



Limit theorem for maximum of the storage process with fractional Brownian motion as input

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Received 12 February 2003; received in revised form 22 April 2004; accepted 1 July 2004

Available online 28 July 2004

Abstract

The maximum M_T of the storage process $Y(t) = \sup_{s \geq t} (X(s) - X(t) - c(s - t))$ in the interval $[0, T]$ is dealt with, in particular, for growing interval length T . Here $X(s)$ is a fractional Brownian motion with Hurst parameter, $0 < H < 1$. For fixed T the asymptotic behaviour of M_T was analysed by Piterbarg (Extremes 4(2) (2001) 147) by determining an approximation for the probability $P\{M_T > u\}$ for $u \rightarrow \infty$. Using this expression the convergence $P\{M_T < u_T(x)\} \rightarrow G(x)$ as $T \rightarrow \infty$ is derived where $u_T(x) \rightarrow \infty$ is a suitable normalization and $G(x) = \exp(-\exp(-x))$ the Gumbel distribution. Also the relation to the maximum of the process on a dense grid is analysed.

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Keywords: Storage process; Maximum; Limit distribution; Fractional Brownian motion; Dense grid

1. Introduction and results

We consider the storage process

$$Y(t) = \sup_{s \geq t} (X(s) - X(t) - c(s - t)),$$

where $X(t)$, $t \geq 0$, is a fractional Brownian Motion (fBM) with Hurst parameter H , $0 < H < 1$, and the constant $c > 0$ is the service rate. The fBM is a centered Gaussian

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process with stationary increments having a.s. continuous sample paths such that $E(X(t) - X(s))^2 = |t - s|^{2H}$, hence with variance $\text{Var}(X(t)) = |t|^{2H}$.

This storage process was considered in Piterbarg [11] who derived results on the large deviations. The particular probability $P\{Y(0) > u\} = P\{\sup_{t \geq 0} X(t) - ct > u\}$ was studied by Duffield and O'Connell [3], Norros [8,9] and Narayan [7]. In particular for $u \rightarrow \infty$ the asymptotic behaviour was derived in Hüsler and Piterbarg [5] and Narayan [7]. Choe and Shroff [2] gave upper bounds of this probability for a general class of integrated Gaussian processes. Albin and Samorodnitsky [1] generalize the result of Piterbarg [10] for infinitely divisible input processes.

Piterbarg [10] analysed the supremum $M(T) = \sup_{t \in [0, T]} Y(t)$ of the process $Y(t)$ in a finite interval $[0, T]$: $P\{M(T) > u\}$ for large u . His proofs showed that T can even depend on u , if T is contained in a certain interval depending on u , without changing the results (see Corollary 2). We continue in this paper to investigate the asymptotic behaviour of the supremum $M(T)$ where T is growing in relation to u , now growing faster, so that T is not included in that interval. However, we write $u = u_T$, thus u depends on T , in the sense of a normalization, such that we get an asymptotic distribution for the supremum $M(T)$. For any $x \in \mathbb{R}$, let $u_T(x) = b(T) + a(T)x$ be the normalization where $a(T)$ and $b(T)$ are

$$b(T) = (2A^{-2} \log T)^{1/(2(1-H))} + \left[\frac{h(2A^{-2})^{1/(2(1-H))} \log(2A^{-2} \log T)}{4(1-H)^2} + \frac{(2A^{-2})^{1/(2(1-H))} \log c_2}{2(1-H)} \right] (\log T)^{-(1-2H)/(2(1-H))} \quad (1)$$

and

$$a(T) = \frac{(2A^{-2})^{1/(2(1-H))}}{2(1-H)} (\log T)^{-(1-2H)/(2(1-H))} \quad (2)$$

where $A := \frac{1}{1-H} \left(\frac{H}{c(1-H)} \right)^{-H}$, $h = 2(1-H)^2/H - 1$ and the constant c_2 is given in (10).

The main result of the paper states that with these normalizing functions, the limiting distribution is the Gumbel one, as follows.

Theorem 1. Let $M_T = \sup_{0 \leq t \leq T} Y(t)$ be the supremum of the storage process $Y(t)$ with fBM as input, with Hurst parameter $H < 1$. Then with the normalizations $a(T)$ and $b(T)$ we have

$$P\{M_T \leq b(t) + xa(T)\} \rightarrow \exp(-e^{-x})$$

as $T \rightarrow \infty$.

The derivation of this result reveals also the complete dependence of M_T and the maximum $M_T^{(\delta)}$ defined with respect to $X(i\delta)$, taken on a sufficiently, fine discrete grid with mesh $\delta = \delta(T) > 0$. This maximum depends on the observations $X(i\delta)$, only, hence $\tilde{Y}(i\delta) = \sup_{l \geq 0} (X((l+i)\delta) - X(i\delta) - cl\delta)$. We use for the grid $\delta = d(2A^{-2} \log T)^{(2H-1)/(2H(1-H))} \sim d u_T^{(2H-1)/H}$ for some $d \rightarrow 0$, slowly, in the following

result. For instance, $d = 1/\log \log T$ is a possible choice. Note that if $H > \frac{1}{2}$, then δ does not tend to 0, but tends to ∞ (with $d \rightarrow 0$ not too fast).

The precise statement is as follows.

Theorem 2. Let $M_T = \sup_{0 \leq t \leq T} Y(t)$ be the supremum of the storage process $Y(t)$ with fBM as input, with Hurst parameter $H < 1$. If

$$\delta = d(2A^{-2} \log T)^{(2H-1)/(2H(1-H))}$$

for some $d \rightarrow 0$, then with the normalizations $a(T)$ and $b(T)$ we have

$$P\{M_T^{(\delta)} \leq b(T) + xa(T), M_T \leq b(T) + ya(T)\} \rightarrow \exp(-\exp(-\min(x, y))).$$

The next section discusses some properties of the storage process needed for the derivation of the two main results treated in Section 3.

2. Preliminaries

We state here some needed relations which were derived in [11]. We begin with the relation

$$P\left\{\sup_{t \in [0, T]} Y(t) \leq u\right\} = P\left\{\sup_{s \in [0, T/u], \tau \geq 0} Z(s, \tau) \leq u^{1-H}\right\},$$

where

$$Z(s, \tau) = \frac{X(u(s + \tau)) - X(su)}{\tau^H u^H v(\tau)}$$

with $v(\tau) = \tau^{-H} + c\tau^{1-H}$. The variance of the field is $v^{-2}(\tau)$. Note that $Z(s, \tau)$ is not dependent on u , that means for any u the Gaussian field $Z(s, \tau)$ has the same distribution. Thus, we do not use u as additional parameter in the notation of $Z(s, \tau)$. This is relation (3) of Piterbarg [11]. It is basic for the derivation of the limit distribution of $M(T)$.

The correlation function $r(s, \tau; s', \tau')$ of $Z(s, \tau)$ equals

$$\begin{aligned} r(s, \tau; s', \tau') &= EZ(s, \tau)Z(s', \tau')v(\tau)v(\tau') \\ &= \frac{-|s - s' + \tau - \tau'|^{2H} + |s - s' + \tau|^{2H} + |s - s' - \tau'|^{2H} - |s - s'|^{2H}}{2\tau^H \tau'^H}. \end{aligned}$$

