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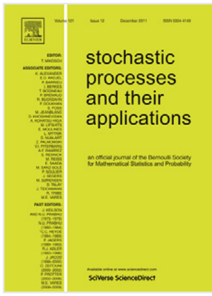

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Backward Stochastic Differential Equations Driven by G-Brownian Motion

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Abstract

In this paper, we study the backward stochastic differential equations driven by a G-Brownian motion $(B_t)_{t\geq 0}$ in the following form:

$$\begin{aligned}Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\langle B\rangle_s \\ - \int_t^T Z_s dB_s - (K_T - K_t),\end{aligned}$$

where K is a decreasing G-martingale. Under Lipschitz conditions of f and g in Y and Z, the existence and uniqueness of the solution (Y, Z, K) of the above BSDE in the G-framework is proved.

Key words: G-expectation, G-Brownian motion, G-martingale, Backward SDEs

MSC-classification: 60H10, 60H30

1 Introduction

Consider a Wiener probability space $(\Omega, \mathcal{F}, P)$ where Ω is the space of continuous paths and P is the Wiener measure. It is well known that the canonical process,

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namely $B_t(\omega) = \omega_t$ for $\omega \in \Omega$, is a standard Brownian motion under P. A typical kind of classical Backward Stochastic Differential Equation (BSDE for short) is defined, on this space, as

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \tag{1.1}$$

where g is a given function, called the generator of (1.1) and ξ is a given $\mathcal{F}_T$-measurable random variable called the terminal condition. The solution of (1.1) consists of a pair of adapted processes (Y, Z). Linear BSDEs were initiated by Bismut [2, 1973]. In 1990, Pardoux and Peng [15, 1990] introduced the general nonlinear BSDEs with Lipschitz continuous generators.

Note that the above classical BSDEs are based on a probability space framework. Recently, there are at least two motivation to drive BSDEs and the corresponding time-consistent nonlinear expectations to develop ahead beyond the probability space framework. The first one is that the classical BSDE can only provide a probabilistic interpretation of a PDE for quasilinear but not for fully nonlinear cases. The second one is that the stochastic control techniques in [1] are difficult to price path-dependent contigent claims in the uncertain volatility model (UVM for short). In the UVM case, one is faced with a family of probability measures which are, in general, mutually singular and nondominated. Thus, we need a new coherent framework on which one can accommodate problems involving a family of nondominated measures (see [4]).

In order to overcome the above shortcomings of classical BSDEs, Peng systemically established a time-consistent fully nonlinear expectation theory. The notion of time-consistent fully nonlinear expectations was first introduced in [20, Peng2004] and [21, Peng2005].

As a typical case, Peng (2006) introduced G-expectation (see [27] and the references therein). Under G-expectation framework (G-framework for short), a new type of Brownian motion called G-Brownian motion was constructed and the corresponding stochastic calculus of Itô's type was established. The existence and uniqueness of solution of a SDE driven by G-Brownian motion can be proved in a way parallel to that in the classical SDE theory. But the BSDE driven by G-Brownian motion $(B_t)_{t\geq 0}$ becomes a challenging and fascinating problem.

Just as in the classical case, G-martingale representation theorem is the key to solve a BSDE in this G-framework. For a dense family of G-martingales, Peng [23] obtained the following result: a G-martingale M has the form

$$\begin{aligned} &M_t = M_0 + \bar{M}_t + K_t, \\ &\bar{M}_t := \int_0^t z_s B_s, \qquad K_t := \int_0^t \eta_s \langle B \rangle_s - \int_0^t 2G(\eta_s)ds. \end{aligned}$$

Here M is decomposed into two incompatible G-martingales. The first one $\bar{M}$ is called symmetric G-martingale. That is, $-\bar{M}$ is also a G-martingale. The second one K is quite unusual since it is a decreasing process. A main concern in the G-framework is how to understand this decreasing G-martingales K, which aroused an interesting open problem (see [23] and [27]).

For a general G-martingale, the first step is to decompose it into a sum of a symmetric G-martingale $\bar{M}$ and a decreasing G-martingale K. This difficult problem was solved after a series of successive efforts of Soner, Touzi & Zhang [33, 2011] and Song [35, 2011], [36, 2012]. The second step is to study that under what condition the decreasing G-martingale K can be uniquely represented as $K_t := \int_0^t \eta_s \langle B \rangle_s - \int_0^t 2G(\eta_s)ds$. Thanks to an original new norm for decreasing G-martingales introduced in Song [36, 2012], a complete representation theorem of G-martingales has been obtained in a complete subspace of $L_G^\alpha(\Omega_T)$ by Peng, Song and Zhang [30, 2012].

Due to the above G-martingale representation theorem, a natural formulation of a BSDE driven by G-Brownian motion consists of a triple of processes (Y, Z, K), satisfying

$$\begin{aligned} Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\langle B\rangle_s \\ - \int_t^T Z_s dB_s - (K_T - K_t). \end{aligned} \tag{1.2}$$

Our main result of this paper is the existence and uniqueness of a solution (Y, Z, K) for (1.2) (see Theorems 4.1 and 4.2) in the G-framework. Two new approaches have been introduced to prove the existence and uniqueness theorems. The first one is to apply the partition of unity theorem to construct a new type of Galerkin approximation instead of the well-known Picard approximation. The second one involves Lemma 3.4 for decreasing G-martingales, which helps us to obtain the uniqueness, as well as the existence part of the proof. Estimate (2.1) originally obtained in [35] also plays an important role in the proof.

Now we compare the results of this paper with the known ones about fully nonlinear BSDEs.

Peng [21, Peng2005] introduced a new type of time consistent fully nonlinear expectations $\mathcal{E}_t^i$, $i = 1, \ldots, n$, through which the existence and uniqueness of a fully nonlinear multi-dimensional BSDE of the following type

$$Y_t^i = \mathcal{E}_t^i[\xi + \int_t^T f_i(s, Y_s)ds], \ i = 1, \ldots, n, \tag{1.3}$$

was obtained, where $\mathcal{E}_t^i$ were not assumed to be sublinear or convex. But this BSDE was not expressed as a classical differential form of BSDE (1.1).

Soner, Touzi and Zhang [34, 2012] have obtained a deep result of existence and uniqueness theorem for a new type of fully nonlinear BSDE, called 2BSDE: to find $(Y, Z, K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_H^\kappa}$ satisfying, for each probability $\mathbb{P} \in \mathcal{P}_H^\kappa$, the following BSDE:

$$Y_t = \xi + \int_t^T \hat{F}_s(Y_s, Z_s)ds - \int_t^T Z_s dB_s - (K_T^{\mathbb{P}} - K_t^{\mathbb{P}}), \quad \mathbb{P}\text{-a.s.}, \tag{1.4}$$

such that the following minimum condition is satisfied

$$K_t^{\mathbb{P}} = \operatorname{ess} \sup_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t+,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[K_T^{\mathbb{P}}], \quad \mathbb{P}\text{-a.s.}, \ \forall \mathbb{P} \in \mathcal{P}_H^\kappa, \ t \in [0, T]. \tag{1.5}$$

The generator $\hat{F}$ in (1.4) is formulated in the following way:

$$\hat{F}(t,\omega,y,z) = F(t,\omega,y,z,\theta_t(\omega))$$

and

$$F(t,\omega,y,z,a) := \sup_{\gamma}[\frac{1}{2}a\gamma - H(t,\omega,y,z,\gamma)]$$

for some function $H(t,\omega,y,z,a)$ (see Section 2 for the definition of θ_t). In [34], to obtain the existence of the solution, the function $F(t,\omega,y,z,a)$ was assumed to be uniformly continuous in ω under the uniform convergence norm. The G-BSDE (1.2) corresponds to the 2BSDE (1.4) with the generator function

$$F(t,\omega,y,z,a) = f(t,\omega,y,z) + g(t,\omega,y,z)a.$$

The main contribution of this paper is about the regularity of the solutions. In this paper, we introduce a quite different method to show that the triple (Y,Z,K) is universally defined in the spaces of the G-framework, in which the processes have strong regularity property (see Section 2 for more details). Up to now, all the representation results for G-martingales are in the G-framework (see Peng [23], Soner, Touzi & Zhang [33, 2011] and Song [35, 2011]). Also, the terminal random variables of 2BSDE considered in [34] are confined in the G-framework. One advantage of the strong regularity is the time consistency of the solution. Precisely, for each $t \in [0,T)$, the solution Y_t of (1.2), is proved to be in the space $Y_t \in L_G^p(\Omega_t)$ if the given terminal condition Y_T is assumed in the space $L_G^p(\Omega_T)$. Consequently, like the classical BSDEs, the solution of G-BSDE can be considered as a time consistent non-linear expectation. Another advantage of the strong regularity is that the process K_t is "aggregated" (not depending on $\mathbb{P}$) into a universal process in the G-framework.

In this paper, we consider the L^p solutions, for $p > 1$, of the BSDEs instead of just the L^2 solutions. Besides, by the method introduced in this paper, the assumption of uniform continuity on $f(t,\omega,y,z), g(t,\omega,y,z)$ in ω is unnecessarily restrictive.

Recently, many authors investigate the relations between BSDEs and path-dependent PDEs, the notion of which was proposed in Peng's ICM2010 lecture. [11] shows that the G-BSDE in the Markovian case corresponds to a fully non-linear PDE. In Peng and Wang [32], it is proved that, under reasonable and concrete regularity assumptions on ξ and g, the classical BSDE is a type of quasilinear path-dependent PDE, in the sense of Dupire derivatives. The notion of viscosity solution of Dupire's type was proposed in [29]. A new notion of viscosity solution, quite different from the original one, was given by [6, 7] through which a new result of existence and uniqueness theorem was obtained. More recently, Peng and Song (2013) introduced a new notion of G-expectation-weighted Sobolev spaces, or in short, G-Sobolev spaces. For the linear case of G corresponding to the classical Wiener probability space $(\Omega,\mathcal{F},P)$, the authors have established a 1-1 correspondence between BSDEs and a type of quasilinear path-dependent PDEs in the corresponding P-Sobolev space. When G is nonlinear, it was proved that the backward SDEs driven by G-Brownian motion corresponds to fully nonlinear path-dependent PDEs in the corresponding G-Sobolev spaces.

The paper is organized as follows. In section 2, we present some preliminaries for stochastic calculus under G-framework. Some estimates for the solution of G-BSDE are established in section 3. In section 4 the existence and uniqueness theorem is provided. In section 5, we present an alternative a priori estimate for the solutions of G-BSDEs, which may be useful in the follow-up work of G-BSDEs theory.

2 Preliminaries

We review some basic notions and results of G-expectation and the related spaces of random variables. The readers may refer to [22], [23], [24], [25], [27] for more details.

Let $\Omega_T = C_0([0,T];\mathbb{R}^d)$, the space of R^d-valued continuous functions on $[0,T]$ with $\omega_0 = 0$, be endowed with the supremum norm and let $B_t(\omega) = \omega_t$ be the canonical process. Set

$$\mathcal{H}_T^0 := \{\varphi(B_{t_1}, ..., B_{t_n}) : n \geq 1, t_1, ..., t_n \in [0,T], \varphi \in C_{l.Lip}(\mathbb{R}^{d\times n})\}.$$

Let $(\Omega_T, \mathcal{H}_T^0, \hat{\mathbb{E}})$ be the G-expectation space. The function $G : \mathbb{S}_d \to \mathbb{R}$ is defined by

$$G(A) := \frac{1}{2}\hat{\mathbb{E}}[\langle AB_1, B_1\rangle].$$

where $\mathbb{S}_d$ denotes the collection of $d \times d$ symmetric matrices. In this paper, we only consider non-degenerate G-normal distribution, i.e., there exists some $\underline{\sigma}^2 > 0$ such that $G(A) - G(B) \geq \underline{\sigma}^2 \mathrm{tr}[A - B]$ for any $A \geq B$.

Define $\|\xi\|_{p,G} = (\hat{\mathbb{E}}[|\xi|^p])^{1/p}$ for $\xi \in \mathcal{H}_T^0$ and $p \geq 1$. Then for all $t \in [0,T]$, $\hat{\mathbb{E}}_t[\cdot]$ is a continuous mapping on $\mathcal{H}_T^0$ w.r.t. the norm $\|\cdot\|_{1,G}$. Therefore it can be extended continuously to the completion $L_G^1(\Omega_T)$ of $\mathcal{H}_T^0$ under the norm $\|\cdot\|_{1,G}$.

