_H^{-1} h = c_H^{-1} s^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} s^{\frac{1}{2}-H} h'. \quad (\text{A.9})$$

The following lemma, combined with the statements in the latter half of Lemma A.1, ensures that \tilde{h}_2 in (A.11) is well-defined.

Lemma A.2. *We have*

$$\int_0^\cdot h_2(s, X_s, Y_s) ds \in I_{0+}^{H+\frac{1}{2}}(L^2([0, T])) \quad \text{almost surely.} \quad (\text{A.10})$$

Proof. The case $H = \frac{1}{2}$ is trivial.

For the case $H < \frac{1}{2}$, by (A.1) we have

$$\int_0^\cdot h_2(s, X_s, Y_s) ds = I_{0+}^1 h_2(\cdot, X_\cdot, Y_\cdot) = I_{0+}^{H+\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} h_2(\cdot, X_\cdot, Y_\cdot).$$

Note that (2.5) in the proof of Theorem 2.1 and the linear growth condition (2.4) on h_2 imply that $h_2(\cdot, X_\cdot, Y_\cdot)$ is in $L^2([0, T])$. Using (i) in this section with $\alpha = \frac{1}{2} - H$, $p = 2$ and $q = \frac{p}{1-\alpha p} = \frac{1}{H} > 2$, we can show that $I_{0+}^{\frac{1}{2}-H} h_2(\cdot, X_\cdot, Y_\cdot) \in L^q([0, T]) \subset L^2([0, T])$ which implies the result.

For the case $H > \frac{1}{2}$, by (A.1), we need $h_2(\cdot, X_\cdot, Y_\cdot) \in I_{0+}^{H-\frac{1}{2}}(L^2([0, T]))$, which is implied by (ii) with $\alpha = H - \frac{1}{2} < \gamma < \frac{1}{2}$ and the fact that $h_2(\cdot, X_\cdot, Y_\cdot) \in C^\gamma([0, T])$ from the result in Theorem 2.1. \square

Since $\int_0^\cdot h_2(s, X_s, Y_s)ds \in I_{0+}^{H+\frac{1}{2}}(L^2([0, T]))$ almost surely and the operator \mathcal{K}_H^{-1} preserves adaptability, there exist adapted stochastic processes $\tilde{h}_1, \tilde{h}_2 \in L^2([0, T])$ such that

$$\tilde{h}_2(t) = \mathcal{K}_H^{-1} \left(\int_0^\cdot h_2(s, X_s, Y_s)ds \right) (t), \quad (\text{A.11})$$

and

$$\rho \tilde{h}_2(t) + \sqrt{1 - \rho^2} \tilde{h}_1(t) = h_1(t, X_t, Y_t). \quad (\text{A.12})$$

Similarly, we have

Lemma A.3. *The two functions \hat{h}_1 and \hat{h}_2 determined by Eqs. (3.10) and (3.11) are well-defined.*

Proof. Note that by applying the Cauchy–Schwarz inequality we have for any $s, t \in [0, T]$

$$\begin{aligned} \left| \int_0^t K_H(T, u)du - \int_0^s K_H(T, u)du \right| &= |\mathbb{E}[(B_t - B_s)B_T^H]| \\ &\leq (\mathbb{E}[|B_t - B_s|^2])^{\frac{1}{2}} (\mathbb{E}[|B_T^H|^2])^{\frac{1}{2}} \\ &\leq T^H |t - s|^{\frac{1}{2}}, \end{aligned}$$

and

$$\begin{aligned} |R_H(t, T) - R_H(s, T)| &= |\mathbb{E}[(B_t^H - B_s^H)B_T^H]| \\ &\leq (\mathbb{E}[|B_t^H - B_s^H|^2])^{\frac{1}{2}} (\mathbb{E}[|B_T^H|^2])^{\frac{1}{2}} = T^H |t - s|^H. \end{aligned}$$

It follows from (3.4) and (3.5) that the expectations $\tilde{\mathbb{E}}[X_t^{x,y}]$ and $\tilde{\mathbb{E}}[Y_t^{x,y}]$ are α -Hölder continuous in t of any order $\alpha < \min\{H, \frac{1}{2}\}$. Hence, as in the proof of Lemma A.2, we can show that $\int_0^\cdot h_2(s, \tilde{\mathbb{E}}[X_s^{x,y}], \tilde{\mathbb{E}}[Y_s^{x,y}])ds \in I_{0+}^{H+\frac{1}{2}}(L^2([0, T]))$. We conclude that the two deterministic functions \hat{h}_1 and \hat{h}_2 are well-defined. \square