We note that $Z(s, \tau)$ is stationary in s , but not in τ . $\sigma_Z(\tau) = v^{-1}(\tau)$ has a single maximum point at $\tau_0 = H/(c(1 - H))$. Taylor expansions show that

$$\sigma_Z(\tau) = v^{-1}(\tau) = \frac{1}{A} - \frac{B}{2A^2}(\tau - \tau_0)^2 + O((\tau - \tau_0)^3) \quad (3)$$

as $\tau \rightarrow \tau_0$, where

$$A := \frac{1}{1-H} \left(\frac{H}{c(1-H)} \right)^{-H} = v(\tau_0),$$

$$B := H \left(\frac{H}{c(1-H)} \right)^{-H-2} = v''(\tau_0)$$

and also

$$r(s, \tau; s', \tau') = 1 - \frac{1 + o(1)}{2\tau_0^{2H}} (|s - s' + \tau - \tau'|^{2H} + |s - s'|^{2H}) \quad (4)$$

as $s - s' \rightarrow 0$, $\tau \rightarrow \tau_0$, $\tau' \rightarrow \tau_0$. These relations are derived in [11]. We need, in addition, an expression of the correlation function for $|s - s'| \rightarrow \infty$. By series expansion we find for any τ, τ' with $0 < \tau_1 < \tau, \tau' < \tau_2 < \infty$, with fixed $\tau_1 < \tau_0 < \tau_2$

$$|r(s, \tau; s', \tau')| \leq C |s - s'|^{2H-2}$$

for some constant $C > 0$ and all s, s' with $|s - s'|$ sufficiently large, since

$$\begin{aligned} |r(s, \tau; s', \tau')| &= \frac{|s - s'|^{2H}}{2(\tau\tau')^H} \left(- \left| 1 + \frac{(\tau - \tau')}{(s - s')} \right|^{2H} + \left| 1 + \frac{\tau}{(s - s')} \right|^{2H} \right. \\ &\quad \left. + \left| 1 - \frac{\tau'}{(s - s')} \right|^{2H} - 1 \right) \\ &\leq \frac{|s - s'|^{2H}}{\tau_1^{2H}} 2H|2H - 1| |s - s'|^{-2} \tau_2^2 \leq C |s - s'|^{2H-2} \end{aligned}$$

if $2H \neq 1$. For $2H = 1$, we have $r(s, \tau, s, \tau') = 0$ for large $|s - s'|$ since the increments of the Brownian motion on disjoint intervals are independent.

3. Asymptotic approximations and proofs

Lemma 2 of Piterbarg [11] says that we can restrict the considered domain of (s, τ) to a domain with $|\tau - \tau_0| \leq \log v/v$, since there exists a constant C such that for any v, T

$$P \left\{ \sup_{\substack{|\tau - \tau_0| \geq \log v/v \\ 0 \leq s \leq T}} AZ(s, \tau) > v \right\} \leq CTv^{2/H} \exp \left(-\frac{1}{2} v^2 - b \log^2 v \right) \quad (5)$$

where $b = B/(2A)$. We will choose $v = Au_T^{1-H}$.

Then we need Lemma 4 from Piterbarg (2001) for the remaining domain (with a correction of a misprint). We use the notations \mathcal{H}_α for Pickands constants with $\alpha = 2H$ which are defined by

$$\mathcal{H}_\alpha = \lim_{T \rightarrow \infty} H_\alpha(T)/T = \lim_{T \rightarrow \infty} \mathbf{E} \exp \left(\max_{0 \leq t \leq T} (\sqrt{2}X(t) - |t|^\alpha) \right) / T$$

and where $X(t)$ is a fBM with parameter of $X(t)$ is $H = \alpha/2$. In addition, we denote by $\phi(u)$ and $\Phi(u)$ the density and the distribution function of unit normal law. Let $\Psi(u) = 1 - \Phi(u)$.

Lemma 1 (Piterbarg [11, Lemma 4]). For any $L > 0$, with $b = B/(2A)$ and $a = 1/(2\tau_0^{2H})$

$$P \left\{ \sup_{\substack{|\tau-\tau_0| \leq \log v/v \\ 0 \leq s \leq L}} AZ(s, \tau) > v \right\} = \sqrt{\pi} a^{2/H} b^{-1/2} \mathcal{H}_{2H}^2 L v^{(2/H)-1} \Psi(v) (1 + o(1))$$

as $v \rightarrow \infty$. This holds also for $L = v^{-1/H'}$, with $1 > H' > H$.

Actually we need a slightly more general result than the one mentioned in Lemma 1. It can be derived also from Lemma 1.

Corollary 2. The assertion of the Lemma 1 holds true for L , depending of v such that $v^{-1/H'} < L < \exp(cv^2)$, for any $H' \in (H, 1)$ and $c \in (0, \frac{1}{2})$.

Proof. Since the proof is immediate for $L = O(1)$, assume that $L \rightarrow \infty$ as $v \rightarrow \infty$ which is needed in the following. We use standard arguments of the double sum method. Using the stationary property of Z relative to s , we have for $n = [L]$,

$$P \left\{ \sup_{\substack{|\tau-\tau_0| \leq \log v/v \\ 0 \leq s \leq L}} AZ(s, \tau) > v \right\} \leq (n+1) P \left\{ \sup_{\substack{|\tau-\tau_0| \leq \log v/v \\ 0 \leq s \leq 1}} AZ(s, \tau) > v \right\} \quad (6)$$

and

$$\begin{aligned} & P \left\{ \sup_{\substack{|\tau-\tau_0| \leq \log v/v \\ 0 \leq s \leq L}} AZ(s, \tau) > v \right\} \\ & \geq n P \left\{ \sup_{\substack{|\tau-\tau_0| \leq \log v/v \\ 0 \leq s \leq 1}} AZ(s, \tau) > v \right\} \\ & \quad - \sum_{k=1}^{n-1} (n-k) P \left\{ \sup_{\substack{|\tau-\tau_0| \leq \log v/v \\ 0 \leq s \leq 1}} AZ(s, \tau) > v, \sup_{\substack{|\tau-\tau_0| \leq \log v/v \\ k \leq s \leq k+1}} AZ(s, \tau) > v \right\} \\ & =: np_1(v) - S_2(v). \end{aligned} \quad (7)$$

Note that $p_1(v) \rightarrow 0$ by Lemma 1 with $L = 1$. Thus it remains to prove the relations

$$S_2(v) = o(np_1(v)) = o(1) \quad \text{as } v \rightarrow \infty. \quad (8)$$

For the first member ($k = 1$) of $S_2(v)$ we have by Lemma 1,

$$\begin{aligned} & P \left\{ \sup_{\substack{|\tau-\tau_0| \leq \log v/v \\ 0 \leq s \leq 1}} AZ(s, \tau) > v, \sup_{\substack{|\tau-\tau_0| \leq \log v/v \\ 1 \leq s \leq 2}} AZ(s, \tau) > v \right\} \\ &= 2P \left\{ \sup_{\substack{|\tau-\tau_0| \leq \log v/v \\ 0 \leq s \leq 1}} AZ(s, \tau) > v \right\} - P \left\{ \sup_{\substack{|\tau-\tau_0| \leq \log v/v \\ 0 \leq s \leq 2}} AZ(s, \tau) > v \right\} \\ &= o(p_1(v)) \quad \text{as } v \rightarrow \infty. \end{aligned}$$

For $k > 1$, use

$$\begin{aligned} & P \left\{ \sup_{\substack{|\tau-\tau_0| \leq \log v/v \\ 0 \leq s \leq 1}} AZ(s, \tau) > v, \sup_{\substack{|\tau-\tau_0| \leq \log v/v \\ k \leq s \leq k+1}} AZ(s, \tau) > v \right\} \\ & \leq P \left\{ \sup_{\substack{|\tau'-\tau_0| \leq \log v/v, |\tau''-\tau_0| \leq \log v/v \\ 0 \leq s' \leq 1, k \leq s'' \leq k+1}} A(Z(s', \tau') + Z(s'', \tau'')) > 2v \right\}. \end{aligned}$$