Let $L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, ..., B_{t_n}) : n \geq 1, t_1, ..., t_n \in [0,T], \varphi \in C_{b.Lip}(\mathbb{R}^{d\times n})\}$, where $C_{b.Lip}(\mathbb{R}^{d\times n})$ denotes the set of bounded Lipschitz functions on $\mathbb{R}^{d\times n}$. Denis et al. [5] proved that the completions of $C_b(\Omega_T)$ (the set of bounded continuous function on Ω_T), $\mathcal{H}_T^0$ and $L_{ip}(\Omega_T)$ under $\|\cdot\|_{p,G}$ are the same and we denote them by $L_G^p(\Omega_T)$.

Definition 2.1 *Let $M_G^0(0,T)$ be the collection of processes in the following form: for a given partition $\{t_0, \cdots, t_N\} = \pi_T$ of $[0,T]$,*

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j, t_{j+1})}(t),$$

where $\xi_i \in L_{ip}(\Omega_{t_i})$, $i = 0, 1, 2, \cdots, N-1$. For $p \geq 1$ and $\eta \in M_G^0(0,T)$, let $\|\eta\|_{H_G^p} = \{\hat{\mathbb{E}}[(\int_0^T |\eta_s|^2 ds)^{p/2}]\}^{1/p}$, $\|\eta\|_{M_G^p} = \{\hat{\mathbb{E}}[\int_0^T |\eta_s|^p ds]\}^{1/p}$ and denote by $H_G^p(0,T)$, $M_G^p(0,T)$ the completions of $M_G^0(0,T)$ under the norms $\|\cdot\|_{H_G^p}$, $\|\cdot\|_{M_G^p}$ respectively.

Let $S_G^0(0,T) = \{h(t, B_{t_1\wedge t}, \cdots, B_{t_n\wedge t}) : t_1, \ldots, t_n \in [0,T], h \in C_{b,Lip}(\mathbb{R}^{n+1})\}$. For $p \geq 1$ and $\eta \in S_G^0(0,T)$, set $\|\eta\|_{S_G^p} = \{\hat{\mathbb{E}}[\sup_{t\in[0,T]} |\eta_t|^p]\}^{\frac{1}{p}}$. Denote by $S_G^p(0,T)$ the completion of $S_G^0(0,T)$ under the norm $\|\cdot\|_{S_G^p}$.

We call $L_G^p(\Omega_T), M_G^p(0,T), H_G^p(0,T)$ and $S_G^p(0,T)$ the spaces of G-framework. In the linear case, such kind of spaces contain amost all processes which we are interested in. Under the G-expection, however, there do exist some less regular processes, which can't be approximated by the "regular" ones by the definitions of the spaces in the G-framework. A typical process which does not belong to $M_G^p(0,T)$ is

$$\theta_s = \limsup_{\varepsilon \downarrow 0} \frac{\langle B\rangle_s - \langle B\rangle_{s-\varepsilon}}{\varepsilon}$$

(see [36, 2012] for more details).

Theorem 2.2 *([5, 12]) There exists a tight subset $\mathcal{P} \subset \mathcal{M}_1(\Omega_T)$, the set of probability measures on $(\Omega_T, \mathcal{B}(\Omega_T))$, such that*

$$\hat{\mathbb{E}}[\xi] = \sup_{P\in\mathcal{P}} E_P[\xi] \quad for \quad all\ \xi \in \mathcal{H}_T^0.$$

$\mathcal{P}$ is called a set that represents $\hat{\mathbb{E}}$.

Remark 2.3 *Denis et al. [5] gave a concrete set $\mathcal{P}_M$ that represents $\hat{\mathbb{E}}$. For simplicity, we only introduce the 1-dimensional case, i.e., $\Omega_T = C_0([0,T];\mathbb{R})$.*

Let $(\Omega^0, \mathcal{F}^0, P^0)$ be a probability space and $\{W_t\}$ be a 1-dimensional Brownian motion under P^0. Let $F^0 = \{\mathcal{F}_t^0\}$ be the augmented filtration generated by W. Denis et al. [5] proved that

$$\mathcal{P}_M := \{P_h : P_h = P^0 \circ X^{-1}, X_t = \int_0^t h_s dW_s, h \in L^2_{F^0}([0,T];[\underline{\sigma}, \overline{\sigma}])\}$$

is a set that represents $\hat{\mathbb{E}}$, where $L^2_{F^0}([0,T];[\underline{\sigma},\overline{\sigma}])$ is the collection of F^0-adapted measurable processes with $\underline{\sigma} \le |h_s| \le \overline{\sigma}$. Here

$$\underline{\sigma}^2 := -\hat{\mathbb{E}}[-B_1^2] \le \hat{\mathbb{E}}[B_1^2] =: \overline{\sigma}^2.$$

For this 1-dimensional case,

$$G(a) = \frac{1}{2}\hat{\mathbb{E}}[aB_1^2] = \frac{1}{2}[\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-].$$

Let $\mathcal{P}$ be a weakly compact set that represents $\hat{\mathbb{E}}$. For this $\mathcal{P}$, we define capacity

$$c(A) := \sup_{P\in\mathcal{P}} P(A),\ A \in \mathcal{B}(\Omega_T).$$

A set $A \subset \Omega_T$ is polar if $c(A) = 0$. A property holds "quasi-surely" (q.s. for short) if it holds outside a polar set. In the following, we do not distinguish two random variables X and Y if $X = Y$ q.s.. We set

$$\mathbb{L}^p(\Omega_t) := \{X \in \mathcal{B}(\Omega_t) : \sup_{P\in\mathcal{P}} E_P[|X|^p] < \infty\} \text{ for } p \ge 1.$$

It is important to note that $L_G^p(\Omega_t) \subset \mathbb{L}^p(\Omega_t)$. We extend G-expectation $\hat{\mathbb{E}}$ to $\mathbb{L}^p(\Omega_t)$ and still denote it by $\hat{\mathbb{E}}$, for each $X \in \mathbb{L}^1(\Omega_T)$, we set

$$\hat{\mathbb{E}}[X] = \sup_{P\in\mathcal{P}} E_P[X].$$

For $p \geq 1$, $\mathbb{L}^p(\Omega_t)$ is a Banach space under the norm $(\hat{\mathbb{E}}[|\cdot|^p])^{1/p}$.

Furthermore, we extend the definition of conditional G-expectation. For each fixed $t \geq 0$, let $(A_i)_{i=1}^n$ be a partition of $\mathcal{B}(\Omega_t)$, and set

$$\xi = \sum_{i=1}^{n} \eta_i I_{A_i},$$

where $\eta_i \in L_G^1(\Omega_T)$, $i = 1, \cdots, n$. We define the corresponding conditional G-expectation, still denoted by $\hat{\mathbb{E}}_s[\cdot]$, by setting

$$\hat{\mathbb{E}}_s[\sum_{i=1}^{n} \eta_i I_{A_i}] := \sum_{i=1}^{n} \hat{\mathbb{E}}_s[\eta_i] I_{A_i} \quad \text{for } s \geq t.$$

The following lemma shows that the above definition of conditional G-expectation is meaningful.

Lemma 2.4 *For each $\xi, \eta \in L_G^1(\Omega_T)$ and $A \in \mathcal{B}(\Omega_t)$, if $\xi I_A \geq \eta I_A$ q.s., then $\hat{\mathbb{E}}_t[\xi] I_A \geq \hat{\mathbb{E}}_t[\eta] I_A$ q.s..*

Proof. Otherwise, we can choose a compact set $\mathcal{B}(\Omega_t) \ni K \subset A$ with $c(K) > 0$ such that $(\hat{\mathbb{E}}_t[\xi] - \hat{\mathbb{E}}_t[\eta])^- > 0$ on K. Since K is compact, we can choose a sequence of nonnegative functions $\{\zeta_n\}_{n=1}^\infty \subset C_b(\Omega_t)$ such that $\zeta_n \downarrow I_K$. By Theorem 31 in [5], we have

$$\hat{\mathbb{E}}[\zeta_n(\xi - \eta)^-] \downarrow \hat{\mathbb{E}}[I_K(\xi - \eta)^-]$$

and

$$\hat{\mathbb{E}}[\zeta_n \hat{\mathbb{E}}_t[(\xi - \eta)^-]] \downarrow \hat{\mathbb{E}}[I_K \hat{\mathbb{E}}_t[(\xi - \eta)^-]].$$

Since

$$\hat{\mathbb{E}}[\zeta_n(\xi - \eta)^-] = \hat{\mathbb{E}}[\zeta_n \hat{\mathbb{E}}_t[(\xi - \eta)^-]],$$

we have

$$\hat{\mathbb{E}}[I_K \hat{\mathbb{E}}_t[(\xi - \eta)^-]] = \hat{\mathbb{E}}[I_K(\xi - \eta)^-] = 0.$$

Noting that

$$(\hat{\mathbb{E}}_t[\xi] - \hat{\mathbb{E}}_t[\eta])^- \leq \hat{\mathbb{E}}_t[(\xi - \eta)^-],$$

we get $\hat{\mathbb{E}}_t[(\xi - \eta)^-] > 0$ on K. Also by $c(K) > 0$ we get $\hat{\mathbb{E}}[I_K \hat{\mathbb{E}}_t[(\xi - \eta)^-] > 0$. This is a contradiction and the proof is complete. $\square$

We set

$$\mathbb{L}_G^{0,p,t}(\Omega_T) := \{\xi = \sum_{i=1}^{n} \eta_i I_{A_i} : A_i \in \mathcal{B}(\Omega_t), \eta_i \in L_G^p(\Omega), n \in \mathbb{N}\}.$$

We have the following properties.

Proposition 2.5 *For each $\xi, \eta \in \mathbb{L}_G^{0,1,t}(\Omega_T)$, we have*

(i) *Monotonicity: If $\xi \leq \eta$, then $\hat{\mathbb{E}}_s[\xi] \leq \hat{\mathbb{E}}_s[\eta]$ for any $s \geq t$;*

(ii) *Constant preserving: If* $\xi \in \mathbb{L}_G^{0,1,t}(\Omega_t)$*, then* $\hat{\mathbb{E}}_t[\xi] = \xi$*;*

(iii) *Sub-additivity:* $\hat{\mathbb{E}}_s[\xi + \eta] \leq \hat{\mathbb{E}}_s[\xi] + \hat{\mathbb{E}}_s[\eta]$ *for any* $s \geq t$*;*

(iv) *Positive homogeneity: If* $\xi \in \mathbb{L}_G^{0,\infty,t}(\Omega_t)$ *and* $\xi \geq 0$*, then* $\hat{\mathbb{E}}_t[\xi\eta] = \xi\hat{\mathbb{E}}_t[\eta]$*;*

(v) *Consistency: For* $t \leq s \leq r$*, we have* $\hat{\mathbb{E}}_s[\hat{\mathbb{E}}_r[\xi]] = \hat{\mathbb{E}}_s[\xi]$*.*

(vi) $\hat{\mathbb{E}}[\hat{\mathbb{E}}_t[\xi]] = \hat{\mathbb{E}}[\xi]$.

Proof. (i) is a direct consequence of Lemma 2.9. (ii)-(v) are obvious from the definition. We only prove $\hat{\mathbb{E}}[\hat{\mathbb{E}}_t[\xi]] = \hat{\mathbb{E}}[\xi]$ for ξ which is bounded and positive.

Step 1. For $\xi = \sum_{i=1}^N I_{K_i}\eta_i$, where K_i, $i = 1, \ldots, N$, are disjoint compact sets and $\eta_i \geq 0$, we can choose $\varphi_m^i \in C_b(\Omega_t)$ such that $\varphi_m^i \downarrow K_i$ and $\varphi_m^i\varphi_m^j = 0$ for $i \neq j$. By the same analysis as that in Lemma 2.4, we can get $\hat{\mathbb{E}}[\sum_{i=1}^N I_{K_i}\hat{\mathbb{E}}_t[\eta_i]] = \hat{\mathbb{E}}[\sum_{i=1}^N I_{K_i}\eta_i]$.