A.2. The conditional expectation of Gaussian random vectors and Gaussian bridges

Let $\mathbf{X} = (X_1, \dots, X_n)'$ and $\mathbf{Y} = (Y_1, \dots, Y_m)'$ be joint Gaussian random vectors and $\mathbf{Z} = (X_1, \dots, X_n, Y_1, \dots, Y_m)'$. Denote the expectations of \mathbf{X} , \mathbf{Y} and the covariance matrix for \mathbf{Z} by

$$\mathbb{E}[\mathbf{X}] = \mu_{\mathbf{X}}, \quad \mathbb{E}[\mathbf{Y}] = \mu_{\mathbf{Y}}$$

and

$$\Sigma = \text{Cov} \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix} = \begin{pmatrix} \Sigma_{\mathbf{XX}} & \Sigma_{\mathbf{XY}} \\ \Sigma_{\mathbf{YX}} & \Sigma_{\mathbf{YY}} \end{pmatrix}.$$

The following lemma gives the conditional distribution of Gaussian random vectors.

Lemma A.4. Suppose that the covariance matrix Σ is positive definite. Then, the conditional distribution of \mathbf{X} given that $\mathbf{Y} = \mathbf{y}$ is n -dimensional Gaussian with expectation

$$\mathbb{E}[\mathbf{X}|\mathbf{Y} = \mathbf{y}] = \mu_{\mathbf{X}} + \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}(\mathbf{y} - \mu_{\mathbf{Y}})$$

and covariance matrix

$$\text{Cov}[\mathbf{X}|\mathbf{Y} = \mathbf{y}] = \Sigma_{\mathbf{XX}} - \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}}.$$

Moreover, the Gaussian vector \mathbf{X} has the following decomposition

$$\mathbf{X} = \mu_{\mathbf{X}} + \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}(\mathbf{Y} - \mu_{\mathbf{Y}}) + \mathbf{V},$$

where the random vector \mathbf{V} is n -dimensional Gaussian with zero expectation and the following covariance matrix

$$\text{Cov}[\mathbf{V}] = \Sigma_{\mathbf{XX}} - \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}}.$$

A.3. Some estimates on \tilde{h}_i and \hat{h}_i , $i = 1, 2$

We will give some important estimates on \tilde{h}_i and \hat{h}_i , $i = 1, 2$, in the cases $H > \frac{1}{2}$ and $H \leq \frac{1}{2}$ respectively.

Lemma A.5.

(1) In the case $H > \frac{1}{2}$, there exists a constant C depending on x_0 , y_0 , ρ and the constants L in (2.3) and K in (2.4) such that, for any $0 < \epsilon < 1 - H$,

(a)

$$\begin{aligned} & \int_0^T (|\tilde{h}_1(t)|^2 + |\tilde{h}_2(t)|^2) dt \\ & \leq C(1 + \|\tilde{\mathbf{B}}\|_{\infty}^2 + \|\tilde{\mathbf{W}}\|_{\infty}^2 + \|\tilde{\mathbf{B}}^H\|_{\infty}^2 + \|\tilde{\mathbf{B}}\|_{\frac{1}{2}-\epsilon}^2 + \|\tilde{\mathbf{W}}\|_{\frac{1}{2}-\epsilon}^2 \\ & \quad + \|\tilde{\mathbf{B}}^H\|_{\frac{1}{2}-\epsilon}^2) T^{2-2H}; \end{aligned} \quad (\text{A.13})$$

(b)

$$\begin{aligned} & \int_0^T (|\hat{h}_1(t)|^2 + |\hat{h}_2(t)|^2) dt \leq C(1 + |x - x_0|^2 + |y - y_0|^2) T^{2-2H} \\ & \quad + C|y - y_0|^2 T^{3-4H}; \end{aligned} \quad (\text{A.14})$$

