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Global fluctuations for 1D log-gas dynamics

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Abstract

We study in this article the hydrodynamic limit in the macroscopic regime of the coupled system of stochastic differential equations,

$$d\lambda_t^i = \frac{1}{\sqrt{N}} dW_t^i - V'(\lambda_t^i)dt + \frac{\beta}{2N} \sum_{j \neq i} \frac{dt}{\lambda_t^i - \lambda_t^j}, \qquad i = 1, \dots, N, \tag{0.1}$$

with $\beta > 1$, sometimes called *generalized Dyson's Brownian motion*, describing the dissipative dynamics of a log-gas of N equal charges with equilibrium measure corresponding to a β-ensemble, with sufficiently regular convex potential V. The limit $N \to \infty$ is known to satisfy a mean-field Mc-Kean–Vlasov equation. We prove that, for suitable initial conditions, fluctuations around the limit are Gaussian and satisfy an explicit PDE.

The proof is very much indebted to the harmonic potential case treated in Israelsson (2001). Our key argument consists in showing that the time-evolution generator may be written in the form of a transport operator on the upper half-plane, plus a bounded non-local operator interpreted in terms of a signed jump process.

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1. Introduction and statement of main results

1.1. Introduction

Let $\beta \geq 1$ be a fixed parameter, and $N \geq 1$ an integer. We consider the following system of coupled stochastic differential equations driven by N independent standard Brownian motions $(W_t^1, \ldots, W_t^N)_{t\geq 0}$,

$$d\lambda_t^i = \frac{1}{\sqrt{N}} dW_t^i - V'(\lambda_t^i)dt + \frac{\beta}{2N} \sum_{j\neq i} \frac{dt}{\lambda_t^i - \lambda_t^j}, \qquad i = 1, \ldots, N. \tag{1.1}$$

Letting

$$\mathcal{W}(\{\lambda^i\}_i) := \sum_{i=1}^N V(\lambda^i) - \frac{\beta}{4N} \sum_{i\neq j} \log(\lambda^i - \lambda^j), \tag{1.2}$$

we can rewrite (1.1) as $d\lambda_t^i = \frac{1}{\sqrt{N}} dW_t^i - \nabla_i \mathcal{W}(\lambda_t^1, \ldots, \lambda_t^N)dt$. Thus the corresponding equilibrium measure,

$$d\mu_{eq}^N(\{\lambda^i\}_i) = \frac{1}{Z_V^N} e^{-2N\mathcal{W}(\{\lambda^i\}_i)} = \frac{1}{Z_V^N} \left(\prod_{j\neq i} |\lambda^j - \lambda^i| \right)^{\beta/2} \times \exp\left(-2N \sum_{i=1}^N V(\lambda^i) \right) d\lambda^1 \cdots d\lambda^N \tag{1.3}$$

is that of a β-log gas with confining potential V.

Let us start with a historical overview of the subject as a motivation for our study. This system of equations was originally considered in a particular case by Dyson [6] who wanted to describe the Markov evolution of a Hermitian matrix M_t with i.i.d. increments dG_t taken from the Gaussian unitary ensemble (GUE). In Dyson's idea, this matrix-valued process was to be a matrix analogue of Brownian motion. The latter time-evolution being invariant through conjugation by unitary matrices, we may project it onto a time-evolution of the set of eigenvalues $\{\lambda_t^1, \ldots, \lambda_t^N\}$ of the matrix, and obtain (1.1) with $\beta = 2$ and $V \equiv 0$. Keeping $\beta = 2$, it is easy to prove that (1.1) is equivalent to a generalized matrix Markov evolution, $dM_t = dG_t - V'(M_t)dt$. The Gibbs measure

$$\mathcal{P}_V^N(M) = \frac{1}{Z_N} e^{-N\mathrm{Tr}V(M)} dM, \qquad dM = \prod_{i=1}^N dM_{ii} \prod_{1\leq i<j\leq n} d\mathrm{Re}\, M_{ij}\, d\mathrm{Im}\, M_{ij}$$

can then be proved to be an equilibrium measure. Such measures, together with their projection onto the eigenvalue set, $\mu_{eq}^N(\{\lambda^1, \ldots, \lambda^N\})$, are the main object of random matrix theory, see e.g. [1,13,18]. The *equilibrium eigenvalue distribution* can be studied by various means, in particular using orthogonal polynomials with respect to the weight $e^{-NV(\lambda)}$. The scaling in N (called *macroscopic scaling* in random matrix theory) ensures the convergence of the random point measure $X^N := \frac{1}{N}\sum_{i=1}^N \delta_{\lambda^i}$ to a deterministic measure μ_V with *compact* support and density ρ when $N \to \infty$ (see e.g. [10], Theorem 2.1). One finds e.g. the well-known semi-circle law, $\rho(x) = \frac{1}{\pi}\sqrt{2 - x^2}$, when $V(x) = x^2/2$. Looking more closely at the limit of the point measure, one finds for arbitrary *polynomial* V (Johansson [10]) Gaussian fluctuations of order $O(1/N)$, contrasting with the $O(1/\sqrt{N})$ scaling of fluctuations for the means of N independent random variables, typical of the central limit theorem. Assuming that the support of the measure is connected (this essential "one-cut" condition holding in particular for V *convex*), Johansson proves that the *covariance* of the limiting law depends on V only through the support of the measure – it is thus *universal* up to a scaling coefficient –, while the means is equal to ρ, plus an apparently non-universal correction in $O(1/N)$.

Then Rogers and Shi [21], disregarding the random matrix background, studied directly for its sake the system (1.1) in the case where V is harmonic (i.e. quadratic) and $\beta = 2$, which we call *Hermite case* henceforth (by reference to the corresponding class of equilibrium orthogonal polynomials), proving in particular the following two facts:

(i) two arbitrary eigenvalues never collide, which implies the non-explosion of (1.1);
(ii) the random point process $X_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_t^i}$ satisfies in the limit $N \to \infty$ a deterministic hydrodynamic equation of Mc-Kean–Vlasov type, namely, the asymptotic density

$$\rho_t \equiv X_t := \text{w-lim}_{N\to\infty} X_t^N \tag{1.4}$$

satisfies the PDE

$$\frac{\partial \rho_t(x)}{\partial t} = \frac{\partial}{\partial x}\left(\left(x - p.v.\int \frac{dy}{x-y}\rho_t(y)\right)\rho_t(x)\right), \tag{1.5}$$

where $p.v. \int \frac{dy}{x-y}\rho_t(y)$ is a principal value integral.

In the case studied by Rogers and Shi, explicit formulas for finite N are known for the Markov generator in the form of a determinant, called *extended kernel*, see e.g. [7], chapter XI, or [15]), whose asymptotics for $N \to \infty$ may in principle be used to study the macroscopic limit. This is accomplished by noting that a simple conjugation trick turns the generator of the process into an N-particle Hamiltonian with a *one-body* potential only, whose eigenfunctions are deduced from those of one-particle Hamiltonians (actually, harmonic oscillators). *For general V, however*, in marked contrast with respect to the equilibrium case, no explicit formulas are known for finite N, even for $\beta = 2$, since the conjugation trick produces a supplementary two-body potential making the spectral problem unsolvable. To be more precise, Macedo and Macedo [12] classified all random matrix dynamics which are unitary equivalent to imaginary-time evolution under a Calogero–Sutherland type Hamiltonian, providing explicit determinantal solutions in connection to classical orthogonal polynomials when $\beta = 2$; however, restricting to SDE's with *additive* noise, the latter class contains only the Hermite case. Related models of diffusions conditioned on non-intersecting, solvable in terms of classical orthogonal polynomials, have been considered in Duits [5], who showed convergence of fluctuation field to inhomogeneous Gaussian free field. Then for $\beta \neq 2$, the finite N equilibrium measure is not fully understood, even in the harmonic case, see [25].

This makes the direct study of (1.1) for general V and β all the more interesting. Whereas the PDE appearing in the hydrodynamical limit is known [11], the law of fluctuations is not known in general, and it is the purpose of this study, and of the forthcoming article [24], to fill this gap. S. Li, X.-D. Li and Y.-X. Xie [11], generalizing properties (i) and (ii) above, prove that the random point process X_t^N satisfies in the limit $N \to \infty$ a generalization of the above Mc-Kean–Vlasov equation, namely,

$$\frac{\partial \rho_t(x)}{\partial t} = \frac{\partial}{\partial x}\left(\left(V'(x) - \frac{\beta}{2}\, p.v.\int \frac{dy}{x-y}\,\rho_t(y)\right)\rho_t(x)\right). \tag{1.6}$$

The equilibrium measure ρ, defined as the solution of the integral equation $\frac{\beta}{2} p.v. \int \frac{dy}{x-y}\rho(y) = V'(x)$, cancels the right-hand side of (1.6). Replacing in (1.1) $\frac{\beta}{2N}\sum_{j\neq i}\frac{1}{\lambda_t^i-\lambda_t^j}$ with $-\frac{1}{N}\sum_{j\neq i}\nabla U(\lambda_t^i-\lambda_t^j)$ where U is some convex two-body potential satisfying some very general properties of regularity and growth at infinity, one may show that there appears in the same limit an equation similar to (1.6),

$$\frac{\partial\rho_t(x)}{\partial t} = \frac{\partial}{\partial x}\left(\left(V'(x) + \frac{\beta}{2}\int dy\, U'(x-y)\rho_t(y)\right)\rho_t(x)\right). \tag{1.7}$$

Solutions of this type of equations, common in plasma theory and the study of granular media [3,8] and in particular, the rate of convergence of these to equilibrium, have been studied in detail using Otto's infinite dimensional differential calculus [16] in a series of papers, see e.g. [4,17,26]. However, as already noted by S. Li, X.-D. Li and Y.-X. Xie, the range of applicability of these papers, written under the assumption that U be *Lipschitz*, does not seem to extend to our case when $U(x) = -c\log|x|$. Since formally the law of fluctuations is obtained by linearizing the system of Eqs. (1.1) around its macroscopic limit ρ, it is clear that one must find some way to deal with (1.6).

Rogers' and Shi's approach to (1.1) has been successfully generalized to the case of a harmonic potential with arbitrary β by Israelsson [9] and Bender [2]. The present study owes very much to these two articles, so let us describe to some extent their contents. There are two main ideas. Let $Y_t^N := N(X_t^N - X_t)$ be the rescaled fluctuation process for finite N; we want to prove that $Y_t^N \stackrel{law}{\to} Y_t$ when $N\to\infty$ and identify the law of the process $(Y_t)_{t\geq 0}$. First, Itô's formula implies that

$$d\langle Y_t^N, f_t\rangle = \frac{1}{2}(1-\frac{\beta}{2})\langle X_t^N, f_t''\rangle dt + \frac{1}{\sqrt{N}}\sum_{i=1}^N f_t'(\lambda_t^i)dW_t^i \tag{1.8}$$

if the test functions $(f_t)_{0\leq t\leq T}$, $f_t:\mathbb{R}\to\mathbb{R}$ solve the following linear PDE

$$\frac{\partial f_t}{\partial t}(x) = V'(x)f_t'(x) - \frac{\beta}{4}\int\frac{f_t'(x)-f_t'(y)}{x-y}(X_t^N(dy)+X_t(dy)) \tag{1.9}$$

(see Proposition 1.3). Eq. (1.9) is a dualized, linearized version of (1.6). Second, Eq. (1.9) may be integrated in the harmonic case by means of a *Stieltjes transform* (see Definition 1.2). Namely, the family of functions $\{\frac{c}{\cdot-z}\}_{c\in\mathbb{C},z\in\mathbb{C}\setminus\mathbb{R}}$ is preserved by (1.9). The solution $\frac{c_t^N}{\cdot-z_t^N}$ at time t with terminal condition $\frac{c_T^N}{\cdot-z_T^N} = \frac{c}{\cdot-z}$ is obtained by solving two coupled ordinary differential equations for c_t^N and z_t^N depending on X and the random point measure X^N (see [9], Lemma 2). Substituting to X^N its deterministic limit X in the r.-h.s. of (1.9), one gets in a natural way a system of two coupled ordinary differential equations for $(z_t)_{0\leq t\leq T}$, $(c_t)_{0\leq t\leq T}$ that describes a solution of the asymptotic limit of (1.9) in the limit $N\to\infty$, namely,

$$\frac{\partial f_t}{\partial t}(x) = V'(x)f_t'(x) - \frac{\beta}{2}\int\frac{f_t'(x)-f_t'(y)}{x-y}X_t(dy). \tag{1.10}$$

Bender interprets these equations as *characteristic equations* for a generalized transport operator (see Appendix A) which is never stated explicitly. Then (at least formally), Itô's formula (see [9], p. 29) makes it possible to find explicitly the Markov kernel in the limit $N\to\infty$. Namely, consider a finite number of points $(z^k)_k$ in $\mathbb{C}\setminus\mathbb{R}$, and the solutions $(z_t^k)_{t\leq T}$ of the corresponding characteristic equations with terminal condition $(z_T^k)_k$. Letting $f_t(x) := \sum_k \frac{c_t^k}{x-z_t^k}$

be the solution of (1.10), and $\phi_{f_t}(Y_t^N) := e^{\mathrm{i}\langle Y_t^N, f_t\rangle}$,

$$\mathbb{E}[\phi_{f_T}(Y_T)|\mathcal{F}_t] = \mathbb{E}[\phi_{f_t}(Y_t)]\exp\left(\frac{1}{2}\int_t^T \left[\mathrm{i}(1-\frac{\beta}{2})\langle X_s, f_s''\rangle - \langle X_s, (f_s')^2\rangle\right] ds\right). \quad (1.11)$$

Since functions f of the above form are dense in some appropriate Sobolev space, formula (1.11) allows to conclude that the limit process is Gaussian. Then Bender solves explicitly the characteristic equations, which take on a particularly simple form in the harmonic case, and deduces first the covariance of the Stieltjes transform of the fluctuation process, $\mathrm{Cov}(U_{t_1}(z_1), U_{t_2}(z_2))$, $U_t(z) := \langle Y_t, \frac{1}{\cdot - z}\rangle$, and then (taking boundary values and using the Plemelj formula, see Appendix B), the (distribution-valued) covariance kernel $\mathrm{Cov}(Y_{t_1}(x_1), Y_{t_2}(x_2))$.

Our approach for the case of a general potential has exactly the same starting point, but dealing with Eq. (1.9) turns out to be more complicated than in the harmonic case. The reason is that the family of functions $\left\{\frac{c}{\cdot - z}\right\}_{c\in\mathbb{C}, z\in\mathbb{C}\setminus\mathbb{R}}$ is no more preserved by (1.9): this is easily seen if V is a polynomial or extends analytically to a strip around the real axis, since

$$-V'(x)\partial_x\left(\frac{1}{x-z}\right) = (V'(z)\partial_z + V''(z))\left(\frac{1}{x-z}\right) + \left(\frac{V^{(3)}(z)}{2!} + \frac{V^{(4)}(z)}{3!}(x-z) + \dots\right) \quad (1.12)$$

features extra unwanted polynomial terms. In practice we need only assume that V is sufficiently regular, and (letting $z =: a + \mathrm{i}b$) write instead, for a in a neighborhood of the support of the random point measure

$$V'(x) = V'(a) + V''(a)(x-a) + V'''(a)\frac{(x-a)^2}{2} + (x-a)^3 W_a(x-a), \quad (1.13)$$

and find for the first three terms,

$$-V'(a)\partial_x\left(\frac{1}{x-z}\right) = V'(a)\partial_a\left(\frac{1}{x-z}\right), \quad -V''(a)(x-a)\partial_x\left(\frac{1}{x-z}\right) = V''(a)\,(1+b\partial_b)\,\frac{1}{x-z}, \quad (1.14)$$

$$-V'''(a)\frac{(x-a)^2}{2}\partial_x\left(\frac{1}{x-z}\right) = \frac{1}{2}V'''(a) + \frac{1}{2}V'''(a)(2\mathrm{i}b + b^2\partial_a)\frac{1}{x-z}, \quad (1.15)$$

defining a generalized transport operator

$$-V'(a)\partial_a - V''(a)(1+b\partial_b) - \frac{1}{2}V'''(a)(2\mathrm{i}b + b^2\partial_a). \quad (1.16)$$

The new piece is the last (Taylor's remainder) term in (1.13). We must give up at this point the idea that the time-evolution is a simple characteristic evolution, and prove that the Taylor remainder produces instead a *non-local kernel*. Let us highlight the main points while avoiding technicalities. The main tool here is the use of *Stieltjes decompositions* of order κ (see Definition 2.3): for any $b_{max} > 0$ and $\kappa = 0, 1, 2, \dots,$ any sufficient regular, integrable function $f : \mathbb{R} \to \mathbb{R}$ may be written as an integral over the strip $\Pi_{b_{max}} := \{a \pm \mathrm{i}b \mid 0 < |b| < b_{max}\}$

$$f(x) = \int_{-\infty}^{+\infty} da \int_{-b_{max}}^{b_{max}} db\,(-\mathrm{i}b)\frac{|b|^\kappa}{(1+\kappa)!}\,\mathfrak{f}_z(x)\,h(a,b) \quad (1.17)$$

$$\text{where} \qquad \mathfrak{f}_z(x) := \frac{1}{x-z}. \quad (1.18)$$

The mapping $f \mapsto h$ is clearly not one-to-one. Explicit Stieltjes decompositions are produced in [9], Lemma 9; part of the job consists precisely in *choosing* Stieltjes decompositions with good properties. Let $\kappa' \geq \kappa \geq 0$. Inserting the time-evolution operator (1.9) into (1.17), we prove that:

– the $(\frac{1}{x})$-potential and the second-order Taylor expansion of the operator $V'(x)\partial_x$, see (1.14), (1.15), act together as a *transport operator* $\mathcal{H}^\kappa : L^1(\Pi_{b_{max}}) \to L^1(\Pi_{b_{max}})$;

– the *Taylor remainder term* (see Section 3.7), to which one must add an inessential off-support contribution (see Section 3.8) and boundary terms (see Section 3.9), may be realized as a non-local operator $|b|^{\kappa'-\kappa}\mathcal{H}^{\kappa';\kappa}_{nonlocal}(t)$ acting on the coefficient function h,

$$|b|^{\kappa'-\kappa}(\mathcal{H}^{\kappa';\kappa}_{nonlocal}(t))(h)(a,b) := |b|^{\kappa'-\kappa}\int_{-\infty}^{+\infty} da_T \int_{-b_{max}}^{b_{max}} db_T \, g^{\kappa';\kappa}_{nonlocal} \times (a, b; a_T, b_T) h(a_T, b_T) \tag{1.19}$$

such that

$$\mathcal{H}^{\kappa+1;\kappa}_{nonlocal}(t) : L^1(\Pi_{b_{max}}) \to L^1(\Pi_{b_{max}}) \tag{1.20}$$

are *bounded*. From (1.20) we get: $b\mathcal{H}^{\kappa+1;\kappa}_{nonlocal}(h)(a,b) = \tilde{h}(a,b)$ with $\tilde{h} \in L^1(\Pi_{b_{max}}, |b|^{-1}\, da\, db)$. In other words, the non-local part of the time evolution is (in some weak sense) *regularizing* near the real axis, and acts therefore as a bounded perturbation of $\mathcal{H}^\kappa_{transport}$.

1.2. Notations and basic facts

In this paragraph, we simply assume that V is convex. The filtration $(\mathcal{F}_t)_{t\geq 0}$ is the filtration of the Brownian $(W^i_t)_{t\geq 0, i=1,\ldots,N}$.

Definition 1.1.

1. Let

$$X^N_t := \frac{1}{N}\sum_{i=1}^N \delta_{\lambda^i_t} \tag{1.21}$$

be the empirical measure process.

2. Call

$$Y^N_t := N(X^N_t - X_t) \tag{1.22}$$

the finite N *fluctuation process*.

The case developed in [9] and [2] is the *harmonic case*, $V(x) = \frac{1}{2}x^2$ (up to normalization), to which we shall often refer. Apart from this very particular case, classical examples include the Landau–Ginzburg potential $V(x) = \frac{1}{2}x^2 + \frac{\lambda}{4}x^4$, $\lambda \geq 0$, for which the support of the equilibrium measure is connected, and the density is the product of the (rescaled) semi-circle law by some explicit polynomial of degree 2 (see e.g. [10], p. 164).

It is proved in (Li–Li–Xie [11], Theorem 1.3) that, provided $X^N_0 \overset{N\to\infty}{\to} \rho_0$, a deterministic density, the empirical measure process (X^N_t) converges in law to a deterministic measure process $(X_t)_{t\geq 0}$ with density ρ_t solution of the non-linear Fokker–Planck equation,

$$\frac{\partial \rho_t(x)}{\partial t} = \frac{\partial}{\partial x}\left(\left(V'(x) - \frac{\beta}{2}\, p.v. \int \frac{dy}{x-y}\rho_t(y)\right)\rho_t(x)\right) \tag{1.23}$$

with initial condition ρ_0. Equivalently, for any test function f,

$$\frac{d}{dt}\langle X_t, f\rangle = \frac{d}{dt}\int \rho_t(x) f(x)\, dx = -\int V'(x) f'(x) X_t(dx) + \frac{\beta}{4}\iint \frac{f'(x)-f'(y)}{x-y} X_t(dx)\, X_t(dy). \tag{1.24}$$

The equilibrium measure μ^N_{eq}, see (1.3) converges weakly when $N \to \infty$ to the stationary, deterministic solution $X_t(dx) = \rho_{eq}(x)dx$ of (1.23), where ρ_{eq} is the solution of the following integral equation, called *cut equation*, [10]

$$p.v. \int \frac{\rho_{eq}(x)\, dx}{x-y} = -\frac{2}{\beta} V'(y). \tag{1.25}$$

Formula (1.24) is formally obtained as in [21] by taking the limit $N \to \infty$ in the finite N Itô formula (eq. (3) in [9]),

$$d\langle X^N_t, f\rangle = \left(\frac{\beta}{4}\iint \frac{f'(x)-f'(y)}{x-y} X^N_t(dx) X^N_t(dy) - \int V'(x) f'(x) X^N_t(dx)\right) dt + \frac{1}{2}(1-\frac{\beta}{2})\frac{1}{N}\langle X^N_t, f''\rangle\, dt + \frac{1}{N\sqrt{N}}\sum_{i=1}^N f'(\lambda^i_t) dW^i_t. \tag{1.26}$$

Roughly speaking, both terms in the second line of (1.26) are $O(\frac{1}{N})$ (the argument for the martingale term relies on an L^2-bound based on the independence of the $(W^i)_{1\le i\le N}$).

Definition 1.2 (*Stieltjes Transform*)**.** Fix $z \in \mathbb{C} \setminus \mathbb{R}$.

(i) Let $\mathfrak{f}_z(x) := \frac{1}{x-z}$ $(x \in \mathbb{R})$.
(ii) Let, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$M^N_t := \langle X^N_t, \mathfrak{f}_z\rangle = \sum_{i=1}^N \frac{1}{\lambda^i_t - z} \tag{1.27}$$

and

$$M_t := \langle X_t, \mathfrak{f}_z\rangle = \int \frac{\rho_t(x)}{x-z}\, dx \tag{1.28}$$

be the Stieltjes transform of X^N_t, resp. X_t.

Starting from the cut equation (1.25) and applying Plemelj's formula (see Appendix B), one finds *at equilibrium*

$$M(x+\mathrm{i}0) = -\frac{2}{\beta} V'(x) + \mathrm{i}\pi \rho_{eq}(x) \tag{1.29}$$

$$M(x+\mathrm{i}0) - M(x-\mathrm{i}0) = 2\mathrm{i}\pi\rho_{eq}(x), \qquad M(x+\mathrm{i}0) + M(x-\mathrm{i}0) = -\frac{4}{\beta}V'(x). \tag{1.30}$$

A PDE for the Stieltjes transform of X_t is determined easily from (1.24),

$$\frac{\partial M_t}{\partial t} = \frac{\partial}{\partial z}\left(\frac{\beta}{4}(M_t(z))^2 + V'(z) M_t(z) + T_t(z)\right), \tag{1.31}$$

where

$$T_t(z) := \int \frac{V'(x)-V'(z)}{x-z} X_t(dx). \tag{1.32}$$

In the harmonic case, T is simply a constant, hence M is the solution of a complex Burgers equation on $\mathbb{C}\setminus\mathbb{R}$, see e.g. [9], Eq. (6). However, this is no more the case in our general setting, and the non-local term T in the right-hand side prevents any explicit solution of the equation. Yet the Stieltjes transform will turn out to be a very convenient technical tool in the computations.