Step 2. For $\xi = \sum_{i=1}^N I_{A_i}\eta_i$, where A_i, $i = 1, \ldots, N$, are disjoint sets and $\eta_i \geq 0$. For each fixed $P \in \mathcal{P}$, we can choose compact sets K_m^i such that $K_m^i \uparrow$ and $P(A_i - K_m^i) \downarrow 0$, then

$$\begin{aligned}
E_P[\sum_{i=1}^N I_{A_i}\hat{\mathbb{E}}_t[\eta_i]] &= \lim_{m\to\infty} E_P[\sum_{i=1}^N I_{K_m^i}\hat{\mathbb{E}}_t[\eta_i]] \\
&\leq \lim_{m\to\infty} \hat{\mathbb{E}}[\sum_{i=1}^N I_{K_m^i}\hat{\mathbb{E}}_t[\eta_i]] \\
&= \lim_{m\to\infty} \hat{\mathbb{E}}[\sum_{i=1}^N I_{K_m^i}\eta_i] \\
&\leq \hat{\mathbb{E}}[\sum_{i=1}^N I_{A_i}\eta_i].
\end{aligned}$$

It follows that $\hat{\mathbb{E}}[\sum_{i=1}^N I_{A_i}\hat{\mathbb{E}}_t[\eta_i]] \leq \hat{\mathbb{E}}[\sum_{i=1}^N I_{A_i}\eta_i]$. Similarly we can prove $\hat{\mathbb{E}}[\sum_{i=1}^N I_{A_i}\eta_i] \leq \hat{\mathbb{E}}[\sum_{i=1}^N I_{A_i}\hat{\mathbb{E}}_t[\eta_i]]$. $\square$

Let $\mathbb{L}_G^{p,t}(\Omega_T)$ be the completion of $\mathbb{L}_G^{0,p,t}(\Omega_T)$ under the norm $(\hat{\mathbb{E}}[|\cdot|^p])^{1/p}$. Clearly, the conditional G-expectation can be extended continuously to $\mathbb{L}_G^{p,t}(\Omega_T)$.

Set

$$\mathbb{M}^{p,0}(0,T) := \{\eta_t = \sum_{i=0}^{N-1} \xi_{t_i} I_{[t_i,t_{i+1})}(t) : 0 = t_0 < \cdots < t_N = T, \xi_{t_i} \in \mathbb{L}^p(\Omega_{t_i})\}.$$

For $p \geq 1$, we denote by $\mathbb{M}^p(0,T)$, $\mathbb{H}^p(0,T)$, $\mathbb{S}^p(0,T)$ the completion of $\mathbb{M}^{p,0}(0,T)$ under the norm $||\eta||_{\mathbb{M}^p} := (\hat{\mathbb{E}}[\int_0^T |\eta_t|^p dt])^{1/p}$, $||\eta||_{\mathbb{H}^p} := \{\hat{\mathbb{E}}[(\int_0^T |\eta_t|^2 dt)^{p/2}]\}^{1/p}$, $||\eta||_{\mathbb{S}^p} := (\hat{\mathbb{E}}[\sup_{t\in[0,T]} |\eta_t|^p])^{1/p}$ respectively. Following Li and Peng [14], for each $\eta \in \mathbb{H}^p(0,T)$ with $p \geq 1$, we can define Itô's integral $\int_0^T \eta_s dB_s$. Moreover, by Proposition 2.10 in [14] and the classical Burkholder-Davis-Gundy Inequality, the following properties hold.

Proposition 2.6 *For each $\eta, \theta \in \mathbb{H}^\alpha(0,T)$ with $\alpha \geq 1$ and $p > 0$, $\xi \in \mathbb{L}^\infty(\Omega_t)$, we have*

$$\hat{\mathbb{E}}[\int_0^T \eta_s dB_s] = 0,$$

$$\underline{\sigma}^p c_p \hat{\mathbb{E}}[(\int_0^T |\eta_s|^2 ds)^{p/2}] \leq \hat{\mathbb{E}}[\sup_{t\in[0,T]} |\int_0^t \eta_s dB_s|^p] \leq \bar{\sigma}^p C_p \hat{\mathbb{E}}[(\int_0^T |\eta_s|^2 ds)^{p/2}],$$

$$\int_t^T (\xi\eta_s + \theta_s) dB_s = \xi \int_t^T \eta_s dB_s + \int_t^T \theta_s dB_s,$$

where $0 < c_p < C_p < \infty$ are constants.

Definition 2.7 *A process $\{M_t\}$ with values in $L^1_G(\Omega_T)$ is called a G-martingale if $\hat{\mathbb{E}}_s[M_t] = M_s$ for any $s \leq t$.*

For $\xi \in L_{ip}(\Omega_T)$, let $\mathcal{E}[\xi] = \hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[\xi]]$, where $\hat{\mathbb{E}}$ is the G-expectation. For convenience, we call $\mathcal{E}$ G-evaluation.

For $p \geq 1$ and $\xi \in L_{ip}(\Omega_T)$, define $\|\xi\|_{p,\mathcal{E}} = \{\mathcal{E}[|\xi|^p]\}^{1/p}$ and denote by $L^p_{\mathcal{E}}(\Omega_T)$ the completion of $L_{ip}(\Omega_T)$ under the norm $\|\cdot\|_{p,\mathcal{E}}$.

The following estimate will be frequently used in this paper.

Theorem 2.8 *([35]) For any $\alpha \geq 1$ and $\delta > 0$, we have $L^{\alpha+\delta}_G(\Omega_T) \subset L^\alpha_{\mathcal{E}}(\Omega_T)$. More precisely, for any $1 < \gamma < \beta := (\alpha+\delta)/\alpha$, $\gamma \leq 2$ and for all $\xi \in L_{ip}(\Omega_T)$, we have*

$$\hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[|\xi|^\alpha]] \leq C\{(\hat{\mathbb{E}}[|\xi|^{\alpha+\delta}])^{\alpha/(\alpha+\delta)} + (\hat{\mathbb{E}}[|\xi|^{\alpha+\delta}])^{1/\gamma}\}, \tag{2.1}$$

where $C = \frac{\gamma}{\gamma-1}(1 + 14\sum_{i=1}^\infty i^{-\beta/\gamma})$.

Remark 2.9 *By $\frac{\alpha}{\alpha+\delta} < \frac{1}{\gamma} < 1$, we have*

$$\hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[|\xi|^\alpha]] \leq 2C\{(\hat{\mathbb{E}}[|\xi|^{\alpha+\delta}])^{\alpha/(\alpha+\delta)} + \hat{\mathbb{E}}[|\xi|^{\alpha+\delta}]\}.$$

Set $C_1 = 2\inf\{\frac{\gamma}{\gamma-1}(1 + 14\sum_{i=1}^\infty i^{-\beta/\gamma}) : 1 < \gamma < \beta, \gamma \leq 2\}$, then

$$\hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[|\xi|^\alpha]] \leq C_1\{(\hat{\mathbb{E}}[|\xi|^{\alpha+\delta}])^{\alpha/(\alpha+\delta)} + \hat{\mathbb{E}}[|\xi|^{\alpha+\delta}]\}, \tag{2.2}$$

where C_1 is a constant only depending on α and δ.

For readers' convenience, we list the main notations of this paper as follows:

- The scalar product and norm of the Euclid space $\mathbb{R}^n$ are respectively denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|$;
- $L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, ..., B_{t_n}) : n \geq 1, t_1, ..., t_n \in [0,T], \varphi \in C_{b.Lip}(\mathbb{R}^{d\times n})\}$;
- $\|\xi\|_{p,G} = (\hat{\mathbb{E}}[|\xi|^p])^{1/p}$, $\quad \|\xi\|_{p,\mathcal{E}} = (\hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[|\xi|^p]])^{1/p}$;

- $L_G^p(\Omega_T)$:=the completion of $L_{ip}(\Omega_T)$ under $\|\cdot\|_{p,G}$;
- $L_{\mathcal{E}}^p(\Omega_T)$:=the completion of $L_{ip}(\Omega_T)$ under $\|\cdot\|_{p,\mathcal{E}}$;
- $M_G^0(0,T)$:=$\{\eta_t = \sum_{j=0}^{N-1}\xi_j I_{[t_j,t_{j+1})}(t) : 0 = t_0 < \cdots < t_N = T,\ \xi_i \in L_{ip}(\Omega_{t_i})\}$;
- $\|\eta\|_{M_G^p} = \{\hat{\mathbb{E}}[\int_0^T |\eta_s|^p ds]\}^{1/p}, \qquad \|\eta\|_{H_G^p} = \{\hat{\mathbb{E}}[(\int_0^T |\eta_s|^2 ds)^{p/2}]\}^{1/p}$;
- $M_G^p(0,T)$:=the completion of $M_G^0(0,T)$ under $\|\cdot\|_{M_G^p}$;
- $H_G^p(0,T)$:=$\{$the completion of $M_G^0(0,T)$ under $\|\cdot\|_{H_G^p}\}$ for $p \geq 1$;
- $\mathbb{L}^p(\Omega_T)$:=$\{X \in \mathcal{B}(\Omega_T) : \sup_{P\in\mathcal{P}} E_P[|X|^p] < \infty\}$ for $p \geq 1$;
- $\mathbb{M}^{p,0}(0,T)$:=$\{\eta_t = \sum_{i=0}^{N-1}\xi_{t_i} I_{[t_i,t_{i+1})}(t) : 0 = t_0 < \cdots < t_N = T,\ \xi_{t_i} \in \mathbb{L}^p(\Omega_{t_i})\}$;
- $||\eta||_{\mathbb{M}^p} := (\hat{\mathbb{E}}[\int_0^T |\eta_t|^p dt])^{1/p}, \qquad ||\eta||_{\mathbb{H}^p} := \{\hat{\mathbb{E}}[(\int_0^T |\eta_t|^2 dt)^{p/2}]\}^{1/p}$;
- $||\eta||_{\mathbb{S}^p} := \{\hat{\mathbb{E}}[\sup_{t\in[0,T]} |\eta_t|^p]\}^{\frac{1}{p}}$;
- $\mathbb{M}^p(0,T)$:=the completion of $\mathbb{M}^{p,0}(0,T)$ under $||\cdot||_{\mathbb{M}^p}$;
- $\mathbb{H}^p(0,T)$:=the completion of $\mathbb{M}^{p,0}(0,T)$ under $||\cdot||_{\mathbb{H}^p}$;
- $\mathbb{S}^p(0,T)$:=the completion of $\mathbb{M}^{p,0}(0,T)$ under $||\cdot||_{\mathbb{S}^p}$;
- $S_G^0(0,T)$ =$\{h(t, B_{t_1\wedge t}, \cdots, B_{t_n\wedge t}) : t_1, \ldots, t_n \in [0,T],\ h \in C_{b,Lip}(\mathbb{R}^{n+1})\}$;
- $\|\eta\|_{S_G^p} = \{\hat{\mathbb{E}}[\sup_{t\in[0,T]} |\eta_t|^p]\}^{\frac{1}{p}}$;
- $S_G^p(0,T)$:=the completion of $S_G^0(0,T)$ under $\|\cdot\|_{S_G^p}$;
- $\mathfrak{S}_G^\alpha(0,T)$:= the collection of processes (Y,Z,K) such that $Y \in S_G^\alpha(0,T)$, $Z \in H_G^\alpha(0,T)$, K is a decreasing G-martingale with $K_0 = 0$ and $K_T \in L_G^\alpha(\Omega_T)$.

3 A priori estimates

For simplicity, we consider the G-expectation space $(\Omega_T, L_G^1(\Omega_T), \hat{\mathbb{E}})$ with $\Omega_T = C_0([0,T],\mathbb{R})$ and $\overline{\sigma}^2 = \hat{\mathbb{E}}[B_1^2] \geq -\hat{\mathbb{E}}[-B_1^2] = \underline{\sigma}^2 > 0$. But our results and methods still hold for the case $d > 1$.

We consider the following type of G-BSDEs for simplicity, and similar estimates hold for equation (1.2).

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s - (K_T - K_t), \tag{3.1}$$

where

$$f(t,\omega,y,z) : [0,T] \times \Omega_T \times \mathbb{R}^2 \to \mathbb{R}$$

satisfies the following properties: there exists some $\beta > 1$ such that

(H1) for any y, z, $f(\cdot,\cdot,y,z) \in M_G^\beta(0,T)$;

(H2) $|f(t,\omega,y,z) - f(t,\omega,y',z')| \leq L(|y-y'| + |z-z'|)$ for some $L > 0$.

For simplicity, we denote by $\mathfrak{S}_G^\alpha(0,T)$ the collection of processes (Y,Z,K) such that $Y \in S_G^\alpha(0,T)$, $Z \in H_G^\alpha(0,T)$, K is a decreasing G-martingale with $K_0 = 0$ and $K_T \in L_G^\alpha(\Omega_T)$.