(c)

$$\begin{aligned} & \int_0^T (|\tilde{h}_1(t) - \hat{h}_1(t)|^2 + |\tilde{h}_2(t) - \hat{h}_2(t)|^2) dt \\ & \leq C(1 + \|\tilde{\mathbf{B}}\|_{\frac{1}{2}-\epsilon}^2 + \|\tilde{\mathbf{W}}\|_{\frac{1}{2}-\epsilon}^2 + \|\tilde{\mathbf{B}}^H\|_{\frac{1}{2}-\epsilon}^2 + \|\tilde{\mathbf{B}}\|_{\infty}^2 + \|\tilde{\mathbf{W}}\|_{\infty}^2 + \|\tilde{\mathbf{B}}^H\|_{\infty}^2) T^{2-2H} \\ & \quad + C(|x - x_0|^2 + |y - y_0|^2) T^{2-2H} + C|y - y_0|^2 T^{3-4H}. \end{aligned} \quad (\text{A.15})$$

(2) In the case $H \leq \frac{1}{2}$, there exists a constant C depending on x_0, y_0, ρ and the constants L in (2.3) and K in (2.4) such that, for any $0 < \epsilon < 1 - H$,

(e)

$$\begin{aligned} \int_0^T (|\tilde{h}_1(s)|^2 + |\tilde{h}_2(s)|^2) ds &\leq C \left(1 + \|\tilde{B}\|_\infty^2 + \|\tilde{W}\|_\infty^2 + \|\tilde{B}^H\|_\infty^2\right) T \\ &\leq C \left(1 + \|\tilde{B}\|_\infty^2 + \|\tilde{W}\|_\infty^2 + \|\tilde{B}^H\|_\infty^2\right) T^{2H}; \end{aligned} \quad (\text{A.16})$$

(f)

$$\int_0^T (|\hat{h}_1(t)|^2 + |\hat{h}_2(t)|^2) dt \leq C(1 + |x - x_0|^2 + |y - y_0|^2) T^{2H}; \quad (\text{A.17})$$

(g)

$$\begin{aligned} &\int_0^T (|\tilde{h}_1(t) - \hat{h}_1(t)|^2 + |\tilde{h}_2(t) - \hat{h}_2(t)|^2) dt \\ &\leq C(\|\tilde{B}\|_\infty^2 + \|\tilde{W}\|_\infty^2 + \|\tilde{B}^H\|_\infty^2 + |x - x_0|^2 + |y - y_0|^2) T^{2H}. \end{aligned} \quad (\text{A.18})$$

Proof. Case $H > \frac{1}{2}$: We choose an arbitrary small $0 < \epsilon < 1 - H$ (note that $1 - H < \frac{1}{2}$).

In the following, we will use C to denote a generic constant which is dependent on x_0, y_0, ρ and the constants L in (2.3) and K in (2.4) but independent of T and (x, y) .

For any $s, t \in [0, T]$, by (2.23), (2.24), the linear growth condition (2.4), the Hölder continuity condition (2.3) and the Lipschitz condition (2.2) on $h_i, i = 1, 2$, we have

$$|h_i(t, X_t, Y_t)| \leq C(1 + \|\tilde{B}\|_\infty + \|\tilde{W}\|_\infty + \|\tilde{B}^H\|_\infty), \quad (\text{A.19})$$

and

$$\begin{aligned} &|h_2(t, X_t, Y_t) - h_2(s, X_s, Y_s)| \\ &\leq C|t - s|^\gamma + C(\|\tilde{B}\|_{\frac{1}{2}-\epsilon} + \|\tilde{W}\|_{\frac{1}{2}-\epsilon} + \|\tilde{B}^H\|_{\frac{1}{2}-\epsilon})|t - s|^{\frac{1}{2}-\epsilon}. \end{aligned} \quad (\text{A.20})$$

Considering (2.12), analogue to (2.13) and (2.14), we get by (A.19) and (A.20)

$$\begin{aligned} |a(t)| &\leq C(1 + \|\tilde{B}\|_\infty + \|\tilde{W}\|_\infty + \|\tilde{B}^H\|_\infty) t^{\frac{1}{2}-H} \\ &\quad + C(1 + \|\tilde{B}\|_{\frac{1}{2}-\epsilon} + \|\tilde{W}\|_{\frac{1}{2}-\epsilon} + \|\tilde{B}^H\|_{\frac{1}{2}-\epsilon}), \end{aligned} \quad (\text{A.21})$$

and by (A.19) and a change of variables we have

$$|b(t)| \leq C(1 + \|\tilde{B}\|_\infty + \|\tilde{W}\|_\infty + \|\tilde{B}^H\|_\infty) t^{\frac{1}{2}-H}. \quad (\text{A.22})$$