It is easy to compute that the variance of the field $A(Z(s', \tau') + Z(s'', \tau''))$ on the considered set does not exceed $2 + 2(1 - \delta') = 4 - 2\delta'$, for some $\delta' > 0$ and any $k > 1$, since the correlation function of $Z(s, \tau)$ is less than 1 for separated points. Further, by elementary arguments and using (4), it follows that:

$$\begin{aligned} & E(Z(s', \tau') + Z(s'', \tau'') - Z(s'_1, \tau'_1) + Z(s''_1, \tau''_1))^2 \\ & \leq G(|s' - s'_1|^\gamma + |s'' - s''_1|^\gamma + |\tau' - \tau'_1|^\gamma + |\tau'' - \tau''_1|^\gamma) \end{aligned}$$

for some positive G and $\gamma \leq 2H$ and all $|s' - s'_1| \leq 1$, $|s'' - s''_1| \leq 1$, $|s' - s''| \geq 1$, $|s'_1 - s''_1| \geq 1$ and $|\tau' - \tau_0| \leq \log v/v$, $|\tau'' - \tau_0| \leq \log v/v$. Thus we may use the inequality of Theorem 8.1 of Piterbarg [10] to get

$$\begin{aligned} & P \left\{ \sup_{\substack{|\tau'-\tau_0| \leq \log v/v, |\tau''-\tau_0| \leq \log v/v \\ 0 \leq s' \leq 1, k \leq s'' \leq k+1}} A(Z(s', \tau') + Z(s'', \tau'')) > 2v \right\} \\ & \leq \text{const.} \cdot v^{8/\gamma-1} \exp \left(-\frac{v^2}{2-\delta'} \right). \end{aligned}$$

We use this inequality for all $k \in [2, k_0)$ where $k_0 = k_0(\varepsilon)$ is an integer sufficiently large such that

$$r(s, \tau; s', \tau') \leq \varepsilon$$

for all $|s - s'| \geq k_0$. For $k \geq k_0$, since the variance of the field does not exceed $2 + 2\varepsilon$, we can get a stronger inequality,

$$P \left\{ \sup_{\substack{|\tau' - \tau_0| \leq \log v/v, |\tau'' - \tau_0| \leq \log v/v \\ 0 \leq s' \leq 1, k \leq s'' \leq k+1}} A(Z(s', \tau') + Z(s'', \tau'')) > 2v \right\} \\ \leq \text{const.} \cdot v^{8/\gamma-1} \exp \left(-\frac{v^2}{1 + \varepsilon} \right).$$

Now, choosing ε small such that

$$\frac{1 - \varepsilon}{2(1 + \varepsilon)} > c$$

and combining all the inequalities for estimating $S_2(v)$, we get (8) and the assertion of the corollary. \square

For any L with L/u satisfying the restriction of Corollary 2, where $v = Au^{1-H} \rightarrow \infty$ with $u \rightarrow \infty$, we get with $\tau^*(u) = \log(Au^{1-H})/Au^{1-H}$, using (5) and Lemma 1

$$P \left\{ \sup_{\substack{s \in [0, L/u] \\ \tau \geq 0}} AZ(s, \tau) > Au^{1-H} \right\} \\ = P \left\{ \sup_{\substack{s \in [0, L/u] \\ |\tau - \tau_0| \leq \tau^*(u)}} AZ(s, \tau) > Au^{1-H} \right\} + O \left(P \left\{ \sup_{\substack{s \in [0, L/u] \\ |\tau - \tau_0| > \tau^*(u)}} AZ(s, \tau) > Au^{1-H} \right\} \right) \\ \sim c_1 (L/u) (Au^{1-H})^{2/H-1} \Psi(Au^{1-H}) \\ \sim c_2 Lu^h \exp \left(-\frac{1}{2} A^2 u^{2-2H} \right) \quad (9)$$

with $h = \frac{2(1-H)^2}{H} - 1$ where

$$c_1 = \sqrt{\pi} a^{2/H} b^{-1/2} \mathcal{H}_{2H}^2 \quad \text{and} \quad c_2 = a^{2/H} (2b)^{-1/2} \mathcal{H}_{2H}^2 A^{2/H-2} \quad (10)$$

are constants evaluated from Lemma 1. We are going to apply (9) for subdomains $\{(s, \tau) : s \leq L/u, \tau > 0\}$ of the domain $\{(s, \tau) : s \leq T/u, \tau > 0\}$ with suitably chosen L such that $L/u = L(T)$ satisfies the restriction of Corollary 2. Obviously $u = u_T$ depends on T as mentioned.

In the next step we show the needed relation for the normalization $u_T = u_T(x) = a(T)x + b(T)$.

Lemma 3. *The normalizing functions $b(T)$ and $a(T)$ given in (1) and (2) are such that*

$$c_2 T [b(T) + xa(T)]^h \exp \left(-\frac{1}{2} A^2 (b(T) + xa(T))^{2-2H} \right) \rightarrow e^{-x} \quad (11)$$

holds for $T \rightarrow \infty$.

Proof. Note that $a(T)$ is a positive function with

$$a(T)/b(T) \rightarrow 0 \quad \text{as } T \rightarrow \infty \quad (12)$$

for any $H < 1$ and that

$$b(T) \sim (2A^{-2} \log T)^{1/(2(1-H))}. \quad (13)$$

To prove (11), observe that

$$\begin{aligned} & \frac{1}{2} A^2 (b(T) + xa(T))^{2(1-H)} \\ &= \log T \left[1 + \left(\frac{h \log(2A^{-2} \log T)}{4(1-H)^2} \right. \right. \\ & \quad \left. \left. + \frac{\log c_2}{2(1-H)} + \frac{x}{2(1-H)} \right) (\log T)^{-1} \right]^{2(1-H)} \\ &= \log T + \left(\frac{h \log(2A^{-2} \log T)}{2(1-H)} + \log c_2 + x + o(1) \right). \end{aligned}$$

With this expression in the exponential term, the left-hand side of (11) is asymptotically equivalent to

$$c_2 T (b(T))^h T^{-1} (2A^{-2} \log T)^{-h/2(1-H)} c_2^{-1} \exp(-x + o(1)) \rightarrow \exp(-x)$$

as $T \rightarrow \infty$. \square

Combining (9) and (11) we find also for $L(T) = L/u$

$$\begin{aligned} & P \left\{ \sup_{s \in [0, L(T)], \tau \geq 0} Z(s, \tau) > (b(T) + xa(T))^{1-H} \right\} \\ & \sim c_2 L(T) (b(T) + xa(T))^{h+1} \exp \left(-\frac{A^2}{2} (b(T) + xa(T))^{2-2H} \right) \\ & \sim c_2 L(T) (b(T))^{h+1} \exp \left(-\frac{A^2}{2} (b(T) + xa(T))^{2-2H} \right) \\ & \sim [L(T)b(T)/T] \exp(-x) \end{aligned} \quad (14)$$

for any fixed x and suitably large $L(T)$ which defines the subdomain $\{(s, \tau) : s \leq L(T), \tau \geq 0\}$. $L(T)$ has to satisfy $A^{1/H'} (b(T) + xa(T))^{-(1-H)/H'} \leq L(T) \leq \exp(cA^2(b(T) + xa(T))^{2(1-H)})$ for some $1 > H' > H$ and $c < \frac{1}{2}$. The condition of Corollary 2 holds for $L(T)$, if

$$(1 + o(1))A^{1/H'} [b(T)]^{-(1-H)/H'} \leq L(T) \leq \exp((2 + o(1))c \log T) = T^{(2+o(1))c}$$

for some $c < \frac{1}{2}$, by using (12) and (13). We choose for the following a slowly increasing $L(T)$:

$$L(T) = v_T = Au_T^{1-H} \sim A(b(T))^{1-H} \sim (2 \log T)^{1/2}$$

which satisfies the condition of Corollary 2. Hence, we can apply (14).

Now, we work in the following tedious, but known way (cf. [6]) for proving Theorem 1. For $L(T)$ and $0 < \delta' < L(T)$ define the two-dimensional intervals

$$I_k = [(k-1)L(T), kL(T) - \delta') \times J(\tau_0)$$

and

$$I_k^* = [kL(T) - \delta', kL(T)) \times J(\tau_0)$$

for any $k \geq 1$, where $J(\tau_0) = \{\tau : |\tau - \tau_0| \leq \tau^*(u)\}$. These are in the first components ‘long’ and ‘short’ intervals, respectively. They depend on T which we do not denote. Then

$$[0, T/u_T] \times J(\tau_0) = \bigcup_{k=1}^{K_T} (I_k \cup I_k^*) \cup I_{K_T+1}$$

where $I_{K_T+1} = [K_T L(T), T/u_T] \times J(\tau_0)$ with $K_T = [T/(u_T L(T))] \in \mathbb{N}$. Hence $|I_{K_T+1}| \leq 2L(T) \times \tau^*(u)$. Thus with the chosen $L(T)$ we get $K_T = [T/u_T L(T)] \sim T/(Au_T^{2-H})$. We select δ' not depending on T , thus we can apply Corollary 2 for the short intervals, also.

