

Comparison theorem, Feynman–Kac formula and Girsanov transformation for BSDEs driven by G -Brownian motion[☆]

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Abstract

In this paper, we study comparison theorem, nonlinear Feynman–Kac formula and Girsanov transformation of the following BSDE driven by a G -Brownian motion:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s, Z_s) d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t),$$

where K is a decreasing G -martingale.

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1. Introduction

A classical Backward Stochastic Differential Equation (BSDE for short) is formulated as follows:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s. \quad (1.1)$$

This BSDE is defined in a Wiener probability space (Ω, \mathcal{F}, P) in which the canonical process $B_t(\omega) = \omega_t$ is a Brownian motion. The solution $(Y_t, Z_t)_{t \in [0, T]}$ is adapted to the natural filtration $(\mathcal{F}_t)_{t \geq 0}$ generated by B . Here g is a given function satisfying the Lipschitz condition in (y, z) , and ξ is a given square-integrable \mathcal{F}_T -measurable random variable. The existence and uniqueness theorem was obtained by Pardoux and Peng [8]. This BSDE is naturally related to a quasilinear parabolic partial differential equation (PDE for short) via the so-called nonlinear Feynman–Kac formula, see [10,11,9]. It is also used to measure risks in finance, under uncertainty of probabilities. We can also use this BSDE to construct a time consistent (or dynamic consistent, filtration $(\mathcal{F}_t)_{t \geq 0}$ -consistent) nonlinear expectation, called g -expectation, introduced in Peng [12].

But it is known that this type of BSDE cannot be used to measure risks of path-dependent derivatives in the case of the uncertain volatility model (UVM). This obstacle is also closely linked to the corresponding Feynman–Kac formula, in which the PDE is not fully nonlinear but only quasi-linear. Such a type of BSDE can be called a quasilinear BSDE. The g -expectation is also called a quasi-linear expectation. Indeed, in a recent study of Peng and Wang [25], it was proved that, under reasonable and concrete regularity assumptions on ξ and g , the corresponding BSDE is a new type of path-dependent PDE (PPDE in short), in the sense of Dupire derivatives. We also refer to [22,3] for different formulations of viscosity solutions of fully nonlinear PPDEs.

Peng [14] introduced a new type of time consistent fully nonlinear expectation $\mathcal{E}_t^i, i = 1, \dots, n$, through which the existence and uniqueness of a fully nonlinear multi-dimensional BSDE of the following type

$$Y_t^i = \mathcal{E}_t^i \left[\xi + \int_t^T f_i(s, Y_s) ds \right], \quad i = 1, \dots, n, \quad (1.2)$$

was obtained, where $Y = (Y^1, \dots, Y^n)$. But this BSDE was not expressed as a classical differential form of BSDE (1.1).

As a special but typical case, Peng (2006) studied a fully nonlinear expectation, called G -expectation $\hat{\mathbb{E}}[\cdot]$ (see [20] and the references therein), and the corresponding time-conditional expectation $\hat{\mathbb{E}}_t[\cdot]$ on a space of random variables completed under the norm $\hat{\mathbb{E}}[\|\cdot\|^p]^{1/p}$, denoted by $L_G^p(\Omega_T)$. Under this G -expectation framework (G -framework for short) a new type of Brownian motion called G -Brownian motion was constructed and the corresponding stochastic calculus of Itô's type was established. The existence and uniqueness of solution of a SDE driven by G -Brownian motion can be proved in a way parallel to that in the classical SDE theory. But the BSDE driven by G -Brownian motion $(B_t)_{t \geq 0}$ becomes a challenging and fascinating problem.

Just as in the classical case, the G -martingale representation theorem is the key to solving a BSDE in this G -framework. If ξ belongs to a family of smooth and finite dimensional path functions, dense in $L_G^p(\Omega_T)$, Peng [17] obtained the following result: the G -martingale $M_t := \hat{\mathbb{E}}_t[\xi]$ can be decomposed into

$$M_t = M_0 + \bar{M}_t + K_t,$$

$$\bar{M}_t := \int_0^t z_s B_s, \quad K_t := \int_0^t \eta_s \langle B \rangle_s - \int_0^t 2G(\eta_s) ds.$$

Namely M is decomposed into two types of G -martingales. The first one \bar{M} is called a symmetric G -martingale. That is, $-\bar{M}$ is also a G -martingale. The second G -martingale K is quite unusual since it is a decreasing process. A main concern in the G -framework involves how to understand this decreasing G -martingales K , which aroused an interesting open problem (see [17,20]).

For a general G -martingale, the first step is to decompose it into a sum of a symmetric G -martingale \bar{M} and a decreasing G -martingale K . This difficult problem was solved after a series of successive efforts of Soner, Touzi and Zhang [26, 2011] and Song [28, 2011], [29, 2012]. The second step is to study under what conditions the decreasing G -martingale K can be uniquely represented as $K_t := \int_0^t \eta_s \langle B \rangle_s - \int_0^t 2G(\eta_s) ds$. Thanks to an original new norm for decreasing G -martingales introduced in Song [29, 2012], a complete representation theorem of G -martingales has been obtained in a complete subspace of $L_G^\alpha(\Omega_T)$ by Peng, Song and Zhang [24, 2012]. We observe that the above results of the G -martingale representation can also be regarded as a non-trivial fully nonlinear BSDE, namely, for a given function G and given random variable ξ , there exists a triple of processes (Y, Z, η) which solves

$$Y_t = \xi + \int_t^T G(\eta_s) ds - \int_t^T \eta_s d\langle B \rangle_s - \int_t^T Z_s dB_s.$$

For a general $\xi \in L_G^p(\Omega_T)$, we also have its weak form: a triple of processes (Y, Z, K) which solves

$$Y_t = \xi + K_T - K_t - \int_t^T Z_s dB_s.$$

We can go further in this direction to introduce a nonlinear (not necessarily sublinear or concave \(\backslash\) convex) time consistent nonlinear \tilde{G} -expectation $\tilde{\mathbb{E}}$, under which the corresponding martingale $Y_t = \tilde{\mathbb{E}}_t[\xi]$ is the solution of

$$Y_t = \xi + \int_t^T \tilde{G}(p_s, \eta_s) ds - \int_t^T \eta_s d\langle B \rangle_s - \int_t^T p_s db_s - \int_t^T Z_s dB_s,$$

where the pair $(B_t, b_t)_{t \geq 0}$ is a nonlinear \tilde{G} -Brownian motion (see [20, Section III.8] and [21, (3.2)]). All the above results are helpful in solving the following very challenging conjecture proposed by Peng:

Conjecture. For a given ‘regular’ time consistent nonlinear expectation $\tilde{\mathbb{E}}$ defined on a continuous path space Ω , there exists a real function G such that $\tilde{\mathbb{E}}$ is a G -expectation.

Within a framework of Wiener probability space (Ω, \mathcal{F}, P) this conjecture was solved by [1] (see also [13]) for nonlinear expectation, and [15] for nonlinear evaluation. Its applications to find and construct a financial pricing mechanism can be seen in [23]. But the method fails for a fully nonlinear expectation.

Soner, Touzi and Zhang [27] obtained a profound result of an existence and uniqueness theorem for a new type of fully nonlinear BSDE, called the 2BSDE: finding $(Y, Z, K^\mathbb{P})_{\mathbb{P} \in \mathcal{P}_H^\kappa}$ satisfying, for each probability $\mathbb{P} \in \mathcal{P}_H^\kappa$, the following BSDE:

$$Y_t = \xi + \int_t^T \hat{F}_s(Y_s, Z_s) ds - \int_t^T Z_s dB_s + (K_T^\mathbb{P} - K_t^\mathbb{P}), \quad \mathbb{P}\text{-a.s.}, \quad (1.3)$$

such that the following minimum condition is satisfied

$$K_t^{\mathbb{P}} = \operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_H^{\kappa}(t+, \mathbb{P})} \mathbb{E}_t^{\mathbb{P}'} [K_T^{\mathbb{P}}], \quad \mathbb{P}\text{-a.s.}, \quad \forall \mathbb{P} \in \mathcal{P}_H^{\kappa}, \quad t \in [0, T]. \quad (1.4)$$

This 2BSDE is also associated with a nonlinear PDE which is convex with respect to the second order term.

But there still remain two open problems. The first is the lack of time consistency in the sense that, not like the classical BSDE [8], for each $t \in [0, T)$, the solution Y_t of (1.3) and (1.4) is not proved to belong to the space $Y_t \in L_G^p(\Omega_t) \subset L_G^p(\Omega_T)$, whereas the given terminal condition Y_T is assumed in the space $L_G^p(\Omega_T)$. Consequently, one cannot consistently treat $(Y_s, Z_s, K_s^{\mathbb{P}})_{0 \leq s \leq t}$ as the same 2BSDE on the time interval $[0, t]$, with the terminal condition Y_t . Namely, the time consistency is not obtained. This problem is also closely involved with another problem that the process $(K^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}_H^{\kappa}}$ was not “aggregated” into a ‘universal K ’. In order to solve these problems, we need to improve the regularity of the solutions.

Recently in [4], the authors of this paper have obtained the existence, uniqueness, time-consistency and a priori estimates of the following fully nonlinear BSDE driven by a given G -Brownian motion B : to find a triple of processes (Y, Z, K) within our G -framework which solves

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s, Z_s) d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t),$$

under standard Lipschitz conditions on $f(s, y, z), g(s, y, z)$ in (y, z) and the $L_G^p(\Omega_T)$ -integrability condition on ξ (see Theorem 2.13, Propositions 2.14 and 2.16). The solution (Y, Z, K) is universally defined in the spaces of the G -framework, in which the processes have a strong regularity property. As a consequence, the decreasing G -martingale K is aggregated and the solution is time consistent (see Corollary 2.15).

Observe that when the G -Brownian motion B is reduced to a standard Brownian motion, we have $K_t \equiv 0$ and $\langle B \rangle_t \equiv t$. Thus the above BSDE becomes the classical BSDE, with $g + f$ in place of g in (1.1).

The main task of this paper is to investigate some fundamental properties of G -BSDE (1.1): its comparison theorem, its fully nonlinear Feynman–Kac formula and the related Girsanov transformation.

To prove the comparison theorem of G -BSDE (1.1), we need to give an explicit solution of the linear G -BSDE. Applying the dual method, a closed formula of Y is obtained by introducing a completely new dual equation in an extension of the G -expectation space. As an interesting by-product, a new Gronwall inequality has been derived.

Then we investigate the link between our G -BSDEs and the corresponding PDEs by considering the following type of FBSDEs:

$$\begin{aligned} dX_s^{t,x} &= b(s, X_s^{t,x}) ds + h_{ij}(s, X_s^{t,x}) d\langle B^i, B^j \rangle_s + \sigma_j(s, X_s^{t,x}) dB_s^j, \quad X_t^{t,x} = x, \\ Y_s^{t,x} &= \Phi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr + \int_s^T g_{ij}(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) d\langle B^i, B^j \rangle_r \\ &\quad - \int_s^T Z_r^{t,x} dB_r - (K_T^{t,x} - K_s^{t,x}). \end{aligned}$$

We have proved that the function defined by $u(t, x) = Y_t^{t, x}$ is deterministic, which is the unique viscosity solution of the following PDE:

$$\begin{cases} \partial_t u + F(D_x^2 u, D_x u, u, x, t) = 0, \\ u(T, x) = \Phi(x), \end{cases}$$

where

$$\begin{aligned} F(D_x^2 u, D_x u, u, x, t) := & G(H(D_x^2 u, D_x u, u, x, t)) + \langle b(t, x), D_x u \rangle \\ & + f(t, x, u, \langle \sigma_1(t, x), D_x u \rangle, \dots, \langle \sigma_d(t, x), D_x u \rangle), \end{aligned}$$

and

$$\begin{aligned} H_{ij}(D_x^2 u, D_x u, u, x, t) = & \langle D_x^2 u \sigma_i(t, x), \sigma_j(t, x) \rangle + 2 \langle D_x u, h_{ij}(t, x) \rangle \\ & + 2 g_{ij}(t, x, u, \langle \sigma_1(t, x), D_x u \rangle, \dots, \langle \sigma_d(t, x), D_x u \rangle). \end{aligned}$$

So the related PDE is sublinear in the second order derivatives. In this sense it is within the framework of [27] as well as within the more general nonlinear expectation framework of Peng in [14,20]. Finally, we study the Girsanov transformation for the process

$$\bar{B}_t := B_t - \int_0^t b_s ds - \int_0^t d_s^{ij} d \langle B^i, B^j \rangle_s.$$

Note that the authors only studied the case $b_s \equiv 0$ in both [7,30]. By a new method which is more direct and simpler we have proved that \bar{B}_t is a G -Brownian motion under a given time consistent sublinear expectation.