Definition 3.1 *Let $\xi \in L_G^\beta(\Omega_T)$ and f satisfy (H1) and (H2) for some $\beta > 1$. A triplet of processes (Y,Z,K) is called a solution of equation (3.1) if for some $1 < \alpha \leq \beta$ the following properties hold:*

(a) $(Y,Z,K) \in \mathfrak{S}_G^\alpha(0,T)$;

(b) $Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds - \int_t^T Z_s dB_s - (K_T - K_t)$.

In order to get a priori estimates for the solution of equation (3.1), we need the following lemmas.

Lemma 3.2 *Let $X \in S_G^\alpha(0,T)$ for some $\alpha > 1$. Set*

$$X_t^n = \sum_{i=0}^{n-1} X_{t_i^n} I_{[t_i^n, t_{i+1}^n)}(t),$$

where $t_i^n = \frac{iT}{n}$, $i = 0, \cdots, n$. Then

$$\hat{\mathbb{E}}[\sup_{t\in[0,T]} |X_t^n - X_t|^\alpha] \to 0 \quad \text{as } n \to \infty. \tag{3.2}$$

Proof. For each given n, $m \geq 1$, it is easy to check that

$$\sup_{i\leq n-1} \sup_{t_k^m \in [t_i^n, t_{i+1}^n]} |B_{t_k^m} - B_{t_i^n}|^\alpha$$

is a convex function. Then by Proposition 11 in Peng [22], we get

$$\hat{\mathbb{E}}[\sup_{i\leq n-1} \sup_{t_k^m \in [t_i^n, t_{i+1}^n]} |B_{t_k^m} - B_{t_i^n}|^\alpha] = E_{P_{\bar\sigma}}[\sup_{i\leq n-1} \sup_{t_k^m \in [t_i^n, t_{i+1}^n]} |B_{t_k^m} - B_{t_i^n}|^\alpha],$$

where $P_{\bar\sigma}$ is a Wiener measure on Ω_T such that $E_{P_{\bar\sigma}}[B_1^2] = \bar\sigma^2$. Noting that

$$\sup_{i\leq n-1} \sup_{t_k^m \in [t_i^n, t_{i+1}^n]} |B_{t_k^m} - B_{t_i^n}|^\alpha \uparrow \sup_{i\leq n-1} \sup_{t \in [t_i^n, t_{i+1}^n]} |B_t - B_{t_i^n}|^\alpha \text{ as } m \uparrow \infty,$$

we have

$$\hat{\mathbb{E}}[\sup_{i\leq n-1} \sup_{t \in [t_i^n, t_{i+1}^n]} |B_t - B_{t_i^n}|^\alpha] = E_{P_{\bar\sigma}}[\sup_{i\leq n-1} \sup_{t \in [t_i^n, t_{i+1}^n]} |B_t - B_{t_i^n}|^\alpha] \to 0.$$

From this we can get $\hat{\mathbb{E}}[\sup_{t\in[0,T]} |\eta_t - \eta_t^n|^\alpha] \to 0$ for each $\eta \in S_G^0(0,T)$. By the definition of $S_G^\alpha(0,T)$, we can choose a sequence $(\eta^m)_{m=1}^\infty \subset S_G^0(0,T)$ such that $\hat{\mathbb{E}}[\sup_{t\in[0,T]} |X_t - \eta_t^m|^\alpha] \to 0$ as $m \to \infty$. Note that

$$\sup_{t\in[0,T]} |X_t - X_t^n| \leq 2 \sup_{t\in[0,T]} |X_t - \eta_t^m| + \sup_{t\in[0,T]} |\eta_t^m - (\eta^m)_t^n|,$$

then we obtain (3.2) by letting $n \to \infty$ first and then $m \to \infty$. $\square$

Lemma 3.3 *Let X_t, X_t^n be as in Lemma 3.2 and $\alpha^* = \frac{\alpha}{\alpha-1}$. Assume that K is a decreasing G-martingale with $K_0 = 0$ and $K_T \in L_G^{\alpha^*}(\Omega_T)$. Then we have*

$$\hat{\mathbb{E}}[\sup_{t\in[0,T]} |\int_0^t X_s^n dK_s - \int_0^t X_s dK_s|] \to 0 \quad \text{as } n \to \infty.$$

Proof.

$$\begin{aligned}&\sup_{t\in[0,T]} |\int_0^t X_s^n dK_s - \int_0^t X_s dK_s| \\ &\le -\int_0^T |X_s^n - X_s| dK_s \\ &\le \sup_{s\in[0,T]} |X_s^n - X_s|(-K_T).\end{aligned}$$

So we have

$$\hat{\mathbb{E}}[\sup_{t\in[0,T]} |\int_0^t X_s^n dK_s - \int_0^t X_s dK_s|] \le \| \sup_{s\in[0,T]} |X_s^n - X_s| \|_{L_G^\alpha} \|K_T\|_{L^{\alpha^*}} \to 0$$

as $n \to \infty$. $\square$

Lemma 3.4 *Let $X \in S_G^\alpha(0,T)$ for some $\alpha > 1$ and $\alpha^* = \frac{\alpha}{\alpha-1}$. Assume that K^j, $j = 1, 2$, are two decreasing G-martingales with $K_0^j = 0$ and $K_T^j \in L_G^{\alpha^*}(\Omega_T)$. Then the process defined by*

$$\int_0^t X_s^+ dK_s^1 + \int_0^t X_s^- dK_s^2$$

is also a decreasing G-martingale.

Proof. Let X^n be as in Lemma 3.2. By Lemma 3.3, it suffices to prove that the process

$$\int_0^t (X_s^n)^+ dK_s^1 + \int_0^t (X_s^n)^- dK_s^2$$

is a G-martingale. By properties of conditional G-expectation, we have, for any $t \in [t_i^n, t_{i+1}^n]$,

$$\begin{aligned}&\hat{\mathbb{E}}_t[X_{t_i^n}^+(K_{t_{i+1}^n}^1 - K_{t_i^n}^1) + X_{t_i^n}^-(K_{t_{i+1}^n}^2 - K_{t_i^n}^2)] \\ &= X_{t_i^n}^+ \hat{\mathbb{E}}_t[K_{t_{i+1}^n}^1 - K_{t_i^n}^1] + X_{t_i^n}^- \hat{\mathbb{E}}_t[K_{t_{i+1}^n}^2 - K_{t_i^n}^2] \\ &= X_{t_i^n}^+(K_t^1 - K_{t_i^n}^1) + X_{t_i^n}^-(K_t^2 - K_{t_i^n}^2).\end{aligned}$$

From this we obtain that $\int_0^t (X_s^n)^+ dK_s^1 + \int_0^t (X_s^n)^- dK_s^2$ is a G-martingale. $\square$

Now we give a priori estimates for the solution of equation (3.1). For this purpose, a weaker version of condition (H2) is enough.

(H2') $|f(t,\omega,y,z) - f(t,\omega,y',z')| \le L^w(|y-y'| + |z-z'| + \varepsilon)$ for some $L^w, \varepsilon > 0$.

In the following three propositions, C_α will always designate a constant depending on $\alpha, T, L^w, \underline{\sigma}$, which may vary from line to line.

Proposition 3.5 *Let f satisfy (H1) and (H2') for some $\beta > 1$. Assume*

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s - (K_T - K_t),$$

where $Y \in \mathbb{S}^\alpha(0,T)$, $Z \in \mathbb{H}^\alpha(0,T)$, K is a decreasing process with $K_0 = 0$ and $K_T \in \mathbb{L}^\alpha(\Omega_T)$ for some $\beta \geq \alpha > 1$. Then there exists a constant $C_\alpha := C(\alpha, T, \underline{\sigma}, L^w) > 0$ such that

$$\hat{\mathbb{E}}[(\int_0^T |Z_s|^2 ds)^{\frac{\alpha}{2}}] \leq C_\alpha\{\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t|^\alpha] + (\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t|^\alpha])^{\frac{1}{2}}(\hat{\mathbb{E}}[(\int_0^T f_s^0 ds)^\alpha])^{\frac{1}{2}}\}, \tag{3.3}$$

$$\hat{\mathbb{E}}[|K_T|^\alpha] \leq C_\alpha\{\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t|^\alpha] + \hat{\mathbb{E}}[(\int_0^T f_s^0 ds)^\alpha]\}, \tag{3.4}$$

where $f_s^0 = |f(s,0,0)| + L^w\varepsilon$.

Proof. Applying Itô's formula to $|Y_t|^2$, we have

$$|Y_0|^2 + \int_0^T |Z_s|^2 d\langle B\rangle_s = |\xi|^2 + \int_0^T 2Y_s f(s)ds - \int_0^T 2Y_s Z_s dB_s - \int_0^T 2Y_s dK_s,$$

where $f(s) = f(s, Y_s, Z_s)$. Then

$$(\int_0^T |Z_s|^2 d\langle B\rangle_s)^{\frac{\alpha}{2}} \leq C_\alpha\{|\xi|^\alpha + |\int_0^T Y_s f(s)ds|^{\frac{\alpha}{2}} + |\int_0^T Y_s Z_s dB_s|^{\frac{\alpha}{2}} + |\int_0^T Y_s dK_s|^{\frac{\alpha}{2}}\}.$$

By Proposition 2.6 and simple calculation, we can obtain

$$\hat{\mathbb{E}}[(\int_0^T |Z_s|^2 ds)^{\frac{\alpha}{2}}] \leq C_\alpha\{\|Y\|_{\mathbb{S}^\alpha}^\alpha + \|Y\|_{\mathbb{S}^\alpha}^{\frac{\alpha}{2}}[(\hat{\mathbb{E}}[|K_T|^\alpha])^{\frac{1}{2}} + (\hat{\mathbb{E}}[(\int_0^T f_s^0 ds)^\alpha])^{\frac{1}{2}}]\}. \tag{3.5}$$

On the other hand,

$$K_T = \xi - Y_0 + \int_0^T f(s)ds - \int_0^T Z_s dB_s.$$

By simple calculation, we get

$$\hat{\mathbb{E}}[|K_T|^\alpha] \leq C_\alpha\{\|Y\|_{\mathbb{S}^\alpha}^\alpha + \hat{\mathbb{E}}[(\int_0^T |Z_s|^2 ds)^{\alpha/2}] + \hat{\mathbb{E}}[(\int_0^T f_s^0 ds)^\alpha]\}. \tag{3.6}$$

By (3.5) and (3.6), it is easy to get (3.3) and (3.4). $\square$

Remark 3.6 *In this proposition, we do not assume that (Y, Z, K) belongs to $\mathfrak{S}_G^\alpha(0,T)$.*

Proposition 3.7 *Let $\xi \in L_G^\beta(\Omega_T)$ and f satisfy (H1) and (H2') for some $\beta > 1$. Assume that $(Y, Z, K) \in \mathfrak{S}_G^\alpha(0,T)$ for some $1 < \alpha < \beta$ is a solution of equation (3.1). Then*

(i) *There exists a constant* $C_\alpha := C(\alpha, T, \underline{\sigma}, L^w) > 0$ *such that*

$$|Y_t|^\alpha \le C_\alpha \hat{\mathbb{E}}_t[|\xi|^\alpha + \int_t^T |f_s^0|^\alpha ds], \tag{3.7}$$

$$\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t|^\alpha] \le C_\alpha \hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[|\xi|^\alpha + \int_0^T |f_s^0|^\alpha ds]], \tag{3.8}$$

where $f_s^0 = |f(s,0,0)| + L^w\varepsilon$.