Lemma 4. *With the definitions of I_k , $k \geq 1$, and some $\delta' > 0$, it follows that for $T \rightarrow \infty$:*

$$P\left\{\sup_{t \leq T} Y(t) > u_T\right\} \sim P\left\{\sup_{(s,\tau) \in \bigcup_{k \leq K_T} I_k} AZ(s, \tau) > Au_T^{1-H}\right\}.$$

Proof. With $v = Au_T^{1-H} = A(b_T + xa(T))^{1-H}$, any x , we have for large T

$$\begin{aligned} P\left\{\sup_{t \leq T} Y(t) > u_T\right\} &\sim P\left\{\sup_{\substack{|\tau - \tau_0| \leq \tau^*(u_T) \\ 0 \leq s \leq T/u_T}} AZ(s, \tau) > Au_T^{1-H}\right\} \\ &\geq P\left\{\sup_{(s,\tau) \in \bigcup_k I_k} AZ(s, \tau) > Au_T^{1-H}\right\} \end{aligned}$$

as lower bound, and with the Bonferroni inequality the upper bound

$$\begin{aligned} &P\left\{\sup_{\substack{|\tau - \tau_0| \leq \tau^*(u_T) \\ 0 \leq s \leq T/u_T}} AZ(s, \tau) > Au_T^{1-H}\right\} \\ &\leq P\left\{\sup_{(s,\tau) \in \bigcup_k I_k} AZ(s, \tau) > Au_T^{1-H}\right\} \\ &\quad + P\left\{\sup_{(s,\tau) \in I_{K_T+1}} AZ(s, \tau) > Au_T^{1-H}\right\} + P\left\{\sup_{(s,\tau) \in \bigcup_k I_k^*} AZ(s, \tau) > Au_T^{1-H}\right\}. \end{aligned}$$

We show that the last two probabilities of the upper bound are asymptotically negligible. For fixed $\delta' > 0$ by Corollary 2

$$\begin{aligned} P\left\{\sup_{(s,\tau) \in \bigcup_k I_k^*} AZ(s,\tau) > Au_T^{1-H}\right\} &\leq \sum_{k \leq K_T} P\left\{\sup_{(s,\tau) \in I_k^*} AZ(s,\tau) > Au_T^{1-H}\right\} \\ &\leq CK_T \delta' u_T^{(1-H)(\frac{2}{H}-1)} \Psi(Au_T^{1-H}) \\ &\sim C \delta' T / (u_T L(T)) u_T^{h+1} \exp(-(1/2) A^2 u_T^{2(1-H)}) \\ &= O(\delta' / L(T)) = o(1) \end{aligned}$$

since $L(T) \rightarrow \infty$ where C and in the following also \tilde{C} denote generic positive constants. We used that the term in (11) tends to a constant by the choice of u_T . In the same way the probability that an exceedance of u_T happens in the interval I_{K_T+1} , is asymptotically negligible, for

$$\begin{aligned} P\left\{\sup_{(s,\tau) \in I_{K_T+1}} AZ(s,\tau) > Au_T^{1-H}\right\} &\leq CL(T) u_T^{(1-H)((2/H)-1)} \Psi(Au_T^{1-H}) \\ &= O(L(T) u_T / T) = o(1) \end{aligned}$$

since $L(T) = o(T/u_T)$. \square

It means we can deal now only with the intervals I_k and want to show that the suprema of $X(t)$ on these intervals are asymptotically independent. To establish this claim we apply Berman's inequality which holds only for sequences of Gaussian r.v.'s. Therefore we define a family of grid points (s, τ) in our domain of interest, depending on T .

For some small $d > 0$ and any T , let

$$q = q(T) = du_T^{-(1-H)/H}$$

and define the grid points

$$s_{k,l} = (k-1)L(T) + lq \quad \text{and} \quad \tau_j = \tau_0 + jq$$

with $(s_{k,l}, \tau_j) \in I_k$ for integers $j \in \mathbb{Z}$, $l \geq 0$, $k \geq 1$. These grid points are denoted simpler by $(s, \tau) \in I_k \cap \mathcal{R}$ for fixed k , without mentioning the dependence on T . We need later to select some $d = d(T) \rightarrow 0$ slowly. We let $d = d(T) = 1/\log \log T$.

For any k the index l of points $s_{k,l}$ is bounded by $L^* = [L(T)/q] \sim Au_T^{(1-H^2)/H}/d \rightarrow \infty$ as $T \rightarrow \infty$. In the whole for $s_{k,l} \leq T/u_T$ we have less than $L(T)K_T/q \sim d^{-1}Tu_T^{(1-2H)/H}$ number of points $s_{k,l}$ in the first component. Since $|\tau - \tau_0| \leq \tau^*(u_T)$ we have also $|j| \leq [(\tau^*(u_T)/q)] \sim \frac{1-H}{Ad} (\log u_T) u_T^{(1-H)^2/H} \rightarrow \infty$ for any $H < 1$.

We investigate now the exceedances in a small domain $\{(s, \tau) \in [s_{k,l}, s_{k,l+1}) \times [\tau_j, \tau_{j+1})\}$ by conditioning on the value $Z(s_{k,l}, \tau_j)$. This is needed for the comparison between the continuous Gaussian field $Z(s, \tau)$ and the related discrete one $Z(s_{k,l}, \tau_j)$.

For any fixed k, l and j we define the Gaussian field

$$\tilde{Z}^{(u)}(t, \xi) = \tilde{Z}_{k,l,j}^{(u)}(t, \xi) = w(Z(s_{k,l} + tq, \tau_j + \xi q) - w)$$

with $0 \leq t, \xi \leq 1$ where

$$E(\tilde{Z}^{(u)}(t, \xi)) = -w^2,$$

$$\text{Var}(\tilde{Z}^{(u)}(t, \xi)) = w^2 v^2(\tau_j + \xi q)$$

and also with $r(s, \tau, s', \tau')$ given in Section 2

$$\begin{aligned} & \text{Corr}(\tilde{Z}^{(u)}(t, \xi), \tilde{Z}^{(u)}(t', \xi')) \\ &= r^{(u)}(t, \xi, t', \xi') \\ &= \frac{-|q(t - t' + \xi - \xi')|^{2H} + |q(t - t') + \tau_j + \xi q|^{2H} + |q(t - t') - \tau_j - \xi' q|^{2H} - |q(t - t')|^{2H}}{\tau_0^{2H}(1 + (j + \xi)q/\tau_0)^H(1 + (j + \xi')q/\tau_0)^H}. \end{aligned}$$

Then the distribution of the supremum of $\tilde{Z}^{(u)}(t, \xi)$ conditioned on the $\tilde{Z}^{(u)}(0, 0)$ can be approximated.

Lemma 5. *With the definition of $\tilde{Z}^{(u)}(t, \xi)$ we get*

$$P\left\{\sup_{0 \leq t, \xi \leq 1} \tilde{Z}^{(u)}(t, \xi) > 0 \mid \tilde{Z}^{(u)}(0, 0) = y\right\} \leq C d^H |y|^{2/H-1} \phi(\tilde{C}|y|/d^H)$$

for $y < -\gamma$ and $T \rightarrow \infty$, with $d = d(T) \rightarrow 0$ and some constants $C, \tilde{C} > 0$, not depending on γ .

Proof. The conditioned centered process $\tilde{Z}^{(u)}(t, \xi) - \mu(t, \xi, y) \mid \tilde{Z}^{(u)}(0, 0)$ is a Gaussian process. To apply Theorem 8.1 of Piterbarg [10] we need to derive the conditional mean, variance and covariance and their approximations.