The paper is organized as follows. In Section 2, we present some preliminaries for stochastic calculus under the G -framework. The explicit solutions of linear G -BSDEs and the comparison theorem are established in Section 3. In Section 4, we obtain the nonlinear Feynman–Kac formula for a fully nonlinear PDE. We prove the Girsanov transformation for G -Brownian motion in Section 5.

2. Preliminaries

We review some basic notions and results of G -expectation, the related spaces of random variables and the backward stochastic differential equations driven by a G -Brownian motion. The readers may refer to [4,16–20] for more details.

Definition 2.1. Let Ω be a given set and let \mathcal{H} be a vector lattice of real valued functions defined on Ω , namely $c \in \mathcal{H}$ for each constant c and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. \mathcal{H} is considered as the space of random variables. A sublinear expectation $\hat{\mathbb{E}}$ on \mathcal{H} is a functional $\hat{\mathbb{E}} : \mathcal{H} \rightarrow \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

- (a) Monotonicity: If $X \geq Y$ then $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$;
- (b) Constant preservation: $\hat{\mathbb{E}}[c] = c$;
- (c) Sub-additivity: $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$;
- (d) Positive homogeneity: $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$ for each $\lambda \geq 0$.

$(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space.

Definition 2.2. Let X_1 and X_2 be two n -dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$. They are called identically

distributed, denoted by $X_1 \stackrel{d}{=} X_2$, if $\hat{\mathbb{E}}_1[\varphi(X_1)] = \hat{\mathbb{E}}_2[\varphi(X_2)]$, for all $\varphi \in C_{l.Lip}(\mathbb{R}^n)$, where $C_{l.Lip}(\mathbb{R}^n)$ is the space of real continuous functions defined on \mathbb{R}^n such that

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y| \quad \text{for all } x, y \in \mathbb{R}^n,$$

where k and C depend only on φ .

Definition 2.3. In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, a random vector $Y = (Y_1, \dots, Y_n)$, $Y_i \in \mathcal{H}$, is said to be independent of another random vector $X = (X_1, \dots, X_m)$, $X_i \in \mathcal{H}$ under $\hat{\mathbb{E}}[\cdot]$, denoted by $Y \perp X$, if for every test function $\varphi \in C_{l.Lip}(\mathbb{R}^m \times \mathbb{R}^n)$ we have $\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}]$.

Definition 2.4 (*G-normal Distribution*). A d -dimensional random vector $X = (X_1, \dots, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called G -normally distributed if for each $a, b \geq 0$ we have

$$aX + b\bar{X} \stackrel{d}{=} \sqrt{a^2 + b^2}X,$$

where \bar{X} is an independent copy of X , i.e., $\bar{X} \stackrel{d}{=} X$ and $\bar{X} \perp X$. Here the letter G denotes the function

$$G(A) := \frac{1}{2} \hat{\mathbb{E}}[\langle AX, X \rangle] : \mathbb{S}_d \rightarrow \mathbb{R},$$

where \mathbb{S}_d denotes the collection of $d \times d$ symmetric matrices.

Peng [19] showed that $X = (X_1, \dots, X_d)$ is G -normally distributed if and only if for each $\varphi \in C_{l.Lip}(\mathbb{R}^d)$, $u(t, x) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X)]$, $(t, x) \in [0, \infty) \times \mathbb{R}^d$, is the solution of the following G -heat equation:

$$\partial_t u - G(D_x^2 u) = 0, \quad u(0, x) = \varphi(x).$$

The function $G(\cdot) : \mathbb{S}_d \rightarrow \mathbb{R}$ is a monotonic, sublinear mapping on \mathbb{S}_d and $G(A) = \frac{1}{2} \hat{\mathbb{E}}[\langle AX, X \rangle] \leq \frac{1}{2} |A| \hat{\mathbb{E}}[|X|^2] =: \frac{1}{2} |A| \bar{\sigma}^2$ implies that there exists a bounded, convex and closed subset $\Gamma \subset \mathbb{S}_d^+$ such that

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \text{tr}[\gamma A],$$

where \mathbb{S}_d^+ denotes the collection of nonnegative elements in \mathbb{S}_d .

In this paper, we only consider a non-degenerate G -normal distribution, i.e., there exists some $\underline{\sigma}^2 > 0$ such that $G(A) - G(B) \geq \underline{\sigma}^2 \text{tr}[A - B]$ for any $A \geq B$.

Definition 2.5. (i) Let $\Omega_T = C_0([0, T]; \mathbb{R}^d)$, the space of real valued continuous functions on $[0, T]$ with $\omega_0 = 0$, be endowed with the supremum norm and let $B_t(\omega) = \omega_t$ be the canonical process. Set

$$\mathcal{H}_T^0 := \{\varphi(B_{t_1}, \dots, B_{t_n}) : n \geq 1, t_1, \dots, t_n \in [0, T], \varphi \in C_{l.Lip}(\mathbb{R}^{d \times n})\}.$$

Let $G : \mathbb{S}_d \rightarrow \mathbb{R}$ be a given monotonic and sublinear function. G -expectation is a sublinear expectation defined by

$$\hat{\mathbb{E}}[X] = \hat{\mathbb{E}} \left[\varphi \left(\sqrt{t_1 - t_0} \xi_1, \dots, \sqrt{t_m - t_{m-1}} \xi_m \right) \right],$$

for all $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_m} - B_{t_{m-1}})$, where ξ_1, \dots, ξ_n are identically distributed d -dimensional G -normally distributed random vectors in a sublinear expectation space $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})$ such that ξ_{i+1} is independent of (ξ_1, \dots, ξ_i) for every $i = 1, \dots, m-1$. The corresponding canonical process $B_t = (B_t^i)_{i=1}^d$ is called a G -Brownian motion.

- (ii) Let us define the conditional G -expectation $\hat{\mathbb{E}}_t$ of $\xi \in \mathcal{H}_T^0$ knowing \mathcal{H}_t^0 , for $t \in [0, T]$. Without loss of generality we can assume that ξ has the representation $\xi = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_m} - B_{t_{m-1}})$ with $t = t_i$, for some $1 \leq i \leq m$, and we put

$$\begin{aligned} \hat{\mathbb{E}}_{t_i}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_m} - B_{t_{m-1}})] \\ = \tilde{\varphi}(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_i} - B_{t_{i-1}}), \end{aligned}$$

where

$$\tilde{\varphi}(x_1, \dots, x_i) = \hat{\mathbb{E}}[\varphi(x_1, \dots, x_i, B_{t_{i+1}} - B_{t_i}, \dots, B_{t_m} - B_{t_{m-1}})].$$

Define $\|\xi\|_{p,G} = (\hat{\mathbb{E}}[|\xi|^p])^{1/p}$ for $\xi \in \mathcal{H}_T^0$ and $p \geq 1$. Then for all $t \in [0, T]$, $\hat{\mathbb{E}}_t[\cdot]$ is a continuous mapping on \mathcal{H}_T^0 w.r.t. the norm $\|\cdot\|_{1,G}$. Therefore it can be extended continuously to the completion $L_G^1(\Omega_T)$ of \mathcal{H}_T^0 under the norm $\|\cdot\|_{1,G}$.

Let $L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, \dots, B_{t_n}) : n \geq 1, t_1, \dots, t_n \in [0, T], \varphi \in C_{b.Lip}(\mathbb{R}^{d \times n})\}$, where $C_{b.Lip}(\mathbb{R}^{d \times n})$ denotes the set of bounded Lipschitz functions on $\mathbb{R}^{d \times n}$. Denis et al. [2] proved that the completions of $C_b(\Omega_T)$ (the set of bounded continuous function on Ω_T), \mathcal{H}_T^0 and $L_{ip}(\Omega_T)$ under $\|\cdot\|_{p,G}$ are the same and we denote them by $L_G^p(\Omega_T)$.

For each fixed $\mathbf{a} \in \mathbb{R}^d$, $B_t^{\mathbf{a}} = \langle \mathbf{a}, B_t \rangle$ is a 1-dimensional $G_{\mathbf{a}}$ -Brownian motion, where $G_{\mathbf{a}}(\alpha) = \frac{1}{2}(\sigma_{\mathbf{a}\mathbf{a}^T}^2 \alpha^+ - \sigma_{-\mathbf{a}\mathbf{a}^T}^2 \alpha^-)$, $\sigma_{\mathbf{a}\mathbf{a}^T}^2 = 2G(\mathbf{a}\mathbf{a}^T)$, $\sigma_{-\mathbf{a}\mathbf{a}^T}^2 = -2G(-\mathbf{a}\mathbf{a}^T)$. Let $\pi_t^N = \{t_0^N, \dots, t_N^N\}$, $N = 1, 2, \dots$, be a sequence of partitions of $[0, t]$ such that $\mu(\pi_t^N) = \max\{|t_{i+1}^N - t_i^N| : i = 0, \dots, N-1\} \rightarrow 0$, the quadratic variation process of $B^{\mathbf{a}}$ is defined by

$$\langle B^{\mathbf{a}} \rangle_t = \lim_{\mu(\pi_t^N) \rightarrow 0} \sum_{j=0}^{N-1} (B_{t_{j+1}^N}^{\mathbf{a}} - B_{t_j^N}^{\mathbf{a}})^2.$$

For each fixed $\mathbf{a}, \bar{\mathbf{a}} \in \mathbb{R}^d$, the mutual variation process of $B^{\mathbf{a}}$ and $B^{\bar{\mathbf{a}}}$ is defined by

$$\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}} \rangle_t = \frac{1}{4}[\langle B^{\mathbf{a}+\bar{\mathbf{a}}} \rangle_t - \langle B^{\mathbf{a}-\bar{\mathbf{a}}} \rangle_t].$$

Definition 2.6. Let $M_G^0(0, T)$ be the collection of processes in the following form: for a given partition $\{t_0, \dots, t_N\} = \pi_T$ of $[0, T]$,

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j, t_{j+1})}(t),$$

where $\xi_i \in L_{ip}(\Omega_{t_i})$, $i = 0, 1, 2, \dots, N-1$. For $p \geq 1$ and $\eta \in M_G^0(0, T)$, let $\|\eta\|_{H_G^p} = \left\{ \hat{\mathbb{E}} \left[\left(\int_0^T |\eta_s|^2 ds \right)^{p/2} \right] \right\}^{1/p}$, $\|\eta\|_{M_G^p} = \left\{ \hat{\mathbb{E}} \left[\int_0^T |\eta_s|^p ds \right] \right\}^{1/p}$ and denote by $H_G^p(0, T)$, $M_G^p(0, T)$ the completions of $M_G^0(0, T)$ under the norms $\|\cdot\|_{H_G^p}$, $\|\cdot\|_{M_G^p}$ respectively.

Theorem 2.7 ([2,5]). *There exists a weakly compact set $\mathcal{P} \subset \mathcal{M}_1(\Omega_T)$, the set of probability measures on $(\Omega_T, \mathcal{B}(\Omega_T))$, such that*

$$\hat{\mathbb{E}}[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi] \quad \text{for all } \xi \in \mathcal{H}_T^0.$$

\mathcal{P} is called a set that represents $\hat{\mathbb{E}}$.

Let \mathcal{P} be a weakly compact set that represents $\hat{\mathbb{E}}$. For this \mathcal{P} , we define capacity

$$c(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega_T).$$

A set $A \subset \Omega_T$ is polar if $c(A) = 0$. A property holds “quasi-surely” (q.s. for short) if it holds outside a polar set. In the following, we do not distinguish two random variables X and Y if $X = Y$ q.s.. We set

$$\mathbb{L}^p(\Omega_t) := \left\{ X \in \mathcal{B}(\Omega_t) : \sup_{P \in \mathcal{P}} E_P[|X|^p] < \infty \right\} \quad \text{for } p \geq 1.$$

It is important to note that $L_G^p(\Omega_t) \subset \mathbb{L}^p(\Omega_t)$. We extend G -expectation $\hat{\mathbb{E}}$ to $\mathbb{L}^p(\Omega_t)$ and still denote it by $\hat{\mathbb{E}}$, for each $X \in \mathbb{L}^1(\Omega_T)$, we set

$$\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} E_P[X].$$

For $p \geq 1$, $\mathbb{L}^p(\Omega_t)$ is a Banach space under the norm $(\hat{\mathbb{E}}[|\cdot|^p])^{1/p}$.