(ii) *For any given* α' *with* $\alpha < \alpha' < \beta$, *there exists a constant* $C_{\alpha,\alpha'}$ *depending on* α, α', T, $\underline{\sigma}$, L^w *such that*

$$\begin{aligned}\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t|^\alpha] \le C_{\alpha,\alpha'}\{&\hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[|\xi|^\alpha]] \\ &+ (\hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[(\int_0^T f_s^0 ds)^{\alpha'}]])^{\frac{\alpha}{\alpha'}} + \hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[(\int_0^T f_s^0 ds)^{\alpha'}]]\}.\end{aligned} \tag{3.9}$$

Proof. For any $\gamma, \epsilon > 0$, set $\tilde{Y}_t = |Y_t|^2 + \epsilon_\alpha$, where $\epsilon_\alpha = \epsilon(1-\alpha/2)^+$, applying Itô's formula to $\tilde{Y}_t^{\alpha/2} e^{\gamma t}$, we have

$$\begin{aligned}&\tilde{Y}_t^{\alpha/2} e^{\gamma t} + \gamma \int_t^T e^{\gamma s}\tilde{Y}_s^{\alpha/2} ds + \frac{\alpha}{2}\int_t^T e^{\gamma s}\tilde{Y}_s^{\alpha/2-1} Z_s^2 d\langle B\rangle_s \\ &= (|\xi|^2 + \epsilon_\alpha)^{\alpha/2} e^{\gamma T} + \alpha(1-\frac{\alpha}{2})\int_t^T e^{\gamma s}\tilde{Y}_s^{\alpha/2-2} Y_s^2 Z_s^2 d\langle B\rangle_s \\ &+ \int_t^T \alpha e^{\gamma s}\tilde{Y}_s^{\alpha/2-1} Y_s f(s) ds - \int_t^T \alpha e^{\gamma s}\tilde{Y}_s^{\alpha/2-1}(Y_s Z_s dB_s + Y_s dK_s) \\ &\le (|\xi|^2 + \epsilon_\alpha)^{\alpha/2} e^{\gamma T} + \alpha(1-\frac{\alpha}{2})\int_t^T e^{\gamma s}\tilde{Y}_s^{\alpha/2-1} Z_s^2 d\langle B\rangle_s \\ &+ \int_t^T \alpha e^{\gamma s}\tilde{Y}_s^{\alpha/2-1/2}|f(s)| ds - (M_T - M_t),\end{aligned} \tag{3.10}$$

where $f(s) = f(s, Y_s, Z_s)$ and

$$M_t = \int_0^t \alpha e^{\gamma s}\tilde{Y}_s^{\alpha/2-1} Y_s Z_s dB_s + \int_0^t \alpha e^{\gamma s}\tilde{Y}_s^{\alpha/2-1} Y_s^+ dK_s.$$

From the assumption of f, we have

$$\begin{aligned}&\int_t^T \alpha e^{\gamma s}\tilde{Y}_s^{\alpha/2-1/2}|f(s)| ds \\ &\le \int_t^T \alpha e^{\gamma s}\tilde{Y}_s^{\alpha/2-1/2}(f_s^0 + L^w|Y_s| + L^w|Z_s|) ds \\ &\le (\alpha L^w + \frac{\alpha(L^w)^2}{\underline{\sigma}^2(\alpha-1)})\int_t^T e^{\gamma s}\tilde{Y}_s^{\alpha/2} ds + \frac{\alpha(\alpha-1)}{4}\int_t^T e^{\gamma s}\tilde{Y}_s^{\alpha/2-1} Z_s^2 d\langle B\rangle_s \\ &+ \int_t^T \alpha e^{\gamma s}\tilde{Y}_s^{\alpha/2-1/2}|f_s^0| ds.\end{aligned} \tag{3.11}$$

(i) By Young's inequality, we have

$$\int_t^T \alpha e^{\gamma s} \tilde{Y}_s{}^{\alpha/2-1/2} |f_s^0| ds \leq (\alpha - 1) \int_t^T e^{\gamma s} \tilde{Y}_s{}^{\alpha/2} ds + \int_t^T e^{\gamma s} |f_s^0|^\alpha ds. \quad (3.12)$$

By (3.10), (3.11) and (3.12), we have

$$\begin{aligned} &\tilde{Y}_t{}^{\alpha/2} e^{\gamma t} + (\gamma - \tilde{\alpha}) \int_t^T e^{\gamma s} \tilde{Y}_s{}^{\alpha/2} ds + \frac{\alpha(\alpha-1)}{4} \int_t^T e^{\gamma s} \tilde{Y}_s{}^{\alpha/2-1} Z_s^2 d\langle B \rangle_s \\ &\leq (|\xi|^2 + \epsilon_\alpha)^{\alpha/2} e^{\gamma T} + \int_t^T e^{\gamma s} |f_s^0|^\alpha ds - (M_T - M_t), \end{aligned}$$

where $\tilde{\alpha} = \alpha L^w + \alpha + \frac{\alpha (L^w)^2}{\underline{\sigma}^2(\alpha-1)} - 1$. Setting $\gamma = \tilde{\alpha} + 1$, we have

$$\begin{aligned} &\tilde{Y}_t{}^{\alpha/2} e^{\gamma t} + M_T - M_t \\ &\leq (|\xi|^2 + \epsilon_\alpha)^{\alpha/2} e^{\gamma T} + \int_t^T e^{\gamma s} |f_s^0|^\alpha ds. \end{aligned}$$

By Lemma 3.4, M_t is a G-martingale, so we have

$$\tilde{Y}_t{}^{\alpha/2} e^{\gamma t} \leq \hat{\mathbb{E}}_t[(|\xi|^2 + \epsilon_\alpha)^{\alpha/2} e^{\gamma T} + \int_t^T e^{\gamma s} |f_s^0|^\alpha ds].$$

By letting $\epsilon \downarrow 0$, there exists a constant $C_\alpha := C_\alpha(T, L^w, \underline{\sigma})$ such that

$$|Y_t|^\alpha \leq C_\alpha \hat{\mathbb{E}}_t[|\xi|^\alpha + \int_t^T |f_s^0|^\alpha ds].$$

It follows that

$$\hat{\mathbb{E}}[\sup_{t \in [0,T]} |Y_t|^\alpha] \leq C_\alpha \hat{\mathbb{E}}[\sup_{t \in [0,T]} \hat{\mathbb{E}}_t[|\xi|^\alpha + \int_0^T |f_s^0|^\alpha ds]].$$

(ii) By (3.10) and (3.11) and setting $\gamma = \alpha L^w + \frac{\alpha (L^w)^2}{\underline{\sigma}^2(\alpha-1)} + 1$, then we get

$$\tilde{Y}_t{}^{\alpha/2} e^{\gamma t} \leq \hat{\mathbb{E}}_t[(|\xi|^2 + \epsilon_\alpha)^{\alpha/2} e^{\gamma T} + \int_t^T \alpha e^{\gamma s} \tilde{Y}_s{}^{\alpha/2-1/2} f_s^0 ds].$$

By letting $\epsilon \downarrow 0$, we get

$$|Y_t|^\alpha \leq C_\alpha \hat{\mathbb{E}}_t[|\xi|^\alpha + \int_t^T |Y_s|^{\alpha-1} f_s^0 ds]. \quad (3.13)$$

From this we get

$$\begin{aligned} |Y_t|^\alpha &\leq C_\alpha \{\hat{\mathbb{E}}_t[|\xi|^\alpha] + \hat{\mathbb{E}}_t[\sup_{s \in [0,T]} |Y_s|^{\alpha-1} \int_0^T f_s^0 ds]\} \\ &\leq C_\alpha \{\hat{\mathbb{E}}_t[|\xi|^\alpha] + (\hat{\mathbb{E}}_t[\sup_{s \in [0,T]} |Y_s|^{(\alpha-1)\alpha'^*}])^{\frac{1}{\alpha'^*}} (\hat{\mathbb{E}}_t[(\int_0^T f_s^0 ds)^{\alpha'}])^{\frac{1}{\alpha'}}\}, \quad (3.14) \end{aligned}$$

where $\alpha'^* = \frac{\alpha'}{\alpha'-1}$. Thus we obtain

$$\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t|^\alpha] \le C_\alpha\{||\xi||^\alpha_{\alpha,\mathcal{E}} + || \sup_{s\in[0,T]} |Y_s|^{\alpha-1}||_{\alpha'^*,\mathcal{E}} || \int_0^T f_s^0 ds||_{\alpha',\mathcal{E}}\}.$$

It is easy to check that $(\alpha-1)\alpha'^* < \alpha$, then by (2.2) there exists a constant C only depending on α and α' such that

$$|| \sup_{s\in[0,T]} |Y_s|^{\alpha-1}||_{\alpha'^*,\mathcal{E}} \le C\{(\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t|^\alpha])^{\frac{\alpha-1}{\alpha}} + (\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t|^\alpha])^{\frac{1}{\alpha'^*}}\}.$$

By Young's inequality, we have

$$CC_\alpha(\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t|^\alpha])^{\frac{\alpha-1}{\alpha}} || \int_0^T f_s^0 ds||_{\alpha',\mathcal{E}} \le \frac{1}{4}\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t|^\alpha] + C_1 C_\alpha || \int_0^T f_s^0 ds||^\alpha_{\alpha',\mathcal{E}}$$

and

$$CC_\alpha(\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t|^\alpha])^{\frac{1}{\alpha'^*}} || \int_0^T f_s^0 ds||_{\alpha',\mathcal{E}} \le \frac{1}{4}\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t|^\alpha] + C_1 C_\alpha || \int_0^T f_s^0 ds||^{\alpha'}_{\alpha',\mathcal{E}},$$

where C_1 is a constant only depending on α and α'. Thus we obtain (3.9). □

Proposition 3.8 *Let f_i, $i = 1, 2$, satisfy (H1) and (H2') for some $\beta > 1$. Assume*

$$Y_t^i = \xi^i + \int_t^T f_i(s, Y_s^i, Z_s^i)ds - \int_t^T Z_s^i dB_s - (K_T^i - K_t^i),$$

where $Y^i \in \mathbb{S}^\alpha(0,T)$, $Z^i \in \mathbb{H}^\alpha(0,T)$, K^i is a decreasing process with $K_0^i = 0$ and $K_T^i \in \mathbb{L}^\alpha(\Omega_T)$ for some $\beta \ge \alpha > 1$. Set $\hat{Y}_t = Y_t^1 - Y_t^2$, $\hat{Z}_t = Z_t^1 - Z_t^2$ and $\hat{K}_t = K_t^1 - K_t^2$. Then there exists a constant $C_\alpha := C(\alpha, T, \underline{\sigma}, L^w) > 0$ such that

$$\hat{\mathbb{E}}[(\int_0^T |\hat{Z}_s|^2 ds)^{\frac{\alpha}{2}}] \le C_\alpha\{\|\hat{Y}\|^\alpha_{\mathbb{S}^\alpha} + \|\hat{Y}\|^{\frac{\alpha}{2}}_{\mathbb{S}^\alpha} \sum_{i=1}^2 [||Y^i||^{\frac{\alpha}{2}}_{\mathbb{S}^\alpha} + || \int_0^T f_s^{i,0} ds||^{\frac{\alpha}{2}}_{\alpha,G}]\}, \tag{3.15}$$

where $f_s^{i,0} = |f_i(s,0,0)| + L^w\varepsilon$, $i = 1, 2$.

Proof. Applying Itô's formula to $|\hat{Y}_t|^2$, by similar analysis as that in Proposition 3.5, we have

$$||\hat{Z}||^\alpha_{\mathbb{H}^\alpha} \le C_\alpha\{\|\hat{Y}\|^\alpha_{\mathbb{S}^\alpha} + \|\hat{Y}\|^{\frac{\alpha}{2}}_{\mathbb{D}^\alpha}[||K_T^1||^{\frac{\alpha}{2}}_{\alpha,G} + ||K_T^2||^{\frac{\alpha}{2}}_{\alpha,G} + || \int_0^T \hat{f}_s ds||^{\frac{\alpha}{2}}_{\alpha,G}]\},$$

where $\hat{f}_s = |f_1(s, Y_s^2, Z_s^2) - f_2(s, Y_s^2, Z_s^2)| + L^w\varepsilon$. By Proposition 3.5, we obtain

$$\begin{aligned} &||K_T^1||^{\frac{\alpha}{2}}_{\alpha,G} + ||K_T^2||^{\frac{\alpha}{2}}_{\alpha,G} + || \int_0^T \hat{f}_s ds||^{\frac{\alpha}{2}}_{\alpha,G} \\ &\le C_\alpha\{||Y^1||^{\frac{\alpha}{2}}_{\mathbb{S}^\alpha} + ||Y^2||^{\frac{\alpha}{2}}_{\mathbb{S}^\alpha} + || \int_0^T f_s^{1,0} ds||^{\frac{\alpha}{2}}_{\alpha,G} + || \int_0^T f_s^{2,0} ds||^{\frac{\alpha}{2}}_{\alpha,G}\}. \end{aligned}$$

Thus we get (3.15). □

Proposition 3.9 *Let $\xi^i \in L_G^\beta(\Omega_T)$, $i = 1, 2$, and f_i satisfy (H1) and (H2') for some $\beta > 1$. Assume that $(Y^i, Z^i, K^i) \in \mathfrak{S}_G^\alpha(0,T)$ for some $1 < \alpha < \beta$ are the solutions of equation (3.1) corresponding to ξ^i and f_i . Set $\hat{Y}_t = Y_t^1 - Y_t^2$, $\hat{Z}_t = Z_t^1 - Z_t^2$ and $\hat{K}_t = K_t^1 - K_t^2$. Then*

(i) *There exists a constant $C_\alpha := C(\alpha, T, \underline{\sigma}, L_1^w) > 0$ such that*

$$|\hat{Y}_t|^\alpha \leq C_\alpha \hat{\mathbb{E}}_t[|\hat{\xi}|^\alpha + \int_t^T |\hat{f}_s|^\alpha ds], \tag{3.16}$$

where $\hat{f}_s = |f_1(s, Y_s^2, Z_s^2) - f_2(s, Y_s^2, Z_s^2)| + L_1^w \varepsilon$.