Then, (2.12), (A.21) and (A.22) imply

$$\begin{aligned} |\tilde{h}_2(t)| &\leq C(1 + \|\tilde{B}\|_\infty + \|\tilde{W}\|_\infty + \|\tilde{B}^H\|_\infty) t^{\frac{1}{2}-H} \\ &\quad + C(1 + \|\tilde{B}\|_{\frac{1}{2}-\epsilon} + \|\tilde{W}\|_{\frac{1}{2}-\epsilon} + \|\tilde{B}^H\|_{\frac{1}{2}-\epsilon}). \end{aligned} \quad (\text{A.23})$$

From (A.12), (A.19) and (A.23), we obtain

$$\begin{aligned} |\tilde{h}_1(t)| &\leq C(1 + \|\tilde{B}\|_\infty + \|\tilde{W}\|_\infty + \|\tilde{B}^H\|_\infty + \|\tilde{B}\|_{\frac{1}{2}-\epsilon} + \|\tilde{W}\|_{\frac{1}{2}-\epsilon} + \|\tilde{B}^H\|_{\frac{1}{2}-\epsilon}) \\ &\quad + C(1 + \|\tilde{B}\|_\infty + \|\tilde{W}\|_\infty + \|\tilde{B}^H\|_\infty) t^{\frac{1}{2}-H}. \end{aligned} \quad (\text{A.24})$$

Thus, by (A.23), (A.24) and noticing the fact that $2 - 2H < 1$, one can obtain (A.13).

Now let us prove Part (b). From (3.6)–(3.9), it implies the following estimates

$$|m_{11}(t; T)| \leq \frac{1 + \rho\rho_H}{\bar{\rho}_H^2}, \quad |m_{12}(t; T)| \leq \frac{\rho + \rho_H}{\bar{\rho}_H^2 T^{H-\frac{1}{2}}}, \quad (\text{A.25})$$

$$|m_{21}(t; T)| \leq \frac{2\rho_H T^{H-\frac{1}{2}}}{\bar{\rho}_H^2}, \quad |m_{22}(t; T)| \leq \frac{1 + \rho_H^2}{\bar{\rho}_H^2}. \quad (\text{A.26})$$

Note that in the case $H > \frac{1}{2}$, we further have

$$|m_{21}(t; T)| \leq \frac{2\rho_H}{\bar{\rho}_H^2}. \quad (\text{A.27})$$

Moreover, for all $s, t \in [0, T]$, there exists a constant C depending on ρ and H such that

$$\begin{aligned} |m_{11}(t; T) - m_{11}(s; T)| &\leq C \left(\frac{|t-s|}{T} + \frac{|\int_s^t K_H(T, u) du|}{T^{H+\frac{1}{2}}} \right) \\ &\leq C \left(\frac{|t-s|}{T} + \frac{|\tilde{\mathbb{E}}[B_T^H(B_t - B_s)]|}{T^{H+\frac{1}{2}}} \right) \\ &\leq C \left(\frac{|t-s|}{T} + \frac{|t-s|^{\frac{1}{2}}}{T^{\frac{1}{2}}} \right) \leq C \frac{|t-s|^{\frac{1}{2}}}{T^{\frac{1}{2}}}, \end{aligned} \quad (\text{A.28})$$

$$\begin{aligned} |m_{12}(t; T) - m_{12}(s; T)| &\leq C \left(\frac{|t-s|}{T^{H+\frac{1}{2}}} + \frac{|\int_s^t K_H(T, u) du|}{T^{2H}} \right) \\ &\leq C \left(\frac{|t-s|}{T^{H+\frac{1}{2}}} + \frac{|t-s|^{\frac{1}{2}}}{T^H} \right) \leq C \frac{|t-s|^{\frac{1}{2}}}{T^H}, \end{aligned} \quad (\text{A.29})$$