Set

$$\mathbb{L}_G^{0,p,t}(\Omega_T) := \left\{ \xi = \sum_{i=1}^n \eta_i I_{A_i} : A_i \in \mathcal{B}(\Omega_t), \eta_i \in L_G^p(\Omega), n \in \mathbb{N} \right\},$$

we define the corresponding conditional G -expectation, still denoted by $\hat{\mathbb{E}}_s[\cdot]$, by setting

$$\hat{\mathbb{E}}_s\left[\sum_{i=1}^n \eta_i I_{A_i}\right] := \sum_{i=1}^n \hat{\mathbb{E}}_s[\eta_i] I_{A_i} \quad \text{for } s \geq t.$$

Proposition 2.8 ([4]). *For each $\xi, \eta \in \mathbb{L}_G^{0,1,t}(\Omega_T)$, we have*

- (i) *Monotonicity: If $\xi \leq \eta$, then $\hat{\mathbb{E}}_s[\xi] \leq \hat{\mathbb{E}}_s[\eta]$ for any $s \geq t$;*
- (ii) *Constant preserving: If $\xi \in \mathbb{L}_G^{0,1,t}(\Omega_t)$, then $\hat{\mathbb{E}}_t[\xi] = \xi$;*
- (iii) *Sub-additivity: $\hat{\mathbb{E}}_s[\xi + \eta] \leq \hat{\mathbb{E}}_s[\xi] + \hat{\mathbb{E}}_s[\eta]$ for any $s \geq t$;*
- (iv) *Positive homogeneity: If $\xi \in \mathbb{L}_G^{0,\infty,t}(\Omega_t)$ and $\xi \geq 0$, then $\hat{\mathbb{E}}_t[\xi \eta] = \xi \hat{\mathbb{E}}_t[\eta]$;*
- (v) *Consistency: For $t \leq s \leq r$, we have $\hat{\mathbb{E}}_s[\hat{\mathbb{E}}_r[\xi]] = \hat{\mathbb{E}}_s[\xi]$.*
- (vi) $\hat{\mathbb{E}}[\hat{\mathbb{E}}_t[\xi]] = \hat{\mathbb{E}}[\xi]$.

Let $\mathbb{L}_G^{p,t}(\Omega_T)$ be the completion of $\mathbb{L}_G^{0,p,t}(\Omega_T)$ under the norm $(\hat{\mathbb{E}}[|\cdot|^p])^{1/p}$. Clearly, the conditional G -expectation can be extended continuously to $\mathbb{L}_G^{1,t}(\Omega_T)$.

Set

$$\mathbb{M}^{p,0}(0, T) := \left\{ \eta_t = \sum_{i=0}^{N-1} \xi_{t_i} I_{[t_i, t_{i+1})}(t) : 0 = t_0 < \dots < t_N = T, \xi_{t_i} \in \mathbb{L}^p(\Omega_{t_i}) \right\}.$$

For $p \geq 1$, we denote by $\mathbb{M}^p(0, T)$, $\mathbb{H}^p(0, T)$, $\mathbb{S}^p(0, T)$ the completion of $\mathbb{M}^{p,0}(0, T)$ under the norm $\|\eta\|_{\mathbb{M}^p} := \left(\hat{\mathbb{E}} \left[\int_0^T |\eta_t|^p dt \right] \right)^{1/p}$, $\|\eta\|_{\mathbb{H}^p} := \left\{ \hat{\mathbb{E}} \left[\left(\int_0^T |\eta_t|^2 dt \right)^{p/2} \right] \right\}^{1/p}$, $\|\eta\|_{\mathbb{S}^p} := (\hat{\mathbb{E}}[\sup_{t \in [0, T]} |\eta_t|^p])^{1/p}$ respectively. Following Li and Peng [6], for each $\eta \in \mathbb{H}^p(0, T)$ with $p \geq 1$, we can define Itô's integral $\int_0^T \eta_s dB_s$. Moreover, by Proposition 2.10 in [6] and classical Burkholder–Davis–Gundy Inequality, the following properties hold.

Proposition 2.9. For each $\eta, \theta \in \mathbb{H}^\alpha(0, T)$ with $\alpha \geq 1$ and $p > 0$, $\xi \in \mathbb{L}^\infty(\Omega_t)$, we have

$$\begin{aligned} \hat{\mathbb{E}} \left[\int_0^T \eta_s dB_s \right] &= 0, \\ \underline{\sigma}^p C_p \hat{\mathbb{E}} \left[\left(\int_0^T |\eta_s|^2 ds \right)^{p/2} \right] \\ &\leq \hat{\mathbb{E}} \left[\sup_{t \in [0, T]} \left| \int_0^t \eta_s dB_s \right|^p \right] \leq \bar{\sigma}^p C_p \hat{\mathbb{E}} \left[\left(\int_0^T |\eta_s|^2 ds \right)^{p/2} \right], \\ \int_t^T (\xi \eta_s + \theta_s) dB_s &= \xi \int_t^T \eta_s dB_s + \int_t^T \theta_s dB_s, \end{aligned}$$

where $0 < c_p < C_p < \infty$ are constants.

Remark 2.10. If $\eta \in H_G^\alpha(0, T)$ with $\alpha \geq 1$ and $p \in (0, \alpha]$, then we can get $\sup_{u \in [t, T]} |\int_t^u \eta_s dB_s|^p \in L_G^1(\Omega_T)$ and

$$\begin{aligned} \underline{\sigma}^p C_p \hat{\mathbb{E}}_t \left[\left(\int_t^T |\eta_s|^2 ds \right)^{p/2} \right] &\leq \hat{\mathbb{E}}_t \left[\sup_{u \in [t, T]} \left| \int_t^u \eta_s dB_s \right|^p \right] \\ &\leq \bar{\sigma}^p C_p \hat{\mathbb{E}}_t \left[\left(\int_t^T |\eta_s|^2 ds \right)^{p/2} \right]. \end{aligned}$$

Definition 2.11. A process $\{M_t\}$ with values in $L_G^1(\Omega_T)$ is called a G -martingale if $\hat{\mathbb{E}}_s[M_t] = M_s$ for any $s \leq t$.

Let $S_G^0(0, T) = \{h(t, B_{t_1 \wedge t}, \dots, B_{t_n \wedge t}) : t_1, \dots, t_n \in [0, T], h \in C_{b, Lip}(\mathbb{R}^{n+1})\}$. For $p \geq 1$ and $\eta \in S_G^0(0, T)$, set $\|\eta\|_{S_G^p} := (\hat{\mathbb{E}}[\sup_{t \in [0, T]} |\eta_t|^p])^{1/p}$. Denote by $S_G^p(0, T)$ the completion of $S_G^0(0, T)$ under the norm $\|\cdot\|_{S_G^p}$.

We consider the following type of G -BSDEs (in this paper we always use Einstein convention):

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g_{ij}(s, Y_s, Z_s) d\langle B^i, B^j \rangle_s$$

$$- \int_t^T Z_s dB_s - (K_T - K_t), \quad (2.1)$$

where

$$f(t, \omega, y, z), g_{ij}(t, \omega, y, z) : [0, T] \times \Omega_T \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$$

satisfy the following properties:

(H1) There exists some $\beta > 1$ such that for any $y, z, f(\cdot, \cdot, y, z), g_{ij}(\cdot, \cdot, y, z) \in M_G^\beta(0, T)$;

(H2) There exists some $L > 0$ such that

$$|f(t, y, z) - f(t, y', z')| + \sum_{i,j=1}^d |g_{ij}(t, y, z) - g_{ij}(t, y', z')| \leq L(|y - y'| + |z - z'|).$$

For simplicity, we denote by $\mathfrak{S}_G^\alpha(0, T)$ the collection of processes (Y, Z, K) such that $Y \in S_G^\alpha(0, T)$, $Z \in H_G^\alpha(0, T; \mathbb{R}^d)$, K is a decreasing G -martingale with $K_0 = 0$ and $K_T \in L_G^\alpha(\Omega_T)$.

Definition 2.12. Let $\xi \in L_G^\beta(\Omega_T)$ and f satisfy (H1) and (H2) for some $\beta > 1$. A triplet of processes (Y, Z, K) is called a solution of Eq. (2.1) if for some $1 < \alpha \leq \beta$ the following properties hold:

(a) $(Y, Z, K) \in \mathfrak{S}_G^\alpha(0, T)$;

(b) $Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g_{ij}(s, Y_s, Z_s) d\langle B^i, B^j \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t)$.

Theorem 2.13 ([4]). Assume that $\xi \in L_G^\beta(\Omega_T)$ and f, g_{ij} satisfy (H1) and (H2) for some $\beta > 1$. Then Eq. (2.1) has a unique solution (Y, Z, K) . Moreover, for any $1 < \alpha < \beta$ we have $Y \in S_G^\alpha(0, T)$, $Z \in H_G^\alpha(0, T; \mathbb{R}^d)$ and $K_T \in L_G^\alpha(\Omega_T)$.

We have the following estimates.

Proposition 2.14 ([4]). Let $\xi \in L_G^\beta(\Omega_T)$ and f, g_{ij} satisfy (H1) and (H2) for some $\beta > 1$. Assume that $(Y, Z, K) \in \mathfrak{S}_G^\alpha(0, T)$ for some $1 < \alpha < \beta$ is a solution of Eq. (2.1). Then

(i) There exists a constant $C_\alpha := C(\alpha, T, G, L) > 0$ such that

$$\begin{aligned} |Y_t|^\alpha &\leq C_\alpha \hat{\mathbb{E}}_t \left[|\xi|^\alpha + \int_t^T |h_s^0|^\alpha ds \right], \\ \hat{\mathbb{E}} \left[\left(\int_0^T |Z_s|^2 ds \right)^{\frac{\alpha}{2}} \right] &\leq C_\alpha \left\{ \hat{\mathbb{E}} \left[\sup_{t \in [0, T]} |Y_t|^\alpha \right] + \left(\hat{\mathbb{E}} \left[\sup_{t \in [0, T]} |Y_t|^\alpha \right] \right)^{\frac{1}{2}} \left(\hat{\mathbb{E}} \left[\left(\int_0^T h_s^0 ds \right)^\alpha \right] \right)^{\frac{1}{2}} \right\}, \\ \hat{\mathbb{E}}[|K_T|^\alpha] &\leq C_\alpha \left\{ \hat{\mathbb{E}} \left[\sup_{t \in [0, T]} |Y_t|^\alpha \right] + \hat{\mathbb{E}} \left[\left(\int_0^T h_s^0 ds \right)^\alpha \right] \right\}, \end{aligned}$$

where $h_s^0 = |f(s, 0, 0)| + \sum_{i,j=1}^d |g_{ij}(s, 0, 0)|$.

(ii) For any given α' with $\alpha < \alpha' < \beta$, there exists a constant $C_{\alpha, \alpha'}$ depending on α, α', T, G, L such that

$$\begin{aligned} \hat{\mathbb{E}} \left[\sup_{t \in [0, T]} |Y_t|^\alpha \right] &\leq C_{\alpha, \alpha'} \left\{ \hat{\mathbb{E}} \left[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t[|\xi|^\alpha] \right] \right. \\ &\quad + \left(\hat{\mathbb{E}} \left[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t \left[\left(\int_0^T h_s^0 ds \right)^{\alpha'} \right] \right] \right)^{\frac{\alpha}{\alpha'}} \\ &\quad \left. + \hat{\mathbb{E}} \left[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t \left[\left(\int_0^T h_s^0 ds \right)^{\alpha'} \right] \right] \right\}. \end{aligned}$$

Corollary 2.15. Let $\xi \in L_G^\beta(\Omega_T)$ and f, g_{ij} satisfy (H1) and (H2) for some $\beta > 1$. Assume that (Y, Z, K) is a solution of Eq. (2.1). Then $Y_t \in L_G^\beta(\Omega_t)$ for any $t \leq T$.