The first idea, coming from [9], is to *transfer the drift in the time-evolution of* Y_t^N *to the test function* f. This is done through a *straightforward generalization* of (Israelsson [9], Lemma 1):

Proposition 1.3 (*see Israelsson [9]*). *Assume the following event holds for some constant* $R>0$,

$$\Omega_R: \sup_{0\le t\le T}\max_{i=1,\dots,N}|\lambda^i|\le R; \qquad \forall t\le T,\ \mathrm{supp}(X_t)\subset[-R,R], \tag{1.33}$$

i.e. that the support of the random point measure X_t^N *and of the measure* X_t *is* $\subset[-R,R]$ *for* $0\le t\le T$. *Let* $(f_t)_{0\le t\le T}$, $f_t:\mathbb{R}\to\mathbb{R}$ *be such that*

$$\frac{\partial f_t}{\partial t}(x)=V'(x)f_t'(x)-\frac{\beta}{4}\int\frac{f_t'(x)-f_t'(y)}{x-y}(X_t^N(dy)+X_t(dy)) \tag{1.34}$$

for all $|x|\le R$. *Then*

$$d\langle Y_t^N,f_t\rangle=\frac12(1-\frac{\beta}{2})\langle X_t^N,f_t''\rangle dt+\frac{1}{\sqrt N}\sum_{i=1}^N f_t'(\lambda_t^i)dW_t^i. \tag{1.35}$$

As emphasized in the above Proposition, (1.34) need only hold on $[-R,R]$, because $\langle X_t^N,f\rangle$ and $\langle X_t,f\rangle$ do *not* depend on the values of f on $\mathbb{C}\setminus[-R,R]$.

The above Proposition is a direct consequence of Itô's formula applied to the fluctuation process (just subtract (1.26) from (1.24)),

$$d\langle Y_t^N,f\rangle=\frac{\beta}{4}\iint\frac{f'(x)-f'(y)}{x-y}\left[X_t^N(dx)+X_t(dx)\right]Y_t^N(dy)-\int V'(x)f'(x)Y_t^N(dx)$$
$$+\frac12(1-\frac{\beta}{2})\langle X_t^N,f''\rangle+\frac{1}{\sqrt N}\sum_{i=1}^N f'(\lambda_t^i)dW_t^i. \tag{1.36}$$

Then Israelsson solves Eq. (1.34) in the harmonic case by using as test functions the $\frac{c}{\cdot-z}$, $c\in\mathbb{C}$, $z\in\mathbb{C}\setminus\mathbb{R}$, on which the generator of time-evolution acts in a particularly simple way. We do not reproduce their results here however, since they do not separate the analysis of the term due to the harmonic potential from that due to the two-body logarithmic potential. We shall analyze (1.34) in Section 3 after we have introduced Stieltjes decompositions.

Normalization: the reader willing to compare our results with those of Israelsson [9] or Bender [2] should take into account the different choices of normalization. Compared to [9], we fix $\sigma=1$ and let $\alpha=\frac{\beta}{2}$, $\gamma=\frac12(1-\frac{\beta}{2})$. After rescaling the λ^i's by a factor $\beta^{-1/2}$, we obtain for V quadratic [2] with $\sigma=\frac12$.

1.3. Main result and outline of the article

Assumptions on V.

We assume V *to be a convex function in* $\mathcal{C}^{11}$.

Main examples are convex polynomials, or suitable, smooth perturbations thereof.

Under our assumptions (see e.g. [10], Theorem 2.1 and Proposition 3.1) the equilibrium measure ρ_{eq} is well-defined and compactly supported, its support $[a, b]$ is connected, and ρ_{eq} is a solution of the cut-equation (1.25).

Assumptions on the initial measure.

Let $\mu_0^N = \mu_0(\{\lambda_0^i\}_i)$ be the initial measure of the stochastic process $\{\lambda_t^i\}_{t\geq 0, i=1,\ldots,N}$, and $X_0^N := \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_0^i}$ be the initial empirical measure. Since N varies, we find it useful here to add an extra upper index $(\lambda_0^{N,i})_{i=1,\ldots,N}$ to denote the initial condition of the process for a given value of N. *We assume that:*

(i) *(large deviation estimate for the initial support) there exist some constants $C_0, c_0, R_0 > 0$ such that, for every $N \geq 1$,*

$$\mathbb{P}[\max_{i=1,\ldots,N} |\lambda_0^{N,i}| > R_0] \leq C_0 e^{-c_0 N}. \tag{1.37}$$

(ii) $X_0^N \overset{law}{\to} \rho_0(x)\, dx$ *when $N \to \infty$, where $\rho_0(x)$ is a deterministic measure*;

(iii) *(rate of convergence)*

$$\left(\mathbb{E}[|M_0^N(z) - M_0(z)|^2]\right)^{1/2} = O(\frac{1}{Nb}) \tag{1.38}$$

for $z = a + ib \in \mathbb{C} \setminus \mathbb{R}$, where $M_0(z) := \int dx\, \frac{\rho_0(x)}{x-z}$ is the Stieltjes transform of ρ_0.

Lemma 5.1 proves that the initial large deviation estimate (i) implies a uniform-in-time large deviation estimate for the support of the random point measure, which is essential for our main result.

Definition 1.4 (*Sobolev Spaces*). Let $H_n := \{f \in L^2(\mathbb{R}) \mid \|f\|_{H_n} < \infty\}$ $(n \geq 0)$, where $\|f\|_{H_n} := \left(\int d\xi\, (1 + |\xi|^2)^n |\mathcal{F}f(\xi)|^2\right)^{1/2}$, and $H_{-n} := (H_n)'$ its dual.

The measure-valued process Y^N may be shown to converge in $C([0, T], H_{-14})$:

Main theorem (Gaussianity of limit fluctuation process).

Let Y_t^N be the finite N fluctuation process (see Definition 1.1). Then:

1. $Y^N \overset{law}{\to} Y$ *when $N \to \infty$, where Y is a Gaussian process. More precisely, Y^N converges to Y weakly in $C([0, T], H_{-14})$;*
2. *let $\phi_h(Y_t^N) := e^{i\langle Y_t^N, \mathcal{C}^0 h\rangle}$, with $\mathcal{C}^0$ (Stieltjes decomposition of order 0 with arbitrary cut-off $b_{max} > 0$) as in Definition 2.3. Then*

$$\mathbb{E}[\phi_{f_T}(Y_T)\big|\mathcal{F}_t] = \phi_{h_t}(Y_t)\, \exp\left(\frac{1}{2}\int_t^T \left[i(1 - \frac{\beta}{2})\langle X_s, f_s''\rangle - \langle X_s, (f_s')^2\rangle\right] ds\right) \tag{1.39}$$

where $(f_s)_{0\leq s\leq T}$ is the solution of the asymptotic equation (1.10).

Scheme of proof. As in Israelsson [9], the main task is to prove a uniform in N Sobolev bound, called "H_8-bound", see (4.10),

$$\mathbb{E}[\sup_{0\leq s\leq T} |\langle Y_s^N, \phi\rangle|] \leq C_T \|\phi\|_{H_8} \tag{1.40}$$

implying in particular tightness in some Sobolev space with negative index. Representing ϕ in terms of its standard Stieltjes decomposition of order 5, $\phi = \mathcal{C}^5 h$, this is shown (by technical arguments developed in [9]) to hold provided

$$\mathbb{E}[|N(M_t^N(z) - M_t(z))|^2] \le C|b|^{-12} \tag{1.41}$$

(see (4.18)) or equivalently $\mathbb{E}[|\langle Y_t^N, \mathfrak{f}_z\rangle|^2] \le C|b|^{-12}$. Apply (1.9): start from terminal condition $f_T := \mathfrak{f}_{z_T}$ and integrate in time, $\langle Y_T^N, \mathfrak{f}_{z_T}\rangle = \Big\langle Y_0^N, f_0 + \frac{1}{2}(1-\frac{\beta}{2})\int_0^T dt\,\Big(\langle X_t^N, f_t''\rangle + \frac{1}{\sqrt{N}}\sum_{i=1}^N f_t'(\lambda_t^i) dW_t^i\Big)\Big\rangle$. Terms in the r.-h.s. are bounded in Section 4 using a control over $(f_t)_{0\le t\le T}$, solution of the evolution equation (1.9). The above equation is solved in the following way: it is proved to be compatible with the Stieltjes decomposition of order κ,

$$f_t(x) \equiv (\mathcal{C}^\kappa h_t)(x) = \int_{-\infty}^{+\infty} da \int_{-b_{max}}^{b_{max}} db\,(-\mathrm{i}b)\frac{|b|^\kappa}{(1+\kappa)!}\,\mathfrak{f}_z(x)\,h_t(a,b) \tag{1.42}$$

see Definition 2.3, if $\frac{\partial h_t}{\partial t}(a,b) = \mathcal{H}_t h_t(a,b)$ for a certain time-dependent operator $\mathcal{H}_t$ – a "Stieltjes transform" of the evolution operator featuring in (1.9) – acting on $L^1(\Pi_{b_{max}})$, which is written down explicitly and analyzed in great details in Section 3.

The article is organized as follows. We first introduce a family of *Stieltjes decompositions* $\mathcal{C}^\kappa$ depending on a regularity index $\kappa = 0, 1, 2, \ldots$ (see Section 2). The main technical section is Section 3, where we rewrite the r.-h.s. of Eq. (1.34) using Stieltjes decompositions as a sum of linear operators which we call *generators*; these are of two types: generalized transport operators, including V-dependent terms sketched above in (1.16), summing up to $\mathcal{H}_{transport}$, and bounded operators summing up to $\mathcal{H}_{nonlocal}$. We prove our Main Theorem in Section 4. Since (1.39) is formally just a consequence of Itô's formula, and most of the technical arguments used in Israelsson's paper to justify this formula hardly depend on V, Section 4 really revolves around a fundamental estimate, Lemma 4.2, which is based on properties of the characteristics, hence is strongly V-dependent. The analysis of the generators made in details in Section 3 allows one to prove the latter estimate. To conclude, one uses as input *large deviation estimates for the support of the measure* proved in Section 5. Finally, Appendices A and B, where we collected some well-known facts and formulas about transport equations and Stieltjes transforms.

In an article in preparation [24], we solve (1.39) and obtain the Gaussian kernel of the limiting fluctuation process Y.

2. Stieltjes decompositions

In (Israelsson [9], Lemma 9) one finds the following decomposition of an L^1 function $f : \mathbb{R} \to \mathbb{R}$ living in the Sobolev space H_2 as a sum of functions of the type

$$\mathfrak{f}_z : x \mapsto \frac{1}{x-z}, \tag{2.1}$$

where $z = a + \mathrm{i}b$, $b \ne 0$ (see Appendix B),

$$f(x) = \int_{-\infty}^{+\infty} da \int_{-\infty}^{+\infty} (-\mathrm{i}b)\,db\,\mathfrak{f}_z(x)h(a), \qquad h(a) = -f''(a). \tag{2.2}$$

The above "reproducing kernel type" decomposition is clearly not unique. The proof is based on the fact that, for $\kappa = 0, 1, 2, \ldots$ (see (B.11))

$$\begin{aligned} \int_{-\infty}^{+\infty} db\,(-\mathrm{i}b)\,|b|^{\kappa} \cdot \mathcal{F}(\mathfrak{f}_{\mathrm{i}b})(s) &= 2\int_0^{+\infty} db\,|b|^{\kappa}\mathcal{F}(\mathrm{Im}\,\mathfrak{f}_{\mathrm{i}b})(s) \\ &= 2\pi \int_{-\infty}^{+\infty} db\,|b|^{1+\kappa}\, e^{-b|s|} \\ &= 2\pi \cdot (1+\kappa)!\,|s|^{-2-\kappa}, \end{aligned} \tag{2.3}$$

where $\mathcal{F}$ is the Fourier transform (see Appendix B for normalization), from which we also get the following family of decompositions, valid for $f \in L^1 \cap H_{2+\kappa}$, $\kappa \in \mathbb{N}$,

$$f(x) = \int_{-\infty}^{+\infty} da \int_{-\infty}^{+\infty} (-\mathrm{i}b)\,db\,\frac{|b|^{\kappa}}{(1+\kappa)!}\,\mathfrak{f}_z(x)h(a),\;\; h(a) = \mathcal{F}^{-1}(|s|^{2+\kappa}\mathcal{F}(f))(a), \tag{2.4}$$

a straightforward generalization of (2.2) obtained by choosing some arbitrary value of κ instead of $\kappa = 0$ in (2.3). Note that h is *real* since $\mathcal{F}^{-1}(|s|\cdot)$ is given by a *real-valued* convolution kernel (see Appendix B).

The reason for introducing this κ-dependent family of decompositions is that the coefficient of $\mathfrak{f}_z(x)$ now *vanishes to order* $1+\kappa$ *instead of* 1 *on the real axis*, a property inherited from the assumed supplementary regularity of f. Note that, for κ *even*, $\mathcal{F}^{-1}(|s|^{2+\kappa}\mathcal{F}(\cdot))$ is the differential operator $(-\partial_s^2)^{1+\kappa/2}$. For κ *odd*, on the other hand, one gets derivatives of the $(\frac{1}{x})$-kernel (see Appendix B).

Since all interesting phenomena appear when $|b|$ is small, and we want to avoid artificial problems arising when $|b|$ is not bounded, we shall actually use analogous decompositions in which $|b|$ ranges from 0 to some maximal value $b_{max} > 0$. This introduces the following changes. First, instead of (2.2), we get

$$f(x) = \int_{-\infty}^{+\infty} da \int_{-b_{max}}^{b_{max}} (-\mathrm{i}b)\,db\,\frac{|b|^{\kappa}}{(1+\kappa)!}\,\mathfrak{f}_z(x)h(a),\;\; h(a) = (\mathcal{F}^{-1}(K^{\kappa}_{b_{max}}) * f)(a), \tag{2.5}$$

where

$$K^{\kappa}_{b_{max}}(s) := \left(2\int_0^{b_{max}} db\,|b|^{1+\kappa} \cdot \mathcal{F}(\mathrm{Im}\,(\mathfrak{f}_{\mathrm{i}b}))(s)\right)^{-1} \tag{2.6}$$

(note that the above integral is > 0 by (B.11)). We now study the convolution operator

$$\mathcal{K}^{\kappa}_{b_{max}} : f \mapsto \mathcal{F}^{-1}(K^{\kappa}_{b_{max}}) * f, \tag{2.7}$$

depending on the parity of κ:

(i) For κ *even*,

$$\begin{aligned} \int_0^{b_{max}} db\,|b|^{1+\kappa} \cdot \mathcal{F}(\mathrm{Im}\,\mathfrak{f}_{\mathrm{i}b})(s) &= \pi \int_0^{b_{max}} db\,|b|^{1+\kappa}\, e^{-b|s|} \\ &= \pi(-\partial_s^2)^{\kappa/2}(k^0_{b_{max}}(|s|)), \end{aligned} \tag{2.8}$$

where

$$k^0_{b_{max}}(|s|) = \frac{1}{s^2}\left(1-(1+b_{max}|s|)e^{-b_{max}|s|}\right). \tag{2.9}$$

When $|s| \to \infty$, $k^0_{b_{max}}(|s|) \sim s^{-2}$; on the other hand, $k^0_{b_{max}}(|s|) \overset{s\to 0}{\sim} b^2_{max}\sum_{k\geq 0}\frac{(-1)^k}{(k+2)!}(k+1)(b_{max}|s|)^k$. Thus $(-\partial_s^2)^{\kappa/2}(k^0_{b_{max}}(|s|)) \overset{|s|\to\infty}{\sim} (-1)^{\kappa/2}\frac{s^{-(2+\kappa)}}{(\kappa+1)!}$, and $(-\partial_s^2)^{\kappa/2}(k^0_{b_{max}}(|s|)) \overset{|s|\to 0}{\sim}$

$(-1)^{\kappa/2}\frac{b_{max}^{2+\kappa}}{2+\kappa}$. It is a simple exercise to prove the following: let

$$\underline{K}^{\kappa}_{b_{max}}(s) := (-1)^{\kappa/2}\left((\kappa+1)!\, s^{2+\kappa} + (2+\kappa)b_{max}^{-(2+\kappa)}\right)^{-1} K^{\kappa}_{b_{max}}(s) - 1. \tag{2.10}$$

Then $(\underline{K}^{\kappa}_{b_{max}})^{(j)}(s)$, $j = 0, 1, 2$ is $O(\frac{b^j_{max}}{1+b^2_{max}s^2})$ uniformly in s and b_{max}. Hence *the convolution operator*

$$\underline{\mathcal{K}}^{\kappa}_{b_{max}} : f \mapsto \mathcal{F}^{-1}(\underline{K}^{\kappa}_{b_{max}}) * f \tag{2.11}$$

is a bounded operator from $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *to* $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Indeed,

$$|||\underline{\mathcal{K}}^{\kappa}_{b_{max}}|||_{L^1(\mathbb{R})\to L^1(\mathbb{R})},\ |||\underline{\mathcal{K}}^{\kappa}_{b_{max}}|||_{L^\infty(\mathbb{R})\to L^\infty(\mathbb{R})} \le \|\mathcal{F}^{-1}(\underline{K}^{\kappa}_{b_{max}})\|_{L^1(\mathbb{R})}$$

and

$$\begin{aligned}|\mathcal{F}^{-1}(\underline{K}^{\kappa}_{b_{max}})(x)| &\le \min\left(\|\underline{K}^{\kappa}_{b_{max}}\|_{L^1}, \frac{1}{x^2}\|(\underline{K}^{\kappa}_{b_{max}})''\|_{L^1}\right)\\ &= O\left(\int \frac{ds}{1+b^2_{max}s^2}\right) \cdot \min(1, (\frac{b_{max}}{x})^2)\\ &= \frac{1}{b_{max}} \cdot \min(1, (\frac{b_{max}}{x})^2),\end{aligned} \tag{2.12}$$

from which $|||\underline{\mathcal{K}}^{\kappa}_{b_{max}}|||_{L^1(\mathbb{R})\to L^1(\mathbb{R})}, |||\underline{\mathcal{K}}^{\kappa}_{b_{max}}|||_{L^\infty(\mathbb{R})\to L^\infty(\mathbb{R})} = O(1)$. We may therefore write

$$\mathcal{K}^{\kappa}_{b_{max}} = (1 + \underline{\mathcal{K}}^{\kappa}_{b_{max}})\left(-(\kappa+1)!\, \partial_x^{2+\kappa} + (-1)^{\kappa/2}(2+\kappa)b_{max}^{-(2+\kappa)}\right). \tag{2.13}$$

(ii) For κ *odd*,

$$\begin{aligned}\int_0^{b_{max}} db\, |b|^{1+\kappa} \cdot \mathcal{F}(\mathrm{Im}\, \mathfrak{f}_{ib})(s) &= \pi \int_0^{+\infty} db\, |b|^{1+\kappa}\, e^{-b|s|}\\ &= \pi(-\partial_s^2)^{(\kappa+1)/2}(k^1_{b_{max}}(|s|)),\end{aligned} \tag{2.14}$$

where

$$k^1_{b_{max}}(|s|) = \frac{1}{|s|}(1 - e^{-b_{max}|s|}). \tag{2.15}$$

When $|s| \to \infty$, $k^1_{b_{max}(|s|)} \sim |s|^{-1}$; on the other hand, $k^1_{b_{max}}(|s|) \overset{s\to 0}{\sim} b_{max}\sum_{k\ge 0}\frac{(-1)^k}{(k+1)!}(b_{max}|s|)^k$. Thus $(-\partial_s^2)^{(\kappa+1)/2}(k^1_{b_{max}}(|s|)) \overset{|s|\to\infty}{\sim} (-1)^{(\kappa+1)/2}\frac{|s|^{-1}s^{-(1+\kappa)}}{(\kappa+1)!}$, and $(-\partial_s^2)^{(\kappa+1)/2}(k^1_{b_{max}}(|s|)) \overset{|s|\to 0}{\sim} (-1)^{(\kappa+1)/2}\frac{b_{max}^{2+\kappa}}{2+\kappa}$. Therefore the previous analysis, from (2.10) to the line before (2.13), remains valid, with $(-1)^{\kappa/2}$, resp. $s^{2+\kappa}$, replaced with $(-1)^{(\kappa+1)/2}$, resp. $|s|s^{1+\kappa}$, and one obtains using (B.20):

$$\mathcal{K}^{\kappa}_{b_{max}} = (1 + \underline{\mathcal{K}}^{\kappa}_{b_{max}})\left(-(\kappa+1)!\, \mathrm{i}H\partial_x^{2+\kappa} + (-1)^{(\kappa+1)/2}(2+\kappa)b_{max}^{-(2+\kappa)}\right) \tag{2.16}$$

where H is the Hilbert transform, defined by the principal value integral

$$Hf(x) := \frac{1}{\pi} p.v. \int_{-\infty}^{+\infty} \frac{1}{x-y} f(y)\, dy \tag{2.17}$$

and $\underline{\mathcal{K}}^{\kappa}_{b_{max}}$, defined by analogy with (i) as the convolution with respect to the inverse Fourier transform of

$$\underline{K}^{\kappa}_{b_{max}}(s) = (-1)^{(\kappa+1)/2}\left((\kappa+1)!\, |s|^{2+\kappa} + (2+\kappa)b_{max}^{-(2+\kappa)}\right)^{-1} K^{\kappa}_{b_{max}}(s) - 1, \tag{2.18}$$

has operator norm $O(1)$ on $L^1(\mathbb{R})$ and on $L^\infty(\mathbb{R})$.

The above formulas are particular instances of *Stieltjes decompositions*, *where* $h = h(a, b)$ *is allowed to be complex-valued and to depend on* b.

Definition 2.1 (*Upper Half-plane*)**.**

1. Let $\Pi^+ := \{z \in \mathbb{C} \mid \mathrm{Im}\,(z) > 0\}$.
2. For $b_{max} > 0$, let $\Pi^+_{b_{max}} := \{z \in \mathbb{C} \mid 0 < \mathrm{Im}\,(z) < b_{max}\}$.
3. Let $\Pi^- := -\Pi^+$, $\Pi^-_{b_{max}} := -\Pi^+_{b_{max}}$ and $\Pi := \Pi^+ \uplus \Pi^-$, $\Pi_{b_{max}} := \Pi^+_{b_{max}} \uplus \Pi^-_{b_{max}}$.

Definition 2.2. Let, for $p \in [1, +\infty]$ and $b_{max} > 0$,

$$L^p(\Pi_{b_{max}}) := \{h : \Pi_{b_{max}} \to \mathbb{C} \mid h(\bar{z}) = \overline{h(z)}\ (z \in \Pi^+_{b_{max}}) \text{ and } \|h\|_{L^p(\Pi_{b_{max}})} < \infty\}, \tag{2.19}$$

where

$$\begin{aligned} \|h\|_{L^p(\Pi_{b_{max}})} &:= \left(\int_{-\infty}^{+\infty} da \int_{-b_{max}}^{b_{max}} db\, |h(a, b)|^p\right)^{1/p} \quad (p < \infty), \\ \|h\|_{L^\infty(\Pi_{b_{max}})} &:= \sup_{z \in \Pi_{b_{max}}} |h(z)|. \end{aligned} \tag{2.20}$$

We will be mostly interested in the extreme cases $p = 1$, $p = \infty$. Letting formally $b_{max} \to +\infty$ one obtains in an obvious way the space $L^p(\Pi)$ with its norm $\|\ \|_{L^p(\Pi)}$. However, we shall actually fix some finite value of b_{max}, say (for convenience only), $0 \le b_{max} \le \frac{1}{2}$, implying: $\ln(1/|b|) \ge \ln 2 > 0$.

Definition 2.3 (*Stieltjes Decomposition*)**.** Let $\kappa = 0, 1, 2, \ldots$

1. Let $h \in L^1(\Pi_{b_{max}})$. We say that $f : \mathbb{R} \to \mathbb{R}$ *has Stieltjes decomposition* h *of order* κ *and cut-off* b_{max} *on* $[-R, R]$ if, for all $|x| \le R$,
$$f(x) = (\mathcal{C}^\kappa h)(x) := \int_{-\infty}^{+\infty} da \int_{-b_{max}}^{b_{max}} (-\mathrm{i}b)\, db\, \frac{|b|^\kappa}{(1+\kappa)!}\ \mathfrak{f}_z(x) h(a, b). \tag{2.21}$$
2. The function $h : (a, b) \mapsto \mathcal{K}^\kappa_{b_{max}}(f)(a)$ is called the *standard Stieltjes decomposition of order* κ and *cut-off* b_{max} of f.

Thanks to the symmetry condition, $h(\bar{z}) = \overline{h(z)}$, (2.21) may be rewritten in the form

$$(\mathcal{C}^\kappa h)(x) = 2\int_{-\infty}^{+\infty} da \int_0^{b_{max}} db\, \frac{|b|^{1+\kappa}}{(1+\kappa)!}\ \mathrm{Im}\,[\mathfrak{f}_z(x) h(a, b)], \tag{2.22}$$

from which it is apparent that f is indeed real-valued.