For the conditional mean we get with $0 \leq t, \xi \leq 1$

$$\begin{aligned} E(\tilde{Z}^{(u)}(t, \xi) \mid \tilde{Z}^{(u)}(0, 0) = y) \\ &= -w^2 + r^{(u)}(t, \xi, 0, 0) \frac{v^{-1}(\tilde{\xi})}{v^{-1}(\tau_j)} (y + v^2) \\ &= y + (y + w^2) \left(\frac{v(\tau_j)}{v(\tilde{\xi})} - 1 \right) - (1 - r^{(u)}(t, \xi, 0, 0)) \frac{v(\tau_j)}{v(\tilde{\xi})} (y + v^2), \end{aligned}$$

where $\tilde{\xi} = \tau_j + \xi q$. Since the lags tq and ξq tend to 0, using the Taylor expansion for $v(\tau)$, we get an approximation for $v(\tau_j)/v(\tilde{\xi})$, and using (4) an approximation for the correlation function. Thus the conditional mean is for fixed y

$$= y - \frac{1 + o(1)}{2\tau_0^2} d^{2H} ((t + \xi)^{2H} + t^{2H}) =: \mu(t, \xi, y). \quad (15)$$

However, for all $y \leq -\gamma$ we have with the same expansions that $\mu(t, \xi, y) = y(1 + O(d^{2H}/\gamma))$, uniformly in y . Now $\gamma \rightarrow 0$ also, so let $\gamma = \gamma(T) = d^H \rightarrow 0$. For

the selected $d = d(T)$ and γ , the term $O(d^{2H}/\gamma)$ tends to 0. This bound is sufficient for our approximations.

Next we derive a bound for the conditional variance. We have by (4)

$$\begin{aligned}\text{Var}(\tilde{Z}^{(u)}(t, \xi) | \tilde{Z}^{(u)}(0, 0) = y) &= \text{Var}(\tilde{Z}^{(u)}(t, \xi))(1 - [r^{(u)}(t, \xi, 0, 0)]^2) \\ &= \frac{w^2}{v^2(\tau_j + \xi q)} \frac{2 + o(1)}{2\tau_0^{2H}} ((t + \xi)^{2H} + t^{2H}) q^{2H} \\ &\leq Cw^2 q^{2H} = Cd^{2H}\end{aligned}\quad (16)$$

for all $t, \xi \leq 1$, with some constant $C > 0$.

We need also an upper bound for the variance of the conditional increments of $\tilde{Z}^{(u)}(t, \xi)$ which is

$$\begin{aligned}\text{Var}(\tilde{Z}^{(u)}(t, \xi) - \tilde{Z}^{(u)}(t', \xi') | \tilde{Z}^{(u)}(0, 0) = y) \\ = \frac{\text{Var}(\tilde{Z}^{(u)}(t, \xi) - \tilde{Z}^{(u)}(t', \xi')) - [\text{Cov}(\tilde{Z}^{(u)}(t, \xi) - \tilde{Z}^{(u)}(t', \xi'), \tilde{Z}^{(u)}(0, 0))]^2}{w^2 v^{-2}(\tau_j)}.\end{aligned}$$

The variance of the increments is approximated first.

$$\begin{aligned}\text{Var}(\tilde{Z}^{(u)}(t, \xi) - \tilde{Z}^{(u)}(t', \xi'))/w^2 \\ = \frac{v^{-2}(\tau_j + \xi q) + v^{-2}(\tau_j + \xi' q) - 2r^{(u)}(t, \xi, t', \xi')}{v(\tau_j + \xi q)v(\tau_j + \xi' q)} \\ \sim A^{-4}[(v(\tau_j + \xi q) - v(\tau_j + \xi' q))^2 + 2(1 - r^{(u)}(t, \xi, t', \xi'))A^2(1 + o(1))].\end{aligned}$$

The first term, the difference of the v -values, is of $o(q|\xi - \xi'|)$ because of the behaviour of v in the neighbourhood of τ_0 , given in (3). The second term is approximated by (4) to get

$$\begin{aligned}A^2 \frac{(1 + o(1))}{\tau_0^{2H}} [|t - t' + \xi - \xi'|^{2H} + |t - t'|^{2H}] q^{2H} \\ \sim w^{-2} A^2 (d/\tau_0)^{2H} [|t - t' + \xi - \xi'|^{2H} + |t - t'|^{2H}].\end{aligned}$$

Combining the two approximations, results in

$$\begin{aligned}\text{Var}(\tilde{Z}^{(u)}(t, \xi) - \tilde{Z}^{(u)}(t', \xi')) \\ \sim A^{-2} (d/\tau_0)^{2H} [|t - t' + \xi - \xi'|^{2H} + |t - t'|^{2H}] + o(|\xi - \xi'|^2) \\ \leq G(|t - t'|^{2H} + |\xi - \xi'|^{2H})\end{aligned}$$

for some $G > 0$. The covariance of the increment and $\tilde{Z}^{(u)}(0, 0)$ is a bit more tedious but straightforward with the same approximations.

$$\begin{aligned}\text{Cov}(\tilde{Z}^{(u)}(t, \xi) - \tilde{Z}^{(u)}(t', \xi'), \tilde{Z}^{(u)}(0, 0)) \\ = \text{Cov}(\tilde{Z}^{(u)}(t, \xi), \tilde{Z}^{(u)}(0, 0)) - \text{Cov}(\tilde{Z}^{(u)}(t', \xi'), \tilde{Z}^{(u)}(0, 0)) \\ \sim \frac{w^2}{A} \frac{v(\tau_j + \xi' q)(r_1 - r_2) - r_2[v(\tau_j + \xi q) - v(\tau_j + \xi' q)]}{v(\tau_j + \xi q)v(\tau_j + \xi' q)}\end{aligned}$$

with $r_1 = r^{(u)}(t, \xi, 0, 0)$ and $r_2 = r^{(u)}(t', \xi', 0, 0)$. By (4) the difference of $r_1 - r_2$ is bounded by

$$O(q^{2H}(|t - t' + \xi - \xi'|^\alpha + |t - t'|^\alpha))$$

with $\alpha = \min(2H, 1)$. The difference of the v -terms is again $O(q|\xi - \xi'|(\log w)/w)$. Together we have for

$$\begin{aligned} & [\text{Cov}(\tilde{Z}^{(u)}(t, \xi) - \tilde{Z}^{(u)}(t', \xi'), \tilde{Z}^{(u)}(0, 0))]/w^2 \\ &= w^2(O(q^{4H}(|t - t' + \xi - \xi'|^\alpha + |t - t'|^\alpha)^2) + o(q^2(\log w)^2|\xi - \xi'|^2/w^2)) \\ &= o(1)(|t - t'|^{2H} + |\xi - \xi'|^{2H}). \end{aligned}$$

Therefore the conditional variance of the increment, being the variance of the increments minus the above squared covariance term divided by the variance of $\tilde{Z}^{(u)}(0, 0)$, is bounded by

$$G(|t - t'|^{2H} + |\xi - \xi'|^{2H})$$

for some $G > 0$.

We can now apply again Theorem 8.1 of Piterbarg [10] for

$$\begin{aligned} & P \left\{ \sup_{0 \leq t, \xi \leq 1} \tilde{Z}^{(u)}(t, \xi) - \mu(t, \xi) > -\mu(t, \xi, y) | \tilde{Z}^{(u)}(0, 0) = y \right\} \\ & \leq C\sigma^* |\mu(t, \xi, y)|^{2/H-1} \phi(|\mu(t, \xi, y)|/\sigma^*) \end{aligned} \quad (17)$$

with $\sigma^{*2} = \sup_{t, \xi \leq 1} \text{Var}(\tilde{Z}^{(u)}(t, \xi) | \tilde{Z}^{(u)}(0, 0))$ and C not depending on γ . Note that the conditional mean $|\mu(t, \xi, y)| = |y|(1 + O(d^{2H}/\gamma)) > |y|(1 - \varepsilon)$, uniformly in $t, \xi \leq 1, y \leq -\gamma$, with d sufficiently small (T large), with the chosen $\gamma = d^H$. By (16) $\sigma^* \leq d^H/\tilde{C}$. Hence we get as upper bound for (17)

$$Cd^H |y|^{2/H-1} \phi(\tilde{C}|y|(1 - \varepsilon)/d^H) = Cd^H |y|^{2/H-1} \phi(\tilde{C}|y|/d^H)$$

with suitable (generic) constants $C, \tilde{C} > 0$, not depending on t, ξ, y and γ which is our statement. \square

This allows now the approximation of the supremum of the process $Z(s, \tau)$ on the continuous points by the maximum on the grid in a small domain in the following way.