$$\begin{aligned} |m_{21}(t; T) - m_{21}(s; T)| &\leq C \left(\frac{|t^{H+\frac{1}{2}} - s^{H+\frac{1}{2}}|}{T} + \frac{|R_H(t, T) - R_H(s, T)|}{T^{H+\frac{1}{2}}} \right) \\ &\leq C \left(\frac{|t-s|}{T^{\frac{3}{2}-H}} + \frac{|\tilde{\mathbb{E}}[B_T^H(B_t - B_s)]|}{t^{H+\frac{1}{2}}} \right) \\ &\leq C \left(\frac{|t-s|}{T^{\frac{3}{2}-H}} + \frac{|t-s|^H}{T^{\frac{1}{2}}} \right) \leq C \frac{|t-s|^H}{T^{\frac{1}{2}}} \leq C \frac{|t-s|^{\frac{1}{2}}}{T^{\frac{1}{2}}}, \end{aligned} \quad (\text{A.30})$$

and

$$\begin{aligned} |m_{22}(t; T) - m_{22}(s; T)| &\leq C \left(\frac{|t^{H+\frac{1}{2}} - s^{H+\frac{1}{2}}|}{T^{H+\frac{1}{2}}} + \frac{|R_H(t, T) - R_H(s, T)|}{T^{2H}} \right) \\ &\leq C \left(\frac{|t-s|}{T} + \frac{|t-s|^H}{T^H} \right) \leq C \frac{|t-s|^H}{T^H}. \end{aligned} \quad (\text{A.31})$$

Similar to (2.12), we can write

$$\hat{h}_2(t) = \frac{c_H^{-1}}{\Gamma(\frac{3}{2} - H)} (\hat{a}(t) + \hat{b}(t)), \quad (\text{A.32})$$

where

$$\hat{a}(t) = t^{\frac{1}{2}-H} \bar{h}_2(t) + (H - \frac{1}{2}) \int_0^t \frac{\bar{h}_2(t) - \bar{h}_2(s)}{(t-s)^{H+\frac{1}{2}}} ds,$$

and

$$\hat{b}(t) = (H - \frac{1}{2}) t^{H-\frac{1}{2}} \int_0^t \frac{(t^{\frac{1}{2}-H} - s^{\frac{1}{2}-H}) \bar{h}_2(s)}{(t-s)^{H+\frac{1}{2}}} ds.$$

By the linear growth condition (2.4), (3.4)–(3.9) and (A.25)–(A.27), we can see

$$|\bar{h}_i(t)| \leq C(1 + |x - x_0| + |y - y_0| + |y - y_0| T^{\frac{1}{2}-H}), \quad i = 1, 2. \quad (\text{A.33})$$

From the Hölder continuity condition (2.3), the Lipschitz condition (2.2) on h_2 , the definition of \bar{h}_2 , (3.4)–(3.9) and (A.28)–(A.31), it is easy to show

$$\begin{aligned} |\bar{h}_2(t) - \bar{h}_2(s)| &\leq C|t - s|^\gamma + C|x - x_0| \frac{|t - s|^{\frac{1}{2}}}{T^{\frac{1}{2}}} \\ &\quad + C|y - y_0| \left(\frac{|t - s|^{\frac{1}{2}} + |t - s|^H}{T^H} \right). \end{aligned} \quad (\text{A.34})$$

Thus, analogue to the proofs of (A.21) and (A.22), we obtain

$$\begin{aligned} |\hat{a}(t)| &\leq C(1 + |x - x_0| + |y - y_0| + |y - y_0| T^{\frac{1}{2}-H}) t^{\frac{1}{2}-H} + C t^{\gamma-H+\frac{1}{2}} \\ &\quad + C|x - x_0| \frac{t^{1-H}}{T^{\frac{1}{2}}} + C|y - y_0| \left(\frac{t^{1-H} + t^{\frac{1}{2}}}{T^H} \right), \end{aligned} \quad (\text{A.35})$$

and

$$|\hat{b}(t)| \leq C(1 + |x - x_0| + |y - y_0| + |y - y_0| T^{\frac{1}{2}-H}) t^{\frac{1}{2}-H}. \quad (\text{A.36})$$

So, from (A.32), (A.35) and (A.36) it implies

$$\begin{aligned} |\hat{h}_2(t)| &\leq C(1 + |x - x_0| + |y - y_0| + |y - y_0| T^{\frac{1}{2}-H}) t^{\frac{1}{2}-H} + C t^{\gamma-H+\frac{1}{2}} \\ &\quad + C|x - x_0| \frac{t^{1-H}}{T^{\frac{1}{2}}} + C|y - y_0| \left(\frac{t^{1-H} + t^{\frac{1}{2}}}{T^H} \right). \end{aligned} \quad (\text{A.37})$$