As already emphasized before, Stieltjes decompositions are not unique. In fact, it turns out to be useful to introduce a larger family of decompositions depending on a further scale parameter, $\rho > 0$. We shall give less details since we apply these to the functions $\mathrm{Im}\,\mathfrak{f}_{z_T}$ with $\mathrm{Im}\, z_T > 0$ only, and concentrate on the case $\kappa = 0$ for computations. For $0 < \rho \le b_{max}$, we write

$$K^\kappa_{b_{max},\rho}(s) := \left(\int_0^{b_{max}} db\, b^{1+\kappa}\, e^{-b/\rho} \cdot \mathcal{F}(\mathrm{Im}\,(\mathfrak{f}_{\mathrm{i}b}))(s)\right)^{-1} \tag{2.23}$$

(compare with (2.6)) and let as before

$$\mathcal{K}^0_{b_{max},\rho} : f \mapsto \mathcal{F}^{-1}(K^\kappa_{b_{max},\rho}) * f, \tag{2.24}$$

so that, for $b_T \neq 0$,

$$\mathrm{Im}\,[\mathfrak{f}_{\mathrm{i}b_T}](x) = \int_{-\infty}^{+\infty} da \int_0^{b_{max}} db\, b^{\kappa+1}\, e^{-b/\rho}\, \mathrm{Im}\,[\mathfrak{f}_z(x)]\, \mathcal{K}^0_{b_{max},\rho}(\mathrm{Im}\,\mathfrak{f}_{\mathrm{i}b_T})(a). \tag{2.25}$$

Then (compare with (2.8)) $(K^0_{b_{max},\rho})^{-1} = \pi k^0_{b_{max}}(\frac{1}{\rho} + |s|)$. Thus (emphasizing only the differences with $K^0_{b_{max}} = \lim_{\rho\to+\infty} K^0_{b_{max},\rho}$) $k^0_{b_{max},\rho}(|s|) \overset{s\to 0}{\sim} C\rho^2$ instead of b^2_{max}, with $C := 1-(1+b_{max}/\rho)e^{-b_{max}/\rho}$ bounded away from 0. Hence

$$\begin{aligned} \underline{K}^0_{b_{max},\rho}(s) &:= (|s|+\rho^{-1})^{-2} K^0_{b_{max},\rho} - 1 \\ &= -\frac{(1+b_{max}(|s|+\rho^{-1}))e^{-b_{max}(|s|+\rho^{-1})}}{1-(1+b_{max}(|s|+\rho^{-1}))e^{-b_{max}(|s|+\rho^{-1})}} \end{aligned} \tag{2.26}$$

is a $O(e^{-\frac{1}{2}b_{max}(|s|+\rho^{-1})})$, defining a bounded operator on $L^1\cap L^\infty$ with operator norm

$$O\left(\int ds\, e^{-\frac{1}{2}b_{max}(|s|+\rho^{-1})}\right)\cdot \int dx\, \min(1, \frac{b_{max}}{x})^2 = O(e^{-\frac{1}{2}b_{max}/\rho}) \tag{2.27}$$

(compare with (2.12)), so that $\mathcal{K}^0_{b_{max},\rho} = \mathcal{F}^{-1}((|s|+\rho^{-1})^2)*$, times (1+plus bounded perturbation). More generally, it may be proved that (for some constant c_κ) $\mathcal{K}^\kappa_{b_{max},\rho} = c_\kappa \mathcal{F}^{-1}((|s|+\rho^{-1})^{2+\kappa})*$, times (1+ bounded perturbation). On the other hand, letting $f = \mathrm{Im}\,[\mathfrak{f}_{z_T}]$, with $z_T = a_T + \mathrm{i}b_T$, $b_T > 0$, we find using (B.4), (B.11)

$$\mathcal{F}^{-1}((|s|+\rho^{-1})^{2+\kappa}\mathcal{F}\mathrm{Im}\,[\mathfrak{f}_{z_T}])(a) = \left(\rho^{-1} - \frac{\partial}{\partial b_T}\right)^{2+\kappa}\left(\frac{b_T}{(a-a_T)^2+|b_T|^2}\right). \tag{2.28}$$

Hence, letting

$$h(a,b) := e^{-b/\rho}\mathcal{K}^\kappa_{b_{max},\rho}(\mathrm{Im}\,\mathfrak{f}_{z_T})(a), \tag{2.29}$$

we get:

$$\begin{aligned} \int_{-\infty}^{+\infty} da \int_{-b_{max}}^{b_{max}} db\, h(a,b) &= O(1)\rho\left\{\frac{1}{\rho^{2+\kappa}} + \frac{1}{|b_T|^{2+\kappa}}\right\} \\ &= O\left(\frac{1}{\rho^{1+\kappa}}(1+O((\frac{\rho}{b_T})^{2+\kappa}))\right) \end{aligned} \tag{2.30}$$

which is minimal, of order

$$\|h\|_{L^1(\Pi_{b_{max}})} \approx b_T^{-1-\kappa}, \tag{2.31}$$

when $\rho \approx b_T$. Choosing $\rho = b_T/C$ for some large enough absolute constant $C > 0$, we further obtain – specifically in the case $\kappa = 0$ – a *positive* function h, which can hence be interpreted as a *density*. Also, one easily checks that, still with $\rho \approx b_T$,

$$\|(a,b)\mapsto \ln(1/|b|)h(a,b)\|_{L^1(\Pi_{b_{max}})} \approx \ln(1/b_T)b_T^{-1-\kappa} \tag{2.32}$$

if $0 < b_T \le \frac{1}{2}$. On the other hand,

$$\|h\|_{L^\infty(\Pi_{b_{max}})} \approx b_T^{-3-\kappa}. \tag{2.33}$$

In Section 4, we consider the time-evolution of $\mathcal{C}^\kappa h_T$, where h_T is essentially as in (2.34),

$$h_T(a,b) := e^{-b/\rho} \mathcal{K}^{\kappa}_{b_{max},\rho}(\chi_R \operatorname{Im} \mathfrak{f}_{z_T})(a), \tag{2.34}$$

for some cut-off function (see Definition 3.3) essentially supported on the ball $B(0, R)$ for some fixed radius $R > 0$. An easy adaptation of the above arguments, and a use of (B.23) when κ is odd in order to deal with the Hilbert transform, show that the above estimates (2.31), (2.32), (2.33) remain correct, while now h_T is $O(1)$, independently of b_T, far from the support, e.g. on $\Big((B(0,2R))^c \times [-b_{max}, b_{max}]\Big) \cup \Big(\mathbb{R} \times ([-b_{max}, b_{max}] \setminus [-\frac{1}{2}b_{max}, \frac{1}{2}b_{max}])\Big)$.

3. Generators

The general purpose of the section is the following: for $\kappa = 0, 1, 2, \ldots$ *fixed*, we want to write down an explicit time-dependent operator $\mathcal{H}(t)$ such that the right-hand side of (1.34) for f_t decomposed as

$$f_t(x) = \int_{-\infty}^{+\infty} da \int_{-b_{max}}^{b_{max}} db\, (-\mathrm{i}b)\, \frac{|b|^\kappa}{(1+\kappa)!}\; f_z(x)\, h_t(a,b) \tag{3.1}$$

(see Definition 2.3) is equal to

$$\int_{-\infty}^{+\infty} da \int_{-b_{max}}^{b_{max}} db\, (-\mathrm{i}b)\, \frac{|b|^\kappa}{(1+\kappa)!}\, \mathfrak{f}_z(x)\, \mathcal{H}(h_t)(t; a, b) \tag{3.2}$$

where $\mathcal{H}(h_t)(t; a, b) \equiv (\mathcal{H}(t)(h_t))(a, b)$.

Given the characteristic evolution in the z-coordinate found by Israelsson (recalled in Section 3.1) – which one may view as a *deterministic* Markov process – it is natural to think of the function h as a density $h(a, b)da\, db$, and then to interpret $\mathcal{H}(t)$ as a *Fokker–Planck operator*, whence (by duality) $\mathcal{L}(t) := (\mathcal{H}(t))^\dagger$ as the generator of a random process (see Appendix A for more). However, contrary to the harmonic case studied by Israelsson and Bender, in general we obtain a truly *random* process, furthermore, a *signed* process, with h a signed function. For lack of references on these notions, we shall refrain from developing this signed Markov process interpretation, and solve instead the evolution equation

$$\frac{dh_t}{dt}(a,b) = \mathcal{H}(h_t)(t; a, b) \tag{3.3}$$

using *semi-group theory*.

We have been voluntarily been vague up to this point about the double dependency of $\mathcal{H}$ on the integer index κ and the cut-off scale b_{max}. Why could not one just set $\kappa = 0$ and let $b_{max} \to +\infty$, as does Israelsson? The reason is, we cannot handle properly the potential-dependent part of the generator (save when V is order ≤ 2, which is the case considered in [9]) without introducing various cut-offs and perturbative arguments – unless maybe if V is analytic (or even better, polynomial), where another strategy is perhaps possible. Since we do not want to make this assumption, we shall:

(1) (*support issues*) in practice replace $V(x)$ by its *Taylor expansion to order 2 around all points in the support of the measures* $(X^N_t)_{0\leq t\leq T}$ *and* $(X_t)_{0\leq t\leq T}$ and *treat the Taylor remainder as a perturbation*. Since V is not bounded at infinity, it is important *not to* Taylor expand around any point on the real line, but only on a *compact interval*; choosing as compact interval the support is natural because the singular kernel term in (1.34) vanishes outside. *For brevity, we shall henceforth call support of the measure the*

random set $\left(\cup_{0\le t\le T}\operatorname{supp}(X_t^N)\right)\cup\left(\cup_{0\le t\le T}\operatorname{supp}(X_t)\right)$ *and denote by* $R>0$ *any number such that the support of the measure is* $\subset[-R,R]$. We rely on the bounds developed in Section 5 to argue that the probability of the support not to be included in $[-R,R]$ for some large enough R is exponentially small in N when R is large enough. Then we naturally decompose $h\in L^1(\Pi_{b_{max}})$ as $h^{int}+h^{ext}$ where $\operatorname{supp}(h^{int})\subset[-3R,3R]\times[-b_{max},b_{max}]$ and $\operatorname{supp}(h^{ext})\subset(\mathbb{R}\setminus[-2R,2R])\times[-b_{max},b_{max}]$, for instance by writing $h(a,b)=\bar\chi_R(a)h(a,b)+(1-\bar\chi_R(a))h(a,b)$, where $\bar\chi_R:\mathbb{R}\to[0,1]$ is some smooth function such that $\operatorname{supp}(\bar\chi_R)\subset[-3R,3R]$ and $\operatorname{supp}(1-\bar\chi_R)\subset\mathbb{R}\setminus[-2R,2R]$. The action of $\mathcal{H}$ on h^{ext} is very simple and can be added to the action of the remainder term $\mathcal{H}_{nonlocal}$ discussed in (2).

(2) (*varying* κ) in order to be able to treat the part (thereafter denoted by $\mathcal{H}_{nonlocal}$) of the generator due to the remainder term as a perturbation, it is important to see that $\mathcal{H}_{nonlocal}h(a,b)=O(|b|)$ when $b\to0$. This being the case, we may also consider $\mathcal{H}_{nonlocal}$ as an operator *intertwining a Stieltjes decomposition of order* κ *with a Stieltjes decomposition of order* $\kappa+1$, leading to a modification of the scheme developed around (3.1): namely, we want the right-hand side of (1.34) for f_t decomposed as (3.1) for some integer index κ to be equal to

$$\int_{-\infty}^{+\infty}da\int_{-b_{max}}^{b_{max}}(-\mathrm{i}b)\,db\,\frac{|b|^{1+\kappa}}{(2+\kappa)!}\,\mathfrak{f}_z(x)\,\mathcal{H}^{\kappa+1,\kappa}(h_t)(t;a,b). \tag{3.4}$$

Thus, instead of a single operator $\mathcal{H}^0$, we deal simultaneously with a family of operators $\mathcal{H}^\kappa$ and a family of intertwining operators $\mathcal{H}^{\kappa+1,\kappa}_{nonlocal}$, for $\kappa=0,1,2,\ldots$. These intertwinings result in an expansion of the Green kernel explicited in (4.22). This strategy yields optimal bounds in Section 4.

Let us collect by anticipation all the terms which will come out of the computations in Section 3.1 through Section 3.9. As a general rule, if $\mathcal{H}:L^1([-3R,3R]\times[-b_{max},b_{max}])\oplus L^1((\mathbb{R}\setminus[-2R,2R])\times[-b_{max},b_{max}])\to L^1([-3R,3R]\times[-b_{max},b_{max}])\oplus L^1((\mathbb{R}\setminus[-2R,2R])\times[-b_{max},b_{max}])$ is an (unbounded) operator, we denote by

$$\mathcal{H}^{int}:=\mathcal{H}\big|_{L^1([-3R,3R]\times[-b_{max},b_{max}])\oplus0},\qquad \mathcal{H}^{ext}:=\mathcal{H}\big|_{0\oplus L^1([-3R,3R]\times[-b_{max},b_{max}])} \tag{3.5}$$

its restrictions to either factor, and $\mathcal{H}^{(int,int)}$, $\mathcal{H}^{(int,ext)}$, $\mathcal{H}^{(ext,int)}$, $\mathcal{H}^{(ext,ext)}$ its four block-components, so that

$$\mathcal{H}=\left(\mathcal{H}^{int}\mid\mathcal{H}^{ext}\right)=\begin{pmatrix}\mathcal{H}^{(int,int)} & \mathcal{H}^{(ext,int)}\\ \mathcal{H}^{(int,ext)} & \mathcal{H}^{(ext,ext)}\end{pmatrix}. \tag{3.6}$$

By explicit computation we show that

$$\mathcal{H}^\kappa=\begin{pmatrix}\mathcal{H}^\kappa_{transport} & 0\\ 0 & 0\end{pmatrix}+\mathcal{H}^\kappa_{nonlocal}; \tag{3.7}$$

$$\mathcal{H}^\kappa_{transport}:=\mathcal{H}^\kappa_0+\sum_{k=0}^{2}\mathcal{H}^{\kappa,(k)}_{pot} \tag{3.8}$$

is a sum of 4 transport operators – with $\mathcal{H}^\kappa_0$ coming from the $\left(\frac{1}{x}\right)$-kernel part (see Section 3.1), and $\mathcal{H}^{\kappa,(k)}_{pot}$, $k=0,1,2$ (see Sections 3.3, 3.4, 3.5) from the Taylor expansion of the potential –, which are *unbounded operators*;

$$\mathcal{H}^{\kappa}_{nonlocal} = \left(\mathcal{H}^{\kappa,(3)}_{pot} + \mathcal{H}^{\kappa}_{\rm bdry} \mid \mathcal{H}^{\kappa,ext}_{pot}\right), \tag{3.9}$$

is a sum of *bounded* operators. The operator $\mathcal{H}^{\kappa,(3)}_{pot}$ – coming from the third order Taylor remainder for the potential – is introduced in Section 3.7. The operator $\mathcal{H}_{\rm bdry} \equiv \mathcal{H}^{\kappa}_{h\text{-}bdry} + \mathcal{H}^{\kappa}_{v\text{-}bdry}$ is itself a sum of boundary terms (see Section 3.9): contributions coming from horizontal boundary $[-3R, 3R] \times \{\pm b_{max}\}$, collected in $\mathcal{H}^{\kappa}_{\text{h-bdry}}$, and contributions coming from vertical boundary $\{\pm 3R\} \times [-b_{max}, b_{max}]$, collected in $\mathcal{H}^{\kappa}_{\text{v-bdry}}$. The eight operators $\mathcal{H}^{\kappa}_0, \mathcal{H}^{\kappa,(0)}_{pot}, \mathcal{H}^{\kappa,(1)}_{pot}, \mathcal{H}^{\kappa,(2)}_{pot}, \mathcal{H}^{\kappa,(3)}_{pot}, \mathcal{H}^{\kappa}_{\text{h-bdry}}, \mathcal{H}^{\kappa}_{\text{v-bdry}}, \mathcal{H}^{\kappa,ext}_{pot}$ are defined resp. in (3.26), (3.41), (3.45), (3.48), (3.64), (3.107), (3.111), (3.91).

We also write down expressions for $\mathcal{H}^{\kappa+1,\kappa}_{nonlocal}$, namely, $\mathcal{H}^{\kappa+1,\kappa,(3)}_{pot}$, $\mathcal{H}^{\kappa+1,\kappa}_{\text{h-bdry}}$, $\mathcal{H}^{\kappa+1,\kappa}_{\text{v-bdry}}$, $\mathcal{H}^{\kappa+1,\kappa,ext}_{pot}$, to be found resp. in (3.65), (3.106), (3.110), (3.92).

The kernels of these operators are denoted by the letter g, for instance,

$$\mathcal{H}^{\kappa}_0(h)(a,b) = \int_{-\infty}^{+\infty} da_T \int_{-b_{max}}^{b_{max}} db_T\ g^{\kappa}_0(a,b;a_T,b_T)h(a_T,b_T) \tag{3.10}$$

and similarly for the other operators.

Let us simply state as general remark that the dependence on R of the bounds of the present section will never be made explicit, since for the proof of our Main Theorem (see Section 4), we shall simply fix $R = R(T)$, where $R(T)$ is a fixed radius depending only on the potential and on the time horizon T, defined in Section 5.

3.1. The $\left(\frac{1}{x}\right)$-kernel part

(1.34) is easily solved by the characteristic method in the case $V \equiv 0$ for test functions f of the form $f(x) = \frac{c}{x-z}$. Up to conjugation we may assume that $b := \mathrm{Im}\, z > 0$. This is done in (Israelsson [9], Lemmas 2–4) – we need only subtract the trivial contribution of the harmonic potential –:

Proposition 3.1 (*See Israelsson [9]*). *Assume $V \equiv 0$. Then Eq.* (1.34) *with terminal condition $f_T(x) = \frac{c}{x-z}$ (Im $z > 0$) is solved as*

$$f^T_t(x) = \frac{C_t}{x - Z_t}, \tag{3.11}$$

where $(C_t)_{0\le t\le T}, (Z_t)_{0\le t\le T}$ solve the following ode's,

$$\frac{dZ_t}{dt} = -\frac{\beta}{4}(M^N_t(Z_t) + M_t(Z_t)), \qquad \frac{dC_t}{dt} = -\frac{\beta}{4}((M^N_t)'(Z_t) + M'_t(Z_t))C_t \tag{3.12}$$

with terminal conditions $Z_T = z$, $C_T = c$. In particular,

$$\mathrm{Im}\,\frac{dZ_t}{dt} = -\frac{\beta}{4}\langle X^N_t + X_t, \mathrm{Im}\,(\mathfrak{f}_{z_T})\rangle \le 0. \tag{3.13}$$

Obviously, the solution of Eq. (1.34) with terminal condition $f_T(x) = \frac{c}{x-\bar{z}}$ is now $f^T_t(x) = \frac{\bar{C}_t}{x-\bar{Z}_t}$.

The last inequality (3.13), a simple consequence of (B.5), implies that Z_t moves *away* from the real axis as t decreases, hence away from singularities. From the above and from general properties of the Stieltjes transform of ρ_t (see Appendix B), one can deduce bounds for the displacement, as in [9]. First $|M_t^N(z)|$, $|M_t(z)| \le 1/b_t$, hence (letting $Z_t =: A_t + \mathrm{i}B_t$, $B_t > 0$), for some large enough constant $C > 0$,

$$B_T \le B_t \le \sqrt{|B_T|^2 + C(T-t)}. \tag{3.14}$$

Similarly, $|A_t - A_T| = O\left(\frac{T-t}{B_T}\right)$. Finally, $\frac{\beta}{4}\left(|(M_t^N)'(Z_t)| + |(M_t)'(Z_t)|\right) \le \frac{|dB_t/dt|}{B_t} = \left|\frac{d}{dt}(\ln(B_t))\right|$, whence $C_t \le \frac{B_t}{B_T} \le \sqrt{|B_T|^2 + C(T-t)} \;/\; B_T$. Summarizing: for a given final condition z, $|\frac{dA_t}{dt}|, |\frac{dB_t}{dt}| \le \frac{\beta}{2}\frac{1}{B_t} \le \frac{\beta}{2}\frac{1}{b}$ is bounded along the characteristics, but may become arbitrarily large when $b \to 0$; $|B_t - B_T| \le \sqrt{C(T-t)}$ is bounded independently of b, while $|A_t - A_T|$ is *not*. On the other hand, starting from Z_T far enough from the support of $(X_t^N)_{0\le t\le T}, (X_T)_{0\le t\le T}$, e.g. $|\mathrm{Re}\ Z_T| \ge CR$, where $C > 1$ and (by assumption) $\mathrm{supp}(X_t^N)$, $\mathrm{supp}(X_t) \subset B(0,R)$, then (by (B.15)) $|A_t| > C'R$ $(1 < C' < C)$ for all $t \in [0,T]$, whence $|\dot{B}_t|, |\dot{A}_t| \le \frac{\beta}{R} < \infty$, provided $T < \frac{R^2}{\beta}(C - C')$.

Definition 3.2. Let $\mathcal{L}_0 = \mathcal{L}_0(t)$ be the time-dependent operator defined by

$$\begin{aligned}(\mathcal{L}_0(t)\phi)(a,b) = -\frac{\beta}{4}\Big(&\mathrm{Re}\ \left[(M_t^N + M_t)(z)\right]\partial_a\phi(a,b) + \mathrm{Im}\ \left[(M_t^N + M_t)(z)\right]\partial_b\phi(a,b) \\ &+ ((M_t^N)' + M_t')(z)\phi(a,b)\Big).\end{aligned} \tag{3.15}$$

The motivation for this definition is the following. Let $f_t^T(x) = \frac{C_t}{x - Z_t}$ as in Proposition 3.1. Then $\frac{\partial f_t^T}{\partial t}\big|_{t=T} = (\mathcal{L}_0 f_T)(T;z) := (\mathcal{L}_0(T)f_T)(z)$. Take, more generally, a terminal condition for (1.34) of the form

$$f_T(x) = \int da_T \int db_T\, \frac{h_T(a_T,b_T)}{x - z_T}. \tag{3.16}$$

Then

$$\begin{aligned}\frac{\partial f_t^T}{\partial t}\Big|_{t=T} &= \int da_T db_T\, (\mathcal{L}_0(\frac{1}{x-\cdot})(z_T)h_T)(a_T,b_T) \\ &= \int da_T db_T\, \mathfrak{f}_{z_T}(x)(\mathcal{L}_0^\dagger h_T)(T,z_T),\end{aligned} \tag{3.17}$$

where $(\mathcal{L}_0^\dagger h_T)(T;z_T) = ((\mathcal{L}_0(T))^\dagger h_T)(z_T)$ is obtained from the adjoint of $\mathcal{L}_0(T)$. Thus $\mathcal{L}_0$, resp. $\mathcal{L}_0^\dagger$ may be considered as the generator of a – here *deterministic* – *generalized* Markov process $Z_\cdot = A_\cdot + \mathrm{i}B_\cdot$, resp. the associated Fokker–Planck generator, where *generalized* refers to the supplementary order 0 term in $\mathcal{L}_0(t)$ (second line of (3.15)), which has on top of that the nasty property of not being even real-valued.

We now need some very general development, which we apply to $\mathcal{L}_0$ in this paragraph. *Let* $w : \Pi \to \mathbb{R}_+^*$ *be some weight function.* The relation $\frac{d}{dt}(wh) = w(\mathcal{L}_0^w(t))^\dagger h$ defines an operator $(\mathcal{L}_0^w(t))^\dagger$,

$$\begin{aligned}(\mathcal{L}_0^w)^\dagger(t;a,b) &:= (w\mathcal{L}_0(t)w^{-1})^\dagger(a,b) \\ &= -\frac{\beta}{4}w(a,b)^{-1}\Big(-\partial_a\ \mathrm{Re}\ \left[(M_t^N + M_t)(z)\right] - \partial_b\ \mathrm{Im}\ \left[(M_t^N + M_t)(z)\right] \\ &\qquad + ((M_t^N)' + M_t')(z)\Big)\, w(a,b)\end{aligned} \tag{3.18}$$

which is the adjoint of $\mathcal{L}_0$ with respect to the measure $w(a,b)da\,db$ on Π. In other words, we see that the solution $(h_t)_{0\le t\le T}$ of the equation

$$\frac{\partial h_t}{\partial t} = (\mathcal{L}_0^w)^\dagger(t)h_t \tag{3.19}$$

with terminal condition h_T is the *density* of $Z_\cdot$ with respect to the measure $w(a,b)da\,db$.