Lemma 6. *For the process $Z(s, \tau)$ we get for T large with $\gamma = d^H$*

$$\begin{aligned} & P \left\{ Z(s_{k,l}, \tau_j) \leq w - \gamma/w, \sup_{0 \leq t, \xi \leq 1} Z(s_{k,l} + tq, \tau_j + \xi q) > w \right\} \\ &= O(d^{H+2})\phi(wv(\tau_j))/w \end{aligned}$$

uniformly in k, l, j , and for any $k \leq K_T$

$$\begin{aligned} & P \left\{ \max_{(s, \tau) \in I_k \cap \mathcal{A}} Z(s, \tau) \leq w - \gamma/w, \sup_{(s, \tau) \in I_k} Z(s, \tau) > w \right\} \\ &= O(d^H L(T) w^{2(1-H)/H} \phi(Aw)) = O(d^H / K_T) \end{aligned}$$

with T large and $d = d(T) > 0$ small.

Proof. For the process $\tilde{Z}^{(u)}(t, \xi)$ we apply Lemma 4

$$\begin{aligned} & P \left\{ Z(s_{k,l}, \tau_j) \leq w - \gamma/w, \sup_{0 \leq t, \xi \leq 1} Z(s_{k,l} + tq, \tau_j + \xi q) > w \right\} \\ &= P \left\{ \tilde{Z}^{(u)}(0, 0) \leq -\gamma, \sup_{0 \leq t, \xi \leq 1} \tilde{Z}^{(u)}(t, \xi) > 0 \right\} \\ &= \int_{-\gamma}^{-\gamma} P \left\{ \sup_{0 \leq t, \xi \leq 1} \tilde{Z}^{(u)}(t, \xi) > 0 \mid \tilde{Z}^{(u)}(0, 0) = y \right\} f_{\tilde{Z}^{(u)}(0,0)}(y) dy \\ &\leq \int_{-\gamma}^{-\gamma} \phi(v(\tau_j)(w + y/w)) C d^H |y|^{2/H-1} \phi(\tilde{C}|y|/d^{2H}) dy v(\tau_j)/w \\ &\leq \frac{O(d^H)}{w} \phi(wv(\tau_j)) \int_{-\gamma}^{-\gamma} |y|^{2/H-1} \exp \left\{ -\frac{\tilde{C}y^2}{2d^{2H}} - yv^2(\tau_j) - \frac{y^2v^2(\tau_j)}{2w^2} \right\} dy \\ &\leq \frac{O(d^H)}{w} \phi(wv(\tau_j)) \int_{\gamma}^{\infty} y^{2/H-1} \exp \left\{ -\frac{y^2}{2} \left(\frac{\tilde{C}}{d^{2H}} + o(1) \right) + yA^2(1 + o(1)) \right\} dy \\ &\leq C d^{H+2} \phi(wv(\tau_j))/w \end{aligned}$$

since the integral can be bounded by d^2 for any $\gamma \geq 0$. The constant C does not depend on k, l, j .

The second claim follows by summing these bounds on l, j for fixed k . We use that $0 \leq v(\tau_j) - v(\tau_0) = (B + o(1))(jq)^2 \geq \tilde{B}(jq)^2$.

$$\begin{aligned} & \sum_{l,j} C d^{H+2} \phi(wv(\tau_j))/w \leq (L(T)/q)(C d^{H+2}/w) \sum_j \phi(wv(\tau_j)) \\ &= (L(T)/q)(C d^{H+2}/w) \phi(Aw) \sum_j e^{-(1/2)(v(\tau_j)-v(\tau_0))^2 w^2 - w^2 A(v(\tau_j)-v(\tau_0))} \\ &\leq O(d^{H+2})(L(T)/qw) \phi(Aw) \sum_j e^{-\tilde{B}Aw^2(jq)^2} \\ &\leq O(d^{H+2})(L(T)/qw) \phi(Aw) O(1/wq) \int_0^{\infty} e^{-\tilde{B}Ax^2} dx \\ &\leq O(d^{H+2} L(T)(d^{-2} w^{2(1-H)/H}) \phi(Aw)) \\ &\leq O(d^H w^{2(1-H)/H} u^{-h-1} K_T^{-1}) [K_T L(T) u^{h+1} \phi(Aw)] \end{aligned}$$

$$\begin{aligned} &\leq O(d^H K_T^{-1})(Tu^h\phi(Aw)) \\ &\leq O(d^H/K_T) \end{aligned}$$

using $Tu_T^h\phi(Aw) = O(1)$ with $w = u_T^{1-H}$. Because of the stationarity (homogeneity) of $Z(s, \tau)$ in the first component, this holds for any k , hence uniformly. \square

In the next step we approximate the supremum on the I_k 's by the supremum based on the discrete time points if $\gamma \rightarrow 0$.

Lemma 7. For $d \rightarrow 0$ with $\gamma = d^H \rightarrow 0$

$$0 \leq P \left\{ \sup_{(s,\tau) \in \cup_k I_k \cap \mathcal{R}} Z(s, \tau) \leq u_T^{1-H} \right\} - P \left\{ \sup_{(s,\tau) \in \cup_k I_k} Z(s, \tau) \leq u_T^{1-H} \right\} \rightarrow 0$$

and also

$$0 \leq \prod_{k=1}^{K_T} P \left\{ \sup_{(s,\tau) \in I_k \cap \mathcal{R}} Z(s, \tau) \leq u_T^{1-H} \right\} - \prod_{k=1}^{K_T} P \left\{ \sup_{(s,\tau) \in I_k} Z(s, \tau) \leq u_T^{1-H} \right\} \rightarrow 0.$$

Proof. (i) We have by (14), Lemma 1 and Corollary 2

$$P \left\{ \max_{(s,\tau) \in I_k} Z(s, \tau) > w \right\} = (1 + o(1))c_2 L(T) \exp \left(-\frac{1}{2} A^2 w^2 \right) w^{2(1-H)/H}$$

for any k . We show now that for any k also

$$P \left\{ \max_{(s,\tau) \in I_k \cap \mathcal{R}} Z(s, \tau) > w \right\} = (1 + o(1))c_2 L(T) \exp \left(-\frac{1}{2} A^2 w^2 \right) w^{2(1-H)/H}$$

holds. This is true since by Lemma 5

$$\begin{aligned} &P \left\{ \sup_{(s,\tau) \in I_k \cap \mathcal{R}} Z(s, \tau) > w \right\} \\ &\leq P \left\{ \sup_{(s,\tau) \in I_k} Z(s, \tau) > w \right\} \\ &\leq P \left\{ \sup_{(s,\tau) \in I_k \cap \mathcal{R}} Z(s, \tau) > w - \gamma/w \right\} \\ &\quad + P \left\{ \sup_{(s,\tau) \in I_k \cap \mathcal{R}} Z(s, \tau) \leq w - \gamma/w, \sup_{(s,\tau) \in I_k} Z(s, \tau) > w \right\} \\ &\leq (1 + O(d^H)) P \left\{ \sup_{(s,\tau) \in I_k} Z(s, \tau) > w - \gamma/w \right\} \\ &= (1 + O(d^H) + O(\gamma)) P \left\{ \sup_{(s,\tau) \in I_k} Z(s, \tau) > w \right\} \end{aligned}$$

using $(w - \gamma/w)^2 = w^2 - 2\gamma + o(1)$ for small γ . With this result it is also straightforward to show that for small γ

$$P\left\{w - \gamma/w < \sup_{(s,\tau) \in I_k \cap \mathcal{R}} Z(s, \tau) \leq w\right\} = O(\gamma L(T)\phi(Aw)w^{2(1-H)/H}) \quad (18)$$

and

$$\begin{aligned} 0 &\leq P\left\{\sup_{(s,\tau) \in I_k \cap \mathcal{R}} Z(s, \tau) \leq w\right\} - P\left\{\sup_{(s,\tau) \in I_k} Z(s, \tau) \leq w\right\} \\ &= O(c_d L(T)\phi(Aw)w^{2(1-H)/H}) \end{aligned} \quad (19)$$

where $c_d = \gamma + d^H = 2d^H \rightarrow 0$ as $d \rightarrow 0$.