Proof. By the proof of Proposition 3.7 in [4], it is easy to check that for any given $\alpha \in (1 + \frac{1}{2}(\beta - 1), \beta)$, there exists a uniform constant $C := C(\beta, T, G, L) > 0$ such that

$$|Y_t|^\alpha \leq C \hat{\mathbb{E}}_t \left[|\xi|^\alpha + \int_t^T |h_s^0|^\alpha ds \right],$$

where $h_s^0 = |f(s, 0, 0)| + \sum_{i,j=1}^d |g_{ij}(s, 0, 0)|$. Then, by Hölder's inequality, we can get

$$\begin{aligned} |Y_t|^\alpha &\leq C \hat{\mathbb{E}}_t[|\xi|^\alpha] + C \hat{\mathbb{E}}_t \left[\int_0^T |h_s^0|^\alpha ds \right] \\ &\leq C (\hat{\mathbb{E}}_t[|\xi|^\beta])^{\alpha/\beta} + C T^{(\beta-\alpha)/\beta} \left(\hat{\mathbb{E}}_t \left[\int_0^T |h_s^0|^\beta ds \right] \right)^{\alpha/\beta}. \end{aligned}$$

Letting $\alpha \rightarrow \beta$, we can get

$$|Y_t|^\beta \leq C \left(\hat{\mathbb{E}}_t[|\xi|^\beta] + \hat{\mathbb{E}}_t \left[\int_0^T |h_s^0|^\beta ds \right] \right).$$

Thus, by Theorem 52 in [2], we can deduce that $Y_t \in L_G^\beta(\Omega_t)$. \square

Proposition 2.16 ([4]). Let $\xi^l \in L_G^\beta(\Omega_T)$, $l = 1, 2$, and f^l, g_{ij}^l satisfy (H1) and (H2) for some $\beta > 1$. Assume that $(Y^l, Z^l, K^l) \in \mathfrak{S}_G^\alpha(0, T)$ for some $1 < \alpha < \beta$ are the solutions of Eq. (2.1) corresponding to ξ^l, f^l and g_{ij}^l . Set $\hat{Y}_t = Y_t^1 - Y_t^2$, $\hat{Z}_t = Z_t^1 - Z_t^2$ and $\hat{K}_t = K_t^1 - K_t^2$. Then

(i) There exists a constant $C_\alpha := C(\alpha, T, G, L) > 0$ such that

$$|\hat{Y}_t|^\alpha \leq C_\alpha \hat{\mathbb{E}}_t \left[|\hat{\xi}|^\alpha + \int_t^T |\hat{h}_s|^\alpha ds \right],$$

where $\hat{\xi} = \xi^1 - \xi^2$, $\hat{h}_s = |f^1(s, Y_s^2, Z_s^2) - f^2(s, Y_s^2, Z_s^2)| + \sum_{i,j=1}^d |g_{ij}^1(s, Y_s^2, Z_s^2) - g_{ij}^2(s, Y_s^2, Z_s^2)|$.

- (ii) For any given α' with $\alpha < \alpha' < \beta$, there exists a constant $C_{\alpha, \alpha'}$ depending on α, α', T, G, L such that

$$\begin{aligned} \hat{\mathbb{E}} \left[\sup_{t \in [0, T]} |\hat{Y}_t|^\alpha \right] &\leq C_{\alpha, \alpha'} \left\{ \hat{\mathbb{E}} \left[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t [|\hat{\xi}|^\alpha] \right] \right. \\ &\quad + \left(\hat{\mathbb{E}} \left[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t \left[\left(\int_0^T \hat{h}_s ds \right)^{\alpha'} \right] \right] \right)^{\frac{\alpha}{\alpha'}} \\ &\quad \left. + \hat{\mathbb{E}} \left[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t \left[\left(\int_0^T \hat{h}_s ds \right)^{\alpha'} \right] \right] \right\}. \end{aligned}$$

3. Comparison theorem of G -BSDEs

For simplicity, we consider the 1-dimensional G -Brownian motion case. The results still hold for the case $d > 1$.

3.1. Explicit solutions of linear G -BSDEs

Let $(\Omega_T, L_G^1(\Omega_T), \hat{\mathbb{E}})$ with $\Omega_T = C_0([0, T], \mathbb{R})$ be a G -expectation space. We consider the explicit solution of the following linear G -BSDE:

$$Y_t = \xi + \int_t^T f_s ds + \int_t^T g_s d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t), \quad (3.1)$$

where $f_s = a_s Y_s + b_s Z_s + m_s$, $g_s = c_s Y_s + d_s Z_s + n_s$ with $\{a_s\}_{s \in [0, T]}$, $\{b_s\}_{s \in [0, T]}$, $\{c_s\}_{0 \leq s \in [0, T]}$, $\{d_s\}_{s \in [0, T]}$ bounded processes in $M_G^\beta(0, T)$ and $\xi \in L_G^\beta(\Omega_T)$, $\{m_s\}_{s \in [0, T]}$, $\{n_s\}_{s \in [0, T]} \in M_G^\beta(0, T)$ with $\beta > 1$. For this purpose we construct an auxiliary extended \tilde{G} -expectation space $(\tilde{\Omega}_T, L_{\tilde{G}}^1(\tilde{\Omega}_T), \tilde{\mathbb{E}}^{\tilde{G}})$ with $\tilde{\Omega}_T = C_0([0, T], \mathbb{R}^2)$ and

$$\tilde{G}(A) = \frac{1}{2} \sup_{\underline{\sigma}^2 \leq v \leq \bar{\sigma}^2} \text{tr} \left[A \begin{bmatrix} v & 1 \\ 1 & v^{-1} \end{bmatrix} \right], \quad A \in \mathbb{S}_2.$$

Let $\{(B_t, \tilde{B}_t)\}$ be the canonical process in the extended space.

Remark 3.1. It is easy to check that $\langle B, \tilde{B} \rangle_t = t$. In particular, if $\underline{\sigma}^2 = \bar{\sigma}^2$, we can further get $\tilde{B}_t = \bar{\sigma}^{-2} B_t$.

Let $\{X_t\}_{t \in [0, T]}$ be the solution of the following \tilde{G} -SDE:

$$X_t = 1 + \int_0^t a_s X_s ds + \int_0^t c_s X_s d\langle B \rangle_s + \int_0^t d_s X_s dB_s + \int_0^t b_s X_s d\tilde{B}_s. \quad (3.2)$$

It is easy to verify that

$$X_t = \exp \left(\int_0^t (a_s - b_s d_s) ds + \int_0^t c_s d\langle B \rangle_s \right) \mathcal{E}_t^B \mathcal{E}_t^{\tilde{B}}, \quad (3.3)$$

where $\mathcal{E}_t^B = \exp \left(\int_0^t d_s dB_s - \frac{1}{2} \int_0^t d_s^2 d\langle B \rangle_s \right)$, $\mathcal{E}_t^{\tilde{B}} = \exp \left(\int_0^t b_s d\tilde{B}_s - \frac{1}{2} \int_0^t b_s^2 d\langle \tilde{B} \rangle_s \right)$.

Theorem 3.2. In the extended \tilde{G} -expectation space, the solution of the G -BSDE (3.1) can be represented as

$$Y_t = (X_t)^{-1} \hat{\mathbb{E}}_t^{\tilde{G}} \left[X_T \xi + \int_t^T m_s X_s ds + \int_t^T n_s X_s d\langle B \rangle_s \right], \quad (3.4)$$

where $\{X_t\}_{t \in [0, T]}$ is the solution of the \tilde{G} -SDE (3.2).

Proof. By applying Itô's formula to $X_t Y_t$, we get

$$\begin{aligned} X_t Y_t + \int_t^T (X_s Z_s + d_s X_s Y_s) dB_s + \int_t^T b_s X_s Y_s d\tilde{B}_s + \int_t^T X_s dK_s \\ = X_T \xi + \int_t^T m_s X_s ds + \int_t^T n_s X_s d\langle B \rangle_s. \end{aligned}$$

By Lemma 3.4 in [4], we have $\left\{ \int_0^t X_s dK_s \right\}_{t \in [0, T]}$ is a \tilde{G} -martingale. Thus we get

$$Y_t = (X_t)^{-1} \hat{\mathbb{E}}_t^{\tilde{G}} \left[X_T \xi + \int_t^T m_s X_s ds + \int_t^T n_s X_s d\langle B \rangle_s \right]. \quad \square$$

Remark 3.3. If $b_t = 0$, the solution of the G -BSDE (3.1) is

$$Y_t = (X_t)^{-1} \hat{\mathbb{E}}_t \left[X_T \xi + \int_t^T m_s X_s ds + \int_t^T n_s X_s d\langle B \rangle_s \right],$$

where $X_t = \exp \left(\int_0^t a_s ds + \int_0^t (c_s - \frac{1}{2} d_s^2) d\langle B \rangle_s + \int_0^t d_s dB_s \right)$. In this case, we do not need to construct an auxiliary space. If $b_t \neq 0$, the form of X_t contains \tilde{B} , but

$$Y_t = \hat{\mathbb{E}}_t^{\tilde{G}} \left[X_T^t \xi + \int_t^T m_s X_s^t ds + \int_t^T n_s X_s^t d\langle B \rangle_s \right]$$

does not contain \tilde{B} , where $X_s^t = X_s / X_t$. For simplicity, we only give an explanation for $\xi = \varphi(B_T)$, $f_s = b_s Z_s$ with $b_s = \psi(B_s)$ and $g_s = 0$ in the G -BSDE (3.1). In this case,

$$\begin{aligned} Y_t &= \hat{\mathbb{E}}_t^{\tilde{G}} \left[\varphi(B_T) \exp \left(\int_t^T \psi(B_s) d\tilde{B}_s - \frac{1}{2} \int_t^T |\psi(B_s)|^2 d\langle \tilde{B} \rangle_s \right) \right] \\ &= \hat{\mathbb{E}}^{\tilde{G}} \left[\varphi(x + B_T^t) \exp \left(\int_t^T \psi(x + B_s^t) d\tilde{B}_s - \frac{1}{2} \int_t^T |\psi(x + B_s^t)|^2 d\langle \tilde{B} \rangle_s \right) \right]_{x=B_t}, \end{aligned}$$

which does not contain \tilde{B} , where $B_s^t = B_s - B_t$.

Note that $\hat{\mathbb{E}}^{\tilde{G}}[\xi] = \hat{\mathbb{E}}[\xi]$ for each $\xi \in L_G^1(\Omega_T)$, thus this Y in Theorem 3.2 is the solution of the G -BSDE (3.1) in $(\Omega_T, L_G^1(\Omega_T), \hat{\mathbb{E}})$. Here \tilde{B} is an auxiliary process and disappears by taking conditional expectation.

Remark 3.4. If $b_s = 0$, $d_s = 0$, we have the following special type of G -BSDE:

$$Y_t = \hat{\mathbb{E}}_t \left[\xi + \int_t^T (a_s Y_s + m_s) ds + \int_t^T (c_s Y_s + n_s) d\langle B \rangle_s \right], \quad (3.5)$$

where $\{a_s\}_{s \in [0, T]}$, $\{c_s\}_{s \in [0, T]}$ are bounded processes in $M_G^1(0, T)$ and $\xi \in L_G^1(\Omega)$, $\{m_s\}_{s \in [0, T]}$, $\{n_s\}_{s \in [0, T]} \in M_G^1(0, T)$. By applying Theorem 3.2 to $\xi^N = (\xi \wedge N) \vee (-N)$, $m_s^N = (m_s \wedge N) \vee (-N)$, $n_s^N = (n_s \wedge N) \vee (-N)$ for each $N > 0$, we obtain that the explicit solution of the G -BSDE (3.5) is

$$Y_t = (X_t)^{-1} \hat{\mathbb{E}}_t \left[X_T \xi + \int_t^T m_s X_s ds + \int_t^T n_s X_s d\langle B \rangle_s \right], \quad (3.6)$$

where $X_t = \exp \left(\int_0^t a_s ds + \int_0^t c_s d\langle B \rangle_s \right)$.

In the following, we explain why we have to extend the space. For simplicity, we only consider

$$Y_t = \xi + \int_t^T Z_s ds - \int_t^T Z_s dB_s - (K_T - K_t).$$

In order to get the explicit solution of the above G -BSDE, we try to find a positive process X (not depending on Y, Z, K) such that XY is a G -martingale. Applying Itô's formula to XY , we have

$$d(X_t Y_t) = X_t Z_t dB_t + X_t dK_t - X_t Z_t dt + Z_t d\langle X, B \rangle_t + Y_t dX_t.$$

So as to guarantee that XY is a G -martingale, $-X_t Z_t dt + Z_t d\langle X, B \rangle_t + Y_t dX_t$ should be a symmetric G -martingale, which implies that X is a symmetric G -martingale and

$$X_t dt = d\langle X, B \rangle_t. \quad (3.7)$$

By the representation theorem of symmetric G -martingales, we assume $X_t = X_0 + \int_0^t h_s dB_s$ for some $h \in M_G^2(0, T)$. Then Eq. (3.7) implies that

$$X_t dt = h_t d\langle B \rangle_t.$$

By Corollary 3.5 in [29], we have $X \equiv 0$ if $\underline{\sigma}^2 < \bar{\sigma}^2$. So generally we cannot find a proper process X in the original G -expectation space. Actually, in Theorem 3.2, we find a process X in the extended \tilde{G} -expectation space such that XY is a \tilde{G} -martingale instead of a G -martingale.

Sometimes we say a process $Y \in S_G^\alpha(0, T)$ with some $\alpha > 1$ is a solution of Eq. (2.1) if there exist processes Z, K such that $(Y, Z, K) \in \mathfrak{S}_G^\alpha(0, T)$ is a solution of Eq. (2.1).

Proposition 3.5. Let K be a decreasing G -martingale with $K_T \in L_G^\alpha(\Omega_T)$ for some $\alpha > 1$. Assume that

$$f(t, K_t, 0) = g(t, K_t, 0) = 0.$$

Then K is a solution of Eq. (2.1).