Then, by (3.10), (A.33) and (A.37) one has

$$\begin{aligned} |\hat{h}_1(t)| &\leq C(1 + |x - x_0| + |y - y_0| + |y - y_0| T^{\frac{1}{2}-H}) (1 + t^{\frac{1}{2}-H}) + C t^{\gamma-H+\frac{1}{2}} \\ &\quad + C|x - x_0| \frac{t^{1-H}}{T^{\frac{1}{2}}} + C|y - y_0| \left(\frac{t^{1-H} + t^{\frac{1}{2}}}{T^H} \right). \end{aligned} \quad (\text{A.38})$$

Therefore, taking into account of (A.37) and (A.38), we can prove

$$\begin{aligned} &\int_0^T (|\hat{h}_1(t)|^2 + |\hat{h}_2(t)|^2) dt \\ &\leq C(1 + |x - x_0|^2 + |y - y_0|^2 + |y - y_0|^2 T^{1-2H}) (T + T^{2-2H}) + C T^{2\gamma-2H+\frac{1}{2}} \\ &\quad + C|x - x_0|^2 T^{2-2H} + C|y - y_0|^2 (T^{3-4H} + T^{2-2H}) \\ &\leq C(1 + |x - x_0|^2 + |y - y_0|^2) T^{2-2H} + C|y - y_0|^2 T^{3-4H}, \end{aligned}$$

since $\gamma > 0$.

Now let us prove Part (c). By the definitions of \tilde{h}_2 and \hat{h}_2 in (A.11) and (3.10), and (A.7) we can write

$$\tilde{h}_2(t) - \hat{h}_2(t) = \frac{c_H^{-1}}{\Gamma(\frac{3}{2} - H)}(\bar{a}(t) + \bar{b}(t)), \quad (\text{A.39})$$

where

$$\begin{aligned} \bar{a}(t) = & \left(H - \frac{1}{2}\right) \int_0^t \frac{(h_2(t, X_t, Y_t) - \tilde{h}_2(t)) - (h_2(s, X_s, Y_s) - \tilde{h}_2(s))}{(t-s)^{H+\frac{1}{2}}} ds \\ & + t^{\frac{1}{2}-H} (h_2(t, X_t, Y_t) - \tilde{h}_2(t)), \end{aligned}$$

and

$$\bar{b}(t) = \left(H - \frac{1}{2}\right) t^{H-\frac{1}{2}} \int_0^t \frac{(t^{\frac{1}{2}-H} - s^{\frac{1}{2}-H})(h_2(s, X_s, Y_s) - \tilde{h}_2(s))}{(t-s)^{H+\frac{1}{2}}} ds.$$

For any $t \in [0, T]$, from the Lipschitz condition on h_i , $i = 1, 2$, (2.23), (2.24), (3.4) and (A.25)–(A.27), it implies

$$\begin{aligned} & |h_i(t, X_t, Y_t) - \tilde{h}_i(t)| \\ & \leq C(|X_t - x_0| + |Y_t - y_0| + (|m_{11}(t; T)| + |m_{21}(t; T)|)|x - x_0| \\ & \quad + (|m_{12}(t; T)| + |m_{22}(t; T)|)|y - y_0|) \\ & \leq C(\|\tilde{B}\|_\infty + \|\tilde{W}\|_\infty + \|\tilde{B}^H\|_\infty) + C(|x - x_0| + |y - y_0| + |y - y_0|T^{\frac{1}{2}-H}). \end{aligned} \quad (\text{A.40})$$

For any $s, t \in [0, T]$, the inequalities (A.40), (A.20), (A.34) and the calculations in (A.21) and (A.35) yield

$$\begin{aligned} |\bar{a}(t)| \leq & C(1 + \|\tilde{B}\|_{\frac{1}{2}-\epsilon} + \|\tilde{W}\|_{\frac{1}{2}-\epsilon} + \|\tilde{B}^H\|_{\frac{1}{2}-\epsilon}) \\ & + C|x - x_0| \frac{t^{1-H}}{T^{\frac{1}{2}}} + C|y - y_0| \left(\frac{t^{1-H} + t^{\frac{1}{2}}}{T^H} \right) \\ & + C(\|\tilde{B}\|_\infty + \|\tilde{W}\|_\infty + \|\tilde{B}^H\|_\infty + |x - x_0| + |y - y_0| \\ & + |y - y_0|T^{\frac{1}{2}-H})t^{\frac{1}{2}-H}. \end{aligned} \quad (\text{A.41})$$