(ii) For the statements of the lemma we have

$$\begin{aligned} 0 &\leq P\left\{\sup_{(s,\tau) \in \bigcup_k I_k \cap \mathcal{R}} Z(s, \tau) \leq w\right\} - P\left\{\sup_{(s,\tau) \in \bigcup_k I_k} Z(s, \tau) \leq w\right\} \\ &\leq \sum_{k=1}^{K_T} \left(P\left\{\sup_{(s,\tau) \in I_k \cap \mathcal{R}} Z(s, \tau) \leq w\right\} - P\left\{\sup_{(s,\tau) \in I_k} Z(s, \tau) \leq w\right\} \right). \end{aligned}$$

Using (19) for each term of the sum, the sum on k is bounded by

$$\begin{aligned} O(K_T L(T) c_d \phi(Aw) w^{2(1-H)/H}) &= O((T/u_T) c_d \phi(Aw) u_T^{2(1-H)/H}) \\ &= O\left(c_d T u_T^h \exp\left(-\frac{1}{2} A^2 u_T^{2(1-H)}\right)\right) \rightarrow 0 \end{aligned}$$

as $d \rightarrow 0$. This shows the first claim. It implies also the second claim using the stationarity of $Z(s, \tau)$ with respect to the first parameter s since the dependence of $Z(s, \tau)$ for $(s, \tau), (s', \tau')$ in different I_k 's is not restricted. \square

Now we are considering the asymptotic independence of suprema on the I_k 's. We begin with the approximation of the sum in Berman's comparison lemma.

Lemma 8. *Under the above definitions and properties of $Z(s, \tau)$ we have*

$$S_T = \sum_{k \neq k'} \sum_{\substack{(s_{k,l}, \tau_j) \in I_k \times \tau_d \\ (s_{k',l'}, \tau_{j'}) \in I_{k'} \times \tau_d}} |r(s_{k,l}, \tau_j, s_{k',l'}, \tau_{j'})| \exp\left\{-\frac{A^2 v^2}{1 + r(s_{k,l}, \tau_j, s_{k',l'}, \tau_{j'})}\right\} \rightarrow 0.$$

Proof. Since $|s_{k,l} - s_{k',l'}| \geq \delta'$ by definition, $r(s_{k,l}, \tau_0, s_{k',l'}, \tau_0) \leq \rho < 1$. Furthermore, we showed that

$$\sup_{|s_{k,l} - s_{k',l'}| \geq s} |r(s_{k,l}, \tau_0, s_{k',l'}, \tau_0)| \leq C s^\lambda$$

for $\lambda = 2H - 2 < 0$ and some constant $C > 0$, since also τ_j and $\tau_{j'}$ tend to τ_0 . If $H = \frac{1}{2}$ we have $r(s_{k,l}, \tau_j, s_{k',l'}, \tau_{j'}) = 0$ if $|s_{k,l} - s_{k',l'}|$ is large. Set $\beta = (1 - \rho)/(1 + \rho)$ and split

the sum into two partial sums $S_{T,1}$ and $S_{T,2}$ with $|s_{k,l} - s_{k',l'}| < \tilde{T}^\beta = (T/u_T)^\beta$ and $|s_{k,l} - s_{k',l'}| \geq \tilde{T}^\beta$, respectively. For the first sum there are $\tilde{T}^{1+\beta}/q^2$ combinations of two points $s_{k,l}, s_{k',l'} \in \bigcup_k I_k$. Together with the τ_j combinations there are $\tilde{T}^{1+\beta}(2\tau^*(u_T))^2/q^4$ terms in the sum $S_{T,1}$. Thus $S_{T,1}$ is bounded by

$$\begin{aligned} & \rho \frac{\tilde{T}^{1+\beta}(2\tau^*(u_T))^2}{q^4} \exp\left\{-\frac{A^2 w^2}{1+\rho}\right\} \\ & \leq 4\rho \exp\left\{(1+\beta)\log \tilde{T} + 2\log(\tau^*(u_T)/q^2) - \frac{(2(1+o(1))\log T)}{1+\rho}\right\} \\ & \leq 4 \exp\left\{-(\log T)\left[\frac{2(1+o(1))}{1+\rho} - (1+\beta)\left(1 - \frac{\log u_T}{\log T}\right) - 2\frac{\log(\tau^*(u_T)/q^2)}{\log T}\right]\right\} \\ & \rightarrow 0 \end{aligned}$$

since $1+\beta < 2/(1+\rho)$ by the choice of β , using $\log(\tau^*(u_T)/q^2) = o(\log T)$ and $\log u_T = O(\log \log T) = o(\log T)$.

For the second sum $S_{T,2}$ with $|s_{k,l} - s_{k',l'}| \geq \tilde{T}^\beta$, we use that

$$\sup_{|s_{k,l} - s_{k',l'}| \geq \tilde{T}^\beta} |r(s_{k,l}, \tau_0, s_{k',l'}, \tau_0)| \leq C\tilde{T}^{\beta\lambda}$$

with $\lambda = 2H - 2 < 0$. In this case there are $(\tilde{T}/q)^2$ many combinations of two points $s_{k,l}, s_{k',l'} \in \bigcup_k I_k$. Hence $S_{T,2}$ has the upper bound

$$\begin{aligned} & C\tilde{T}^{\beta\lambda} \frac{(2\tilde{T}\tau^*(u_T))^2}{q^4} \exp\left\{-\frac{A^2 w^2}{1+C\tilde{T}^{\beta\lambda}}\right\} \\ & \leq C \exp\left\{\beta\lambda \log \tilde{T} + 2\log \tilde{T} + 2\log(\tau^*(u_T)/q^2) - \frac{2(1+o(1))\log T}{1+C\tilde{T}^{\beta\lambda}}\right\} \\ & \leq C \exp\{(\log \tilde{T})[\beta\lambda + o(1)]\} \\ & \rightarrow 0 \end{aligned}$$

since $\lambda < 0$. If $H = 1/2$, the sum $S_{T,2} = 0$ obviously. \square

Berman's comparison lemma implies now the independence of the suprema based on the I_k 's.

Lemma 9. *Under the above definitions and properties of $Z(s, \tau)$ we have*

$$P\left\{\sup_{(s,\tau) \in \bigcup_k I_k \cap \mathcal{R}} Z(s, \tau) \leq w\right\} - \prod_{k=1}^{K_T} P\left\{\sup_{(s,\tau) \in I_k \cap \mathcal{R}} Z(s, \tau) \leq w\right\} \rightarrow 0$$

as $T \rightarrow \infty$ with $d \rightarrow 0$.