Proof. It is easy to check that $(K, 0, K)$ is a solution of Eq. (2.1). \square

3.2. Comparison theorem of G -BSDEs

Theorem 3.6. Let $(Y_t^i, Z_t^i, K_t^i)_{t \leq T}$, $i = 1, 2$, be the solutions of the following G -BSDEs:

$$Y_t^i = \xi^i + \int_t^T f_i(s, Y_s^i, Z_s^i) ds + \int_t^T g_i(s, Y_s^i, Z_s^i) d\langle B \rangle_s - \int_t^T Z_s^i dB_s - (K_T^i - K_t^i),$$

where $\xi^i \in L_G^\beta(\Omega_T)$, f_i, g_i satisfy (H1) and (H2) with $\beta > 1$. If $\xi^1 \geq \xi^2$, $f_1 \geq f_2$, $g_1 \geq g_2$, then $Y_t^1 \geq Y_t^2$.

Proof. We have

$$\hat{Y}_t + K_t^2 = \hat{\xi} + K_T^2 + \int_t^T \hat{f}_s ds + \int_t^T \hat{g}_s d\langle B \rangle_s - \int_t^T \hat{Z}_s dB_s - (K_T^1 - K_t^1),$$

where $\hat{Y}_t = Y_t^1 - Y_t^2$, $\hat{Z}_t = Z_t^1 - Z_t^2$, $\hat{\xi} = \xi^1 - \xi^2 \geq 0$, $\hat{f}_s = f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2)$, $\hat{g}_s = g_1(s, Y_s^1, Z_s^1) - g_2(s, Y_s^2, Z_s^2)$. For each given $\varepsilon > 0$, we can choose Lipschitz function $l(\cdot)$ such that $I_{[-\varepsilon, \varepsilon]} \leq l(x) \leq I_{[-2\varepsilon, 2\varepsilon]}$. Thus we have

$$f_1(s, Y_s^1, Z_s^1) - f_1(s, Y_s^2, Z_s^1) = (f_1(s, Y_s^1, Z_s^1) - f_1(s, Y_s^2, Z_s^1))l(\hat{Y}_s) + a_s^\varepsilon \hat{Y}_s,$$

where $a_s^\varepsilon = (1 - l(\hat{Y}_s))(f_1(s, Y_s^1, Z_s^1) - f_1(s, Y_s^2, Z_s^1))\hat{Y}_s^{-1}$ if \hat{Y}_s is not equal to 0, whereas $a_s^\varepsilon = 0$, otherwise. Also, $b_s^\varepsilon = (1 - l(\hat{Z}_s))(f_1(s, Y_s^2, Z_s^1) - f_1(s, Y_s^2, Z_s^2))\hat{Z}_s^{-1}$ if \hat{Z}_s is not equal to 0, whereas $b_s^\varepsilon = 0$, otherwise.

a_s^ε and b_s^ε belong to $M_G^\varepsilon(0, T)$ and $|a_s^\varepsilon| \leq L$, $|b_s^\varepsilon| \leq L$. It is easy to verify that

$$|(f_1(s, Y_s^1, Z_s^1) - f_1(s, Y_s^2, Z_s^1))l(\hat{Y}_s)| \leq L|\hat{Y}_s|l(\hat{Y}_s) \leq 2L\varepsilon.$$

Thus we can get

$$\hat{f}_s = a_s^\varepsilon \hat{Y}_s + b_s^\varepsilon \hat{Z}_s + m_s - m_s^\varepsilon, \quad \hat{g}_s = c_s^\varepsilon \hat{Y}_s + d_s^\varepsilon \hat{Z}_s + n_s - n_s^\varepsilon,$$

where $|m_s^\varepsilon| \leq 4L\varepsilon$, $|n_s^\varepsilon| \leq 4L\varepsilon$, $m_s = f_1(s, Y_s^2, Z_s^2) - f_2(s, Y_s^2, Z_s^2) \geq 0$ and $m_s^\varepsilon = a_s^\varepsilon \hat{Y}_s - (f_1(s, Y_s^1, Z_s^1) - f_1(s, Y_s^2, Z_s^1)) + b_s^\varepsilon \hat{Z}_s - (f_1(s, Y_s^2, Z_s^1) - f_1(s, Y_s^2, Z_s^2))$. The definitions of $c_s^\varepsilon, d_s^\varepsilon, n_s$ and n_s^ε for $\hat{g}_s = g_1(s, Y_s^1, Z_s^1) - g_2(s, Y_s^2, Z_s^2)$ are similar to the definitions of $a_s^\varepsilon, b_s^\varepsilon, m_s$ and m_s^ε .

By Theorem 3.2, in the extended space, we have

$$\begin{aligned} \hat{Y}_t + K_t^2 &= (X_t^\varepsilon)^{-1} \hat{\mathbb{E}}_t^{\tilde{G}} \left[X_T^\varepsilon (\hat{\xi} + K_T^2) + \int_t^T (m_s - m_s^\varepsilon - a_s^\varepsilon K_s^2) X_s^\varepsilon ds \right. \\ &\quad \left. + \int_t^T (n_s - n_s^\varepsilon - c_s^\varepsilon K_s^2) X_s^\varepsilon d\langle B \rangle_s \right] \\ &\geq (X_t^\varepsilon)^{-1} \hat{\mathbb{E}}_t^{\tilde{G}} \left[X_T^\varepsilon K_T^2 - \int_t^T (m_s^\varepsilon + a_s^\varepsilon K_s^2) X_s^\varepsilon ds - \int_t^T (n_s^\varepsilon + c_s^\varepsilon K_s^2) X_s^\varepsilon d\langle B \rangle_s \right] \\ &= (X_t^\varepsilon)^{-1} \left\{ \hat{\mathbb{E}}_t^{\tilde{G}} \left[X_T^\varepsilon K_T^2 - \int_t^T a_s^\varepsilon K_s^2 X_s^\varepsilon ds - \int_t^T c_s^\varepsilon K_s^2 X_s^\varepsilon d\langle B \rangle_s \right] \right. \\ &\quad \left. - \hat{\mathbb{E}}_t^{\tilde{G}} \left[\int_t^T m_s^\varepsilon X_s^\varepsilon ds + \int_t^T n_s^\varepsilon X_s^\varepsilon d\langle B \rangle_s \right] \right\}, \end{aligned}$$

where $\{X_t^\varepsilon\}_{t \in [0, T]}$ is the solution of the following \tilde{G} -SDE:

$$X_t^\varepsilon = 1 + \int_0^t a_s^\varepsilon X_s^\varepsilon ds + \int_0^t c_s^\varepsilon X_s^\varepsilon d\langle B \rangle_s + \int_0^t d_s^\varepsilon X_s^\varepsilon dB_s + \int_0^t b_s^\varepsilon X_s^\varepsilon d\tilde{B}_s.$$

By Theorem 3.2 and Proposition 3.5, we get

$$(X_t^\varepsilon)^{-1} \hat{\mathbb{E}}_t^{\tilde{G}} \left[X_T^\varepsilon K_T^2 - \int_t^T a_s^\varepsilon K_s^2 X_s^\varepsilon ds - \int_t^T c_s^\varepsilon K_s^2 X_s^\varepsilon d\langle B \rangle_s \right] = K_t^2.$$

Thus

$$\hat{Y}_t \geq -4L\varepsilon (X_t^\varepsilon)^{-1} \hat{E}^{\tilde{G}} \left[\int_t^T |X_s^\varepsilon| ds + \int_t^T |X_s^\varepsilon| d\langle B \rangle_s \right],$$

which completes the proof by letting $\varepsilon \rightarrow 0$. \square

Theorem 3.7. Let $(Y_t^i, Z_t^i, K_t^i)_{t \leq T}$, $i = 1, 2$, be the solutions of the following G -BSDEs:

$$Y_t^i = \xi^i + \int_t^T f_i(s) ds + \int_t^T g_i(s) d\langle B \rangle_s + V_T^i - V_t^i - \int_t^T Z_s^i dB_s - (K_T^i - K_t^i),$$

where $f_i(s) = f_i(s, Y_s^i, Z_s^i)$, $g_i(s) = g_i(s, Y_s^i, Z_s^i)$, $\xi^i \in L_G^\beta(\Omega_T)$, f_i, g_i satisfy (H1) and (H2), $(V_t^i)_{t \leq T}$ are RCLL processes such that $\hat{\mathbb{E}}[\sup_{t \in [0, T]} |V_t^i|^\beta] < \infty$ with $\beta > 1$. If $\xi^1 \geq \xi^2$, $f_1 \geq f_2$, $g_1 \geq g_2$, $V_t^1 - V_t^2$ is an increasing process, then $Y_t^1 \geq Y_t^2$.

Proof. The proof is similar to that of Theorem 3.6. \square

Remark 3.8. If f_i, g_i , $i = 1, 2$, do not contain Z , we get the following special G -BSDEs:

$$Y_t^i = \hat{\mathbb{E}}_t \left[\xi^i + \int_t^T f_i(s, Y_s^i) ds + \int_t^T g_i(s, Y_s^i) d\langle B \rangle_s \right].$$

The same as in Remark 3.4, here we suppose that $\xi \in L_G^1(\Omega)$, $\{f_i(s, y)\}_{s \in [0, T]} \in M_G^1(0, T)$ and $\{g_i(s, y)\}_{s \in [0, T]} \in M_G^1(0, T)$ for each $y \in \mathbb{R}$, f_i and g_i satisfy the Lipschitz condition with respect to y . The comparison theorem still holds for this case.

In the following, we give an example to show that the strict comparison theorem does not hold.

Example 3.9. We consider the simplest G -BSDE:

$$Y_t = \xi - \int_t^T Z_s dB_s - (K_T - K_t),$$

the solution $Y_t = \hat{\mathbb{E}}_t[\xi]$, $t \in [0, T]$. Let $\xi^1 = 0$ and $\xi^2 = \langle B \rangle_T - \bar{\sigma}^2 T$. It is easy to verify that $\xi^1 \geq \xi^2$ and $\hat{\mathbb{E}}[\xi^1 - \xi^2] > 0$ for the case $\underline{\sigma} < \bar{\sigma}$. But $\hat{\mathbb{E}}[\xi^1] = \hat{\mathbb{E}}[\xi^2] = 0$.

We now give an application of the comparison theorem.

Theorem 3.10 (Gronwall Inequality). Let $(Y_t)_{t \leq T} \in S_G^1(0, T)$ satisfy

$$Y_t \leq \hat{\mathbb{E}}_t \left[\xi + \int_t^T f(s, Y_s) ds + \int_t^T g(s, Y_s) d\langle B \rangle_s \right],$$

where $\xi \in L_G^1(\Omega)$, $\{f(s, y)\}_{s \in [0, T]} \in M_G^1(0, T)$ and $\{g(s, y)\}_{s \in [0, T]} \in M_G^1(0, T)$ for each $y \in \mathbb{R}$, f and g satisfy the Lipschitz condition with respect to y , $f(\cdot, y_1) \leq f(\cdot, y_2)$ and

$g(\cdot, y_1) \leq g(\cdot, y_2)$ for each $y_1 \leq y_2$. Then $Y_t \leq \tilde{Y}_t$, where $(\tilde{Y}_t)_{t \leq T}$ is the solution of the following G -BSDE:

$$\tilde{Y}_t = \hat{\mathbb{E}}_t \left[\xi + \int_t^T f(s, \tilde{Y}_s) ds + \int_t^T g(s, \tilde{Y}_s) d\langle B \rangle_s \right].$$

In particular, if $f(s, y) = a_s y + m_s$, $g(s, y) = c_s y + n_s$, where $a_s \geq 0$, $c_s \geq 0$, then

$$Y_t \leq (X_t)^{-1} \hat{\mathbb{E}}_t \left[X_T \xi + \int_t^T m_s X_s ds + \int_t^T n_s X_s d\langle B \rangle_s \right], \quad (3.8)$$

where $X_t = \exp \left(\int_0^t a_s ds + \int_0^t c_s d\langle B \rangle_s \right)$.