Let us consider specifically the cases

$$w(a,b) := b\,|b|^\kappa, \qquad \kappa = 0,1,2,\dots \tag{3.20}$$

for which we write

$$\mathcal{L}_0^w \equiv \mathcal{L}_0^\kappa. \tag{3.21}$$

These cases allow a direct connection to Stieltjes decompositions of order κ with $b_{max} = +\infty$, namely: *if*

$$f_T(x) = \mathcal{C}^\kappa h_T(x) = \int da \int_0^{+\infty} db\,(-\mathrm{i}b)\,\frac{|b|^\kappa}{(1+\kappa)!}\,h_T(a,b)\,\mathfrak{f}_z(x) \tag{3.22}$$

then

$$f_t(x) := \mathcal{C}^\kappa h_t(x), \tag{3.23}$$

where $(h_t)_{0\le t\le T}$ *is the solution of*

$$\frac{\partial h_t}{\partial t} = (\mathcal{L}_0^\kappa)^\dagger(t)h_t. \tag{3.24}$$

The generator

$$\mathcal{L}_0^\kappa = \left((b\,|b|^\kappa)^{-1}\mathcal{L}_0^\dagger b\,|b|^\kappa\right)^\dagger = |b|^{1+\kappa}\mathcal{L}_0(|b|^{1+\kappa})^{-1} \tag{3.25}$$

is obtained from (3.15) by replacing the multiplicative term $\frac{\beta}{4}((M_t^N)' + M_t')(z)\phi(a,b)$ with $\frac{\beta}{4}\left(((M_t^N)' + M_t')(z) - \frac{1+\kappa}{b}\mathrm{Im}\,[(M_t^N + M_t)(z)]\right)\phi(t;a,b)$, from which

$$\begin{aligned}\mathcal{H}_0^\kappa(h)(t;a,b) &:= (\mathcal{L}_0^\kappa)^\dagger(t)(h_t)(a,b)\\ &= \frac{\beta}{4}\Big(\partial_a\left[\mathrm{Re}\,\left((M_t^N + M_t)(z)\right)h(t;a,b)\right]\\ &\quad + \partial_b\left[\mathrm{Im}\,\left((M_t^N + M_t)(z)\right)h(t;a,b)\right]\\ &\quad + \left[\frac{1+\kappa}{b}\mathrm{Im}\,\left((M_t^N + M_t)(z)\right) - ((M_t^N)' + M_t')(z)\right]h(t;a,b)\Big).\end{aligned} \tag{3.26}$$

Consider the extended characteristics $(z_t, c_t) := (a_t + \mathrm{i}b_t, c_t^\kappa)$ of the operator $\mathcal{H}_0^\kappa$ as in Appendix A: they are defined as the solution of

$$\frac{da_t}{dt} = \frac{\beta}{4}\mathrm{Re}\,(M_t^N + M_t)(a_t + \mathrm{i}b_t), \qquad \frac{db_t}{dt} = \frac{\beta}{4}\mathrm{Im}\,(M_t^N + M_t)(a_t + \mathrm{i}b_t) \tag{3.27}$$

$$\frac{dc_t^\kappa}{dt} = \frac{\beta}{4}\left(\frac{1+\kappa}{b_t}\mathrm{Im}\,\left[(M_t^N + M_t)(a_t + \mathrm{i}b_t)\right] + \left((\bar{M}_t^N)' + \bar{M}_t'\right)(a_t + \mathrm{i}b_t)\right)c_t^\kappa. \tag{3.28}$$

Here we used the fact that $\partial_a\,\mathrm{Re}\,(M_t^N + M_t)(z) = \partial_b\,\mathrm{Im}\,(M_t^N + M_t)(z) = \mathrm{Re}\,((M_t^N)' + M_t')(z)$ since $z \mapsto (M_t^N + M_t)(z)$ is holomorphic. *Mind the sign changes* with respect to Proposition 3.1: characteristics of the dual operator $\mathcal{H}_0^\kappa$ have reversed velocities with respect to those of $\mathcal{L}$, with characteristic curves $(Z_t)_{0\le t\le T}$ running backwards in time (see Appendix A for more details). By convention, characteristics are killed upon touching the real axis.

Now it follows from (B.14) that Re $(-\tau(t,z_t)) \equiv$ Re $\left(-(c_t^\kappa)^{-1}\frac{dc_t^\kappa}{dt}\right) \le 0$ *for* $\kappa \ge 0$, *whence (see* Appendix A*)* $\mathcal{H}_0^\kappa$ *is a generator of a semi-group of* L^∞*-contractions. Because the first line of* (3.26) *is in divergence form, and the second line has positive real part,* $\mathcal{H}_0^\kappa$ *is also the generator of a semi-group of* L^1*-contractions.*

Let us now see what happens for R and b_{max} *finite*. It is clear that the cut-off does not change the characteristic equations for $(a_t + \mathrm{i}b_t, c_t)$. On the other hand, we get two supplementary *boundary terms*. This can be proved as follows. Assume

$$f_T(x) = \mathcal{C}^\kappa h_T(x) = \int_{-3R}^{3R} da_T \int_{-b_{max}}^{b_{max}} db_T\,(-\mathrm{i}b_T)\,\frac{|b|^\kappa}{(1+\kappa)!}\,h_T(a_T,b_T)\,\mathfrak{f}_{z_T}(x). \tag{3.29}$$

Then (coming back directly to the characteristics equations of Proposition 3.1)

$$\begin{aligned}
&-\frac{\beta}{4}\iint \frac{f_T'(x)-f_T'(y)}{x-y}(X_t^N(dy)+X_t(dy))\\
&= -\frac{\beta}{4}\int_{-3R}^{+3R} da_T \int_{-b_{max}}^{b_{max}} (-\mathrm{i}b_T)\,db_T\,\frac{|b_T|^\kappa}{(1+\kappa)!}\,\Big\{\mathrm{Im}\,[(M_T^N+M_T)(z_T)]\,(\partial_{b_T}\mathfrak{f}_{z_T})(x)+\\
&\qquad \mathrm{Re}\,[(M_T^N+M_T)(z_T)]\,(\partial_{a_T}\mathfrak{f}_{z_T})(x)\Big\}\;h_T(a_T,b_T)+\cdots\\
&= \frac{\beta}{4}\int_{-3R}^{+3R} da_T \int_{-b_{max}}^{b_{max}} (-\mathrm{i}b_T)\,db_T\,\frac{|b_T|^\kappa}{(1+\kappa)!}\,\Big\{\partial_{b_T}\big(\mathrm{Im}\,[(M_T^N+M_T)(z_T)]\,h_T(a_T,b_T)\big)+\\
&\qquad \partial_{a_T}\big(\mathrm{Re}\,[(M_T^N+M_T)(z_T)]\,h_T(a_T,b_T)\big)\Big\}\;\mathfrak{f}_{z_T}(x)\;+\cdots\\
&\qquad + \mathrm{bdry}
\end{aligned} \tag{3.30}$$

where "$\cdots$" denote the contribution of the c-characteristics (which we can ignore), so (by integration by parts) we get a boundary term bdry $\equiv$ h-bdry$_0$+ v-bdry$_0$ on the support of the measure, with

$$\begin{aligned}
\text{h-bdry}_0 &= -\frac{\beta}{4}\frac{b_{max}^{1+\kappa}}{(1+\kappa)!}\int_{-\infty}^{+\infty} da_T\;\cdot\chi_R(x)\cdot\;\big(\mathrm{Im}\,[(M_T^N+M_T)(a_T+\mathrm{i}b_{max})]\,\cdot\\
&\quad \cdot f_{a_T+\mathrm{i}b_{max}}(x)h(a_T,b_{max}) - \mathrm{Im}\,[(M_T^N+M_T)(a_T-\mathrm{i}b_{max})]\\
&\quad \times f_{a_T-\mathrm{i}b_{max}}(x)h(a_T,-b_{max})\big)
\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}
\text{v-bdry}_0 &= -\frac{\beta}{4}\int_{-b_{max}}^{b_{max}} (-\mathrm{i}b_T)\,db_T\,\frac{|b_T|^\kappa}{(1+\kappa)!}\;\cdot\chi_R(x)\;\cdot\;\big\{\mathrm{Re}\,[(M_T^N+M_T)(3R+\mathrm{i}b_T)]\;\cdot\\
&\quad \cdot f_{3R+\mathrm{i}b_T}(x)\,h(3R,b_T) - \mathrm{Re}\,[(M_T^N+M_T)(-3R+\mathrm{i}b_T)]\\
&\quad \times f_{-3R+\mathrm{i}b_T}(x)\,h(-3R,b_T)\big\}\,.
\end{aligned} \tag{3.32}$$

3.2. The potential-dependent part: general introduction

Consider now the potential-dependent part in (1.34). *Without further mention we fix for the discussion* $R \ge 1$ *and some arbitrary* $b_{max} \in (0, \frac{1}{2}]$. Generally speaking we want to *write the action of the operator* $V'(x)\frac{\partial}{\partial x}$ *on a function* $f \equiv f_T$ *with Stieltjes decomposition of order* κ

on $[-R,R]$

$$f(x)=\int_{-\infty}^{+\infty}da_T\int_{-b_{max}}^{b_{max}}db_T\,(-\mathrm{i}b_T)\,\frac{|b_T|^\kappa}{(1+\kappa)!}\;\mathfrak{f}_{z_T}(x)h(a_T,b_T),\qquad |x|\le R \tag{3.33}$$

(see Definition 2.3) . In principle (see below though) it may be done in the following way.

Definition 3.3 (*Cut-offs*)**.** Let $\chi_R:\mathbb{R}\to\mathbb{R}$ be a smooth cut-off function such that:

(i) $\chi_R\equiv 1$ on $[-R,R]$;
(ii) $\chi_R\big|_{B(0,\frac{3}{2}R)^c}\equiv 0$

and $\bar{\chi}_R$ be the function $x\mapsto\chi_R(\frac{x}{2})$.

Definition 3.4 (g-*kernel*)**.** Let, for $\kappa,\kappa'=0,1,2,\dots$

$$g_{pot}^{\kappa';\kappa}(a,b;a_T,b_T):=\mathbf{1}_{|b|<b_{max}}\frac{(-\mathrm{i}b_T)|b_T|^\kappa}{(1+\kappa)!}\,\mathcal{K}_{b_{max}}^{\kappa'}\left(x\mapsto\chi_R(x)V'(x)f'_{z_T}(x)\right)(a) \tag{3.34}$$

with $\mathcal{K}_{b_{max}}^{\kappa'}$ as in (2.7).

In practice we are only interested in couples of indices $(\kappa,\kappa'=\kappa)$ and $(\kappa,\kappa'=\kappa+1)$, and let $g_{pot}^{\kappa}\equiv g_{pot}^{\kappa;\kappa}$. Let f be a function as in (3.33). Then, *using the standard order κ' Stieltjes decomposition with cut-off b_{max} of* $V'(x)\frac{\partial}{\partial x}(\mathfrak{f}_{z_T}(x))$, we get

$$\begin{aligned}V'(x)\frac{\partial}{\partial x}f(x)=&\int_{-\infty}^{+\infty}da\int_{-b_{max}}^{b_{max}}db\,(-\mathrm{i}b)\,\frac{|b|^{\kappa'}}{(1+\kappa')!}\;\mathfrak{f}_z(x)\\&\int_{-\infty}^{+\infty}da_T\int_{-b_{max}}^{b_{max}}db_T\;g_{pot}^{\kappa';\kappa}(a,b;a_T,b_T)h(a_T,b_T),\end{aligned} \tag{3.35}$$

thus defining (unbounded) operators $\mathcal{H}_{pot}^{\kappa},\mathcal{H}_{pot}^{\kappa+1;\kappa}:L^1(\Pi_{b_{max}},da_T\,db_T)\to L^1(\Pi_{b_{max}},da\,db)$,

$$\mathcal{H}_{pot}^{\kappa}(h)(a,b)=\int_{-\infty}^{+\infty}da_T\int_{-b_{max}}^{b_{max}}db_T\;g_{pot}^{\kappa}(a,b;a_T,b_T)h(a_T,b_T)$$

$$\mathcal{H}_{pot}^{\kappa+1;\kappa}(h)(a,b)=\int_{-\infty}^{+\infty}da_T\int_{-b_{max}}^{b_{max}}db_T\;g_{pot}^{\kappa+1;\kappa}(a,b;a_T,b_T)h(a_T,b_T).$$

However, these Stieltjes representations of the vector field $V'(x)\frac{\partial}{\partial x}$ will be used directly only when $h=0\oplus h^{ext}$ has support in $\mathbb{R}\setminus[-2R,2R]$, producing the $\mathcal{H}^{ext}$-term. When $h=h^{int}\oplus 0$ has support $\subset[-3R,3R]$, we separate first the Taylor expansion of order 2 of $V'(x)$ around a_T, as explained in (1.13) or (3.37), which is directly analyzed without further Stieltjes decomposition.

Let us discuss in details how we proceed when $h=h^{int}$. As we have just mentioned, the first step is to use a second-order Taylor-expansion of V' around a_T for $|a_T|\le 3R$ and $x\in\mathrm{supp}(\chi_R)$, i.e. $|x|\le\frac{3}{2}R$,

$$\begin{aligned}V'(x)=&\;V'(a_T)+V''(a_T)(x-a_T)+V'''(a_T)\frac{(x-a_T)^2}{2}\\&+(x-a_T)^3W_{a_T}(x-a_T),\end{aligned} \tag{3.36}$$

where W_{a_T} is C^7. Thus

$$\begin{aligned}V'(x)\frac{\partial}{\partial x}=&\;V'(a_T)\partial_x+V''(a_T)(x-a_T)\partial_x+V'''(a_T)\frac{(x-a_T)^2}{2}\partial_x\\&+(x-a_T)^3W_{a_T}(x-a_T)\partial_x\end{aligned} \tag{3.37}$$

makes four different contributions to the generator, $\mathcal{H}^{\kappa}_{pot}$, $\kappa = 0, 1, 2, 3$, resp. called *constant, linear, quadratic and Taylor remainder term*, plus some boundary contributions. Computations show the following: $\mathcal{H}^{\kappa,(k)}_{pot}$, $k = 0, 1, 2$, are directly of the adjoint form

$$h \mapsto \Big((a_T, b_T) \mapsto \partial_{a_T}(v_{hor}(a_T, b_T)h(a_T, b_T)) + \partial_{b_T}(v_{vert}(a_T, b_T)h(a_T, b_T)) \\ - \tau(a_T, b_T)h(a_T, b_T) \Big) \tag{3.38}$$

with Re $\tau(\cdot) \le 0$, implying that *they generate L^1-contractions* (see Appendix A, and recall that we go *backwards* in time). As a nice feature of this problem, $\mathcal{H}^{\kappa,(k)}_{pot}$, $k = 0, 1, 2$ *also generate L^∞-contractions*. Replacing the vector field $\chi_R(x)V'(x)\partial_x$ in (3.34) by $\chi_R(x)(x - a_T)^3 W_{a_T}(x - a_T)\partial_x$ produces a kernel $g^{(3)}_{pot}(a, b; a_T, b_T)$ discussed in Section 3.8. As for the first three terms, they are directly shown to be equivalent to the action of a transport operator (see Sections 3.3, 3.4, 3.5). We sum up in Section 3.6 the contributions of the transport operators introduced in Sections 3.1, 3.3, 3.4 and 3.5. The operators $\mathcal{H}^{\kappa,(3)}_{pot}$ and $\mathcal{H}^{\kappa,ext}_{pot}$ are studied in Sections 3.7 and 3.8. Finally, the contribution of the boundary terms is analyzed in Section 3.9.

The terms in $\mathcal{H}_{nonlocal}$, on the other hand, *do not* generate neither L^∞- nor L^1-contractions. Thanks to the horizontal and vertical cut-offs however, they are bounded, hence generate by integration some exponentially increasing time factor $e^{C_R t}$, with C_R depending on R.

3.3. Constant term

Inserting the constant operator $V'(a_T)\frac{\partial}{\partial x}$ inside the Stieltjes decomposition

$$f_T(x) = \int_{-3R}^{+3R} da_T \int_{-b_{max}}^{b_{max}} (-\mathrm{i}b_T)\, db_T\, |b_T|^\kappa\, \mathfrak{f}_{z_T}(x)h(a_T, b_T),$$

we get

$$\int_{-3R}^{3R} da_T \int_{-b_{max}}^{b_{max}} (-\mathrm{i}b_T)\, db_T\, |b_T|^\kappa V'(a_T) f'_{z_T}(x)h(a_T, b_T) \\ = -\int_{-3R}^{3R} da_T \int_{-b_{max}}^{b_{max}} (-\mathrm{i}b_T)\, db_T\, |b_T|^\kappa\, V'(a_T)\frac{\partial}{\partial a_T}(\mathfrak{f}_{z_T}(x))h(a_T, b_T) \\ = \int_{-3R}^{3R} da_T \int_{-b_{max}}^{b_{max}} (-\mathrm{i}b_T)\, db_T\, |b_T|^\kappa\, \mathfrak{f}_{z_T}(x)\frac{\partial}{\partial a_T}\left(V'(a_T)h(a_T, b_T)\right) + \mathrm{bdry} \tag{3.39}$$

where

$$\mathrm{bdry} \equiv \text{v-bdry}^{(0)}_{pot} = -\int_{-b_{max}}^{b_{max}} (-\mathrm{i}b_T)\, db_T\, |b_T|^\kappa \cdot \chi_R(x) \cdot \\ \cdot \left(V'(3R) f_{3R+\mathrm{i}b_T}(x)\, h(3R, b_T) - V'(-3R) f_{-3R+\mathrm{i}b_T}(x)\, h(-3R, b_T)\right) \tag{3.40}$$

is a vertical boundary term coming from integration by parts, which we shall not discuss till Section 3.9.

This defines a new operator $\mathcal{H}^{(0)}_{pot}$ in divergence form,

$$\mathcal{H}^{\kappa,(0)}_{pot}(h)(a_T, b_T) = \frac{\partial}{\partial a_T}\left(V'(a_T)h(a_T, b_T)\right), \tag{3.41}$$

with corresponding extended characteristics

$$\frac{da_t}{dt} = V'(a_t), \qquad \frac{db_t}{dt} = 0, \qquad \frac{dc^\kappa_t}{dt} = V''(a_t). \tag{3.42}$$

3.4. Linear term

Proceeding as in the *constant term* case, we insert the operator $V''(a_T)(x-a_T)\frac{\partial}{\partial x}$ inside the Stieltjes decomposition of f_T, getting

$$\begin{aligned}
&\int da_T \int_{-b_{max}}^{b_{max}} (-\mathrm{i}b_T)\, db_T\, |b_T|^{\kappa}\, V''(a_T)\left[(x-a_T)\partial_x \mathfrak{f}_{z_T}(x)\right] h(a_T,b_T) \\
&\quad = -\int da_T \int_{-b_{max}}^{b_{max}} (-\mathrm{i}b_T)\, db_T\, |b_T|^{\kappa}\, V''(a_T)\ \cdot \partial_{b_T}(b_T \mathfrak{f}_{z_T}(x))\, h(a_T,b_T) \\
&\quad = \mathrm{bdry} + \int da_T \int_{-b_{max}}^{b_{max}} (-\mathrm{i}b_T)\, db_T\, |b_T|^{\kappa} \\
&\qquad \times \mathfrak{f}_{z_T}(x)\, V''(a_T)(b_T\partial_{b_T} + 1 + \kappa) h(a_T,b_T),
\end{aligned} \tag{3.43}$$

where

$$\begin{aligned}
\mathrm{bdry} \equiv \text{h-bdry}^{(1)}_{pot} &= -b_{max}^{2+\kappa} \int da_T\, V''(a_T)\ \cdot \chi_R(x) \\
&\cdot \left(f_{a_T+\mathrm{i}b_{max}}(x)\, h(a_T,b_{max}) - f_{a_T-\mathrm{i}b_{max}}(x)\, h(a_T,-b_{max})\right)
\end{aligned} \tag{3.44}$$

is a horizontal boundary term coming from integration by parts. This defines an operator, $\mathcal{H}^{\kappa,(1)}_{pot}$,

$$\mathcal{H}^{\kappa,(1)}_{pot}(h)(a_T,b_T) = \left(\partial_{b_T} b_T + \kappa\right)(V''(a_T)h(a_T,b_T)) \tag{3.45}$$

with associated characteristics,

$$\frac{da_t}{dt} = 0, \qquad \frac{db_t}{dt} = V''(a_t)b_t, \qquad \frac{dc^{\kappa}_t}{dt} = (1+\kappa)V''(a_t)c^{\kappa}_t. \tag{3.46}$$

3.5. Quadratic term

Proceeding as in the previous paragraph, we insert the operator $\frac{1}{2}V'''(a_T)(x-a_T)^2\frac{\partial}{\partial x}$ inside the Stieltjes decomposition. Since $(x-a_T)^2\frac{\partial}{\partial x} = \partial_x(x-a_T)^2 - 2(x-a_T)$, and

$$\begin{aligned}
&\partial_x\left[(x-a_T)^2 \mathfrak{f}_{z_T}(x)\right] = 1 + b_T^2 \partial_{a_T}\mathfrak{f}_{z_T}(x), \\
&-2(x-a_T)\mathfrak{f}_{z_T}(x) = -2 - 2\mathrm{i}b_T \mathfrak{f}_{z_T}(x)
\end{aligned} \tag{3.47}$$

this term produces a new operator

$$\mathcal{H}^{\kappa,(2)}_{pot}(h)(a_T,b_T) = \frac{1}{2}\left(-\partial_{a_T} b_T^2 - 2\mathrm{i}b_T\right)(V'''(a_T)h(a_T,b_T)). \tag{3.48}$$

with associated characteristics

$$\frac{da_t}{dt} = -\frac{1}{2}V'''(a_t)b_t^2, \qquad \frac{db_t}{dt} = 0, \qquad \frac{dc^{\kappa}_t}{dt} = -\mathrm{i}V'''(a_t)b_t\, c^{\kappa}_t \tag{3.49}$$

plus a vertical boundary term,

$$\begin{aligned}
\mathrm{bdry} \equiv \text{v-bdry}^{(2)}_{pot} &= -\int_{-b_{max}}^{b_{max}} (-\mathrm{i}b_T)\, db_T\, |b_T|^{\kappa+2}\ \cdot \chi_R(x) \\
&\cdot \left(V'''(3R) f_{3R+\mathrm{i}b_T}(x)\, h(3R,b_T) - V'''(-3R) f_{-3R+\mathrm{i}b_T}(x)\, h(-3R,b_T)\right).
\end{aligned} \tag{3.50}$$

The remaining term due to the constant $\frac{1}{2}V'''(a_T)(1-2)$ in (3.47), integrated with respect to the measure $|b_T|^{1+\kappa} da_T\, db_T$, adds a time-independent constant C to f_t, which however disappears from the computations since: (i) the right-hand side of (1.34) features only f_t'; (ii) $d\langle Y^N_t, C\rangle = dC = 0$ in (1.8).

3.6. Recapitulating: the transport contribution

We can now write down the action of the sum of our three transport operators,

$$\mathcal{H}^{\kappa}_{transport}(t) := \mathcal{H}^{\kappa}_0(t) + \mathcal{H}^{\kappa,(0)}_{pot} + \mathcal{H}^{\kappa,(1)}_{pot} + \mathcal{H}^{\kappa,(2)}_{pot} \tag{3.51}$$

(note that only $\mathcal{H}^{\kappa}_0$ depends on the time variable), defined respectively in (3.26), (3.41), (3.45) and (3.48). Summing the contributions in (3.28), (3.42), (3.46) and (3.49), we obtain the following equation for the characteristics on $[-3R, 3R] \times [-b_{max}, b_{max}]$,

$$\frac{da_t}{dt} = \frac{\beta}{4}\mathrm{Re}\,(M^N_t + M_t)(a_t + \mathrm{i}b_t) + V'(a_t) - \frac{1}{2}V'''(a_t)b_t^2 \tag{3.52}$$

$$\frac{db_t}{dt} = \frac{\beta}{4}\mathrm{Im}\,(M^N_t + M_t)(a_t + \mathrm{i}b_t) + V''(a_t)b_t \tag{3.53}$$

$$\frac{dc^{\kappa}_t}{dt} = \Big[\frac{\beta}{4}\Big(\frac{1+\kappa}{b_t}\mathrm{Im}\,(M^N_t + M_t)(a_t + \mathrm{i}b_t) + ((\bar{M}^N_t)' + (\bar{M}_t)')(a_t + \mathrm{i}b_t)\Big) \\ + (2+\kappa)V''(a_t) - \mathrm{i}V'''(a_t)b_t\Big]\, c^{\kappa}_t. \tag{3.54}$$

Let us make the following observations (see Section 3.1):

(i) $t \mapsto |b_t|$ increases, whence $|b_t| \le |b_T|$ $(0 \le t \le T)$.

(ii) velocities are $O(1)$ far from the support, e.g. on $\{|a| > CR\} \cup \{|b| > b_{max}/2\}$ (see discussion below Proposition 3.1);

(iii) as is already true of each individual generator $\mathcal{H}^{\kappa}_0, \mathcal{H}^{\kappa,(i)}_{pot}$ $(i = 0, 1, 2)$, the sum $\mathcal{H}^{\kappa}_{transport}(t)$ generates a semi-group of L^1-contractions. The same holds if one considers L^{∞}- norms, since the real part of (3.54) is positive.

As a side remark, we may choose T small enough so that characteristics $(a_t + \mathrm{i}b_t)_{0\le t\le T}$ started at time T on the boundary $\Big(\{\pm 3R\} \times [-b_{max}, b_{max}]\Big) \cup \Big([-3R, 3R] \times \{\pm b_{max}\}\Big)$ always remains far from the support, in the sense of (ii).