Proof. To apply Berman's comparison lemma (cf. [4] or [6] for this general form) we have to standardize the Gaussian field yielding nonconstant boundaries $v(\tau)w$.

$$\begin{aligned}
 & P\left\{\sup_{(s,\tau)\in\bigcup_k I_k\cap\mathcal{R}} Z(s,\tau)\leq w\right\}-\prod_{k=1}^{K_T}P\left\{\sup_{(s,\tau)\in I_k\cap\mathcal{R}} Z(s,\tau)\leq w\right\} \\
 &= P\left\{\sup_{(s,\tau)\in\bigcup_k I_k\cap\mathcal{R}} Z(s,\tau)v(\tau)\leq v(\tau)w\right\}-\prod_{k=1}^{K_T}P\left\{\sup_{(s,\tau)\in I_k\cap\mathcal{R}} Z(s,\tau)v(\tau)\leq v(\tau)w\right\} \\
 &\leq \sum_{k\neq k'}\sum_{\substack{(s_k,l,\tau_j)\in I_k\cap\mathcal{R} \\ (s_{k'},l',\tau'_j)\in I_{k'}\cap\mathcal{R}}} |r(s_k,l,\tau_j,s_{k'},l',\tau'_j)|\exp\left\{-\frac{(v^2(\tau_j)+v^2(\tau'_j))w^2}{2(1+r(s_k,l,\tau_j,s_{k'},l',\tau'_j))}\right\} \\
 &\leq \sum_{k\neq k'}\sum_{\substack{(s_k,l,\tau_j)\in I_k\cap\mathcal{R} \\ (s_{k'},l',\tau'_j)\in I_{k'}\cap\mathcal{R}}} |r(s_k,l,\tau_j,s_{k'},l',\tau'_j)|\exp\left\{-\frac{v^2(\tau_0)w^2}{(1+r(s_k,l,\tau_j,s_{k'},l',\tau'_j))}\right\}
 \end{aligned}$$

which tends to 0 by Lemma 8. \square

With these lemmas we are ready to prove the statement of Theorem 1.

Proof of Theorem 1. The proof consists of showing with $w = u_T^{1-H}$ that

$$\begin{aligned}
 P\left\{\sup_{t\leq T} Y_t\leq u_T\right\} &\sim P\left\{\sup_{(s,\tau)\in\bigcup_k I_k} AZ(s,\tau)\leq Au_T^{1-H}\right\} \\
 &\sim P\left\{\sup_{(s,\tau)\in\bigcup_k I_k\cap\mathcal{R}} Z(s,\tau)\leq w\right\} \tag{20}
 \end{aligned}$$

$$\sim \prod_{k=1}^{K_T} P\left\{\sup_{(s,\tau)\in I_k\cap\mathcal{R}} Z(s,\tau)\leq w\right\} \tag{21}$$

$$\sim \prod_{k=1}^{K_T} P\left\{\sup_{(s,\tau)\in I_k} Z(s,\tau)\leq w\right\} \tag{22}$$

$$\sim \exp\left(-K_TP\left\{\sup_{(s,\tau)\in I_1} Z(s,\tau)>w\right\}\right) \tag{23}$$

$$\rightarrow \exp(-e^{-x}). \tag{24}$$

Eqs. (20) and (22) hold by the statements of Lemma 7. Eq. (21) is shown by Berman's inequality in Lemma 9. Note that $P\{\sup_{(s,\tau)\in I_k} Z(s,\tau)\leq w\}$ is the same for each k , since the fBM $X(t)$ has stationary increments, implying the mentioned stationarity in the first component. Hence (23) is immediate using (8). Finally, Lemma 3 shows the convergence (24) by the proper choice of u_T . \square

The proof reveals the mentioned statement of Theorem 2. We considered the maximum on the discrete process $\tilde{M}_T^{(q)} = \sup_{(s,\tau)\in\bigcup_k I_k\cap\mathcal{R}} Z(s,\tau)$ to approximate the

maximum of the continuous process $\sup_{(s,\tau) \in \cup_k I_k} Z(s, \tau)$. The proof shows that they are asymptotically completely dependent. Obviously, this holds also for the maxima $M_T^{(q)}$ on the whole time domain not only on the $\cup_k I_k$ since the grid points are dense by the chosen $q(T)$ and $d(T)$. However, this statement holds for any $d \rightarrow 0$, not only for the chosen $d(T) = 1/\log \log T$. Therefore we state the more general result in Theorem 2 with $\delta = d(2A^{-2} \log T)^{-1/(2H)}$ (with $d \rightarrow 0$) instead of $q = q(T, x)$. Note also that the assumption $q = d(2A^{-2} \log T)^{-1/(2H)} \sim du_T^{-(1-H)/H}$ for any x in the normalization $u_T = u_T(x)$.

Proof of Theorem 2. Since $u_T(y) \leq u_T(x)$ for all T and $y \leq x$, we have for $y \leq x$

$$\begin{aligned} P\{M_T^{(\delta)} \leq u_T(x), M_T \leq u_T(y)\} &= P\{M_T^{(\delta)} \leq u_T(y), M_T \leq u_T(y)\} \\ &= P\{M_T \leq u_T(y)\} - P\{M_T^{(\delta)} \leq u_T(y), M_T > u_T(y)\} \end{aligned}$$

and for $x \leq y$

$$P\{M_T^{(\delta)} \leq u_T(x), M_T \leq u_T(y)\} = P\{M_T^{(\delta)} \leq u_T(x)\} - P\{M_T^{(\delta)} \leq u_T(x), M_T > u_T(y)\}.$$

The statement follows by using

$$P\{M_T^{(\delta)} \leq u_T(x)\} \sim P\{M_T \leq u_T(x)\}$$

and

$$P\{M_T^{(\delta)} \leq u_T(x), M_T > u_T(x)\} = o(1)$$

by Lemma 6 for any dense grid with $d \rightarrow 0$. \square

Note that the grid is dense for the transformed storage process, for the Gaussian field $Z(s, \tau)$. However, considering the grid for the storage process $Y(t)$ we have the grid points $uq = du_T^{1-(1-H)/H} = du_T^{(2H-1)/H}$ which tends to ∞ , for $H > \frac{1}{2}$ and suitable d . It means that we have to observe quite rarely the storage process to get the complete information on the maximum of the continuous storage process.

Remark. The case $H = 1$ was not dealt with because the fBM $X(t)$ with $H = 1$ is a simple process: $X(t) = tX$ with a standardized Gaussian r.v. X . This yields for the storage process $Y(t) = \sup_{s \geq t} (X(s) - X(t) - c(s - t)) = \sup_{s \geq t} (s - t)(X - c) = \infty$, if $X > c$, or 0, else. Thus this storage process is degenerated and not interesting. Knowing one $X(t)$, is sufficient for the knowledge of supremum of this storage process.

References

- [1] J.M.P. Albin, G. Samorodnitsky, On overload in a storage model with self-similar and infinitely divisible input, *Ann. Appl. Probab.* 14 (2004) 820–844.
- [2] J. Choe, N.B. Shroff, On the supremum distribution of integrated stationary Gaussian processes with negative drift, *Adv. Appl. Probab.* 31 (1999) 135–157.

- [3] N.G. Duffield, O'Connell Nell, Large deviations and overflow probabilities for the general single-server queue, with applications. *Mathematical Proceedings of the Cambridge Philosophical Society*, 1996.
- [4] J. Hüslér, Asymptotic approximation of crossing probabilities of random sequences, *Z. Wahrscheinlichkeitstheorie Verw. Geb.* 63 (1983) 257–270.
- [5] J. Hüslér, V. Piterbarg, Extremes of a certain class of Gaussian processes, *Stochastic Process Appl.* 83 (1999) 257–271.
- [6] M.R. Leadbetter, G. Lindgren, H. Rootzén, *Extremes and Related Properties of Random Sequences and Processes*, Springer Series in Statistics, Springer, New York, 1983.
- [7] O. Narayan, Exact asymptotic queue length distribution for fractional Brownian traffic, *Adv. Performance Anal.* 1 (1998) 39–63.
- [8] I. Norros, Four approaches to the fractional Brownian storage, in: J. Lévy Véhel, E. Lutten, C. Tricot (Eds.), *Fractals in Engineering*, Springer, Berlin, 1997.
- [9] I. Norros, Busy periods of fractional Brownian storage: a large deviations approach, *Adv. Performance Anal.* 1 (1999) 1–19.
- [10] V.I. Piterbarg, *Asymptotic Methods in the Theory of Gaussian Processes and Fields*, AMS, Providence, 1996.
- [11] V. Piterbarg, Large deviations of a storage process with fractional Brownian motion as input, *Extremes* 4 (2) (2001) 147–164.