Proof. We set

$$\delta_t = \hat{\mathbb{E}}_t \left[\xi + \int_t^T f(s, Y_s) ds + \int_t^T g(s, Y_s) d\langle B \rangle_s \right] - Y_t \geq 0,$$

then

$$\begin{aligned} Y_t + \delta_t &= \hat{\mathbb{E}}_t \left[\xi + \int_t^T f(s, Y_s) ds + \int_t^T g(s, Y_s) d\langle B \rangle_s \right] \\ &= \hat{\mathbb{E}}_t \left[\xi + \int_t^T f(s, Y_s + \delta_s - \delta_s) ds + \int_t^T g(s, Y_s + \delta_s - \delta_s) d\langle B \rangle_s \right]. \end{aligned}$$

Thus $(Y_t + \delta_t)_{t \leq T}$ is the solution of the following G -BSDE:

$$\bar{Y}_t = \hat{\mathbb{E}}_t \left[\xi + \int_t^T f(s, \bar{Y}_s - \delta_s) ds + \int_t^T g(s, \bar{Y}_s - \delta_s) d\langle B \rangle_s \right].$$

By the comparison theorem of G -BSDEs, we get $\bar{Y}_t \leq \tilde{Y}_t$. Thus $Y_t \leq \tilde{Y}_t$. By formula (3.6), we get (3.8). \square

4. Nonlinear Feynman–Kac formula

In this section, we give the nonlinear Feynman–Kac Formula which was studied in Peng [20] for a special type of G -BSDEs. Let $G : \mathbb{S}_d \rightarrow \mathbb{R}$ be a given monotonic and sublinear function such that $G(A) - G(B) \geq \sigma^2 \text{tr}[A - B]$ for any $A \geq B$ and $B_t = (B_t^i)_{i=1}^d$ be the corresponding G -Brownian motion. We consider the following type of G -FBSDEs:

$$dX_s^{t,\xi} = b(s, X_s^{t,\xi}) ds + h_{ij}(s, X_s^{t,\xi}) d\langle B^i, B^j \rangle_s + \sigma_j(s, X_s^{t,\xi}) dB_s^j, \quad X_t^{t,\xi} = \xi, \quad (4.1)$$

$$\begin{aligned} Y_s^{t,\xi} &= \Phi(X_T^{t,\xi}) + \int_s^T f(r, X_r^{t,\xi}, Y_r^{t,\xi}, Z_r^{t,\xi}) dr + \int_s^T g_{ij}(r, X_r^{t,\xi}, Y_r^{t,\xi}, Z_r^{t,\xi}) d\langle B^i, B^j \rangle_r \\ &\quad - \int_s^T Z_r^{t,\xi} dB_r - (K_T^{t,\xi} - K_s^{t,\xi}), \end{aligned} \quad (4.2)$$

where $b, h_{ij}, \sigma_j : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$, $f, g_{ij} : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ are deterministic functions and satisfy the following conditions:

(A1) $h_{ij} = h_{ji}$ and $g_{ij} = g_{ji}$ for $1 \leq i, j \leq d$;

(A2) $b, h_{ij}, \sigma_j, f, g_{ij}$ are continuous in t ;

(A3) There exist a positive integer m and a constant $L > 0$ such that

$$\begin{aligned} & |b(t, x) - b(t, x')| + \sum_{i,j=1}^d |h_{ij}(t, x) - h_{ij}(t, x')| \\ & + \sum_{j=1}^d |\sigma_j(t, x) - \sigma_j(t, x')| \leq L|x - x'|, \\ & |\Phi(x) - \Phi(x')| \leq L(1 + |x|^m + |x'|^m)|x - x'|, \\ & |f(t, x, y, z) - f(t, x', y', z')| + \sum_{i,j=1}^d |g_{ij}(t, x, y, z) - g_{ij}(t, x', y', z')| \\ & \leq L[(1 + |x|^m + |x'|^m)|x - x'| + |y - y'| + |z - z'|]. \end{aligned}$$

We have the following estimates of G -SDEs, which can be found in Chapter V in Peng [20].

Proposition 4.1. Let $\xi, \xi' \in L_G^p(\Omega_t; \mathbb{R}^n)$ with $p \geq 2$. Then we have, for each $\delta \in [0, T - t]$,

$$\begin{aligned} \hat{\mathbb{E}}_t[|X_{t+\delta}^{t,\xi} - X_{t+\delta}^{t,\xi'}|^p] & \leq C|\xi - \xi'|^p, \\ \hat{\mathbb{E}}_t[|X_{t+\delta}^{t,\xi}|^p] & \leq C(1 + |\xi|^p), \\ \hat{\mathbb{E}}_t\left[\sup_{s \in [t, t+\delta]} |X_s^{t,\xi} - \xi|^p\right] & \leq C(1 + |\xi|^p)\delta^{p/2}, \end{aligned}$$

where the constant C depends on L, G, p, n and T .

Proof. For convenience of the reader, we sketch the proof. It is easy to verify that $(X_s^{t,\xi})_{s \in [t, T]}$, $(X_s^{t,\xi'})_{s \in [t, T]} \in M_G^p(0, T; \mathbb{R}^n)$. By Remark 2.10, we can get

$$\begin{aligned} \hat{\mathbb{E}}_t[|X_{t+\delta}^{t,\xi} - X_{t+\delta}^{t,\xi'}|^p] & \leq C_1 \left(|\xi - \xi'|^p + \hat{\mathbb{E}}_t \left[\int_t^{t+\delta} |X_s^{t,\xi} - X_s^{t,\xi'}|^p ds \right] \right) \\ & \leq C_1 \left(|\xi - \xi'|^p + \int_t^{t+\delta} \hat{\mathbb{E}}_t[|X_s^{t,\xi} - X_s^{t,\xi'}|^p] ds \right), \end{aligned}$$

where the constant C_1 depends on L, G, p, n and T . By the Gronwall inequality, we obtain

$$\hat{\mathbb{E}}_t[|X_{t+\delta}^{t,\xi} - X_{t+\delta}^{t,\xi'}|^p] \leq C_1 \exp(C_1 T) |\xi - \xi'|^p.$$

Then we get the first inequality. The other inequalities can be proved similarly. \square

Proposition 4.2. For each $\xi, \xi' \in L_G^{4m+1}(\Omega_t; \mathbb{R}^n)$, we have

$$\begin{aligned} |Y_t^{t,\xi} - Y_t^{t,\xi'}| & \leq C(1 + |\xi|^m + |\xi'|^m) |\xi - \xi'|, \\ |Y_t^{t,\xi}| & \leq C(1 + |\xi|^{m+1}), \end{aligned}$$

where the constant C depends on L, G, n and T .

Proof. It follows from Propositions 2.16 and 4.1 that

$$\begin{aligned} |Y_t^{t,\xi} - Y_t^{t,\xi'}|^2 &\leq C_1 \left\{ \hat{\mathbb{E}}_t[(1 + |X_T^{t,\xi}|^m + |X_T^{t,\xi'}|^m)^2 |X_T^{t,\xi} - X_T^{t,\xi'}|^2] \right. \\ &\quad \left. + \int_t^T \hat{\mathbb{E}}_t[(1 + |X_s^{t,\xi}|^m + |X_s^{t,\xi'}|^m)^2 |X_s^{t,\xi} - X_s^{t,\xi'}|^2] ds \right\} \\ &\leq C_2(1 + |\xi|^{2m} + |\xi'|^{2m}) \left\{ (\hat{\mathbb{E}}_t[|X_T^{t,\xi} - X_T^{t,\xi'}|^4])^{1/2} \right. \\ &\quad \left. + \int_t^T (\hat{\mathbb{E}}_t[|X_s^{t,\xi} - X_s^{t,\xi'}|^4])^{1/2} ds \right\} \\ &\leq C_3(1 + |\xi|^{2m} + |\xi'|^{2m}) |\xi - \xi'|^2, \end{aligned}$$

where C_1 , C_2 and C_3 depend on L , G , n and T . Thus we get $|Y_t^{t,\xi} - Y_t^{t,\xi'}| \leq C(1 + |\xi|^m + |\xi'|^m) |\xi - \xi'|$. By Proposition 2.14, we can get $|Y_t^{t,\xi}| \leq C(1 + |\xi|^{m+1})$ by using a similar analysis. \square

We are more interested in the case when $\xi = x \in \mathbb{R}^n$. We define

$$u(t, x) := Y_t^{t,x}, \quad (t, x) \in [0, T] \times \mathbb{R}^n.$$

By Proposition 4.2, we immediately have the following estimates:

$$\begin{aligned} |u(t, x) - u(t, x')| &\leq C(1 + |x|^m + |x'|^m) |x - x'|, \\ |u(t, x)| &\leq C(1 + |x|^{m+1}), \end{aligned}$$

where the constant C depends on L , G , n and T .

Remark 4.3. It is important to note that $u(t, x)$ is a deterministic function of (t, x) , because $b, h_{ij}, \sigma_j, \Phi, f, g_{ij}$ are deterministic functions and $\tilde{B}_s := B_{t+s} - B_t$ is a G -Brownian motion.

The following theorem plays a key role in proving the Feynman–Kac formula.

Theorem 4.4. For each $\xi \in L_G^{4m+1}(\Omega_t; \mathbb{R}^n)$, we have

$$u(t, \xi) = Y_t^{t,\xi}.$$

Proof. By Proposition 4.2, we only need to prove Theorem 4.4 for bounded $\xi \in L_G^{4m+1}(\Omega_t; \mathbb{R}^n)$. Thus for each $\varepsilon > 0$, we can choose a simple function

$$\eta^\varepsilon = \sum_{i=1}^N x_i I_{A_i},$$

where $(A_i)_{i=1}^N$ is a $\mathcal{B}(\Omega_t)$ -partition and $x_i \in \mathbb{R}^n$, such that $|\eta^\varepsilon - \xi| \leq \varepsilon$. It follows from Proposition 4.2 that

$$|Y_t^{t,\xi} - u(t, \eta^\varepsilon)| = \left| Y_t^{t,\xi} - \sum_{i=1}^n u(t, x_i) I_{A_i} \right|$$

$$\begin{aligned}
&= \left| Y_t^{t,\xi} - \sum_{i=1}^N Y_t^{t,x_i} I_{A_i} \right| \\
&= \sum_{i=1}^N |Y_t^{t,\xi} - Y_t^{t,x_i}| I_{A_i} \\
&\leq \sum_{i=1}^N C(1 + |\xi|^m) |\xi - x_i| I_{A_i} \\
&= C(1 + |\xi|^m) \left| \xi - \sum_{i=1}^N x_i I_{A_i} \right| \\
&\leq C(1 + |\xi|^m) \varepsilon,
\end{aligned}$$

where the constant C depends on L, G, n and T . Noting that

$$|u(t, \xi) - u(t, \eta^\varepsilon)| \leq C(1 + |\xi|^m) |\xi - \eta^\varepsilon| \leq C(1 + |\xi|^m) \varepsilon,$$

we get $|Y_t^{t,\xi} - u(t, \xi)| \leq 2C(1 + |\xi|^m) \varepsilon$. Since ε can be arbitrarily small, we obtain $Y_t^{t,\xi} = u(t, \xi)$. \square

We now give the Feynman–Kac formula.

Theorem 4.5. Let $u(t, x) := Y_t^{t,x}$ for $(t, x) \in [0, T] \times \mathbb{R}^n$. Then $u(t, x)$ is the unique viscosity solution of the following PDE:

$$\begin{cases} \partial_t u + F(D_x^2 u, D_x u, u, x, t) = 0, \\ u(T, x) = \Phi(x), \end{cases} \quad (4.3)$$

where

$$\begin{aligned}
F(D_x^2 u, D_x u, u, x, t) &= G(H(D_x^2 u, D_x u, u, x, t)) + \langle b(t, x), D_x u \rangle \\
&\quad + f(t, x, u, \langle \sigma_1(t, x), D_x u \rangle, \dots, \langle \sigma_d(t, x), D_x u \rangle), \\
H_{ij}(D_x^2 u, D_x u, u, x, t) &= \langle D_x^2 u \sigma_i(t, x), \sigma_j(t, x) \rangle + 2 \langle D_x u, h_{ij}(t, x) \rangle \\
&\quad + 2g_{ij}(t, x, u, \langle \sigma_1(t, x), D_x u \rangle, \dots, \langle \sigma_d(t, x), D_x u \rangle).
\end{aligned}$$

Proof. The uniqueness of viscosity solution of Eq. (4.3) can be found in Appendix C in Peng [20], we only prove that u is a viscosity solution of Eq. (4.3). By $Y_{t+\delta}^{t,x} = Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t,x}}$ and Theorem 4.4, we get $Y_{t+\delta}^{t,x} = u(t+\delta, X_{t+\delta}^{t,x})$ for $\delta \in [0, T-t]$ and

$$\begin{aligned}
Y_t^{t,x} &= u(t+\delta, X_{t+\delta}^{t,x}) + \int_t^{t+\delta} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr \\
&\quad + \int_t^{t+\delta} g_{ij}(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) d\langle B^i, B^j \rangle_r - \int_t^{t+\delta} Z_r^{t,x} dB_r - (K_{t+\delta}^{t,x} - K_t^{t,x}).
\end{aligned}$$