Appendix A therefore implies:

Lemma 3.5. *Let $u_T \in (L^1 \cap L^{\infty})([-3R, 3R] \times (0, b_{max}])$ and $\kappa = 0, 1, 2 \dots$. Then the backward evolution equation $\frac{du}{dt} = \mathcal{H}^{\kappa}_{transport}(t)u(t)$, $u\big|_{t=T} = u_T$ $(0 \le t \le T)$ has a unique solution $u(t) := U_{transport}(t, T)u_T$, such that*

$$\|u_t\|_{L^1([-3R,3R]\times(0,b_{max}])} \le \|u_T\|_{L^1([-3R,3R]\times(0,b_{max}])} \tag{3.55}$$

$$\|u_t\|_{L^{\infty}([-3R,3R]\times(0,b_{max}])} \le \|u_T\|_{L^{\infty}([-3R,3R]\times(0,b_{max}])}. \tag{3.56}$$

In Section 4, we will separate (3.54) into its real and imaginary parts. Solving the (a, b)-characteristics coupled with the real characteristic $\tilde{c}^{\kappa}$,

$$\frac{d\tilde{c}^{\kappa}_t}{dt} = \mathrm{Re}\,\Big[\frac{\beta}{4}\Big(\frac{1+\kappa}{b_t}\mathrm{Im}\,(M^N_t + M_t)(a_t + \mathrm{i}b_t) + ((\bar{M}^N_t)' + (\bar{M}_t)')(a_t + \mathrm{i}b_t)\Big) \\ + (2+\kappa)V''(a_t) - \mathrm{i}V'''(a_t)b_t\Big]\, \tilde{c}^{\kappa}_t, \tag{3.57}$$

one gets a backward evolution equation,

$$\frac{d\tilde{u}}{dt} = [\mathrm{Re}\,\mathcal{H}^{\kappa}_{transport}(t)]\tilde{u}(t), \quad \tilde{u}\big|_{t=T} = \tilde{u}_T \equiv u_T \qquad (0 \le t \le T), \tag{3.58}$$

solved as $\tilde{u}(t) := \tilde{U}_{transport}(t,T)\tilde{u}_T$, such that

$$\|\tilde{u}_t\|_{L^1([-3R,3R]\times(0,b_{max}])} \le \|\tilde{u}_T\|_{L^1([-3R,3R]\times(0,b_{max}])} \tag{3.59}$$

$$\|\tilde{u}_t\|_{L^\infty([-3R,3R]\times(0,b_{max}])} \le \|\tilde{u}_T\|_{L^\infty([-3R,3R]\times(0,b_{max}])}. \tag{3.60}$$

Assuming $u_T \ge 0$, $\tilde{u}$ is simply the modulus of u,

$$\tilde{u}_t(a,b) = |u_t(a,b)|. \tag{3.61}$$

In particular, $\tilde{u}_t \ge 0$. The *adjoint evolution* with generator $(\mathrm{Re}\,\mathcal{H}^{\kappa}_{transport})^\dagger$ is *sub-Markovian*, i.e. $\tilde{u}_t$ is the density at time t of a (deterministic) time-reversed Markov $(\tilde{A}_t, \tilde{B}_t)_{0\le t\le T}$ process (whose trajectories run backwards w.r. to (a_t, b_t)) with kernel $p(t,\tilde{a}_t,\tilde{b}_t; s,\tilde{a}_s,\tilde{b}_s) \ge 0$ $(t \le s)$ such that $\int d\tilde{a} \int d\tilde{b}\, p(t,\tilde{a},\tilde{b}; s,\tilde{a}_s,\tilde{b}_s) \le 1$.

Remark. It is instructive to look at terms that would be produced by continuing the Taylor expansion to infinity. Note that the boundary value of the operator $\mathcal{H}^{\kappa,(2)}_{pot}(h)$ vanishes ($\mathcal{H}^{\kappa,(2)}_{pot}(h)(a_T, b_T = 0) \equiv 0$ vanishes to order ≥ 1). It may be proven in general that the contribution of order j vanishes to order $\ge j-1$. Summing up the whole series would yield (up to κ-dependent terms)

$$\frac{da_t}{dt} = \sum_{n=2p} \frac{V^{(n+1)}(a_t)}{n!}(\mathrm{i}b_t)^n, \qquad \frac{db_t}{dt} = -\mathrm{i}\sum_{n=2p+1} \frac{V^{(n+1)}(a_t)}{n!}(\mathrm{i}b_t)^n \tag{3.62}$$

$$\frac{dc_t}{dt} = \sum_{n\ge 1} V^{(n+1)}(a_t)\frac{(\mathrm{i}b_t)^{n-1}}{(n-1)!}. \tag{3.63}$$

Note that, for V *holomorphic*, this is equivalent to the vector field $V'(Z_t)\partial_{Z_t} + V''(Z_t)$. *However*, terms of order ≥ 3 also produce *polynomials* in x (instead of linear combination of $\mathfrak{f}_z(x)$, $z \in \mathbb{C}\setminus\mathbb{R}$). Integrating the generator would yield power series in x which are maybe controllable, but this would require totally different techniques with respect to ours.

3.7. Main remainder term

We study in this subsection the g-kernels $g^{\kappa';\kappa,(3)}_{pot}$ obtained by replacing $\chi_R(x)V'(x)f'_{z_T}(x)$ with $\chi_R(x)(x-a_T)^3 W_{a_T}(x-a_T)f'_{z_T}(x)$ in Definition 3.4. We want to prove that the operators

$$\mathcal{H}^{\kappa,(3)}_{pot} : h \mapsto \mathcal{H}^{\kappa,(3)}_{pot}(h) : \Big((a,b) \mapsto \int_{-3R}^{3R} da_T \int_{-b_{max}}^{b_{max}} db_T\, g^{\kappa,(3)}_{pot} \times (a,b;a_T,b_T)h(a_T,b_T)\Big) \tag{3.64}$$

and

$$\mathcal{H}^{\kappa+1;\kappa,(3)}_{pot} : h \mapsto \mathcal{H}^{\kappa+1;\kappa,(3)}_{pot}(h) : \Big((a,b) \mapsto \int_{-3R}^{3R} da_T \int_{-b_{max}}^{b_{max}} db_T\, g^{\kappa+1;\kappa,(3)}_{pot} \times (a,b;a_T,b_T)h(a_T,b_T)\Big) \tag{3.65}$$

are bounded operators in $L^1(\Pi_{b_{max}})$.

Definition 3.6. For $f \in C^k(\mathbb{R}, \mathbb{R})$ and $R > 0$, let

$$\|f\|_{k,[-R,R]} := \sup_{0\le j\le k} \sup_{[-R,R]} |f^{(j)}|. \tag{3.66}$$

Lemma 3.7.

(i) $|||\mathcal{H}_{pot}^{\kappa+1;\kappa,(3)}|||_{L^1(\Pi_{b_{max}})\to L^1(\Pi_{b_{max}})} = O(b_{max}^{-1}\|V'\|_{8+\kappa,[-3R,3R]})$;

(ii) $|||\mathcal{H}_{pot}^{\kappa,(3)}|||_{L^1(\Pi_{b_{max}})\to L^1(\Pi_{b_{max}})} = O(\|V'\|_{7+\kappa,[-3R,3R]})$.

The same estimates hold for the L^∞-operator norms $|||\cdot|||_{L^\infty(\Pi_{b_{max}})\to L^\infty(\Pi_{b_{max}})}$.

As can be checked by looking at the details of the proof, norms $|||\cdot|||_{L^1(\Pi_{b_{max}})\to L^\infty(\Pi_{b_{max}})}$ are deduced from these by dividing by a volume factor $\mathrm{Vol}([-3R, 3R] \times [-b_{max}, b_{max}]) \approx b_{max}$, which follows from the fact that the kernel $g_{pot}^{\kappa';\kappa,(3)}$ is regular on the diagonal.

We shall actually only prove the statement for $|||\mathcal{H}_{pot}^{\kappa+1;\kappa,(3)}|||_{L^1\to L^1}$. The proof of (ii) reduces immediately to (i) by substituting $\kappa \to \kappa - 1$ and taking into account the extra b-prefactor.

Remark. Had we Taylor expanded V' to order 2 instead of order 3, would we have obtained an *unbounded* operator instead of $\mathcal{H}^{\kappa+1;\kappa,(3)}$, as testified by the bound (3.76) (with x^2 instead of x^3 in the numerator, the integral would diverge in the limit $b_T \to 0$ for $m = 3 + \kappa$). On the other hand, for the same reason, it is easy to see by looking at the details of the proof that $\mathcal{H}^{\kappa,(3)} : L^1(\Pi_{b_{max}}) \to L^\infty(\Pi_{b_{max}})$ is bounded, typically because in (3.76) one obtains instead of the L^1-kernel $x \mapsto \frac{b_T}{x^2+b_T^2}$ the bounded function $x \mapsto \frac{b_T^2}{x^2+b_T^2}$.

Proof. By definition,

$$g_{pot}^{\kappa+1;\kappa,(3)}(a, b; a_T, b_T) = \frac{(-\mathrm{i}b_T)|b_T|^\kappa}{(1+\kappa)!}\,\mathcal{K}_{b_{max}}^{\kappa+1} \times \big(x \mapsto \chi_R(x)(x-a_T)^3 W_{a_T}(x-a_T) f'_{z_T}(x)\big)(a).$$

We consider in the computations only the part g^κ of the kernel $g_{pot}^{\kappa+1;\kappa,(3)}$ obtained by replacing $\mathcal{K}^\kappa$ with the operators in factor of the bounded operator $1 + \underline{\mathcal{K}}_{b_{max}}^\kappa$ in (2.13), (2.16). For some numerical constants $c_1 = c_1(\kappa)$, $c_2 = c_2(\kappa)$,

$$\begin{aligned} g^\kappa(a, b; a_T, b_T) &= (-\mathrm{i}b_T)|b_T|^\kappa \left\{c_1[\mathcal{F}^{-1}(|s|^{3+\kappa})*] + c_2 b_{max}^{-(3+\kappa)}\right\}(F_{a_T+\mathrm{i}b_T})(x) \\ &=: (-\mathrm{i}b_T)|b_T|^\kappa (c_1 G^1_{a_T+\mathrm{i}b_T}(x) + c_2 G^2_{a_T+\mathrm{i}b_T}(x)), \end{aligned} \tag{3.67}$$

with $x := a - a_T$ and

$$F_{a_T+\mathrm{i}b_T}(y) := \tilde{W}_{a_T}(y)\frac{y^3}{(y-\mathrm{i}b_T)^2} \tag{3.68}$$

where $\tilde{W}_{a_T}(y) = \chi_R(a_T + y)W_{a_T}(y)$ has support $\subset [-\frac{9}{2}R, \frac{9}{2}R]$ since $|a_T| \le 3R$. Thus (3.65) looks like a convolution formula in the coordinates a, a_T – but not quite since $\tilde{W}_{a_T}$ depends on a_T –, which leads us to use the following bound (where $G_{a_T+\mathrm{i}b_T} := c_1 G^1_{a_T+\mathrm{i}b_T} + c_2 G^2_{a_T+\mathrm{i}b_T}$)

$$\|\mathcal{H}^{\kappa+1;\kappa,(3)}(h)\|_{L^1(\Pi_{b_{max}})} \le b_{max}\left(\sup_{(a_T,b_T)\in[-3R,3R]\times[-b_{max},b_{max}]} |b_T|^{1+\kappa} \times \int da\, |G_{a_T+\mathrm{i}b_T}(a-a_T)|\right) \|h\|_{L^1(\Pi_{b_{max}})} \tag{3.69}$$

whence

$$\|\!|\mathcal{H}^{\kappa+1;\kappa,(3)}\|\!|_{L^1\to L^1} \le b_{max}\left(\sup_{(a_T,b_T)\in[-3R,3R]\times[-b_{max},b_{max}]} \|\, |b_T|^{1+\kappa} G_{a_T+\mathrm{i}b_T}\|_{L^1}\right). \quad (3.70)$$

We must therefore bound $\|\, |b_T|^{1+\kappa} G_{a_T+\mathrm{i}b_T}\|_{L^1}$. *The main issue, on which we shall now concentrate, is to bound* $\|\, |b_T|^{1+\kappa} G^1_{a_T+\mathrm{i}b_T}\|_{L^1([-5R,5R])}$*;* as shown later on, the missing terms are less singular and may be bounded similarly. Thus from now on and till below (3.87), $|a_T| \le 3R$, $|x| \le 5R$ and $|a| := |x + a_T| \le 8R$ are bounded.

If κ is *odd* then $|s|^{3+\kappa} = s^{3+\kappa}$ is the Fourier symbol of a differential operator, otherwise $|s|^{3+\kappa} = \mathrm{sgn}(s)s^{3+\kappa}$ involves a further convolution by a singular kernel. Let us accordingly distinguish two cases. But first of all we let, for $\ell \ge 3$ and $|a_T| \le 3R$,

$$C_\ell := \sup_{3\le\ell'\le\ell} \|\tilde{W}^{(\ell'-3)}_{a_T}\|_\infty \quad (3.71)$$

and note that

$$C_\ell = O\left(\sup_{3\le\ell'\le\ell}\ \sup_{[-3R,3R]} |V^{(\ell'+1)}|\right) = O(\|V'\|_{\ell,[-3R,3R]}). \quad (3.72)$$

More generally, if $\ell \ge \ell' \ge \ell'' \ge 3$ and $r \ge 0$,

$$\|y \mapsto (\tilde{W}^{(\ell''-3)}_{a_T}(y)y^r)^{(\ell'-\ell'')}\|_\infty = O(C_\ell). \quad (3.73)$$

(i) (κ odd) First

$$\|\, |b_T|^{1+\kappa} G^1_{a_T+\mathrm{i}b_T}\|_{L^1(\mathbb{R})} = O(C_{6+\kappa}|b_T|^{1+\kappa})\sum_{m=0}^{3+\kappa}\int_{-5R}^{5R} dx\, \left|\partial_x^m\left(\frac{x^3}{(x-\mathrm{i}b_T)^2}\right)\right|. \quad (3.74)$$

Then

$$\partial_x^m\left(\frac{x^3}{(x-\mathrm{i}b_T)^2}\right) = \sum_{p-q=1-m,\, p\le 3,\, q\ge 2} C^m_{p,q}x^p(x-\mathrm{i}b_T)^{-q} \quad (3.75)$$

for some coefficients $C^m_{p,q}$. If $3 \le m \le 3+\kappa$ then $|x^p(x-\mathrm{i}b_T)^{-q}| = O(\frac{|x-\mathrm{i}b_T|^{3-m}}{x^2+|b_T|^2}) = O(\frac{|b_T|^{3-m}}{x^2+|b_T|^2})$. Multiplying with respect to $|b_T|^{1+\kappa}$ and integrating, one gets (using $b_T \le b_{max} \le 1$)

$$\|\, |b_T|^{1+\kappa}\partial_x^m\left(\frac{x^3}{(x-\mathrm{i}b_T)^2}\right)\|_{L^1(\mathbb{R})} \le C'|b_T|^{3+\kappa-m}\int dx\, \frac{b_T}{x^2+|b_T|^2} = O(1). \quad (3.76)$$

If $m \le 2$ then $b_T\left|\partial_x^m\left(\frac{x^3}{(x-\mathrm{i}b_T)^2}\right)\right| = O(|b_T|\frac{|x|^{3-m}}{x^2+|b_T|^2}) = O(|x|^{2-m})$. Thus $\|\, |b_T|^{1+\kappa}\partial_x^m \left(\frac{x^3}{(x-\mathrm{i}b_T)^2}\right)\|_{L^1([-5R,5R])} = O(|b_T|^\kappa) = O(1)$.
All together:

$$\|\, |b_T|^{1+\kappa} G^1_{a_T+\mathrm{i}b_T}\|_{L^1([-5R,-5R])} = O(C_{6+\kappa}). \quad (3.77)$$

(ii) (κ even) Let, for $0 \le m \le 3+\kappa$,

$$I(x) := p.v.(\frac{1}{x}) * \left[x \mapsto \tilde{W}^{(3+\kappa-m)}_{a_T}(x)\partial_x^m\left(\frac{x^3}{(x-\mathrm{i}b_T)^2}\right)\right]. \quad (3.78)$$

We must bound $|b_T|^{1+\kappa}\int dx\, |I(x)|$.

We rewrite $I(x)$ as the sum of two contributions (we refer to Appendix B without further mention for computations and bounds related to the principal value integral), $I(x) =: I_{reg}(x) + I_{sing}(x)$, where

$$I_{reg}(x) := \int dy\, \frac{\tilde{W}^{(3+\kappa-m)}_{a_T}(x) - \tilde{W}^{(3+\kappa-m)}_{a_T}(y)}{x-y} \partial_x^m \left(\frac{x^3}{(x-\mathrm{i}b_T)^2}\right) \tag{3.79}$$

and

$$I_{sing}(x) := \int dy\, \tilde{W}^{(3+\kappa-m)}_{a_T}(y) \frac{\partial_x^m \left(\frac{x^3}{(x-\mathrm{i}b_T)^2}\right) - \partial_y^m \left(\frac{y^3}{(y-\mathrm{i}b_T)^2}\right)}{x-y}. \tag{3.80}$$

Using $\left|\frac{\tilde{W}^{(3+\kappa-m)}_{a_T}(x) - \tilde{W}^{(3+\kappa-m)}_{a_T}(y)}{x-y}\right| \le \|\tilde{W}^{(4+\kappa-m)}_{a_T}\|_\infty \le C_{7+\kappa}$, and noting that, for $|x| \le 5R$, the integral $\int dy\,(\cdots) = \int_{B(x,\frac{19}{2}R)}(\cdots)$ (by symmetry) simply produces an extra factor $O(1)$, one obtains, using (i)

$$|b_T|^{1+\kappa} \int dx\, |I_{reg}(x)| = O(C_{7+\kappa}). \tag{3.81}$$

Considering now $I_{sing}(x)$, we expand the numerator of (3.80) as in (3.75) and rewrite

$$\begin{aligned} \frac{x^p(x-\mathrm{i}b_T)^{-q} - y^p(y-\mathrm{i}b_T)^{-q}}{x-y} &= \frac{x^p - y^p}{x-y}(y-\mathrm{i}b_T)^{-q} \\ &\quad + x^p \frac{(y-\mathrm{i}b_T)^q - (x-\mathrm{i}b_T)^q}{(x-y)(x-\mathrm{i}b_T)^q(y-\mathrm{i}b_T)^q} \\ &= \sum_{r=0}^{p-1} x^{p-1-r} \frac{y^r}{(y-\mathrm{i}b_T)^q} - \sum_{r'=1}^{q} \frac{x^p}{(x-\mathrm{i}b_T)^{r'}} \frac{1}{(y-\mathrm{i}b_T)^{q+1-r'}}. \end{aligned} \tag{3.82}$$

The integrals $J_1 := \int dy\, \frac{\tilde{W}^{(3+\kappa-m)}_{a_T}(y) y^r}{(y-\mathrm{i}b_T)^q}$, $J_2 := \int dy\, \frac{\tilde{W}^{(3+\kappa-m)}_{a_T}(y)}{(y-\mathrm{i}b_T)^{q+1-r'}}$ may be bounded using (B.19) since $q \ge 2 \ge 1$ and $q+1-r' \ge 1$. Thus

$$|J_1| = O(C_{8+\kappa}),\ |J_2| = O(C_{8+\kappa}) \tag{3.83}$$

Then $\int_{-5R}^{5R} dx\, |x|^{p-1-r} = O(1)$ and

$$\begin{aligned} |b_T|^{1+\kappa} \int dx\, \frac{|x|^p}{|x-\mathrm{i}b_T|^{r'}} &= O(|b_T|^{(1+\kappa)-(r'-p-1)}) = O(b_{max}^{2-r'+\kappa+p}) \\ &\times (r' \ge p+2) \end{aligned} \tag{3.84}$$

(note that $2 - r' + \kappa + p \ge 0$);

$$\begin{aligned} |b_T|^{1+\kappa} \int_{-5R}^{5R} dx\, \frac{|x|^p}{|x-\mathrm{i}b_T|^{r'}} &\le |b_T|^{1+\kappa} \int_{-5R}^{5R} \frac{dx}{|x-\mathrm{i}b_T|} = O(b_{max}^{1+\kappa} \ln(1/b_{max})) \\ &\times (r' = p+1) \end{aligned} \tag{3.85}$$

$$|b_T|^{1+\kappa} \int_{-5R}^{5R} dx\, \frac{|x|^p}{|x-\mathrm{i}b_T|^{r'}} \le b_{max}^{1+\kappa} \int_{-5R}^{5R} dx\, |x|^{p-r'} = O(b_{max}^{1+\kappa}) \qquad (r' \le p). \tag{3.86}$$

All together, one obtains, summing all terms:

$$|b_T|^{1+\kappa} \int dx\, |I_{sing}(x)| = O(C_{8+\kappa}). \tag{3.87}$$

Let us now quickly deal with the missing terms. First

$$\begin{aligned}\| \, |b_T|^{1+\kappa} G^2_{a_T+\mathrm{i}b_T} \|_{L^1} &\le b_{max}^{-2} \int dx \, \frac{|x|^3}{|x-\mathrm{i}b_T|^2} |\tilde{W}_{a_T}(x)| \\ &\le b_{max}^{-2} \, \|\tilde{W}_{a_T}\|_\infty \int_{-\frac{9}{2}R}^{\frac{9}{2}R} dx \, |x| = O\left(b_{max}^{-2} C_3\right). \end{aligned} \tag{3.88}$$

Then one must bound $|b_T|^{1+\kappa} \int_{|x|\ge 5R} dx \, |G^1_{a_T+\mathrm{i}b_T}(x)|$; because $\mathrm{supp}(\tilde{W}_{a_T}) \subset [-\frac{9}{2}R, \frac{9}{2}R]$, this contribution vanishes except if κ is *even*, see (ii) above, in which case (by integration by parts)

$$\begin{aligned}|b_T|^{1+\kappa} \int_{|x|\ge 5R} dx \, |G^1_{a_T+\mathrm{i}b_T}(x)| &= |b_T|^{1+\kappa} \int_{|x|\ge 5R} dx \\ &\times \left| \int_{-\frac{9}{2}R}^{\frac{9}{2}R} dy \; p.v.(\frac{1}{(x-y)^2}) F^{(2+\kappa)}_{a_T+\mathrm{i}b_T}(y) \right| \\ &\le |b_T| \int_{|x|\ge 5R} dx \; O(\frac{1}{x^2}) \, \| \, |b_T|^\kappa F^{(2+\kappa)}_{a_T+\mathrm{i}b_T} \|_{L^1([-5R,5R])}. \end{aligned} \tag{3.89}$$

The integral $\| \, |b_T|^\kappa F^{(2+\kappa)}_{a_T+\mathrm{i}b_T} \|_{L^1([-5R,5R])}$ is (up to the replacement $\kappa \to \kappa - 1$) exactly the one which has been computed in case (i) above. Hence we find:

$$|b_T|^{1+\kappa} \int_{|x|\ge 5R} dx \, |G^1_{a_T+\mathrm{i}b_T}(x)| = O\left(C_{5+\kappa}\right). \tag{3.90}$$

Since $C_3 \le C_{5+\kappa} \le C_{6+\kappa} \le C_{8+\kappa} = O(\|V'\|_{8+\kappa,[-3R,3R]})$ and $b_{max} \le 1$, the sum of estimates (3.77), (3.81), (3.87), (3.88), (3.90) is $O(b_{max}^{-2} \|V'\|_{8+\kappa,[-3R,3R]})$. □

3.8. *Away from the support*

We study in this subsection the g-kernels $g^{\kappa';\kappa,ext}_{pot}$ obtained by assuming that $h = h^{ext}$, whence $|a_T| \ge 2R$. We want to prove that the operators

$$\mathcal{H}^{\kappa,ext}_{pot} : h \mapsto \left(\mathcal{H}^{\kappa,ext}(h) : (a,b) \mapsto \int da_T \int_{-b_{max}}^{b_{max}} db_T \; g^{\kappa,ext}_{pot}(a,b;a_T,b_T) h(a_T,b_T) \right) \tag{3.91}$$

and

$$\mathcal{H}^{\kappa+1;\kappa,ext}_{pot} : h \mapsto \left(\mathcal{H}^{\kappa+1;\kappa,ext}(h) : (a,b) \mapsto \int da_T \int_{-b_{max}}^{b_{max}} db_T \; g^{\kappa+1;\kappa,ext}_{pot}(a,b;a_T,b_T) h(a_T,b_T) \right) \tag{3.92}$$

are bounded operators in $L^1(\Pi_{b_{max}})$. We shall actually only prove the statement for $\mathcal{H}^{\kappa+1;\kappa,ext}_{pot}$, and leave the similar proof of the statement for $\mathcal{H}^{\kappa,ext}_{pot}$ to the reader.

Lemma 3.8.

(i) $|||\mathcal{H}^{\kappa+1;\kappa,ext}|||_{L^1(\Pi_{b_{max}})\to L^1(\Pi_{b_{max}})} = O\left(b_{max}^{-1} \|V'\|_{4+\kappa,[-\frac{3}{2}R,\frac{3}{2}R]}\right)$;

(ii) $|||\mathcal{H}^{\kappa,ext}|||_{L^1(\Pi_{b_{max}})\to L^1(\Pi_{b_{max}})} = O\left(\|V'\|_{3+\kappa,[-\frac{3}{2}R,\frac{3}{2}R]}\right)$.