(ii) *For any given α' with $\alpha < \alpha' < \beta$, there exists a constant $C_{\alpha,\alpha'}$ depending on α, α', T, $\underline{\sigma}$, L^w such that*

$$\hat{\mathbb{E}}[\sup_{t\in[0,T]} |\hat{Y}_t|^\alpha] \leq C_{\alpha,\alpha'}\{\hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[|\hat{\xi}|^\alpha]]$$
$$+ (\hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[(\int_0^T \hat{f}_s ds)^{\alpha'}]])^{\frac{\alpha}{\alpha'}} + \hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[(\int_0^T \hat{f}_s ds)^{\alpha'}]]\}. \tag{3.17}$$

Proof. For any $\gamma, \epsilon > 0$, applying Itô's formula to $(|\hat{Y}_t|^2 + \epsilon_\alpha)^{\alpha/2} e^{\gamma t}$, where $\epsilon_\alpha = \epsilon(1 - \alpha/2)^+$, by similar analysis as in Proposition 3.7, we have by setting $\gamma = \alpha L^w + \alpha + \frac{\alpha (L^w)^2}{\underline{\sigma}^2(\alpha-1)}$

$$(|\hat{Y}_t|^2 + \epsilon_\alpha)^{\alpha/2} e^{\gamma t} + \int_t^T \alpha e^{\gamma s}(|\hat{Y}_s|^2 + \epsilon_\alpha)^{\alpha/2-1} \hat{Y}_s \hat{Z}_s dB_s + J_T - J_t$$
$$\leq (|\hat{\xi}|^2 + \epsilon_\alpha)^{\alpha/2} e^{\gamma T} + \int_t^T e^{\gamma s} |\hat{f}_s|^\alpha ds$$

and

$$(|\hat{Y}_t|^2 + \epsilon_\alpha)^{\alpha/2} e^{\gamma t} + \int_t^T \alpha e^{\gamma s}(|\hat{Y}_s|^2 + \epsilon_\alpha)^{\alpha/2-1} \hat{Y}_s \hat{Z}_s dB_s + J_T - J_t$$
$$\leq (|\hat{\xi}|^2 + \epsilon_\alpha)^{\alpha/2} e^{\gamma T} + \int_t^T \alpha e^{\gamma s}(|\hat{Y}_s|^2 + \epsilon_\alpha)^{\alpha/2-1/2} \hat{f}_s ds,$$

where

$$J_t = \int_0^t \alpha e^{\gamma s}(|\hat{Y}_s|^2 + \epsilon_\alpha)^{\alpha/2-1}(\hat{Y}_s^+ dK_s^1 + \hat{Y}_s^- dK_s^2).$$

By Lemma 3.4, J_t is a G-martingale. Taking conditional G-expectation and letting $\epsilon \downarrow 0$, we obtain a constant $C_\alpha := C_\alpha(T, L_1^w, \underline{\sigma}) > 0$ such that

$$|\hat{Y}_t|^\alpha \leq C_\alpha \hat{\mathbb{E}}_t[|\hat{\xi}|^\alpha + \int_t^T |\hat{f}_s|^\alpha ds]$$

and

$$|\hat{Y}_t|^\alpha \leq C_\alpha \hat{\mathbb{E}}_t[|\hat{\xi}|^\alpha + \int_t^T |\hat{Y}_s|^{\alpha-1} \hat{f}_s ds].$$

By the same analysis as that in Proposition 3.7, we get (3.17). □

Remark 3.10 *Noting that $\int_0^T \eta_s d\langle B\rangle_s \leq \bar{\sigma}^2 \int_0^T \eta_s ds$ for any $\eta \in M_G^1(0,T)$, thus Propositions 3.7 and 3.9 still hold for G-BSDE (1.2).*

4 Existence and uniqueness of G-BSDEs

In order to prove the existence of equation (3.1), we start with the simple case $f(t,\omega,y,z)=h(y,z)$, $\xi=\varphi(B_T)$. Here $h\in C_0^\infty(\mathbb{R}^2)$, $\varphi\in C_{b.Lip}(\mathbb{R}^2)$. For this case, we can obtain the solution of equation (3.1) from the following nonlinear partial differential equation:

$$\partial_t u+G(\partial^2_{xx}u)+h(u,\partial_x u)=0, u(T,x)=\varphi(x). \tag{4.1}$$

Then we approximate the solution of equation (3.1) with more complicated f by those of equations (3.1) with much simpler $\{f_n\}$. More precisely, assume that $\|f_n-f\|_{M_G^\beta}\to 0$ and (Y^n,Z^n,K^n) is the solution of the following G-BSDE

$$Y_t^n=\xi+\int_t^T f_n(s,Y_s^n,Z_s^n)ds-\int_t^T Z_s^n dB_s-(K_T^n-K_t^n).$$

We try to prove that (Y^n,Z^n,K^n) converges to (Y,Z,K) and (Y,Z,K) is the solution of the following G-BSDE

$$Y_t=\xi+\int_t^T f(s,Y_s,Z_s)ds-\int_t^T Z_s dB_s-(K_T-K_t).$$

One of the main results of this paper is

Theorem 4.1 *Assume that $\xi\in L_G^\beta(\Omega_T)$ and f satisfies (H1) and (H2) for some $\beta>1$. Then equation (3.1) has a unique solution (Y,Z,K). Moreover, for any $1<\alpha<\beta$ we have $Y\in S_G^\alpha(0,T)$, $Z\in H_G^\alpha(0,T)$ and $K_T\in L_G^\alpha(\Omega_T)$.*

Proof. The uniqueness of the solution is a direct consequence of the a priori estimates in Proposition 3.8 and Proposition 3.9. By these estimates it also suffices to prove the existence for the case $\xi\in L_{ip}(\Omega_T)$ and then pass to the limit for the general situation.

Step 1. $f(t,\omega,y,z)=h(y,z)$ with $h\in C_0^\infty(\mathbb{R}^2)$.

Part 1. We first consider the case $\xi=\varphi(B_T-B_{t_1})$ with $\varphi\in C_{b,Lip}(\mathbb{R})$ and $t_1<T$. Let u be the solution of equation (4.1) with terminal condition φ. By Theorem 6.4.3 in Krylov [13] (see also Theorem 4.4 in Appendix C in Peng [27]), there exists a constant $\alpha\in(0,1)$ such that for each $\kappa>0$,

$$||u||_{C^{1+\alpha/2,2+\alpha}([0,T-\kappa]\times\mathbb{R})}<\infty.$$

Applying Itô's formula to $u(t,B_t-B_{t_1})$ on $[t_1,T-\kappa]$, we get

$$\begin{aligned}u(t,B_t-B_{t_1})=&u(T-\kappa,B_{T-\kappa}-B_{t_1})+\int_t^{T-\kappa}h(u,\partial_x u)(s,B_s-B_{t_1})ds\\&-\int_t^{T-\kappa}\partial_x u(s,B_s-B_{t_1})dB_s-(K_{T-\kappa}-K_t),\end{aligned} \tag{4.2}$$

where $K_t=\frac{1}{2}\int_{t_1}^t\partial^2_{xx}u(s,B_s-B_{t_1})d\langle B\rangle_s-\int_{t_1}^t G(\partial^2_{xx}u(s,B_s-B_{t_1}))ds$ is a non-increasing G-martingale. We now prove that there exists a constant $L_1>0$ such that

$$|u(t,x)-u(s,y)|\le L_1(\sqrt{|t-s|}+|x-y|),\ t,s\in[0,T],x,y\in\mathbb{R}. \tag{4.3}$$

For each fixed $x_0 \in \mathbb{R}$, set $\tilde{u}(t,x) = u(t,x+x_0)$, it is easy to check that $\tilde{u}$ is the solution of the following PDE:

$$\partial_t \tilde{u} + G(\partial^2_{xx}\tilde{u}) + h(\tilde{u}, \partial_x \tilde{u}) = 0, \ \tilde{u}(T,x) = \varphi(x+x_0). \tag{4.4}$$

Define $\hat{u}(t,x) = u(t,x) + L_\varphi |x_0| \exp(L_h(T-t))$, where L_φ and L_h are the Lipschitz constants of φ and h respectively, it is easy to verify that $\hat{u}$ is a supersolution of PDE (4.4). Thus by comparison theorem (see Theorem 2.4 in Appendix C in Peng [27]) we get

$$u(t,x+x_0) \le u(t,x) + L_\varphi |x_0| \exp(L_h(T-t)), \ t \in [0,T], x \in \mathbb{R}.$$

Since x_0 is arbitrary, we get $|u(t,x) - u(t,y)| \le \hat{L}|x-y|$, where $\hat{L} = L_\varphi \exp(L_h T)$. From this we can get $|\partial_x u(t,x)| \le \hat{L}$ for each $t \in [0,T]$, $x \in \mathbb{R}$. On the other hand, for each fixed $\bar{t} < \hat{t} < T$ and $x \in \mathbb{R}$, applying Itô's formula to $u(s, x + B_s - B_{\bar{t}})$ on $[\bar{t}, \hat{t}]$, we get

$$u(\bar{t}, x) = \hat{\mathbb{E}}[u(\hat{t}, x + B_{\hat{t}} - B_{\bar{t}}) + \int_{\bar{t}}^{\hat{t}} h(u, \partial_x u)(s, x + B_s - B_{\bar{t}}) ds].$$

From this we deduce that

$$|u(\bar{t}, x) - u(\hat{t}, x)| \le \hat{\mathbb{E}}[\hat{L}|B_{\hat{t}} - B_{\bar{t}}| + \tilde{L}|\hat{t} - \bar{t}|] \le (\hat{L}\bar{\sigma} + \tilde{L}\sqrt{T})\sqrt{|\hat{t} - \bar{t}|},$$

where $\tilde{L} = \sup_{(x,y)\in\mathbb{R}^2} |h(x,y)|$. Thus we get (4.3) by taking $L_1 = \max\{\hat{L}, \hat{L}\bar{\sigma} + \tilde{L}\sqrt{T}\}$. Letting $\kappa \downarrow 0$ in equation (4.2), it is easy to verify that

$$\hat{\mathbb{E}}[|Y_{T-\kappa} - \xi|^2 + \int_{T-\kappa}^T |Z_t|^2 dt + (K_{T-\kappa} - K_T)^2] \to 0,$$

where $Y_t = u(t, B_t - B_{t_1})$ and $Z_t = \partial_x u(t, B_t - B_{t_1})$. Thus $(Y_t, Z_t, K_t)_{t\in[t_1,T]}$ is a solution of equation (3.1) with terminal value $\xi = \varphi(B_T - B_{t_1})$. Furthermore, it is easy to check that $Y \in S^\alpha_G(t_1,T)$, $Z \in H^\alpha_G(t_1,T)$ and $K_T \in L^\alpha_G(\Omega_T)$ for any $\alpha > 1$.