Taking G -expectation, we get

$$u(t, x) = \hat{\mathbb{E}} \left[u(t+\delta, X_{t+\delta}^{t,x}) + \int_t^{t+\delta} f_r dr + \int_t^{t+\delta} g_r^{ij} d\langle B^i, B^j \rangle_r \right],$$

where $f_r = f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})$, $g_r^{ij} = g_{ij}(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})$. In order to prove that u is a viscosity solution, we first show that u is a continuous function. By Proposition 4.2, we know

that $|u(t, x) - u(t, x')| \leq C(1 + |x|^m + |x'|^m)|x - x'|$. By Propositions 4.1 and 2.14, we have $\hat{\mathbb{E}}_t[|X_{t+\delta}^{t,x} - x|^2] \leq C(1 + |x|^2)\delta$ and $\hat{\mathbb{E}}_t\left[|Y_r^{t,x}|^2 + \int_t^T |Z_r^{t,x}|^2 dr\right] \leq C(1 + |x|^{2m+2})$, where C depends on L, G, n and T . Thus we get

$$\begin{aligned} |u(t, x) - u(t + \delta, x)| &\leq C \left\{ (1 + |x|^m)(\hat{\mathbb{E}}[|X_{t+\delta}^{t,x} - x|^2])^{1/2} \right. \\ &\quad \left. + \left(\hat{\mathbb{E}} \left[\int_t^T (|f_r|^2 + |g_r^{ij}|^2) dr \right] \right)^{1/2} \delta^{1/2} \right\} \\ &\leq C(1 + |x|^{m+1})\delta^{1/2}. \end{aligned}$$

It follows that u is a continuous function. For any fixed $(t, x) \in (0, T) \times \mathbb{R}^n$, let $\psi \in C^{2,3}([0, T] \times \mathbb{R}^n)$ be such that $\psi \geq u$, $\psi(t, x) = u(t, x)$ and $|\partial_{t x_i}^2 \psi(t, x)| + |\partial_{x_i} \psi(t, x)| + |\partial_{x_i x_j}^2 \psi(t, x)| + |\partial_{x_i x_j x_k}^3 \psi(t, x)| \leq C(1 + |x|^{m_1})$ for some $m_1 > 0$. Let $(\tilde{Y}, \tilde{Z}, \tilde{K})$ be the solution of G -BSDE (4.2) on $[t, t + \delta]$ with terminal condition $\psi(t + \delta, X_{t+\delta}^{t,x})$. Set $\hat{Y}_s^1 = \tilde{Y}_s - \psi(s, X_s^{t,x})$, $\hat{Z}_s^1 = \tilde{Z}_s - (\langle \sigma_1(s, X_s^{t,x}), D_x \psi(s, X_s^{t,x}) \rangle, \dots, \langle \sigma_d(s, X_s^{t,x}), D_x \psi(s, X_s^{t,x}) \rangle)$, $\hat{K}_s^1 = \tilde{K}_s$, applying Itô's formula to $\tilde{Y}_s - \psi(s, X_s^{t,x})$, we obtain that $(\hat{Y}^1, \hat{Z}^1, \hat{K}^1)$ is the solution of the following G -BSDE:

$$\begin{aligned} \hat{Y}_s^1 &= \int_s^{t+\delta} F_1(r, X_r^{t,x}, \hat{Y}_r^1, \hat{Z}_r^1) dr + \int_s^{t+\delta} F_2^{ij}(r, X_r^{t,x}, \hat{Y}_r^1, \hat{Z}_r^1) d\langle B^i, B^j \rangle_r \\ &\quad - \int_s^{t+\delta} \hat{Z}_r^1 dB_r - (\hat{K}_{t+\delta}^1 - \hat{K}_s^1), \end{aligned}$$

where

$$\begin{aligned} F_1(r, x, y, z) &= f(r, x, y + \psi(r, x), z + (\langle \sigma_1, D_x \psi \rangle, \dots, \langle \sigma_d, D_x \psi \rangle)(r, x)) \\ &\quad + \partial_t \psi(r, x) + \langle b(r, x), D_x \psi(r, x) \rangle, \\ F_2^{ij}(r, x, y, z) &= g_{ij}(r, x, y + \psi(r, x), z + (\langle \sigma_1, D_x \psi \rangle, \dots, \langle \sigma_d, D_x \psi \rangle)(r, x)) \\ &\quad + \langle D_x \psi(r, x), h_{ij}(r, x) \rangle + \frac{1}{2} \langle D_x^2 \psi(r, x) \sigma_i(r, x), \sigma_j(r, x) \rangle. \end{aligned}$$

Let $(\hat{Y}, \hat{Z}, \hat{K})$ be the solution of the following G -BSDE:

$$\begin{aligned} \hat{Y}_s &= \int_s^{t+\delta} F_1(r, x, \hat{Y}_r, \hat{Z}_r) dr + \int_s^{t+\delta} F_2^{ij}(r, x, \hat{Y}_r, \hat{Z}_r) d\langle B^i, B^j \rangle_r \\ &\quad - \int_s^{t+\delta} \hat{Z}_r dB_r - (\hat{K}_{t+\delta} - \hat{K}_s). \end{aligned}$$

It is easy to check that $\hat{Z}_s = 0$, \hat{Y}_s is the solution of the following ODE:

$$\begin{aligned} \hat{Y}_s &= \int_s^{t+\delta} [F_1(r, x, \hat{Y}_r, 0) + 2G(F_2(r, x, \hat{Y}_r, 0))] dr, \\ \hat{K}_s &= \int_t^s F_2^{ij}(r, x, \hat{Y}_r, 0) d\langle B^i, B^j \rangle_r - \int_t^s 2G(F_2(r, x, \hat{Y}_r, 0)) dr, \end{aligned}$$

where $F_2(r, x, \hat{Y}_r, 0) = (F_2^{ij}(r, x, \hat{Y}_r, 0))_{i,j=1}^d$. By Proposition 2.16, we have for any fixed $p > 2$

$$\begin{aligned} |\hat{Y}_t^1 - \hat{Y}_t|^2 &\leq \hat{\mathbb{E}} \left[\sup_{s \in [t, t+\delta]} |\hat{Y}_s^1 - \hat{Y}_s|^2 \right] \\ &\leq C \left\{ \left(\hat{\mathbb{E}} \left[\sup_{s \in [t, t+\delta]} \hat{\mathbb{E}}_s \left[\left(\int_t^{t+\delta} \hat{F}_r dr \right)^p \right] \right] \right)^{2/p} \right. \\ &\quad \left. + \hat{\mathbb{E}} \left[\sup_{s \in [t, t+\delta]} \hat{\mathbb{E}}_s \left[\left(\int_t^{t+\delta} \hat{F}_r dr \right)^p \right] \right] \right\}, \end{aligned}$$

where $\hat{F}_r = |F_1(r, X_r^{t,x}, \hat{Y}_r, 0) - F_1(r, x, \hat{Y}_r, 0)| + \sum_{i,j=1}^d |F_2^{ij}(r, X_r^{t,x}, \hat{Y}_r, 0) - F_2^{ij}(r, x, \hat{Y}_r, 0)|$. It is easy to verify that there exists a constant $m_2 > 0$ such that

$$\hat{F}_r \leq C(1 + |x|^{m_2} + |X_r^{t,x}|^{m_2})|X_r^{t,x} - x|.$$

Then by Theorem 2.13 in [4] and Proposition 4.1 we can deduce that $|\hat{Y}_t^1 - \hat{Y}_t| \leq C(1 + |x|^{m_2+2})\delta^{\frac{3}{2}}$. By the comparison theorem of G -BSDEs, we know that $\tilde{Y}_t \geq u(t, x)$, that is $\hat{Y}_t^1 \geq 0$. Then we get

$$-C(1 + |x|^{m_2+2})\delta^{1/2} \leq \delta^{-1}\hat{Y}_t = \delta^{-1} \int_t^{t+\delta} [F_1(r, x, \hat{Y}_r, 0) + 2G(F_2(r, x, \hat{Y}_r, 0))]dr.$$

Letting $\delta \rightarrow 0$, we obtain $F_1(t, x, 0, 0) + 2G(F_2(t, x, 0, 0)) \geq 0$, which implies that u is a viscosity subsolution. Similarly we can prove that u is a viscosity supersolution. \square

5. Girsanov transformation

5.1. Nonlinear expectations generated by G -BSDEs

For simplicity, we consider the following G -BSDE driven by 1-dimensional G -Brownian motion. The results still hold for the case $d > 1$.

$$\begin{aligned} Y_t^{T,\xi} &= \xi + \int_t^T f(s, Y_s^{T,\xi}, Z_s^{T,\xi})ds + \int_t^T g(s, Y_s^{T,\xi}, Z_s^{T,\xi})d\langle B \rangle_s \\ &\quad - \int_t^T Z_s^{T,\xi}dB_s - (K_T^{T,\xi} - K_t^{T,\xi}), \end{aligned} \quad (5.1)$$

where f and g satisfy the Lipschitz condition. We further suppose that $f(s, y, 0) = g(s, y, 0) = 0$. We define, for each $\xi \in L_G^\beta(\Omega_T)$ with $\beta > 1$,

$$\tilde{\mathbb{E}}_{t,T}[\xi] := Y_t^{T,\xi}.$$

It is easy to verify that for each $T_1 < T_2$ and $\xi \in L_G^\beta(\Omega_{T_1})$ with $\beta > 1$, $\tilde{\mathbb{E}}_{t,T_1}[\xi] = \tilde{\mathbb{E}}_{t,T_2}[\xi]$. Thus we use the notation $\tilde{\mathbb{E}}_t[\xi]$.

Theorem 5.1. *We have*

- (1) For each $\xi^1 \geq \xi^2$, we have $\tilde{\mathbb{E}}_t[\xi^1] \geq \tilde{\mathbb{E}}_t[\xi^2]$;
- (2) For each $\xi \in L_G^\beta(\Omega_t)$ with $\beta > 1$, $\tilde{\mathbb{E}}_t[\xi] = \xi$;

- (3) $\tilde{\mathbb{E}}_t[\tilde{\mathbb{E}}_s[\xi]] = \tilde{\mathbb{E}}_{t \wedge s}[\xi]$;
 (4) If f and g are positively homogeneous, then for each $\lambda_t \in L_G^\infty(\Omega_t)$, we have $\tilde{\mathbb{E}}_t[\lambda_t \xi] = \lambda_t \tilde{\mathbb{E}}_t[\xi]$;
 (5) If f and g are subadditive, then $\tilde{\mathbb{E}}_t[\xi^1 + \xi^2] \leq \tilde{\mathbb{E}}_t[\xi^1] + \tilde{\mathbb{E}}_t[\xi^2]$;
 (6) If f and g are convex, then $\tilde{\mathbb{E}}_t[\lambda_t \xi^1 + (1 - \lambda_t) \xi^2] \leq \lambda_t \tilde{\mathbb{E}}_t[\xi^1] + (1 - \lambda_t) \tilde{\mathbb{E}}_t[\xi^2]$ for each $\lambda_t \in L_G^\infty(\Omega_t)$ and $\lambda_t \in [0, 1]$;
 (7) For each $\xi \in L_G^1(\Omega_t; \mathbb{R}^m)$, $\eta \in L_G^1(\Omega_T; \mathbb{R}^n)$, $\Phi \in C_{b.Lip}(\mathbb{R}^{m+n})$, we have
- $$\tilde{\mathbb{E}}_t[\Phi(\xi, \eta)] = \tilde{\mathbb{E}}_t[\Phi(x, \eta)]_{x=\xi}.$$

- (8) Let K be a decreasing G -martingale with $K_T \in L_G^\alpha(\Omega_T)$ for some $\alpha > 1$. Then we have

$$\tilde{\mathbb{E}}_s[K_t] = K_s, \quad \text{for any } s \leq t.$$

Proof. It is easy to get (1)–(3). (8) is straightforward from Proposition 3.5. First we prove (6). (4) and (5) can be proved similarly. Let (Y^i, Z^i, K^i) , $i = 1, 2$, be the solutions of G -BSDE (5.1) corresponding to ξ^i . We have for $r \in [t, T]$

$$\tilde{Y}_r = \tilde{\xi} + \int_r^T \tilde{f}_s ds + \int_r^T \tilde{g}_s d\langle B \rangle_s - \tilde{K}_T^2 + \tilde{K}_r^2 - \int_r^T \tilde{Z}_s dB_s - (\tilde{K}_T^1 - \tilde{K}_r^1),$$

where $\tilde{Y}_r = \lambda_t Y_r^1 + (1 - \lambda_t) Y_r^2$, $\tilde{\xi} = \lambda_t \xi^1 + (1 - \lambda_t) \xi^2$, $\tilde{f}_s = \lambda_t f(s, Y_s^1, Z_s^1) + (1 - \lambda_t) f(s, Y_s^2, Z_s^2)$, $\tilde{g}_s = \lambda_t g(s, Y_s^1, Z_s^1) + (1 - \lambda_t) g(s, Y_s^2, Z_s^2)$, $\tilde{Z}_s = \lambda_t Z_s^1 + (1 - \lambda_t) Z_s^2$, $\tilde{K}_r^1 = \lambda_t K_r^1$, $\tilde{K}_r^2 = (1 - \lambda_t) K_r^2$. By the convexity of f and g , we get $\tilde{f}_s \geq f(s, \tilde{Y}_s, \tilde{Z}_s)$ and $\tilde{g}_s \geq g(s, \tilde{Y}_s, \tilde{Z}_s)$. Note that $-\tilde{K}_r$ is an increasing process, then by Theorem 3.7 we obtain $\mathbb{E}_t[\xi] \leq \tilde{Y}_t$, which implies (6).