By (A.40) and a change of variables we can prove that

$$\begin{aligned} |\bar{b}(t)| \leq & C(\|\tilde{B}\|_\infty + \|\tilde{W}\|_\infty + \|\tilde{B}^H\|_\infty + |x - x_0| + |y - y_0| \\ & + |y - y_0|T^{\frac{1}{2}-H})t^{\frac{1}{2}-H}. \end{aligned} \quad (\text{A.42})$$

Thus, by (A.39)–(A.42), we have

$$\begin{aligned} & \int_0^T |\tilde{h}_2(t) - \hat{h}_2(t)|^2 dt \\ & \leq C(1 + \|\tilde{B}\|_{\frac{1}{2}-\epsilon}^2 + \|\tilde{W}\|_{\frac{1}{2}-\epsilon}^2 + \|\tilde{B}^H\|_{\frac{1}{2}-\epsilon}^2 + \|\tilde{B}\|_\infty^2 + \|\tilde{W}\|_\infty^2 + \|\tilde{B}^H\|_\infty^2)T^{2-2H} \\ & \quad + C(|x - x_0|^2 + |y - y_0|^2)T^{2-2H} + C|y - y_0|^2T^{3-4H}, \end{aligned} \quad (\text{A.43})$$

which implies together from (A.12), (3.11) and (A.40)

$$\begin{aligned} & \int_0^T |\tilde{h}_1(t) - \hat{h}_1(t)|^2 dt \\ & \leq C(1 + \|\tilde{B}\|_{\frac{1}{2}-\epsilon}^2 + \|\tilde{W}\|_{\frac{1}{2}-\epsilon}^2 + \|\tilde{B}^H\|_{\frac{1}{2}-\epsilon}^2 + \|\tilde{B}\|_{\infty}^2 + \|\tilde{W}\|_{\infty}^2 + \|\tilde{B}^H\|_{\infty}^2) T^{2-2H} \\ & \quad + C(|x - x_0|^2 + |y - y_0|^2) T^{2-2H} + C|y - y_0|^2 T^{3-4H}. \end{aligned} \quad (\text{A.44})$$

Hence, by (A.43) and (A.44) we obtain (A.15).

Case $H \leq \frac{1}{2}$: Proof of Part (e): From (A.9), (2.4), (2.23), (2.24) and a change of variables it implies

$$\begin{aligned} |\tilde{h}_2(t)| &= c_H^{-1} t^{H-\frac{1}{2}} \left| \int_0^t (t-s)^{-\frac{1}{2}-H} s^{\frac{1}{2}-H} h_2(s, X_s, Y_s) ds \right| \\ &\leq C(1 + \|\tilde{B}\|_{\infty} + \|\tilde{W}\|_{\infty} + \|\tilde{B}^H\|_{\infty}) B \left(\frac{1}{2} - H, \frac{3}{2} - H \right) t^{\frac{1}{2}-H} \\ &= C(1 + \|\tilde{B}\|_{\infty} + \|\tilde{W}\|_{\infty} + \|\tilde{B}^H\|_{\infty}). \end{aligned} \quad (\text{A.45})$$

From (A.12), (A.45), the linear growth condition (2.4) on h_1 , (2.23) and (2.24), we obtain

$$|\tilde{h}_1(t)| \leq C(1 + \|B\|_{\infty} + \|W\|_{\infty} + \|B^H\|_{\infty}). \quad (\text{A.46})$$

Thus, (A.45) and (A.46) imply

$$\int_0^T (|\tilde{h}_1(s)|^2 + |\tilde{h}_2(s)|^2) ds \leq C(1 + \|B\|_{\infty}^2 + \|W\|_{\infty}^2 + \|B^H\|_{\infty}^2) T.$$