Part 2. We now consider the case $\xi = \psi(B_{t_1}, B_T - B_{t_1})$ with $\psi \in C_{b,Lip}(\mathbb{R}^2)$, and the more general case can be proved similarly. For each fixed $x \in \mathbb{R}$, let $u(\cdot, x, \cdot)$ be the solution of equation (4.1) with terminal condition $\psi(x, \cdot)$. By Part 1, we have

$$\begin{aligned} u(t,x,B_t - B_{t_1}) =& u(T,x,B_T - B_{t_1}) + \int_t^T h(u, \partial_y u)(s,x,B_s - B_{t_1}) ds \\ & - \int_t^T \partial_y u(s,x,B_s - B_{t_1}) dB_s - (K^x_T - K^x_t), \end{aligned} \tag{4.5}$$

where $K^x_t = \frac{1}{2}\int_{t_1}^t \partial^2_{yy} u(s,x,B_s - B_{t_1}) d\langle B\rangle_s - \int_{t_1}^t G(\partial^2_{yy} u(s,x,B_s - B_{t_1})) ds$. We replace x by B_{t_1} and get

$$Y_t = Y_T + \int_t^T h(Y_s, Z_s) ds - \int_t^T Z_s dB_s - (K_T - K_t),$$

where $Y_t = u(t, B_{t_1}, B_t - B_{t_1})$, $Z_t = \partial_y u(t, B_{t_1}, B_t - B_{t_1})$ and

$$K_t = \frac{1}{2}\int_{t_1}^{t} \partial_{yy}^2 u(s, B_{t_1}, B_s - B_{t_1}) d\langle B\rangle_s - \int_{t_1}^{t} G(\partial_{yy}^2 u(s, B_{t_1}, B_s - B_{t_1}))ds.$$

Now we are in a position to prove $(Y, Z, K) \in \mathfrak{S}_G^{\alpha}(0, T)$. We use the following argument, for each given $n \in \mathbb{N}$, by partition of unity theorem, there exist $h_i^n \in C_0^{\infty}(\mathbb{R})$ with the diameter of support $\lambda(\text{supp}(h_i^n)) < 1/n$, $0 \le h_i^n \le 1$, $I_{[-n,n]}(x) \le \sum_{i=1}^{k_n} h_i^n \le 1$. Choose x_i^n such that $h_i^n(x_i^n) > 0$. Through equation (4.5), we have

$$Y_t^n = Y_T^n + \int_t^T \sum_{i=1}^{n} h(y_s^{n,i}, z_s^{n,i}) h_i^n(B_{t_1}) ds - \int_t^T Z_s^n dB_s - (K_T^n - K_t^n),$$

where $y_t^{n,i} = u(t, x_i^n, B_t - B_{t_1})$, $z_t^{n,i} = \partial_y u(t, x_i^n, B_t - B_{t_1})$, $Y_t^n = \sum_{i=1}^{n} y_t^{n,i} h_i^n(B_{t_1})$, $Z_t^n = \sum_{i=1}^{n} z_t^{n,i} h_i^n(B_{t_1})$ and $K_t^n = \sum_{i=1}^{n} K_t^{x_i^n} h_i^n(B_{t_1})$.

By the same analysis as that in Part 1, we can obtain a constant $L_2 > 0$ such that for each t, $s \in [0, T]$, x, x', y, $y' \in \mathbb{R}$,

$$|u(t, x, y) - u(s, x', y')| \le L_2(\sqrt{|t - s|} + |x - x'| + |y - y'|).$$

From this we get

$$\begin{aligned} |Y_t - Y_t^n| &\le \sum_{i=1}^{k_n} h_i^n(B_{t_1}) |u(t, x_i^n, B_t - B_{t_1}) - u(t, B_{t_1}, B_t - B_{t_1})| + |Y_t| I_{[|B_{t_1}|>n]} \\ &\le \frac{L_2}{n} + \frac{||u||_{\infty}}{n} |B_{t_1}|. \end{aligned}$$

Thus

$$\hat{\mathbb{E}}[\sup_{t\in[t_1,T]} |Y_t - Y_t^n|^{\alpha}] \le \hat{\mathbb{E}}[(\frac{L_2}{n} + \frac{||u||_{\infty}}{n} |B_{t_1}|)^{\alpha}] \to 0.$$

By Proposition 3.8, we have

$$\hat{\mathbb{E}}[(\int_{t_1}^{T} |Z_s - Z_s^n|^2 ds)^{\alpha/2}] \le C_{\alpha}\{\hat{\mathbb{E}}[\sup_{t\in[t_1,T]} |Y_t - Y_t^n|^{\alpha}] + (\hat{\mathbb{E}}[\sup_{t\in[t_1,T]} |Y_t - Y_t^n|^{\alpha}])^{1/2}\},$$

where $C_{\alpha} > 0$ is a constant depending only on α, T, L^w and $\underline{\sigma}$, thus we obtain $\hat{\mathbb{E}}(\int_{t_1}^{T} |Z_s - Z_s^n|^2 ds)^{\alpha/2} \to 0$, which implies that $Z \in H_G^{\alpha}(t_1, T)$ for any $\alpha > 1$. By $K_t = Y_t - Y_{t_1} + \int_{t_1}^{t} h(Y_s, Z_s) ds - \int_{t_1}^{t} Z_s dB_s$, we obtain $K_t \in L_G^{\alpha}(\Omega_t)$ for any $\alpha > 1$. We now proceed to prove that K is a G-martingale. Following the framework in Li and Peng [14], we take

$$h_i^n(x) = I_{[-n+\frac{i}{n}, -n+\frac{i+1}{n})}(x), i = 0, \ldots, 2n^2 - 1,$$

$h_{2n^2}^n = 1 - \sum_{i=0}^{2n^2-1} h_i^n$. Through equation (4.5), we get

$$\tilde{Y}_t^n = \tilde{Y}_T^n + \int_t^T h(\tilde{Y}_s^n, \tilde{Z}_s^n) ds - \int_t^T \tilde{Z}_s^n dB_s - (\tilde{K}_T^n - \tilde{K}_t^n),$$

where $\tilde{Y}_t^n = \sum_{i=0}^{2n^2} u(t, -n+\frac{i}{n}, B_t - B_{t_1})h_i^n(B_{t_1})$, $\tilde{Z}_t^n = \sum_{i=0}^{2n^2} \partial_y u(t, -n+\frac{i}{n}, B_t - B_{t_1})h_i^n(B_{t_1})$ and $\tilde{K}_t^n = \sum_{i=0}^{2n^2} K_t^{-n+\frac{i}{n}} h_i^n(B_{t_1})$. By Proposition 3.8, we have $\hat{\mathbb{E}}[(\int_{t_1}^T |Z_s - \tilde{Z}_s^n|^2 ds)^{\alpha/2}] \to 0$ for any $\alpha > 1$. Thus we get $\hat{\mathbb{E}}[|K_t - \tilde{K}_t^n|^\alpha] \to 0$ for any $\alpha > 1$. By Proposition 2.5, we obtain for each $t_1 \leq t < s \leq T$,

$$\begin{aligned}
\hat{\mathbb{E}}[|\hat{\mathbb{E}}_t[K_s] - K_t|] &= \hat{\mathbb{E}}[|\hat{\mathbb{E}}_t[K_s] - \hat{\mathbb{E}}_t[\tilde{K}_s^n] + \tilde{K}_t^n - K_t|] \\
&\leq \hat{\mathbb{E}}[\hat{\mathbb{E}}_t[|K_s - \tilde{K}_s^n|]] + \hat{\mathbb{E}}[|\tilde{K}_t^n - K_t|] \\
&= \hat{\mathbb{E}}[|K_s - \tilde{K}_s^n|] + \hat{\mathbb{E}}[|\tilde{K}_t^n - K_t|] \to 0.
\end{aligned}$$

Thus we get $\hat{\mathbb{E}}_t[K_s] = K_t$. For $Y_{t_1} = u(t_1, B_{t_1}, 0)$, we can use the same method as Part 1 on $[0, t_1]$.

Step 2. $f(t, \omega, y, z) = \sum_{i=1}^N f^i h^i(y, z)$ with $f^i \in M_G^0(0, T)$ and $h^i \in C_0^\infty(\mathbb{R}^2)$.

The analysis is similar to Part 2 of Step 1.

Step 3. $f(t, \omega, y, z) = \sum_{i=1}^N f^i h^i(y, z)$ with $f^i \in M_G^\beta(0, T)$ bounded and $h^i \in C_0^\infty(\mathbb{R}^2)$, $h^i \geq 0$ and $\sum_{i=1}^N h^i \leq 1$.

Choose $f_n^i \in M_G^0(0, T)$ such that $|f_n^i| \leq \|f^i\|_\infty$ and $\sum_{i=1}^N \|f_n^i - f^i\|_{M_G^\beta} < 1/n$. Set $f_n = \sum_{i=1}^N f_n^i h^i(y, z)$, which are uniformly Lipschitz. Let (Y^n, Z^n, K^n) be the solution of equation (3.1) with generator f_n.

Noting that

$$\hat{f}_s^{m,n} := |f_m(s, Y_s^n, Z_s^n) - f_n(s, Y_s^n, Z_s^n)| \leq \sum_{i=1}^N |f_n^i - f^i| + \sum_{i=1}^N |f_m^i - f^i| =: \hat{f}_n + \hat{f}_m,$$

we have, for any $1 < \alpha < \beta$,

$$\hat{\mathbb{E}}_t[(\int_0^T \hat{f}_s^{m,n} ds)^\alpha] \leq \hat{\mathbb{E}}_t[(\int_0^T (|\hat{f}_n(s)| + |\hat{f}_m(s)|) ds)^\alpha].$$

Thus by Theorem 2.8, we get $|| \int_0^T \hat{f}_s^{m,n} ds||_{\alpha, \mathcal{E}} \to 0$ as $m, n \to \infty$ for any $\alpha \in (1, \beta)$. By Proposition 3.9 we know that $\{Y^n\}$ is a cauchy sequence under the norm $\|\cdot\|_{S_G^\alpha}$. By Proposition 3.7 and Proposition 3.8, $\{Z^n\}$ is a cauchy sequence under the norm $\|\cdot\|_{H_G^\alpha}$. In order to show that $\{K_T^n\}$ is a cauchy sequence under the norm $\|\cdot\|_{L_G^\alpha}$, it suffices to prove $\{\int_0^T f_n(s, Y_s^n, Z_s^n) ds\}$ is a cauchy sequence under the norm $\|\cdot\|_{L_G^\alpha}$. In fact,

$$\begin{aligned}
&|f_n(s, Y^n, Z^n) - f_m(s, Y^m, Z^m)| \\
&\leq |f_m(s, Y^n, Z^n) - f_m(s, Y^m, Z^m)| + |f_n(s, Y^n, Z^n) - f_m(s, Y^n, Z^n)| \\
&\leq L(|\hat{Y}_s| + |\hat{Z}_s|) + \hat{f}_n + \hat{f}_m,
\end{aligned}$$

which implies the desired result.

Step 4. f is bounded, Lipschitz. $|f(t, \omega, y, z)| \leq C I_{B(R)}(y, z)$ for some $C, R > 0$. Here $B(R) = \{(y, z) | y^2 + z^2 \leq R^2\}$.

For any n, by the partition of unity theorem, there exists $\{h_n^i\}_{i=1}^{N_n}$ such that $h_n^i \in C_0^\infty(\mathbb{R}^2)$, the diameter of support $\lambda(\text{supp}(h_n^i)) < 1/n$, $0 \leq h_n^i \leq 1$,

$I_{B(R)} \leq \sum_{i=1}^N h_n^i \leq 1$. Then $f(t,\omega,y,z) = \sum_{i=1}^N f(t,\omega,y,z)h_n^i$. Choose y_n^i, z_n^i such that $h_n^i(y_n^i, z_n^i) > 0$. Set $f_n(t,\omega,y,z) = \sum_{i=1}^N f(t,\omega,y_n^i,z_n^i)h_n^i$. Then

$$|f(t,\omega,y,z) - f_n(t,\omega,y,z)| \leq \sum_{i=1}^N |f(t,\omega,y,z) - f(t,\omega,y_n^i,z_n^i)|h_n^i \leq L/n$$

and

$$|f_n(t,\omega,y,z) - f_n(t,\omega,y',z')| \leq L(|y-y'| + |z-z'| + 2/n).$$

Noting that $|f_m(s,Y_s^n,Z_s^n) - f_n(s,Y_s^n,Z_s^n)| \leq (L/n + L/m)$, we have

$$\hat{\mathbb{E}}_t[|\int_0^T (|f_m(s,Y_s^n,Z_s^n) - f_n(s,Y_s^n,Z_s^n)| + \frac{2L}{m})ds|^\alpha] \leq T^\alpha(\frac{L}{n} + \frac{3L}{m})^\alpha.$$

So by Proposition 3.9 we conclude that $\{Y^n\}$ is a cauchy sequence under the norm $\|\cdot\|_{S_G^\alpha}$. Consequently, $\{Z^n\}$ is a cauchy sequence under the norm $\|\cdot\|_{H_G^\alpha}$ by Proposition 3.7 and Proposition 3.8. Now we shall prove $\{\int_0^T f_n(s,Y_s^n,Z_s^n)ds\}$ is a cauchy sequence under the norm $\|\cdot\|_{L_G^\alpha}$. In fact,

$$\begin{aligned}
&|f_n(s,Y^n,Z^n) - f_m(s,Y^m,Z^m)| \\
&\leq |f_m(s,Y^n,Z^n) - f_m(s,Y^m,Z^m)| + |f_n(s,Y^n,Z^n) - f_m(s,Y^n,Z^n)| \\
&\leq L(|\hat{Y}_s| + |\hat{Z}_s| + 2/m) + L/n + L/m,
\end{aligned}$$

which implies the desired result.