The same estimates hold for the L^∞-operator norms $||| \cdot |||_{L^\infty(\Pi_{b_{max}})\to L^\infty(\Pi_{b_{max}})}$.

As can be checked by looking at the details of the proof, norms $||| \cdot |||_{L^1(\Pi_{b_{max}})\to L^\infty(\Pi_{b_{max}})}$ are deduced from these by dividing by a volume factor $\mathrm{Vol}([-3R,3R] \times [-b_{max}, b_{max}]) \approx b_{max}$, which follows from the fact that the kernel $g^{\kappa';\kappa,ext}_{pot}$ is regular on the diagonal.

Proof.

(1) (operator norm of $\mathcal{H}_{pot}^{\kappa+1;\kappa,ext}$) By definition,

$$g_{pot}^{\kappa+1;\kappa,ext}(a,b;a_T,b_T) = \mathbf{1}_{|b|<b_{max}}\mathbf{1}_{|a_T|\ge 2R}\frac{1}{(1+\kappa)!}(-\mathrm{i}b)|b_T|^{\kappa} \times \mathcal{K}_{b_{max}}^{\kappa+1}(\chi_R V' f'_{z_T})(a). \tag{3.93}$$

Note that, since $|a_T| \ge 2R$ and $\mathrm{supp}(\chi_R) \subset B(0,\frac{3}{2}R)$, the function $\chi_R V' f'_{z_T}$ is regular and bounded. Furthermore, for every $n = 0, 1, \dots$, $\|(\chi_R V' f'_{z_T})^{(n)}\|_\infty = O(\|V'\|_{n,[-\frac{3}{2}R,\frac{3}{2}R]}$ $|a_T|^{-2})$ is integrable at infinity in a_T.
We consider in the computations only the part g^κ of the kernel $g_{pot}^{\kappa+1;\kappa,ext}$ obtained by replacing $\mathcal{K}^\kappa$ with the operators in factor of the bounded operator $1+\underline{\mathcal{K}}_{b_{max}}^{\kappa}$ in (2.13), (2.16), and distinguish the cases κ odd, κ even. Assume first κ is *odd*. Then

$$\begin{aligned}&|g^\kappa(a,b;a_T,b_T)|\\ &= \mathbf{1}_{|b|<b_{max}}\mathbf{1}_{|a_T|\ge 2R}\, O(|b_T|^{1+\kappa})\left(\left|(\chi_R V' f'_{z_T})^{(3+\kappa)}(a)\right| + b_{max}^{-(3+\kappa)}|(\chi_R V' f'_{z_T})(a)|\right)\\ &= O\left(b_{max}^{1+\kappa}\|V'\|_{3+\kappa,[-\frac{3}{2}R,\frac{3}{2}R]} + b_{max}^{-2}\|V'\|_{0,[-\frac{3}{2}R,\frac{3}{2}R]}\right)|a_T|^{-2}\\ &\qquad \mathbf{1}_{(a,b)\in[-\frac{3}{2}R,\frac{3}{2}R]\times[-b_{max},b_{max}]}.\end{aligned} \tag{3.94}$$

Hence $\mathcal{H}_{pot}^{\kappa+1;\kappa,ext}$ is a bounded operator on $L^1(\Pi_{b_{max}})$, with L^1-operator norm

$$\begin{aligned}\|\!|\mathcal{H}_{pot}^{\kappa+1;\kappa,ext}\|\!|_{L^1\to L^1} &= O\left(b_{max}^{1+\kappa}\|V'\|_{3+\kappa,[-\frac{3}{2}R,\frac{3}{2}R]} + |b|_{max}^{-2}\|V'\|_{0,[-\frac{3}{2}R,\frac{3}{2}R]}\right)\\ &\qquad\cdot \mathrm{Vol}([-3R,3R]\times[-b_{max},b_{max}])\\ &= O\left(b_{max}^{3+\kappa}\|V'\|_{3+\kappa,[-\frac{3}{2}R,\frac{3}{2}R]} + \|V'\|_{0,[-\frac{3}{2}R,\frac{3}{2}R]}\right)b_{max}^{-1}.\end{aligned} \tag{3.95}$$

Assume now that κ is *even*. Then (using (3.94) and (B.23)) $|g^\kappa(a,b;a_T,b_T)| \le O(b_{max}^{-2}$ $\|V'\|_{0,[-\frac{3}{2}R,\frac{3}{2}R]}\mathbf{1}_{(a,b)\in[-\frac{3}{2}R,\frac{3}{2}R]\times[-b_{max},b_{max}]})|a_T|^{-2}$, plus

$$\begin{aligned}&\mathbf{1}_{|b|<b_{max}}O(|b_T|^{1+\kappa})\left|\mathbf{1}_{|a_T|\ge 2R}\, p.v.(\frac{1}{x}) * (\chi_R V' f'_{z_T})^{(3+\kappa)}(a)\right|\\ &= \mathbf{1}_{|b|<b_{max}}O(|b_T|^{1+\kappa})\left|\mathbf{1}_{|a_T|\ge 2R}\, p.v.(\frac{1}{x^2}) * (\chi_R V' f'_{z_T})^{(2+\kappa)}(a)\right|\\ &= O(|b_T|^{1+\kappa})\left[\mathbf{1}_{(a,b)\in[-3R,3R]\times[-b_{max},b_{max}]}\|\mathbf{1}_{|a_T|\ge 2R}(\chi_R V' f'_{z_T})^{(4+\kappa)}\|_\infty\right.\\ &\qquad\left.+\mathbf{1}_{(a,b)\in(\mathbb{R}\setminus[-3R,3R])\times[-b_{max},b_{max}]}\frac{1}{a^2}\|\mathbf{1}_{|a_T|\ge 2R}(\chi_R V' f'_{z_T})^{(2+\kappa)}\|_\infty\right].\end{aligned} \tag{3.96}$$

We conclude by multiplying by $\bar\chi_R + (1-\bar\chi_R)$ that $\mathcal{H}_{pot}^{\kappa+1;\kappa,ext}h = \tilde h^{int} + \tilde h^{ext}$, where

$$\|\tilde h^{int}\|_{L^1} \le O\left(b_{max}^{3+\kappa}\|V'\|_{4+\kappa,[-\frac{3}{2}R,\frac{3}{2}R]} + \|V'\|_{0,[-\frac{3}{2}R,\frac{3}{2}R]}\right) b_{max}^{-1}\,\|h\|_{L^1} \tag{3.97}$$

and

$$\|\tilde h^{ext}\|_{L^1} \le O(b_{max}^{2+\kappa}\|V'\|_{2+\kappa,[-\frac{3}{2}R,\frac{3}{2}R]})\,\|h\|_{L^1}. \tag{3.98}$$

(2) (operator norm of $\mathcal{H}^{\kappa,ext}_{pot}$) With respect to (3.95), (3.97) and (3.98), we remove one order of differentiation and one power of b^{-1}_{max}, and exchange parities. Thus

$$|||\mathcal{H}^{\kappa,ext}_{pot}|||_{L^1\to L^1} = O\left(b^{2+\kappa}_{max}\|V'\|_{2+\kappa,[-\frac{3}{2}R,\frac{3}{2}R]} + \|V'\|_{0,[-\frac{3}{2}R,\frac{3}{2}R]}\right) \tag{3.99}$$

for κ even, while for κ odd, $\mathcal{H}^{\kappa,ext}_{pot}h = \tilde{h}^{int} + \tilde{h}^{ext}$, with

$$\|\tilde{h}^{int}\|_{L^1} \le O\left(b^{2+\kappa}_{max}\|V'\|_{3+\kappa,[-\frac{3}{2}R,\frac{3}{2}R]} + \|V'\|_{0,[-\frac{3}{2}R,\frac{3}{2}R]}\right)\|h\|_{L^1} \tag{3.100}$$

and

$$\|\tilde{h}^{ext}\|_{L^1} \le O(b^{2+\kappa}_{max}\|V'\|_{1+\kappa,[-\frac{3}{2}R,\frac{3}{2}R]})\|h\|_{L^1}. \tag{3.101}$$

Remark. When $b_{max} = +\infty$, the adjoint of $\mathcal{H}^{\kappa,ext}_{pot}$ or $\mathcal{H}^{\kappa+1;\kappa,ext}_{pot}$ is the generator of a *signed jump Markov process* with good properties. Namely, for all a_T, b_T,

$$\int_{-\infty}^{+\infty} da \int_{-\infty}^{+\infty} db\, g^{\kappa,ext}_{pot}(a,b;a_T,b_T) = 0 \tag{3.102}$$

since $K^{\kappa}_{+\infty}(s=0) = 0$, hence $\mathcal{L}^{\kappa,ext}_{pot} := (\mathcal{H}^{\kappa,ext}_{pot})^{\dagger}$ may we written in the following form,

$$\mathcal{L}^{\kappa,ext}_{pot}(\phi)(a_T,b_T) = \int_{-\infty}^{+\infty} da \int_{-\infty}^{+\infty} db\,(\phi(a,b)-\phi(a_T,b_T))g^{\kappa,ext}_{pot} \times (a,b;a_T,b_T). \tag{3.103}$$

For a bona fide Markov process, the function $(a,b) \mapsto -g^{\kappa,ext}_{pot}(a,b;a_T,b_T) \ge 0$ would be the jump rate from (a_T,b_T) to (a,b), and one would have $-\int da\, db\, g^{\kappa,ext}_{pot}(a,b;a_T,b_T) = 1$. Here g is a signed kernel, so the probabilistic interpretation fails stricto sensu. However, the L^1 semi-group generated by $\mathcal{H}^{\kappa,ext}_{pot}$ has good properties because $\mathcal{H}^{\kappa,ext}_{pot}$ is a bounded operator (see Section 4).

3.9. Boundary terms

Recall the *horizontal boundary terms* h-bdry$_0$ (3.31), h-bdry$^{(1)}_{pot}$ (3.44), and the *vertical boundary terms*, v-bdry$_0$ (3.32), v-bdry$^{(0)}_{pot}$ (3.40) and v-bdry$^{(2)}_{pot}$ (3.50). We adopt the following notations. First let

$$\partial_R \Pi_{b_{max}} := \partial([-3R,3R]\times[-b_{max},b_{max}]) = ([-3R,3R]\times\{\pm b_{max}\}) \cup (\{\pm 3R\} \times [-b_{max},b_{max}]), \tag{3.104}$$

and, for $h : [-3R,3R]\times[-b_{max},b_{max}] \to \mathbb{C}$, let $\partial_R h := h\big|_{\partial_R \Pi_{b_{max}}}$ be the restriction of h to the boundary. Horizontal boundary terms h-bdry$^{\cdot}_{\cdot}$ (where the dots stand for various lower and upper indices) are of the form h-bdry$^{\cdot}_{\cdot}(b_{max};\partial_R h)$ − h-bdry$^{\cdot}_{\cdot}(-b_{max};\partial_R h)$; similarly, vertical boundary terms v-bdry$^{\cdot}_{\cdot}$ are of the form v-bdry$^{\cdot}_{\cdot}(3R;\partial_R h)$ − v-bdry$^{\cdot}_{\cdot}(-3R;\partial_R h)$.

These terms depend a priori on the restriction of h to $\partial_R \Pi_{b_{max}}$, which is incompatible with the L^1-norms. In order to avoid that, we replace $\mathfrak{f}_{a_T\pm ib_{max}}$ with $\mathcal{C}^{\kappa'}\left((a,b)\mapsto \mathcal{K}^{\kappa'}_{b_{max}}(\mathfrak{f}_{a_t\pm ib_{max}})(a)\right)$, $\kappa' = \kappa, \kappa+1$ so that

$$(\text{h-bdry}_0 + \text{h-bdry}^{(1)}_{pot})(b_{max};\partial_R h) - (b_{max} \leftrightarrow -b_{max}) = \mathcal{C}^{\kappa+1}(\mathcal{H}^{\kappa+1;\kappa,h-bdry}(\partial_R h)) = \mathcal{C}^{\kappa}(\mathcal{H}^{\kappa,h-bdry}(\partial_R h)) \tag{3.105}$$

with $\mathcal{H}^{\kappa+1;\kappa,h-bdry}, \mathcal{H}^{\kappa,h-bdry} : L^\infty(\partial_R \Pi_{b_{max}}) \to (L^1 \cap L^\infty)(\Pi_{b_{max}})$,

$$\mathcal{H}^{\kappa+1;\kappa}_{\text{h-bdry}}(\partial_R h)(a,b) = \int_{-3R}^{3R} da_T\, g^{\kappa+1;\kappa,}_{\text{h-bdry}}(a,b;a_T)\partial_R h(a_T, b_{max}) - (b_{max} \leftrightarrow -b_{max}) \quad (3.106)$$

$$\mathcal{H}^{\kappa}_{\text{h-bdry}}(\partial_R h)(a,b) = \int_{-3R}^{3R} da_T\, g^{\kappa}_{\text{h-bdry}}(a,b;a_T)\partial_R h(a_T, b_{max}) - (b_{max} \leftrightarrow -b_{max}) \quad (3.107)$$

and

$$g^{\kappa';\kappa}_{\text{h-bdry}}(a,b;a_T) = -\left\{\frac{\beta}{4}\frac{b_{max}^{1+\kappa}}{(1+\kappa)!}\text{Im}\,[(M_T^N + M_T)(a_T + \mathrm{i}b_{max})] + b_{max}^{2+\kappa}V''(a_T)\right\} \mathcal{K}^{\kappa'}_{b_{max}}(\mathfrak{f}_{a_T+\mathrm{i}b_{max}})(a). \quad (3.108)$$

Similarly, we replace $f_{\pm 3R+\mathrm{i}b}$ with $\mathcal{C}^{\kappa'}\Big((a,b) \mapsto \mathcal{K}^{\kappa'}_{b_{max}}(\mathfrak{f}_{\pm 3R+\mathrm{i}b})(a)\Big)$, $\kappa' = \kappa, \kappa+1$, so that

$$(\text{v-bdry}_0 + \text{v-bdry}^{(0)}_{pot} + \text{v-bdry}^{(2)}_{pot})(3R;\partial_R h) - (R \leftrightarrow -R) = \mathcal{C}^{\kappa+1}(\mathcal{H}^{\kappa+1;\kappa}_{v-bdry}(\partial_R h)) = \mathcal{C}^{\kappa}(\mathcal{H}^{\kappa}_{v-bdry}(\partial_R h)) \quad (3.109)$$

with $\mathcal{H}^{\kappa+1;\kappa}_{\text{v-bdry}}, \mathcal{H}^{\kappa}_{\text{v-bdry}} : L^\infty(\partial_R \Pi_{b_{max}}) \to (L^1 \cap L^\infty)(\Pi_{b_{max}})$,

$$\mathcal{H}^{\kappa+1;\kappa}_{\text{v-bdry}}(\partial_R h)(a,b) = \int_{-b_{max}}^{b_{max}} db_T\, g^{\kappa+1;\kappa}_{\text{v-bdry}}(a,b;b_T)\partial_R h(3R, b_T) - (R \leftrightarrow -R) \quad (3.110)$$

$$\mathcal{H}^{\kappa}_{\text{v-bdry}}(\partial_R h)(a,b) = \int_{-b_{max}}^{b_{max}} db_T\, g^{\kappa}_{\text{v-bdry}}(a,b;b_T)\partial_R h(3R, b_T) - (R \leftrightarrow -R) \quad (3.111)$$

and

$$g^{\kappa';\kappa}_{\text{v-bdry}}(a,b;b_T) = -(-\mathrm{i}b_T)\left\{\frac{\beta}{4}\frac{|b_T|^\kappa}{(1+\kappa)!}\text{Re}\,[(M_T^N + M_T)(3R + \mathrm{i}b_T)] + |b_T|^\kappa V'(3R) + |b_T|^{\kappa+2}V'''(3R)\right\} \mathcal{K}^{\kappa'}_{b_{max}}(\chi_{|R|}\,\mathfrak{f}_{3R+\mathrm{i}b_T})(a). \quad (3.112)$$

Lemma 3.9. *Let $\kappa \geq 0$ and $\kappa' = \kappa, \kappa+1$. Then*

(i) $|||\mathcal{H}^{\kappa';\kappa}_{\text{h-bdry}}|||_{L^\infty(\partial_R \Pi_{b_{max}}) \to L^\infty(\Pi_{b_{max}})} = b_{max}^{-3-(\kappa'-\kappa)}\left\{O(\frac{1}{b_{max}}) + O(b_{max}\|V''\|_{0,[-3R,3R]}\right\}$ *and* $|||\mathcal{H}^{\kappa';\kappa}_{\text{h-bdry}}|||_{L^\infty(\partial_R \Pi_{b_{max}}) \to L^1(\Pi_{b_{max}})} = b_{max}^{-2-(\kappa'-\kappa)}\left\{O(\frac{1}{b_{max}}) + O(b_{max}\|V''\|_{0,[-3R,3R]}\right\}$;

(ii)

$$|||\mathcal{H}^{\kappa';\kappa}_{\text{v-bdry}}|||_{L^\infty(\partial_R \Pi_{b_{max}}) \to L^\infty(\Pi_{b_{max}})} = b_{max}^{-1-(\kappa'-\kappa)}\left\{O(\frac{1}{b_{max}}) + O(\|V'\|_{0,[-3R,3R]}) + O(b_{max}^2\|V'''\|_{0,[-3R,3R]})\right\} \quad (3.113)$$

and

$$|||\mathcal{H}^{\kappa}_{\text{v-bdry}}|||_{L^\infty(\partial_R \Pi_{b_{max}}) \to L^1(\Pi_{b_{max}})}$$

$$= b_{max}^{-(\kappa'-\kappa)} \left\{ O(\frac{1}{b_{max}}) + O(\|V'\|_{0,[-3R,3R]}) \right.$$
$$\left. + \; O(b_{max}^2 \|V'''\|_{0,[-3R,3R]}) \right\}. \tag{3.114}$$

As in the two previous subsections, L^1-estimates and L^∞-estimates differ only by a volume factor $\approx b_{max}$.

Proof.

(i) Immediate consequence of the bounds $\left|\mathrm{Im}\,[(M_T^N + M_T)(a_T + \mathrm{i}b_{max})]\right| \le 2/b_{max}$, b_{max} $|V''(a_T)| \le b_{max}\|V''\|_{0,[-3R,3R]}$ and $|\mathcal{K}_{b_{max}}^{\kappa'}(\chi_R \mathfrak{f}_{a_T \pm \mathrm{i}b_{max}})(a)| = O(b_{max}^{-(4+\kappa')})\; O(\frac{1}{(1+|a|)^2})$ (as seen by using (2.13) when κ' is even, and (2.16), (B.23) when κ' is odd).

(ii) Immediate consequence of the bounds $\left|\mathrm{Re}\,[(M_T^N + M_T)(a_T + \mathrm{i}b_{max})]\right| \le 2/b_{max}$, $|V'(3R)| \le \|V'\|_{0,[-3R,3R]}$, $b_T^2|V'''(3R)| \le b_{max}^2\|V'''\|_{0,[-3R,3R]}$, and $|\mathcal{K}_{b_{max}}^{\kappa'}(\chi_R\, \mathfrak{f}_{\pm 3R+\mathrm{i}b_T})(a)| = O(b_{max}^{-(2+\kappa')})\; O(\frac{1}{(1+|a|)^2})$. $\square$

4. Gaussianity of the fluctuation process

In this section, we prove our Main Theorem (see Section 1.3), namely, we prove that the finite-N fluctuation process $(Y_t^N)_{t\ge 0}$ converges weakly in $C([0,T], H_{-14})$ to a fluctuation process $(Y_t)_{t\ge 0}$, which is the unique solution of a martingale problem that we solve explicitly in terms of the solution of (1.34).

We fix once and for all: $b_{max} = \frac{1}{2}$.

Summarizing what we have found up to now and applying Proposition 1.3, we find for $\kappa = 0, 1, 2, \ldots$:

$$d\langle Y_t^N, \mathcal{C}^\kappa h_t\rangle = \frac{1}{2}(1 - \frac{\beta}{2})\langle X_t^N, (\mathcal{C}^\kappa h_t)''\rangle\, dt + \frac{1}{\sqrt{N}} \sum_i (\mathcal{C}^\kappa h_t)'(\lambda_t^i) dW_t^i \tag{4.1}$$

where $(h_t)_{t\le T}$ is the solution of the evolution equation

$$\frac{dh_t}{dt} = \mathcal{H}^\kappa(t)h(t). \tag{4.2}$$

On the other hand, the process $(Y_t^N)_{t\ge 0}$ is a solution of the following *martingale problem*: if $\bar\phi = \{\phi_j\}_{1\le j\le k}$ is a family of test functions in $C_c^\infty(\mathbb{R},\mathbb{R})$, $F \in C_b^2(\mathbb{R}^k,\mathbb{R})$, then, letting $F_{\bar\phi}(Y_\cdot^N) := F(\langle Y_\cdot^N, \phi_1\rangle, \ldots, \langle Y_\cdot^N, \phi_k\rangle)$,

$$\Phi_t^{T,N}(Y^N) := F_{\bar\phi}(Y_T^N) - F_{\bar\phi}(Y_t^N) - \int_t^T ds\, L_s^N F_{\bar\phi}(Y_s^N) \tag{4.3}$$

is a martingale, where

$$L_s^N F_{\bar\phi}(Y_s^N) := \sum_{j=1}^k \frac{\partial F_{\bar\phi}}{\partial x_j}(Y_s^N) \Bigg(\langle Y_s^N, \frac{\beta}{4} \int \frac{\phi_j'(\cdot) - \phi_j'(y)}{\cdot - y}(X_t + X_t^N)(dy) - V'(\cdot)\phi_j'(\cdot)\rangle$$
$$+ \frac{1}{2}(1 - \frac{\beta}{2})\langle X_t^N, \phi_j''\rangle \Bigg) + \frac{1}{2}\sum_{j,l=1}^k \partial_{jl}^2 F_{\bar\phi}(Y_s^N)\, \langle X_t^N, \phi_j'\phi_l'\rangle \tag{4.4}$$

(see [9], p. 28–29).

We now use an exponential functional of the process to derive the limit law.

Definition 4.1. For $\kappa = 0, 1, 2, \ldots$ and $h \in (L^1 \cap L^\infty)(\Pi_{b_{max}})$, let

$$\phi_h(Y_t^N) := e^{\mathrm{i}\langle Y_t^N, \mathcal{C}^\kappa h\rangle}. \tag{4.5}$$

Itô's formula implies as in ([9], p. 29)

$$d\phi_{h_t}(Y_t^N) = \phi_{h_t}(Y_t^N)\left(\frac{1}{2}\mathrm{i}(1-\frac{\beta}{2})\langle X_t^N, (\mathcal{C}^\kappa h_t)''\rangle - \frac{1}{2}\langle X_t^N, ((\mathcal{C}^\kappa h_t)')^2\rangle\right)dt, \tag{4.6}$$

where $(\mathcal{C}^\kappa h_t)_{0\le t\le T}$ is the solution of (1.9) for N finite, from which for $0 \le t \le T$ (letting *formally* $N \to \infty$)

$$\mathbb{E}[\phi_{h_T}(Y_T)\big|\mathcal{F}_t] = \phi_{h_t}(Y_t)\ \exp\left(\frac{1}{2}\int_t^T \left[\mathrm{i}(1-\frac{\beta}{2})\big\langle X_s, (\mathcal{C}^\kappa h_s)''\big\rangle - \big\langle X_s, ((\mathcal{C}^\kappa h_s)')^2\big\rangle\right] ds\right) \tag{4.7}$$

where $(h_t)_{0\le t\le T}$ is now the solution of the asymptotic equation (1.10). Since h_s, $t \le s \le T$ are linear in h_T, the term in the exponential in (4.7) is a sum of a linear and a quadratic term in h_T, giving resp. the expectation and the variance of a Gaussian process (see Israelsson, Section 2.6 for more details).