Proof of Part (f): Similar to the proof of (A.45), by (A.9), the linear growth condition (2.4) on h_2 and (A.25)–(A.26), we can show

$$|\bar{h}_2(t)| \leq C(1 + |x - x_0| + |y - y_0| + |x - x_0| T^{H-\frac{1}{2}}) \quad (\text{A.47})$$

and hence,

$$\begin{aligned} |\hat{h}_2(t)| &= c_H^{-1} t^{H-\frac{1}{2}} \left| \int_0^t (t-s)^{-\frac{1}{2}-H} s^{\frac{1}{2}-H} \bar{h}_2(s) ds \right| \\ &\leq C(1 + |x - x_0| + |y - y_0| + |x - x_0| T^{H-\frac{1}{2}}). \end{aligned} \quad (\text{A.48})$$

From (3.11), (A.48), the linear growth condition (2.4) on h_1 , we have

$$|\hat{h}_1(t)| \leq C(1 + |x - x_0| + |y - y_0| + |x - x_0| T^{H-\frac{1}{2}}). \quad (\text{A.49})$$

Thus, we can obtain

$$\begin{aligned} \int_0^T (|\hat{h}_1(t)|^2 + |\hat{h}_2(t)|^2) dt &\leq C(1 + |x - x_0|^2 + |y - y_0|^2) T + C|x - x_0|^2 T^{2H} \\ &\leq C(1 + |x - x_0|^2 + |y - y_0|^2) T^{2H}. \end{aligned}$$

Proof of Part (g): From the Lipschitz condition on h_i , $i = 1, 2$, (2.23), (2.24), (3.4) and (A.25)–(A.26), it implies

$$\begin{aligned} & |h_i(t, X_t, Y_t) - \bar{h}_i(t)| \\ & \leq C(|X_t - x_0| + |Y_t - y_0| + (|m_{11}(t; T)| + |m_{21}(t; T)|)|x - x_0| \\ & \quad + (|m_{12}(t; T)| + |m_{22}(t; T)|)|y - y_0|) \\ & \leq C(\|\tilde{B}\|_{\infty} + \|\tilde{W}\|_{\infty} + \|\tilde{B}^H\|_{\infty}) + C(|x - x_0| + |y - y_0| + |x - x_0| T^{H-\frac{1}{2}}). \end{aligned} \quad (\text{A.50})$$

From (A.50), the definitions of \tilde{h}_2 and \hat{h}_2 and (A.9), one can easily see

$$\begin{aligned} & |\tilde{h}_2(t) - \hat{h}_2(t)| \\ &= c_H^{-1} t^{H-\frac{1}{2}} \left| \int_0^t (t-s)^{-\frac{1}{2}-H} s^{\frac{1}{2}-H} (h_2(s, X_s, Y_s) - \tilde{h}_2(s)) ds \right| \\ &\leq C(\|\tilde{B}\|_\infty + \|\tilde{W}\|_\infty + \|\tilde{B}^H\|_\infty) + C(|x - x_0| + |y - y_0| + |x - x_0| T^{H-\frac{1}{2}}). \end{aligned} \quad (\text{A.51})$$

From (A.12), (3.11), (A.50) and (A.51), one can also easily get

$$\begin{aligned} & |\tilde{h}_1(t) - \hat{h}_1(t)| \\ &\leq C(\|\tilde{B}\|_\infty + \|\tilde{W}\|_\infty + \|\tilde{B}^H\|_\infty) + C(|x - x_0| + |y - y_0| + |x - x_0| T^{H-\frac{1}{2}}). \end{aligned} \quad (\text{A.52})$$

Therefore, one can obtain

$$\begin{aligned} & \int_0^T (|\tilde{h}_1(t) - \hat{h}_1(t)|^2 + |\tilde{h}_2(t) - \hat{h}_2(t)|^2) dt \\ &\leq C(\|\tilde{B}\|_\infty^2 + \|\tilde{W}\|_\infty^2 + \|\tilde{B}^H\|_\infty^2 + |x - x_0|^2 + |y - y_0|^2) T \\ &\quad + C|x - x_0|^2 T^{2H} \\ &\leq C(\|\tilde{B}\|_\infty^2 + \|\tilde{W}\|_\infty^2 + \|\tilde{B}^H\|_\infty^2 + |x - x_0|^2 + |y - y_0|^2) T^{2H}. \end{aligned}$$

The proof is completed. \square

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