Step 5. f is bounded, Lipschitz.

For any $n \in \mathbb{N}$, choose $h^n \in C_0^\infty(\mathbb{R}^2)$ such that $I_{B(n)} \leq h^n \leq I_{B(n+1)}$ and $\{h^n\}$ are uniformly Lipschitz w.r.t. n. Set $f_n = fh^n$, which are uniformly Lipschitz. Noting that for $m > n$

$$\begin{aligned}
&|f_m(s,Y_s^n,Z_s^n) - f_n(s,Y_s^n,Z_s^n)| \\
&\leq |f(s,Y_s^n,Z_s^n)|I_{[|Y_s^n|^2+|Z_s^n|^2>n^2]} \\
&\leq \|f\|_\infty \frac{|Y_s^n| + |Z_s^n|}{n},
\end{aligned}$$

we have

$$\begin{aligned}
&\hat{\mathbb{E}}_t[(\int_0^T |f_m(s,Y_s^n,Z_s^n) - f_n(s,Y_s^n,Z_s^n)|ds)^\alpha] \\
&\leq \frac{\|f\|_\infty^\alpha}{n^\alpha}\hat{\mathbb{E}}_t[(\int_0^T |Y_s^n| + |Z_s^n|ds)^\alpha] \\
&\leq \frac{\|f\|_\infty^\alpha}{n^\alpha}C(\alpha,T)\hat{\mathbb{E}}_t[\int_0^T |Y_s^n|^\alpha ds + (\int_0^T |Z_s^n|^2 ds)^{\alpha/2}],
\end{aligned}$$

where $C(\alpha,T) := 2^{\alpha-1}(T^{\alpha-1} + T^{\alpha/2}])$.

So by Theorem 2.8 and Proposition 3.7 we get $\|\int_0^T \hat{f}_s^{m,n}ds\|_{\alpha,\mathcal{E}} \to 0$ as $m,n \to \infty$ for any $\alpha \in (1,\beta)$. By Proposition 3.9, we conclude that $\{Y^n\}$ is a cauchy sequence under the norm $\|\cdot\|_{S_G^\alpha}$. Consequently, $\{Z^n\}$ is a cauchy sequence under the norm $\|\cdot\|_{H_G^\alpha}$. Now it suffices to prove $\{\int_0^T f_n(s, Y_s^n, Z_s^n)ds\}$ is a cauchy sequence under the norm $\|\cdot\|_{L_G^\alpha}$. In fact,

$$\begin{aligned}
&|f_n(s, Y^n, Z^n) - f_m(s, Y^m, Z^m)| \\
&\leq |f_m(s, Y^n, Z^n) - f_m(s, Y^m, Z^m)| + |f_n(s, Y^n, Z^n) - f_m(s, Y^n, Z^n)| \\
&\leq L(|\hat{Y}_s| + |\hat{Z}_s|) + |f(s, Y_s^n, Z_s^n)| 1_{[|Y_s^n| + |Z_s^n| > n]},
\end{aligned}$$

which implies the desired result by Proposition 3.7.

Step 6. For the general f.

Set $f_n = [f \vee (-n)] \wedge n$, which are uniformly Lipschitz. Choose $0 < \delta < \frac{\beta - \alpha}{\alpha} \wedge 1$. Then $\alpha < \alpha' = (1+\delta)\alpha < \beta$. Since for $m > n$

$$|f_n(s, Y^n, Z^n) - f_m(s, Y^n, Z^n)| \leq |f(s, Y_s^n, Z_s^n)| I_{[|f(s, Y_s^n, Y_s^n)| > n]} \leq \frac{1}{n^\delta} |f(s, Y_s^n, Z_s^n)|^{1+\delta},$$

we have

$$\begin{aligned}
&\hat{\mathbb{E}}_t[(\int_0^T |f_n(s, Y^n, Z^n) - f_m(s, Y^n, Z^n)| ds)^\alpha] \\
&\leq \frac{1}{n^{\alpha\delta}} \hat{\mathbb{E}}_t[(\int_0^T |f(s, Y_s^n, Z_s^n)|^{1+\delta} ds)^\alpha], \\
&\leq \frac{C(\alpha, T, L, \delta)}{n^{\alpha\delta}} \hat{\mathbb{E}}_t[\int_0^T |f(s, 0, 0)|^{\alpha'} ds + \int_0^T |Y_s^n|^{\alpha'} ds + (\int_0^T |Z_s^n|^2 ds)^{\frac{\alpha'}{2}}],
\end{aligned}$$

where $C(\alpha, T, L, \delta) := 3^{\alpha'-1}(T^{\alpha-1} + L^{\alpha'} T^{\frac{\alpha(1-\delta)}{2}} + T^{\alpha-1} L^{\alpha'})$. So by Theorem 2.8 and Proposition 3.7 we get $\|\int_0^T \hat{f}_s^{m,n} ds\|_{\alpha, \mathcal{E}} \to 0$ as $m, n \to \infty$ for any $\alpha \in (1, \beta)$. By Proposition 3.9, we know that $\{Y^n\}$ is a cauchy sequence under the norm $\|\cdot\|_{S_G^\alpha}$. And consequently $\{Z^n\}$ is a cauchy sequence under the norm $\|\cdot\|_{H_G^\alpha}$. Now we prove $\{\int_0^T f_n(s, Y_s^n, Z_s^n)ds\}$ is a cauchy sequence under the norm $\|\cdot\|_{L_G^\alpha}$. In fact,

$$\begin{aligned}
&|f_n(s, Y^n, Z^n) - f_m(s, Y^m, Z^m)| \\
&\leq |f_m(s, Y^n, Z^n) - f_m(s, Y^m, Z^m)| + |f_n(s, Y^n, Z^n) - f_m(s, Y^n, Z^n)| \\
&\leq L(|\hat{Y}_s| + |\hat{Z}_s|) + \frac{3^\delta}{n^\delta}(|f_s^0|^{1+\delta} + |Y_s^n|^{1+\delta} + |Z_s^n|^{1+\delta}),
\end{aligned}$$

which implies the desired result by Proposition 3.7. $\square$

Moreover, we have the following result.

Theorem 4.2 *Assume that $\xi \in L_G^\beta(\Omega_T)$ and f, g satisfy (H1) and (H2)for some $\beta > 1$. Then equation (1.2) has a unique solution (Y, Z, K). Moreover, for any $1 < \alpha < \beta$ we have $Y \in S_G^\alpha(0, T)$, $Z \in H_G^\alpha(0, T)$ and $K_T \in L_G^\alpha(\Omega_T)$.*

Proof. The proof is similar to that of Theorem 4.1. $\square$

Remark 4.3 *The above results still hold for the case $d > 1$.*

5 An alternative estimates of G-BSDEs

In this section, we present an alternative a priori estimate for the solutions of G-BSDEs, which may be useful in the follow-up work of G-BSDEs theory.

For simplicity, we only consider the case $d = 1$. The results still hold for the case $d > 1$.

We consider the following type of G-BSDEs:

$$
\begin{aligned}
Y_t^i = \xi^i + \int_t^T f_i(s, Y_s^i, Z_s^i)ds + \int_t^T g_i(s, Y_s^i, Z_s^i)d\langle B\rangle_s \\
- \int_t^T Z_s^i dB_s - (K_T^i - K_t^i),
\end{aligned}
\tag{5.1}
$$

where $\xi^i \in L_G^\beta(\Omega_T)$ and f_i, g_i satisfy (H1) and (H2) for some $\beta > 1$, $i = 1, 2$.

Proposition 5.1 *Assume that $(Y^i, Z^i, K^i) \in \mathfrak{S}_G^\alpha(0, T)$ for some $1 < \alpha < \beta$, $i = 1, 2$, are the solutions of equation (5.1) corresponding to ξ^i, f_i and g_i. Set $\hat{Y}_t = Y_t^1 - Y_t^2$. Then there exists a constant $C_\alpha > 0$ depending on α, T, G and L such that*

$$
|\hat{Y}_t|^\alpha \le C_\alpha \hat{\mathbb{E}}_t[(|\hat{\xi}| + \int_t^T (|\hat{f}_s| + |\hat{g}_s|)ds)^\alpha], \tag{5.2}
$$

where $\hat{\xi} = \xi^1 - \xi^2$, $\hat{f}_s = f_1(s, Y_s^2, Z_s^2) - f_2(s, Y_s^2, Z_s^2)$, $\hat{g}_s = g_1(s, Y_s^2, Z_s^2) - g_2(s, Y_s^2, Z_s^2)$.

Proof. For each fixed $t < T$, we consider the following SDE:

$$
\begin{aligned}
X_r = &\int_t^r (f_1(s, Y_s^2 - X_s, Z_s^2) - f_2(s, Y_s^2, Z_s^2))ds \\
&+ \int_t^r (g_1(s, Y_s^2 - X_s, Z_s^2) - g_2(s, Y_s^2, Z_s^2))d\langle B\rangle_s.
\end{aligned}
$$

By the comparison theorem, we obtain that

$$
\begin{aligned}
|X_r| \le &\int_t^r |\hat{f}_s| \exp\{L(r - s + \langle B\rangle_r - \langle B\rangle_s)\}ds \\
&+ \int_t^r |\hat{g}_s| \exp\{L(r - s + \langle B\rangle_r - \langle B\rangle_s)\}d\langle B\rangle_s \\
\le &\, C \int_t^r (|\hat{f}_s| + |\hat{g}_s|)ds,
\end{aligned}
$$

where C depends on T, G and L. Set $\tilde{Y}_r^1 = Y_r^1 + X_r$ for $r \in [t, T]$, by applying Itô's formula to $Y_r^1 + X_r$, we get

$$
\begin{aligned}
\tilde{Y}_r^1 = \xi^1 + X_T + &\int_r^T (f_1(s, \tilde{Y}_s^1 - X_s, Z_s^1) - f_1(s, Y_s^2 - X_s, Z_s^2) + f_2(s, Y_s^2, Z_s^2))ds \\
+ &\int_r^T (g_1(s, \tilde{Y}_s^1 - X_s, Z_s^1) - g_1(s, Y_s^2 - X_s, Z_s^2) + g_2(s, Y_s^2, Z_s^2))d\langle B\rangle_s \\
- &\int_r^T Z_s^1 dB_s - (K_T^1 - K_r^1).
\end{aligned}
$$

By Proposition 3.9 we obtain that

$$|\hat{Y}_t|^\alpha = |\tilde{Y}_t^1 - Y_t^2|^\alpha \leq C_\alpha \hat{\mathbb{E}}_t[|\hat{\xi} + X_T|^\alpha]$$
$$\leq C_\alpha \hat{\mathbb{E}}_t[(|\hat{\xi}| + \int_t^T (|\hat{f}_s| + |\hat{g}_s|)ds)^\alpha].$$

□

Corollary 5.2 *Assume that $(Y, Z, K) \in \mathfrak{S}_G^\alpha(0,T)$ for some $1 < \alpha < \beta$ is the solution of equation (5.1) corresponding to ξ, f and g. Then there exists a constant $C_\alpha > 0$ depending on α, T, G and L such that*

$$|Y_t|^\alpha \leq C_\alpha \hat{\mathbb{E}}_t[(|\xi| + \int_t^T (|f(s,0,0)| + |g(s,0,0)|)ds)^\alpha]. \tag{5.3}$$

Proof. Letting $\xi^2 = 0$, $f_2 = g_2 = 0$, it is easy to check that $Y_t^2 = 0$. By Proposition 5.1 we get equation (5.3). □

Remark 5.3 *Noting that C_α is bounded for $\alpha \geq \frac{1}{2}(1+\beta)$, then equations (5.2) and (5.3) still hold for $\alpha = \beta$ by taking $\alpha \uparrow \beta$.*

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