We now prove (7). For each given $n \in \mathbb{N}$, we can choose $A_i^n \in \mathcal{B}(\mathbb{R}^m)$, $i = 1, \dots, k_n$, such that $A_i^n \cap A_j^n = \emptyset$ for $i \neq j$, $\cup_{i=1}^{k_n} A_i^n = \mathbb{R}^m$, $\{x : |x| \leq n\} \subset \cup_{i=1}^{k_n-1} A_i^n$ and $\lambda(A_i^n) \leq 1/n$ for $i \leq k_n - 1$, where $\lambda(A_i^n)$ denotes the diameter of A_i . Let $x_i^n \in A_i^n$, by Proposition 2.16, we have

$$\begin{aligned} \left| \sum_{i=1}^{k_n} \tilde{\mathbb{E}}_t[\Phi(x_i^n, \eta)] I_{A_i^n}(\xi) - \tilde{\mathbb{E}}_t[\Phi(\xi, \eta)] \right|^2 &= \sum_{i=1}^{k_n} I_{A_i^n}(\xi) |\tilde{\mathbb{E}}_t[\Phi(x_i^n, \eta)] - \tilde{\mathbb{E}}_t[\Phi(\xi, \eta)]|^2 \\ &\leq C \sum_{i=1}^{k_n} I_{A_i^n}(\xi) |\hat{\mathbb{E}}_t[\Phi(x_i^n, \eta) - \Phi(\xi, \eta)]|^2 \\ &= C \hat{\mathbb{E}}_t \left[\sum_{i=1}^{k_n} I_{A_i^n}(\xi) |\Phi(x_i^n, \eta) - \Phi(\xi, \eta)|^2 \right], \end{aligned}$$

where C is a constant independent of n . Note that

$$\sum_{i=1}^{k_n} I_{A_i^n}(\xi) |\Phi(x_i^n, \eta) - \Phi(\xi, \eta)|^2 \leq \frac{L^2}{n^2} + 4\|\Phi\|_\infty^2 I_{\{|\xi| > n\}},$$

where L is the Lipschitz constant of Φ , then we get

$$\begin{aligned} \hat{\mathbb{E}} \left[\left| \sum_{i=1}^{k_n} \tilde{\mathbb{E}}_t[\Phi(x_i^n, \eta)] I_{A_i^n}(\xi) - \tilde{\mathbb{E}}_t[\Phi(\xi, \eta)] \right|^2 \right] &\leq C \hat{\mathbb{E}} \left[\frac{L^2}{n^2} + 4\|\Phi\|_\infty^2 I_{\{|\xi| > n\}} \right] \\ &\leq C \left\{ \frac{L^2}{n^2} + \frac{4\|\Phi\|_\infty^2}{n} \hat{\mathbb{E}}[|\xi|] \right\} \rightarrow 0. \end{aligned}$$

On the other hand, by Proposition 2.16, we know that there exists a constant $C > 0$ such that

$$|\tilde{\mathbb{E}}_t[\Phi(x, \eta)] - \tilde{\mathbb{E}}_t[\Phi(y, \eta)]| \leq C|x - y| \quad \text{for } x, y \in \mathbb{R}^m.$$

Thus

$$\begin{aligned} & \hat{\mathbb{E}} \left[\left| \sum_{i=1}^{k_n} \tilde{\mathbb{E}}_t[\Phi(x_i^n, \eta)] I_{A_i^n}(\xi) - \tilde{\mathbb{E}}_t[\Phi(x, \eta)]_{x=\xi} \right|^2 \right] \\ &= \hat{\mathbb{E}} \left[\sum_{i=1}^{k_n} I_{A_i^n}(\xi) |\tilde{\mathbb{E}}_t[\Phi(x_i^n, \eta)] - \tilde{\mathbb{E}}_t[\Phi(x, \eta)]_{x=\xi}|^2 \right] \\ &\leq \hat{\mathbb{E}} \left[\frac{C^2}{n^2} + 4\|\Phi\|_\infty^2 I_{\{|\xi| > n\}} \right] \\ &\leq \frac{C^2}{n^2} + \frac{4\|\Phi\|_\infty^2}{n} \hat{\mathbb{E}}[|\xi|] \rightarrow 0, \end{aligned}$$

which implies $\tilde{\mathbb{E}}_t[\Phi(\xi, \eta)] = \tilde{\mathbb{E}}_t[\Phi(x, \eta)]_{x=\xi}$. \square

5.2. Girsanov transformation

We first consider the following G -BSDE driven by 1-dimensional G -Brownian motion:

$$Y_t = \xi + \int_t^T b_s Z_s ds + \int_t^T d_s Z_s d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t),$$

where $(b_t)_{t \leq T}$ and $(d_t)_{t \leq T}$ are bounded processes. For each $\xi \in L_G^\beta(\Omega_T)$ with $\beta > 1$, define

$$\tilde{\mathbb{E}}_t[\xi] = Y_t.$$

By Theorem 5.1, we know that $\tilde{\mathbb{E}}_t[\cdot]$ is a consistent sublinear expectation.

Theorem 5.2 (Girsanov Theorem). Let $(b_t)_{t \leq T}$ and $(d_t)_{t \leq T}$ be bounded processes. Then $\bar{B}_t := B_t - \int_0^t b_s ds - \int_0^t d_s d\langle B \rangle_s$ is a G -Brownian motion under $\tilde{\mathbb{E}}$.

Proof. We only need to show that for each $\Phi \in C_{b.Lip}(\mathbb{R}^n)$, $t_1 < \dots < t_n$,

$$\tilde{\mathbb{E}}[\Phi(\bar{B}_{t_1}, \bar{B}_{t_2} - \bar{B}_{t_1}, \dots, \bar{B}_{t_n} - \bar{B}_{t_{n-1}})] = \hat{\mathbb{E}}[\Phi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})].$$

Step 1. We consider the case $b_s \equiv b$ and $d_s \equiv d$. For each $\varphi \in C_{b.Lip}(\mathbb{R})$, we define

$$\tilde{u}(t, x) = \tilde{\mathbb{E}}[\varphi(x + \bar{B}_t)].$$

Set $u(t, x) = \tilde{u}(T - t, x)$ for fixed $T > 0$, by Theorem 4.5, we obtain u satisfies the following PDE:

$$\partial_t u - b \partial_x u + b \partial_x u + 2G \left(-d \partial_x u + \frac{1}{2} \partial_{xx}^2 u + d \partial_x u \right) = 0, \quad u(T, x) = \varphi(x),$$

i.e. $\partial_t u + G(\partial_{xx}^2 u) = 0$, $u(T, x) = \varphi(x)$. Thus $\tilde{\mathbb{E}}[\varphi(\bar{B}_t)] = \hat{\mathbb{E}}[\varphi(B_t)]$ for any $t \geq 0$, $\varphi \in C_{b.Lip}(\mathbb{R})$.

Step 2. We consider the case $b_s^n = \sum_{i=0}^{n-1} \xi_i I_{[t_i^n, t_{i+1}^n)}(s)$, $d_s^n = \sum_{i=0}^{n-1} \eta_i I_{[t_i^n, t_{i+1}^n)}(s)$, where $\xi_i, \eta_i \in Lip(\Omega_{t_i^n}^n)$. For each $\varphi \in C_{b,Lip}(\mathbb{R})$, we have

$$\tilde{\mathbb{E}}[\varphi(\bar{B}_{t_{i+1}^n})] = \tilde{\mathbb{E}}[\varphi(\bar{B}_{t_i^n} + B_{t_{i+1}^n} - B_{t_i^n} - \xi_i(t_{i+1}^n - t_i^n) - \eta_i(\langle B \rangle_{t_{i+1}^n} - \langle B \rangle_{t_i^n}))].$$

By (7) in [Theorem 5.1](#), we get

$$\begin{aligned} \tilde{\mathbb{E}}[\varphi(\bar{B}_{t_{i+1}^n})] &= \tilde{\mathbb{E}}[\varphi(x + B_{t_{i+1}^n} - B_{t_i^n} - b(t_{i+1}^n - t_i^n) - d(\langle B \rangle_{t_{i+1}^n} - \langle B \rangle_{t_i^n}))]_{x=\bar{B}_{t_i^n}, b=\xi_i, d=\eta_i} \\ &= \tilde{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x + B_{t_{i+1}^n} - B_{t_i^n})]_{x=\bar{B}_{t_i^n}}]. \end{aligned}$$

Repeat this process, we obtain $\tilde{\mathbb{E}}[\varphi(\bar{B}_{t_{i+1}^n})] = \hat{\mathbb{E}}[\varphi(B_{t_{i+1}^n})]$. Similarly, we can get

$$\tilde{\mathbb{E}}[\Phi(\bar{B}_{t_1}, \bar{B}_{t_2} - \bar{B}_{t_1}, \dots, \bar{B}_{t_n} - \bar{B}_{t_{n-1}})] = \hat{\mathbb{E}}[\Phi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})].$$

Step 3. For general bounded processes (b_t) and (d_t) , we can choose uniformly bounded processes $(b_t^n), (d_t^n) \in M_{G^{2,0}}(0, T)$ such that $\|b^n - b\|_{M_G^2} + \|d^n - d\|_{M_G^2} \rightarrow 0$. By [Proposition 2.16](#), we obtain the result by letting $n \rightarrow \infty$. \square

Remark 5.3. If $b_s = 0$, we know by [Remark 3.3](#)

$$\tilde{\mathbb{E}}_t[\xi] = \hat{\mathbb{E}}_t \left[\xi \exp \left(\int_t^T d_s dB_s - \frac{1}{2} \int_t^T |d_s|^2 d\langle B \rangle_s \right) \right].$$

This type of Girsanov transformation was studied in [\[30,7\]](#), but here we give a simple proof. If $b_s \neq 0$, we know by [Theorem 3.2](#)

$$\begin{aligned} \tilde{\mathbb{E}}_t[\xi] &= \hat{\mathbb{E}}_t^{\tilde{G}} \left[\xi \exp \left(\int_t^T d_s dB_s - \frac{1}{2} \int_t^T |d_s|^2 d\langle B \rangle_s - \int_t^T b_s ds \right. \right. \\ &\quad \left. \left. + \int_t^T b_s d\tilde{B}_s - \frac{1}{2} \int_t^T |b_s|^2 d\langle \tilde{B} \rangle_s \right) \right], \end{aligned}$$

where (B, \tilde{B}) is an auxiliary extended \tilde{G} -Brownian motion and

$$\tilde{G}(A) = \frac{1}{2} \sup_{\sigma^2 \leq v \leq \bar{\sigma}^2} \text{tr} \left[A \begin{bmatrix} v & 1 \\ 1 & v^{-1} \end{bmatrix} \right], \quad A \in \mathbb{S}_2.$$

We now consider the Girsanov transformation for the case $d > 1$. Let $B_t = (B_t^i)_{i=1}^d$ be a d -dimensional G -Brownian motion. We consider the following G -BSDE:

$$Y_t = \xi + \int_t^T b_s Z_s ds + \int_t^T d_s^{ij} Z_s d\langle B^i, B^j \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t),$$

where $(b_t)_{t \leq T}$ and $(d_t^{ij})_{t \leq T}$ are \mathbb{R}^d -valued bounded processes. By [Theorem 5.1](#), $\tilde{\mathbb{E}}_t[\xi] := Y_t$ is a consistent sublinear expectation.

Theorem 5.4 (Girsanov Theorem). Let $(b_t)_{t \leq T}$ and $(d_t^{ij})_{t \leq T}$ be \mathbb{R}^d -valued bounded processes. Then $\tilde{B}_t := B_t - \int_0^t b_s ds - \int_0^t d_s^{ij} d\langle B^i, B^j \rangle_s$ is a d -dimensional G -Brownian motion under $\tilde{\mathbb{E}}$.

Proof. The proof is similar to [Theorem 5.2](#). \square

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