The strategy of the proof, following closely the proof in (Israelsson [9], Section 2), is the following:

(A) *find bounds* for $\mathbb{E}[\sup_{s\le T}|\langle Y_s^N, \phi\rangle|]$, $\mathbb{E}[\sup_{s\le T}|\langle Y_s^N, \int \frac{\phi'(\cdot)-\phi'(y)}{\cdot - y} X_s^N(dy)\rangle|]$ and $\mathbb{E}[\sup_{s\le T}|\langle Y_s^N, V'(\cdot)\phi'(\cdot)\rangle|]$ (see first line in (1.36) or (4.4));

(B) prove a *tightness property* for the family of processes Y^N, implying the existence of a (non necessarily unique) limit in law;

(C) prove that *any* weak limit Y of the $(Y^N)_{n\ge 1}$ *satisfies the limit martingale problem*, i.e.

$$\Phi_t^T(Y) = F_{\bar\phi}(Y_T) - F_{\bar\phi}(Y_t) - \int_t^T ds\, L_s F_{\bar\phi}(Y_s) \tag{4.8}$$

is a martingale, where

$$\begin{aligned} L_s F_{\bar\phi}(Y_s) := \sum_{j=1}^k \frac{\partial F_{\bar\phi}}{\partial x_j}(Y_s)\Bigg(\langle Y_s, \frac{\beta}{2}\int \frac{\phi_j'(\cdot)-\phi_j'(y)}{\cdot - y} X_t(dy) - V'(\cdot)\phi_j'(\cdot)\rangle + \\ + \frac{1}{2}(1-\frac{\beta}{2})\langle X_t, \phi_j''\rangle\Bigg) + \frac{1}{2}\sum_{j,l=1}^k \partial_{jl}^2 F_{\bar\phi}(Y_s)\ \langle X_t, \phi_j'\phi_l'\rangle \end{aligned} \tag{4.9}$$

(obtained formally from (4.4) by letting $N \to \infty$);

(D) prove that there exists only *one* measure with given initial measure satisfying (4.9), and that it is Gaussian, and satisfies (4.7).

To prove (C) one must essentially prove that $(\Phi_t^T - \Phi_t^{T,N})(Y^N)$ is small, since Φ_t^T is continuous (see [9], pp. 47–48).

The proof is borrowed from Israelsson, where it takes up a few pages, see [9]. The main bound, compare with [9], Lemma 15, is the following "H_8"-bound,

$$\mathbb{E}[\sup_{s\le T}|\langle Y_s^N, \phi\rangle|] \le C_T\|\phi\|_{H_8}, \tag{4.10}$$

The above H_8-bound is obtained by integrating (1.36), namely,

$$\mathbb{E}\Big[\sup_{s\le T}|\langle Y_s^N,\phi\rangle|\Big] \le C\Big(\mathbb{E}[|\langle Y_0^N,\phi\rangle|] + \int_0^T ds\,\mathbb{E}\Big[|\langle Y_s^N, V'(\cdot)\phi'(\cdot)\rangle| + |\langle Y_s^N, \int \frac{\phi'(x)-\phi'(\cdot)}{x-\cdot}(X_s^N(dx)+X_s(dx))\rangle|\Big] + \int_0^T ds\,\mathbb{E}\Big[|\int \phi''(x)X_s^N(dx)|\Big] + \mathbb{E}\Big[\sup_{s\le T}|M_s|\Big]\Big) \tag{4.11}$$

where $M_s := \frac{1}{\sqrt{N}}\int_0^s \sum_{i=1}^N \phi'(\lambda_s^i)dW_s^i$ is a martingale. All these terms are bounded as in [9], Lemma 15, using easy, V-independent inequalities *and* the following *fundamental estimate*,

Lemma 4.2. *There exists a constant C depending on T such that, for all $t \le T$,*

$$\mathbb{E}[|\langle Y_t^N,\phi\rangle|^2] \le C\|\phi\|_{H_7}^2, \tag{4.12}$$

itself an immediate consequence of

$$\sqrt{\mathbb{E}[|\langle Y_T^N,\mathfrak{f}_z\rangle|^2]} \equiv \sqrt{\mathbb{E}[|N(M_t^N - M_t)(z)|^2]} \le C|b|^{-6}, \quad b := \mathrm{Im}\, z \in [-b_{max}, b_{max}] \tag{4.13}$$

see (4.15) .

Note the loss of regularity with respect to ([9], Lemma 14), where one has $\|\phi\|_{H_2}$ in the r.-h.s. This changes the bounds in the course of the proof of ([9], Lemma 15), see in particular p. 43, l. 5, where estimates are proved using Israelsson's Lemma 8 for $q = 2$ (with $q-1$ playing the same rôle as our κ): the latter lemma yields (with our notations) a bound on $\|\mathcal{K}_{b_{max}}^{\kappa}(f)\|_{L^1}$ in terms of $\|f\|_{L^1} + \|f\|_{H_{\kappa+2}}$, resp. $\|f\|_{L^1} + \|f\|_{H_{\kappa+3}}$, depending whether κ is even, resp. odd; because of the loss of regularity one must take $\kappa = 5$ ($q = 6$, same exponent as in (4.13)), hence the "H_8"$=H_{\kappa+3}$-bound.

The proof of (C), see [9], Lemmas 17 and 20, mainly depends on a bound for $C(\phi) := \mathbb{E}\Big[\sup_{0\le s\le T}\Big|\Big\langle Y_s^N, \int \frac{\phi'(\cdot)-\phi'(y)}{\cdot - y} X_s^N(dy)\Big\rangle\Big|\Big]$. Assume $\phi = \mathcal{C}^{\kappa}h$, $\kappa \in \mathbb{N}$; then $C(\phi)$ is bounded in terms of the integral against the measure $|b|^{\kappa+1}|h(a,b)|\,da\,db$ on $\Pi_{b_{max}}$ of the random function $N(M_t^N(z) - M_t(z))^2$, averaging to $O(\frac{1}{N}|b|^{-12})$ by (4.13). Therefore the integral converges if $\kappa \ge 11$, and is bounded, as recalled in the previous paragraph, by $O(1/N)\|\phi\|_{H_{14}}$.

Then the tightness property (B) is proved using a lemma due to Mitoma [14] and the above estimates (see [9], Section 2.4); the Sobolev space H_{-14} in our Main Theorem is such that there exists a nuclear mapping $H_{14} \to H_8$ (see (4.10)), as follows from Treves [23], which is a requirement in Mitoma's hypotheses.

Finally, letting $\phi_i = \mathcal{C}^0 h_i$, $i = 1,\dots,k$, (D) is proved by computing $\mathbb{E}\left[\exp \mathrm{i}\left(\langle Y_{t_1},\phi_1\rangle + \cdots + \langle Y_{t_k},\phi_k\rangle\right)\right]$, $0 \le t_k \le \dots \le t_1 \le T$ by induction on k using the assumed martingale property of the limit(s) and solving in terms of the time-evolved functions $h_i(t)$, $i = 1,\dots,k$. For $k = 2$ we obtain (4.7).

So everything boils down to the proof of the above lemma.

Proof of Lemma 4.2. Let $\phi \in C_c^{\infty}$. Consider its standard Stieltjes decomposition of order 5, $\phi = \mathcal{C}^5 h$ (take $b_{max} = \frac{1}{2}$). Then (using (2.21) and Cauchy–Schwarz's inequality)

$$|\langle Y_t^N,\phi\rangle|^2 \le \|h\|_{L^2(\Pi_{b_{max}})}^2 \int_{\Pi_{b_{max}}} da\,db\,b^{12}(N|M_t(z) - M_t^N(z)|)^2 \tag{4.14}$$

and $\|h\|_{L^2(\Pi_{b_{max}})} = O(\|\phi\|_{H^7})$ (as follows from the kernel representation (2.13) of the standard Stieltjes decomposition, together with Parseval–Bessel's formula). Hence (4.12) follows if we can show that

$$\mathbb{E}[|N(M_T^N(z) - M_T(z))|^2] = \mathbb{E}[|\langle Y_T^N, \mathfrak{f}_z\rangle|^2] \le Cb^{-12} \tag{4.15}$$

for $0 < b := \mathrm{Im}\, z \le \frac{1}{2}$; compare with Israelsson [9], Proposition 1, where a much better bound in $O((\ln(1+1/b)/b)^2)$ is proved. Note, however, that there is, to the best of our understanding, a flaw in Israelsson's proof, see (4.51), whence (despite some efforts) we find in fact a bound in $O(b^{-12})$ in the harmonic case too. Further, introducing the stopping time

$$\tau := \inf\{0 < t \le T : \sup_{1\le i\le N} |\lambda_t^i| > R(T)\}, \tag{4.16}$$

(and letting by convention $\tau \equiv T$ if $\sup_{0\le t\le T}\sup_{1\le i\le N}|\lambda_t^i| \le R(T)$), see Lemma 5.1, we have, using the large deviation bound of Section 5, and the obvious deterministic bound $M_t^N(z) \le |\mathrm{Im}\,(z)|^{-1}$,

$$\mathbb{E}[|N(M_T^N(z) - M_T(z))|^2] \le \mathbb{E}[|N(M_\tau^N(z) - M_\tau(z))|^2] + Ce^{-cN}N^2|\mathrm{Im}\, z|^{-2}. \tag{4.17}$$

So (4.15) holds provided we show that

$$\mathbb{E}[|N(M_\tau^N(z) - M_\tau(z))|^2] \le Cb^{-12} \tag{4.18}$$

where now by construction $\sup_i |\lambda_t^{N,i}| \le R(T)$ for all $t \le \tau$, a support condition which is essential for the subsequent computations.

Before we can do that, however, we need a long preliminary discussion. Indeed, Israelsson's proof of this fact in his Proposition 1 does not carry through immediately to the case of a general V, because it relies in an essential way on the bounds on characteristics. As explained in the Introduction though, the deterministic characteristics due to the $\left(\frac{1}{x}\right)$-potential, see Section 3.1, to which we can safely add the other transport generators without much change, yield the most singular contribution, so our strategy is to treat the non-local term $\mathcal{H}_{nonlocal}$ as a *perturbation* of $\mathcal{H}_{transport}$, by using a Green function expansion. Note however the *twist* here: the operators $\mathcal{H}^{\kappa}_{transport}$ and $\mathcal{H}^{\kappa}_{nonlocal}$ are endomorphisms of $L^1(\Pi_{b_{max}})$, but $\mathcal{H}^{\kappa+1;\kappa}_{nonlocal}$ intertwines in some sense *two different copies* of $L^1(\Pi_{b_{max}})$, with different mappings, $\mathcal{C}^\kappa$, vs. $\mathcal{C}^{\kappa+1}$ to $L^1(\mathbb{R})$. The intertwining is not trivial, in the sense that $b\mathcal{H}^{\kappa+1;\kappa}_{nonlocal} \ne \mathcal{H}^{\kappa}_{nonlocal}$. This leads us to introduce the following operator-valued matrices.

Definition 4.3.

1. Let $L^1[\varepsilon](\Pi_{b_{max}}) := (\mathbb{R}[\varepsilon]/\varepsilon^3) \otimes L^1(\Pi_{b_{max}}) \simeq \varepsilon^0 \otimes L^1(\Pi_{b_{max}}) \oplus \varepsilon^1 \otimes L^1(\Pi_{b_{max}}) \oplus \varepsilon^2 \otimes L^1(\Pi_{b_{max}})$.
2. Let $\mathcal{H}[\varepsilon] : L^1[\varepsilon](\Pi_{b_{max}}) \to L^1[\varepsilon](\Pi_{b_{max}})$ be represented by the operator-valued matrix $\mathcal{H}[\varepsilon] := \begin{pmatrix} \mathcal{H}^0_{transport} & 0 & 0 \\ \mathcal{H}^{1,0}_{nonlocal} & \mathcal{H}^1_{transport} & 0 \\ 0 & \mathcal{H}^{2,1}_{nonlocal} & \mathcal{H}^2 \end{pmatrix}$.
3. Let Ev: $L^1[\varepsilon](\Pi_{b_{max}}) \to L^1(\Pi_{b_{max}})$ be the *evaluation mapping*,

$$\mathrm{Ev}(\varepsilon^0 \otimes h^0 + \varepsilon^1 \otimes h^1 + \varepsilon^2 \otimes h^2)(a,b) = h^0(a,b) + bh^1(a,b) + b^2h^2(a,b). \tag{4.19}$$

The $(2,2)$-coefficient $\mathcal{H}^2$ – the sum $\mathcal{H}^2_{transport} + \mathcal{H}^2_{nonlocal}$ – is coherent with the truncation. Another possibility (also coherent with Lemma 4.4, but introducing pointless complications) would be to consider the un-truncated infinite-dimensional matrix

$$\tilde{\mathcal{H}}[\varepsilon] := \begin{pmatrix} \mathcal{H}^0_{transport} & 0 & 0 & 0 & \cdots \\ \mathcal{H}^{1,0}_{nonlocal} & \mathcal{H}^1_{transport} & 0 & 0 & \cdots \\ 0 & \mathcal{H}^{2,1}_{nonlocal} & \mathcal{H}^2 & 0 & \cdots \\ 0 & 0 & \mathcal{H}^{3,2}_{nonlocal} & \mathcal{H}^3_{transport} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \text{acting on } \tilde{L}^1[\varepsilon](\Pi_{b_{max}}) \equiv \mathbb{R}[\varepsilon] \otimes L^1(\Pi_{b_{max}}),$$

with evaluation mapping $\mathrm{Ev}(\sum_{j\ge 0}\varepsilon^j \otimes h^j)(a,b) = \sum_{j\ge 0} b^j h^j$.

Lemma 4.4. *Let $(h_t)_{0\le t\le T} \in L^1[\varepsilon](\Pi_{b_{max}})$ be the solution of the time-evolution problem $\frac{dh_t}{dt} = \mathcal{H}[\varepsilon](t)h_t$ with terminal condition $h_T \equiv \varepsilon^0 \otimes h_T$.*
Then $f_t := \mathcal{C}^0 \circ \mathrm{Ev}(h_t)$ solves (1.34) with initial condition $\mathcal{C}^0(h_T)$.

Proof. By definition. □

Thus *our time-evolution operator is $\mathcal{H}[\varepsilon]$*. Let $\mathcal{H}_{transport}[\varepsilon] := \begin{pmatrix} \mathcal{H}^0_{transport} & 0 & 0 \\ 0 & \mathcal{H}^1_{transport} & 0 \\ 0 & 0 & \mathcal{H}^2_{transport} \end{pmatrix}$ and $\mathcal{H}_{nonlocal}[\varepsilon] := \begin{pmatrix} 0 & 0 & 0 \\ \mathcal{H}^{1,0}_{nonlocal} & 0 & 0 \\ 0 & \mathcal{H}^{2,1}_{nonlocal} & \mathcal{H}^2_{nonlocal} \end{pmatrix}$. The Green function first-order expansion then reads as follows,

$$U[\varepsilon](t,T) = U_{transport}[\varepsilon](t,T) - \int_t^T ds\, U[\varepsilon](t,s)\mathcal{H}_{nonlocal}[\varepsilon](s) \times U_{transport}[\varepsilon](s,T), \tag{4.20}$$

$U[\varepsilon](t,T)$, resp. $U_{transport}[\varepsilon](t,T)$ being the Green kernels (or evolution operators) obtained by integrating the time-inhomogeneous evolution systems generated by $\mathcal{H}[\varepsilon]$, resp. $\mathcal{H}_{transport}[\varepsilon]$, i.e. $u(t) = U[\varepsilon](t,T)u(T)$, resp. $u_{transport}(t) = U_{transport}[\varepsilon](t,T)u_{transport}(T)$, solves the linear equation $\frac{du(t)}{dt} = \mathcal{H}[\varepsilon](t)u(t)$, resp. $\frac{du_{transport}}{dt} = \mathcal{H}_{transport}[\varepsilon](t)u_{transport}(t)$. We shall actually require a *second-order* expansion of the Green kernel, obtained by iterating (4.20),

$$\begin{aligned} U[\varepsilon](t,T) &= U_{transport}[\varepsilon](t,T) - \int_t^T ds \\ &\times U_{transport}[\varepsilon](t,s)\mathcal{H}_{nonlocal}[\varepsilon](s)U_{transport}[\varepsilon](s,T) \\ &+ \int_t^T ds \int_t^s ds'\, U[\varepsilon](t,s')\mathcal{H}_{nonlocal}[\varepsilon](s')U_{transport}[\varepsilon](s,'s) \\ &\times \mathcal{H}_{nonlocal}[\varepsilon](s)U_{transport}[\varepsilon](s,T). \end{aligned} \tag{4.21}$$

Thus (considering a terminal condition $h_T \equiv \varepsilon^0 \otimes h_T$)

$$\begin{aligned} (\mathrm{Ev} \circ U[\varepsilon](t,T))(h_T)(a,b) &= U^0_{transport}(t,T)h_T(a,b) \\ &- b\int_t^T ds\, U^1_{transport}(t,s)\mathcal{H}^{1,0}_{nonlocal}(s)U^0_{transport}(s,T)h_T(a,b) \\ &+ b^2 \int_t^T ds \int_t^s ds'\, U^2(t,s')\mathcal{H}^{2,1}_{nonlocal}(s')U^1_{transport}(s,'s)\mathcal{H}^{1,0}_{nonlocal}(s) \\ &\times U^0_{transport}(s,T)h_T(a,b). \end{aligned} \tag{4.22}$$

Define

$$\|f\|_{(L^1\cap L^\infty)(\Pi_{b_{max}})} := \sup\left(\|f\|_{L^1(\Pi_{b_{max}})}, \|f\|_{L^\infty(\Pi_{b_{max}})}\right) \tag{4.23}$$

and, for an operator $\mathcal{H} : (L^1 \cap L^\infty)(\Pi_{b_{max}}) \to (L^1 \cap L^\infty)(\Pi_{b_{max}})$,

$$|||\mathcal{H}|||_{(L^1\cap L^\infty)(\Pi_{b_{max}})} := \sup_{\|f\|_{(L^1\cap L^\infty)(\Pi_{b_{max}})}=1} \|\mathcal{H}f\|_{(L^1\cap L^\infty)(\Pi_{b_{max}})}. \tag{4.24}$$

From the estimates proved in Section 3, that is, from Lemma 3.5 on the one hand, and Lemmas 3.7, 3.8, 3.9 on the other, we know that, for all $\kappa \in \mathbb{N}$ and $0 \le t \le s$:

$$|||U^\kappa_{transport}(t,s)|||_{L^1(\Pi_{b_{max}})\to L^1(\Pi_{b_{max}})}, |||U^\kappa_{transport}(t,s)|||_{L^\infty(\Pi_{b_{max}})\to L^\infty(\Pi_{b_{max}})} \le 1; \tag{4.25}$$

$$|||\mathcal{H}^{\kappa+1;\kappa}_{nonlocal}|||_{(L^1\cap L^\infty)(\Pi_{b_{max}})\to(L^1\cap L^\infty)(\Pi_{b_{max}})} = O(\|V'\|_{8+\kappa,[-3R,3R]}), \tag{4.26}$$

$$|||\mathcal{H}^{\kappa}_{nonlocal}|||_{(L^1\cap L^\infty)(\Pi_{b_{max}})\to(L^1\cap L^\infty)(\Pi_{b_{max}})} = O(\|V'\|_{7+\kappa,[-3R,3R]}). \tag{4.27}$$

Hence

Lemma 4.5. *Let $T > 0$ fixed, and $0 \le t \le s \le T$. Then*

$$|||U^\kappa(t,s)|||_{(L^1\cap L^\infty)(\Pi_{b_{max}})\to(L^1\cap L^\infty)(\Pi_{b_{max}})} \le e^{c\|V'\|_{7+\kappa,[-3R,3R]}t} \tag{4.28}$$

for some constant $c > 0$.

Proof. Results from (4.27), Tanabe [22], Theorem 4.4.1 (construction of fundamental solutions of temporally inhomogeneous equations) and Proposition 4.3.3 (bounded perturbations of generators of "stable" strongly continuous semi-groups, here of $\mathcal{H}^\kappa_{transport}$ by $\mathcal{H}^\kappa_{nonlocal}$). □

Proof of (4.18). We start from Itô's formula (1.8),

$$d\langle Y^N_t, f_t\rangle = \frac{1}{2}(1-\frac{\beta}{2})\langle X^N_t, f''_t\rangle dt + \frac{1}{\sqrt{N}}\sum_{i=1}^N f'_t(\lambda^i_t)dW^i_t \tag{4.29}$$

where

$$f_\tau(x) = \frac{\chi_R(x)}{x - z_\tau}, \qquad z_\tau \equiv z, \tag{4.30}$$

represented as $(\mathcal{C}^\kappa h_\tau)(x)$ for some κ (to be chosen later), with $h_\tau \ge 0$ defined as in (2.28), and f_t satisfies the finite-N evolution equation (1.9). Recall (see (2.31), (2.33)) that $\|h_\tau\|_{L^1(\Pi_{b_{max}})} \approx 1/b_\tau^{1+\kappa}$, $\|h_\tau\|_{L^\infty(\Pi_{b_{max}})} \approx 1/b_\tau^{3+\kappa}$.

Integrating, we must bound three terms:

(1) (initial condition) $\mathbb{E}|\langle Y^N_0, f_0\rangle|^2$;
(2) (drift term) $\mathbb{E}\left(\int_0^\tau dt\, |\langle X^N_t, f''_t\rangle|\right)^2$;
(3) ("martingale term") $\mathbb{E}\left(\frac{1}{\sqrt{N}}\int_0^\tau dt \sum_{i=1}^N f'_t(\lambda^i_t)dW^i_t\right)^2$

where $f_t = \mathcal{C}^\kappa h_t$.

Bounding (1) is easy. We use the 0-th order Stieltjes decomposition,

$f_0(x) = (\mathcal{C}^0 h_0)(x) = \int da \int_{-b_{max}}^{b_{max}} (-\mathrm{i}b)\, db\, \mathfrak{f}_z(x)\, h_0(a,b)$, together with the Cauchy–Schwarz inequality, and obtain as in (4.14)

$$|\langle Y_0^N, f_0\rangle|^2 \le \|h_0\|^2_{L^2(\Pi_{b_{max}})} \int_{\Pi_{b_{max}}} da\, db\, b^2 (N|M_0(z) - M_0^N(z)|)^2. \tag{4.31}$$

We use the obvious $L^1 - L^\infty$-bound, $\|h_0\|^2_{L^2(\Pi_{b_{max}})} \le \|h_0\|_{L^1(\Pi_{b_{max}})} \|h_0\|_{L^\infty(\Pi_{b_{max}})}$, and Lemma 4.5,

$$\|h_0\|_{L^1(\Pi_{b_{max}})} \|h_0\|_{L^\infty(\Pi_{b_{max}})} \le C \|h_\tau\|^2_{(L^1\cap L^\infty)(\Pi_{b_{max}})} \le C'/b_\tau^6 \tag{4.32}$$

with C, C' are constants depending on T, $R \equiv R(T)$ and $\|V'\|_{7,[-3R,3R]}$.

There remains to bound the integral in the r.-h.s. of (4.31), using our Assumption (iii) on the initial measure, see (1.38). We split $\int_{\Pi_{b_{max}}}(\cdots)$ into $\int_{[-2R,2R]\times[-b_{max},b_{max}]}(\cdots) + \int_{(\mathbb{R}\setminus[-2R,2R])\times[-b_{max},b_{max}]}(\cdots)$. The integral over $[-2R, 2R] \times [-b_{max}, b_{max}]$ is $O(1)$. As for the integral over $z \in (\mathbb{R} \setminus [-2R, 2R]) \times [-b_{max}, b_{max}]$, we first remark that

$$N|M_0^N(z) - M_0(z)| = \langle Y_0^N, \chi_R \mathfrak{f}_z\rangle \tag{4.33}$$

and

$$\chi_R(x)\mathfrak{f}_z(x) = \int da' \int_{-b_{max}}^{b_{max}} (-\mathrm{i}b')\, db'\, \frac{1}{x - z'} \mathcal{K}^0_{b_{max}}(\chi_R \mathfrak{f}_z)(a') \tag{4.34}$$

with (see (2.13)) $\mathcal{K}^0_{b_{max}}(\chi_R\mathfrak{f}_z)(a') = O(\frac{1}{|a|}) O(\frac{1}{1+|a'|^2})$. Hence (using Cauchy–Schwarz's inequality and (1.38) once again)

$$\begin{aligned}
&\mathbb{E} \int_{|a|>2R} da \int_{|b|<b_{max}} db\, b^2 \left(N|M_0^N(z) - M_0(z)|\right)^2 \\
&\le C\, \mathbb{E} \int_{(\mathbb{R}\setminus[-2R,2R])\times[-b_{max},b_{max}]} \frac{b^2\, da\, db}{|a|^2} \left| \int da' \int_{-b_{max}}^{b_{max}} db'\, \frac{N|b'|\, |M_0^N(z') - M_0(z')|}{1+|a'|^2} \right|^2 \\
&\le C' \int \frac{da'}{1+|a'|^2} \int_{-b_{max}}^{b_{max}} db'\, \mathbb{E}[(N|b'|\, |M_0^N(z') - M_0(z')|)^2] = O(1).
\end{aligned} \tag{4.35}$$

All together we have proved: $\mathbb{E}|\langle Y_0^N, f_0\rangle|^2 = O(b_T^{-6})$.

The bound for (2) is essentially pathwise, but more subtle and relies on our perturbative expansion for the Green kernel, which yields the optimal exponent of $1/b_\tau$. *Main term* is obtained by replacing f_t'' in (2) with $(\mathcal{C}^0 u_t)''$, where $u_t := U_{transport}(t,\tau)h_\tau$. By assumption $h_\tau \ge 0$, so (see (3.58)) $\tilde{u}_t := \tilde{U}_{transport}(t,\tau)h_\tau = |u_t| \ge 0$ for $0 \le t \le \tau$, and these terms may be bounded as in Israelsson in a probabilistic way, by using the characteristic estimates proved in Section 3.

First (using $|(\mathfrak{f}_z)''(x)| \le \frac{2}{|x-z|^3} \le \frac{2}{|b|}\frac{1}{|x-z|^2}$), we have pathwise

$$\begin{aligned}
\int_0^\tau dt\, |\langle X_t^N, (\mathcal{C}^0 u_t)''\rangle| &\le \int_0^\tau dt \left| \left\langle X_t^N, \int da \int_{-b_{max}}^{b_{max}} db\, |b| |(\mathfrak{f}_z)''(x)| \tilde{u}_t(a,b) \right\rangle \right| \\
&\le 2 \int_0^\tau dt \int_{-3R}^{3R} da \int_{-b_{max}}^{b_{max}} db\, \frac{1}{b} \mathrm{Im}\, (M_t^N(z))\, \tilde{u}_t(a,b) \\
&\le 2 \int_0^\tau dt \int_{-3R}^{3R} da \int_{-b_{max}}^{b_{max}} db\, \frac{1}{b} \mathrm{Im} \left[(M_t^N + M_t)(z)\right] \tilde{u}_t(a,b)
\end{aligned}$$

$$= 2\int_0^\tau dt \int_{-3R}^{3R} da \int_{-b_{max}}^{b_{max}} db\, \partial_b \Big[\mathrm{Im}\, \big[(M_t^N + M_t)(z) \big] \tilde{u}_t(a,b) \Big] \ln(1/|b|) \\ + \mathrm{bdry}_1 \tag{4.36}$$

where

$$\mathrm{bdry}_1 := -2\int_0^\tau dt \int_{-3R}^{3R} da\, \big[\mathrm{Im}\, (M_t^N + M_t)(a + \mathrm{i}b_{max}) \tilde{u}_t(a, b_{max}) \\ - (b_{max} \leftrightarrow -b_{max}) \big] \ln(1/|b_{max}|) \tag{4.37}$$

is a boundary term.

Now, we compare the first term in the r.-h.s. of (4.36) to

$$\frac{8}{\beta} \int_{-3R}^{3R} da \int_{-b_{max}}^{b_{max}} db\, (\tilde{u}_\tau(a,b) - \tilde{u}_0(a,b)) \ln(1/|b|) \\ \equiv \frac{8}{\beta} \int_0^\tau dt\, \frac{d}{dt} \left(\int_{-3R}^{3R} da \int_{-b_{max}}^{b_{max}} db\, \tilde{u}_t(a,b) \ln(1/|b|) \right). \tag{4.38}$$

The main terms in $\frac{8}{\beta}\frac{d}{dt}\tilde{u}_t(a,b)$ are those due to the $(\frac{1}{x})$-kernel,

$$2\left[\partial_a \left(\mathrm{Re}\, (M_t^N + M_t)(z) \tilde{u}_t(a,b) \right) + \partial_b \left(\mathrm{Im}\, (M_t^N + M_t)(z) \tilde{u}_t(a,b) \right) \right].$$

The horizontal drift term $\partial_a\, \mathrm{Re}\, (M_t^N + M_t)$ vanishes by integration by parts up to a boundary term,

$$\mathrm{bdry}_2 := 2\int_0^\tau dt \int_{-b_{max}}^{b_{max}} db\, \ln(1/|b|) \\ \times \left[\mathrm{Re}\, (M_t^N + M_t)(3R + \mathrm{i}b) \tilde{u}_t(3R + \mathrm{i}b) - (R \leftrightarrow -R) \right] \tag{4.39}$$

(note that $\mathrm{Re}\, (M_t^N + M_t)(\pm 3R + \mathrm{i}b) = O(1/R)$ is bounded), while the vertical drift term is identical to (4.36). The other terms in $\mathcal{H}_{transport}(t)$, see (3.52), (3.53),

$$\partial_a \left((V'(a) - \frac{1}{2} V'''(a) b^2) \tilde{u}_t(a,b) \right) + \left(\partial_b \left(V''(a) b \tilde{u}_t(a,b) \right) \right) + \mathrm{bdry} \tag{4.40}$$

contribute respectively: yet another boundary term,

$$\mathrm{bdry}_3 := \frac{8}{\beta} \int_0^\tau dt \int_{-b_{max}}^{b_{max}} db\, \ln(1/|b|) \\ \times \left[(V'(3R) - \frac{1}{2} V'''(3R) b^2) \tilde{u}_t(3R, b) - (R \leftrightarrow -R) \right]; \tag{4.41}$$

and

$$\frac{8}{\beta} \int_0^\tau dt \int_{-3R}^{3R} da\, V''(a) \int_{-b_{max}}^{b_{max}} db\, \tilde{u}_t(a,b) \\ = O\left(\sup_{[-3R,3R]} |V''| \right) e^{c\|V'\|_{7,[-3R,3R]} T} \|u_T\|_{L^1(\Pi_{b_{max}})} \tag{4.42}$$

by Lemma 4.5, plus a boundary term,

$$\mathrm{bdry}_4 := \frac{8}{\beta} \int_0^\tau dt \int_{-3R}^{3R} da\, V''(a) \left[b_{max} \ln(1/|b_{max}|) \tilde{u}_t(a, b_{max}) \\ - (b_{max} \leftrightarrow -b_{max}) \right], \tag{4.43}$$

and also the integral over the domain $[-3R, 3R] \times [-b_{max}, b_{max}]$ of $(\frac{8}{\beta}\ln(1/|b|)$ times the boundary terms of Section 3.9. Finally, the contribution of the $\tilde{c}$-characteristic is known from the contraction property to be of the form $\frac{8}{\beta}\int_0^\tau dt \int_{3R}^{3R} da \int_{-b_{max}}^{b_{max}} db\, \sigma_t(a,b)\tilde{u}_t(a,b)\ln(1/|b|) \geq 0$ with $\sigma_t(\cdot,\cdot) \geq 0$, hence positive.

Using the $(L^1 \cap L^\infty)$-bound of $\tilde{u}_t$, one sees that all boundary terms are $O(1)\|u_\tau\|_{(L^1\cap L^\infty)(\Pi_{b_{max}})} = O(|b_\tau|^{-3})$, times some derivative of V, $\|V^{(j)}\|_{0,[-3R,3R]}$, $j = 1,2,3$, times possibly $\int_{-b_{max}}^{b_{max}} db\, \ln(1/|b|) = O(1)$. But, actually, we have a much better bound *for T small enough*: because $h_\tau(z)$ is $O(1)$, independent of b_τ, far from the support $[-R,R]\times\{0\}$, say, on $\Pi_{b_{max}} \setminus \Big([-2R,2R]\times[-\frac{1}{2}b_{max}, \frac{1}{2}b_{max}]\Big)$, we shall have

$$\|u_t\|_{L^\infty(\partial_R \Pi_{b_{max}})} = O(1) \tag{4.44}$$

for all $t \in [0,\tau]$, as explained in the side remark before Lemma 3.5. Anticipating on the next terms featuring in the second-order expansion of the Green kernel (see (4.45)), it is easy to see that $\mathcal{H}^{1,0}_{nonlocal}(s)u_s$, whence $U^1_{transport}(t,s)(|\mathcal{H}^{1,0}_{nonlocal}(s)u_s|)$ and $U^2(t,s')\mathcal{H}^{2,1}_{nonlocal}(s')$ $U^1_{transport}(s',s)\mathcal{H}^{1,0}_{nonlocal}(s)u_s$ too, enjoy the same property (4.44). Incidentally, this implies $\|h_0\|_{L^1(\Pi_{b_{max}})}\|h_0\|_{L^\infty(\Pi_{b_{max}})} \leq C\|h_\tau\|_{L^1(\Pi_{b_{max}})}\|h_\tau\|_{L^\infty(\Pi_{b_{max}})} \leq C'/b_\tau^4$instead of C'/b_τ^6 in (4.32).

Consider now the *left-hand side* of (4.38). Considering the *adjoint evolution*, we get a time-reversed sub-Markov process $(\tilde{A}_t, \tilde{B}_t)$ with kernel $p(s, \tilde{a}_s; t, \tilde{z}_t)$, $s \leq t$ (see Section 3.6). Since $t \mapsto |\tilde{B}_t|$ *decreases*, we obtain

$$\begin{aligned} \left|\int da \int_{-b_{max}}^{b_{max}} db\, \tilde{u}_0(a,b)\ln(1/|b|)\right| &= \int da_\tau \int db_\tau\, u_\tau(a_\tau, b_\tau) \\ &\qquad \int da_0 \int db_0\, p(0, \tilde{a}_0, \tilde{b}_0; \tau, a_\tau, b_\tau)\ln(1/|\tilde{b}_0|) \\ \leq \int da_\tau \int db_\tau\, \ln(1/|b_\tau|)u_\tau(a_\tau,b_\tau) &= O(\ln(1/|b_\tau|)b_\tau^{-1}) \end{aligned} \tag{4.45}$$

by the log-estimate (2.32). So much for the contribution of $U^0_{transport}$ to (2), which we have shown to be overall $O((b_\tau^{-3})^2) = O(b_\tau^{-6})$, and even $O((\ln(1+1/b_\tau)/b_\tau)^2)$ for T small enough, as in [9].

We now use the second-order expansion of the Green kernel (4.22). The *second term* in the expansion,

$$v(a,b) := \int_t^\tau ds\, U^1_{transport}(t,s)\mathcal{H}^{1,0}_{nonlocal}(s)U^0_{transport}(s,\tau)h_\tau(a,b) \tag{4.46}$$

leads to a development similar to (4.36):

$$\begin{aligned} &\int_0^\tau dt\, |\langle X_t^N, (\mathcal{C}^0 v_t)''\rangle| \\ &\quad \leq \int_0^\tau dt\, \Big|\langle X_t^N, \int_{-3R}^{3R} da \int_{-b_{max}}^{b_{max}} db\, b^2|(\mathfrak{f}_z)''(x)| \\ &\quad \times \int_t^\tau ds (U^1_{transport}(t,s)|\mathcal{H}^{1,0}_{nonlocal}(s)u_s|)(a,b)\rangle\Big| \\ &\quad \leq 2\int_0^\tau ds \left[\int_0^s dt \int_{-3R}^{3R} da \int_{-b_{max}}^{b_{max}} db\, |\mathrm{Im}\,(M_t^N + M_t)(z)|\tilde{u}_t^s(a,b)\right] \end{aligned}$$

$$= -2\int_0^\tau ds\left[\int_0^s dt \int_{-3R}^{3R} da \int_{-b_{max}}^{b_{max}} db\, \partial_b \left[\mathrm{Im}\,(M_t^N + M_t)(z)\tilde{u}_t^s(a,b)\right] b\right] + \mathrm{bdry}'_1 \tag{4.47}$$

where $\tilde{u}_t^s(a,b) := \tilde{U}^1_{transport}(t,s)(|\mathcal{H}^{1,0}_{nonlocal}(s)u_s|)(a,b)$ (≥ 0), which we compare to

$$\int_0^\tau ds\left[\frac{8}{\beta}\int_{-3R}^{3R} da \int_{-b_{max}}^{b_{max}} db\,(\tilde{u}_s^s(a,b) - \tilde{u}_0^s(a,b))b\right] \equiv \int_0^\tau ds\left[\frac{8}{\beta}\int_0^s dt\,\frac{d}{dt}\left(\int_{-3R}^{3R} da \int_{-b_{max}}^{b_{max}} db\,\tilde{u}_t^s(a,b)b\right)\right]. \tag{4.48}$$

The right-hand side in (4.48) decomposes in the same way as explained below (4.40) — but with $\kappa = 1$ now . Compared to the main term studied in the previous two pages, $\ln(b)$ has been replaced with b (which may simply be bounded by a constant, $b \leq \frac{1}{2}$), so logarithms disappear in the estimates, while the replacement of u_s by $|\mathcal{H}^{1,0}_{nonlocal}(s)u_s|$ produces the supplementary factor $\|\!|\mathcal{H}^{1,0}_{nonlocal}(s)\|\!|_{(L^1\cap L^\infty)(\Pi_{b_{max}})} = O(1)$. The total contribution to (2) is therefore $O(b_\tau^{-6})$ or even $O((\ln(1+1/b)/b)^2)$ as for the main term.

The *last* term in the Green kernel expansion,

$$w(a,b) := \int_t^\tau ds \int_t^s ds'\, U^2(t,s')\mathcal{H}^{2,1}_{nonlocal}(s')U^1_{transport}(s,'s)\mathcal{H}^{1,0}_{nonlocal}(s) \times U^0_{transport}(s,\tau)h_\tau(a,b), \tag{4.49}$$

leads now to a third contribution which is bounded in a very simple way,

$$\int_0^\tau dt\left|\langle X_t^N, (\mathcal{C}^0 w_t)''\rangle\right| \leq \int_0^\tau dt\left|\langle X_t^N, \int_{-3R}^{3R} da \int_{-b_{max}}^{b_{max}} db\,|b|^3|(\mathfrak{f}_z)''(x)| \times \int_t^\tau ds \int_t^s ds'\left|U^2(t,s')\mathcal{H}^{2,1}_{nonlocal}(s') \times (U^1_{transport}(s',s)\mathcal{H}^{1,0}_{nonlocal}(s)u_s)(a,b)\right|\rangle\right|. \tag{4.50}$$

Since $|b|^3|(\mathfrak{f}_z)''(x)| = O(1)$, (4.50) is simply bounded in the end for arbitrary T by $\|u_\tau\|_{(L^1\cap L^\infty)(\Pi_{b_{max}})} = O(|b_\tau^{-3}|)$, times the product of the $(L^1\cap L^\infty)(\Pi_{b_{max}})$- operator norms $\|\!|U^i_{transport}(\cdot,\cdot)\|\!|$, $\|\!|\mathcal{H}^{i+1,i}_{nonlocal}(\cdot)\|\!|$ ($i = 0, 1$) figuring in the integral, yielding once again a total contribution $O(b_\tau^{-6})$, or (for T short enough) $O((\ln(1+1/b)/b)^2)$ to (2).

We finally proceed to bound the "martingale term" (3).

A caveat is required here: for finite N, $f_t(\cdot)$ is *not* $\mathcal{F}_t$-measurable, since it is obtained by integrating the ordinary differential equation with *random coefficients* (1.9) *backwards* from time τ to time t. Hence

$$\mathbb{E}\Big[\Big(\frac{1}{\sqrt{N}}\sum_{i=1}^N I_i(\tau)\Big)^2\Big], \qquad I_i(\tau) := \int_0^\tau dt\; f_t'(\lambda_t^i)dW_t^i \tag{4.51}$$

cannot be bounded by $\mathbb{E}\int_0^\tau dt\,(f_t'(\lambda_t^i))^2$ using standard tools of stochastic calculus. By the way, this points out to a mistake in the proof of the estimate for $\mathbb{E}[|N(M_T^N(z) - M_T(z))|^2]$ given in Proposition 1 of Israelsson's article. Our arguments below yield a bound in $O(1/b^{12})$ independently of V – in particular in the harmonic case, instead of the bound in $O((\ln(1+1/b)/b)^2)$ found by Israelsson. See remark at the end of this section for some after-thoughts.

The correct way to cope with the stochastic integral $I_i(\tau)$ (4.51) is the following. Since $s \mapsto f_s'(\lambda)$ is C^1 for λ *fixed*, it can be considered as the *non-adapted* finite variation part of a semi-martingale, in the extended definition briefly mentioned below Definition (1.17) of [20], Chapter 4. Hence the integration-by-parts lemma of standard differential calculus holds, and we can rewrite $I_i(\tau)$ as $\int_0^\tau ds\, f_0'(\lambda_s^i) dW_s^i - \int_0^\tau dt\, J_i(t)$, where

$$J_i(t) := \int_t^\tau \frac{\partial f_t'}{\partial t}(\lambda_s^i) dW_s^i. \tag{4.52}$$

Then, *considering the standard Stieltjes decomposition of h_t of order $\kappa = 3$ this time* (which turns out in the end of the ensuing computations to be the minimum possible order yielding finite results in the neighborhood of the real axis),

$$f_t'(\lambda_s^i) = \frac{\partial}{\partial x}\mathcal{C}^3(h_t)(x) = \int da \int_{-b_{max}}^{b_{max}} db\,(-\mathrm{i}b)\,|b|^3 (\mathfrak{f}_z)'(\lambda_s^i) h_t(a,b) \tag{4.53}$$

and

$$J_i(t) = \int da \int_{-b_{max}}^{b_{max}} db\,(-\mathrm{i}b)\,|b|^3 (\mathcal{H}(t)h_t)(a,b) J_i^z(t), \tag{4.54}$$

where the stochastic integral

$$J_i^z(t) := \int_t^\tau (\mathfrak{f}_z)'(\lambda_s^i) dW_s^i \tag{4.55}$$

is now a standard (i.e. non-anticipative) Itô integral, whence (using primed integration variables t', a', b' for $I_{i'}(\tau)$ in the averaged squared quantity $\mathbb{E}\Big[(\int_0^\tau dt \sum_i J_i(t))(\int_0^\tau dt' \sum_{i'} J_{i'}(t'))\Big]$)

$$\mathbb{E}[J_i^z(t) J_j^{z'}(t')] = \delta_{i,j} \int_{\max(t,t')}^\tau ds\, (\mathfrak{f}_z)'(\lambda_s^i)(\mathfrak{f}_{z'})'(\lambda_s^i). \tag{4.56}$$

The first step consists in transferring to the $\mathfrak{f}_z'$-factors the derivatives ∂_a, ∂_b coming from the action of $\mathcal{H}_{transport}(t)$ on h_t. We concentrate on the most singular terms coming from $\mathcal{H}_0^\kappa(t)$, namely, $(\mathcal{H}_0^\kappa(t)h_t)(a,b) = \frac{\beta}{4}\Big[\partial_a(\mathrm{Re}\,(M_t + M_t^N)(z))h(t;a,b) + \partial_b(\mathrm{Im}\,(M_t + M_t^N)(z))h(t;a,b)\Big] + \cdots$, where the missing order 0 part $(\cdots)$ is as in (3.26). Integrating these two terms w.r. to the measure $\int da\, db\,(-\mathrm{i}b)|b|^3\, \mathfrak{f}_z'(\lambda_s^i)$ yields by integration by parts $-\frac{\beta}{4}\int da\,(-\mathrm{i}db)\, h_t(a,b)\Big(\pm b^4 \mathrm{Re}\,(M_t + M_t^N)(z)\mathfrak{f}_z''(\lambda_s^i) \pm \mathrm{i}\mathrm{Im}\,(M_t + M_t^N)(z)\partial_b\Big(b^4 \mathfrak{f}_z'(\lambda_s^i)\Big)\Big)$.

For finite N, $\mathcal{H}(t)h_t(\cdot)$ is random and not $\mathcal{F}_t$-measurable, hence $\mathbb{E}\Big[(\mathcal{H}(t)h_t(a,b)\mathcal{H}(t')h_{t'}(a',b'))(J_i^z(t)J_{i'}^{z'}(t'))\Big]$ may not directly be bounded using (4.56) (see Remark below). Instead, we use the bounds

$$|h_t(a,b)|, |h_{t'}(a',b')| = O(\|h_\tau\|_{L^\infty(\Pi_{b_{max}})}) = O(b_\tau^{-3-\kappa}) = O(b_\tau^{-6}) \tag{4.57}$$

$$|b^4 \mathrm{Re}\,(M_t + M_t^N)(z)| = O(|b|^3), \qquad \|\mathfrak{f}_z''\|_\infty \le |b|^{-3} \tag{4.58}$$

and

$$\mathbb{E}\Big|\sum_{i,i'} \int_t^\tau \mathfrak{f}_z''(\lambda_s^i) dW_s^i \int_{t'}^\tau \mathfrak{f}_{z'}'(\lambda_s^{i'}) dW_s^{i'}\Big|$$
$$\le \Big[\mathbb{E}\Big(\sum_i \int_t^\tau \mathfrak{f}_z''(\lambda_s^i) dW_s^i\Big)^2\Big]^{1/2} \Big[\mathbb{E}\Big(\sum_{i'} \int_{t'}^\tau \mathfrak{f}_{z'}'(\lambda_s^{i'}) dW_s^{i'}\Big)^2\Big]^{1/2}$$

$$= \Big[\sum_i \mathbb{E}\int_t^\tau ds\, |\mathfrak{f}''_z(\lambda_s^i)|^2\Big]^{1/2}\Big[\sum_{i'} \mathbb{E}\int_{t'}^\tau ds\, |\mathfrak{f}'_{z'}(\lambda_s^{i'})|^2\Big]^{1/2}$$
$$\le \frac{1}{2}\mathbb{E}\Big[\Big(\frac{b}{b'}\Big)^3 \sum_i \int_t^\tau ds\, |\mathfrak{f}''_z(\lambda_s^i)|^2 + \Big(\frac{b'}{b}\Big)^3 \sum_i \int_{t'}^\tau ds\, |\mathfrak{f}'_{z'}(\lambda_s^i)|^2\Big]$$
$$= \frac{N}{2}\mathbb{E}\Big[\Big(\frac{b}{b'}\Big)^3 \int_t^\tau ds\, \langle X_s^N, (\mathfrak{f}''_z)^2\rangle + \Big(\frac{b'}{b}\Big)^3 \int_t^\tau ds\, \langle X_s^N, (\mathfrak{f}'_{z'})^2\rangle\Big]$$
$$= O(N(|bb'|)^{-3}). \tag{4.59}$$

Considering instead the terms of the type $4b^3\mathfrak{f}'_z(\lambda_s^i)$ coming from $\partial_b(b^4\mathfrak{f}'_z(\lambda_s^i))$, or those coming from the missing order 0 part $(\cdots)$ above, leads to the same scaling in b, b' when $b, b' \to 0$, as can easily be seen, while terms coming from the bounded operator $\mathcal{H}_{nonlocal}$ or from the time 0 contribution $\int_0^\tau ds\, f_0'(\lambda_s^i)dW_s^i$ are less singular. Thus we finally find, as expected:

$$\mathbb{E}\Big[\Big(\frac{1}{\sqrt{N}}\sum_{i=1}^N I_i(\tau)\Big)^2\Big] \le \Big(\int_{-3R}^{3R} da \int_{-b_{max}}^{b_{max}} db\, O(b_\tau^{-6})\Big)^2 = O(b_\tau^{-12}), \tag{4.60}$$

plus an $O(1)$-contribution coming from $(\mathcal{H}(t)h_t)_{ext}$. $\square$

Remark. Israelsson's bounds in $O((\ln(1+1/b)/b)^2)$ are recovered if one (somewhat carelessly and out of the blue!) *replaces* f_t, solution of the finite-N evolution equation (1.9), with the *deterministic* solution f_t^∞ of the asymptotic evolution equation (1.10). Namely, in that case, Itô's formula applies, see (4.51). Using $|(\mathfrak{f}_z)'(x)| \le \frac{1}{|b|}\frac{1}{|x-z|}$ and (B.12), we get, letting $f_t^\infty = \mathcal{C}^0(h_t^\infty)$ and $I_i^\infty(\tau) := \int_0^\tau dt\, (f_t^\infty)'(\lambda_t^i)dW_t^i$:

$$\mathbb{E}\Big[\Big(\frac{1}{\sqrt{N}}\sum_{i=1}^N I_i^\infty(\tau)\Big)^2\Big] = \int_0^\tau dt |\langle X_t^N, ((\mathcal{C}^0 h_t^\infty)')^2\rangle| dt$$
$$\le \int_0^\tau dt \int da \int_{-b_{max}}^{b_{max}} db \cdot \int da' \int_{-b_{max}}^{b_{max}} db'\, |h_t(a,b)||h_t(a',b')|$$
$$\times |\langle X_t^N, |bf_z'(\cdot)|\, |b' f_{z'}'|(\cdot)\rangle|$$
$$\le \int_0^\tau dt \int da \int_{-b_{max}}^{b_{max}} db \cdot \int da' \int_{-b_{max}}^{b_{max}} db'\, |h_t(a,b)||h_t(a',b')|\langle X_t^N, b^2|(f_z'(\cdot))|^2\rangle$$
$$\le \sup_{0\le t\le\tau}\Big(\int da' \int_{-b_{max}}^{b_{max}} db'\, |h_t(a',b')|\Big) \cdot \int_0^\tau dt$$
$$\times \Big(\int da \int_{-b_{max}}^{b_{max}} db\, \frac{1}{b}\mathrm{Im}\, M_t^N(z)|h_t(a,b)|\Big). \tag{4.61}$$

The second factor in (4.61) is bounded exactly like the drift term (2), while the first one is just $\|h_t\|_{L^1(\Pi_{b_{max}})}$. All together, $I^\infty(\tau)$ is bounded by $O(b_\tau^{-6})$, or even by $O(\ln(1+1/b_\tau)/b_\tau^2)$ for T small enough.

A way to improve our poor estimates (4.60) would be to separate h_t into h_t^∞ plus a $O(1/N)$ fluctuation δh_t^∞, whose contribution to (4.18) would be hopefully $O(1/N)$ times some inverse power of b, and would therefore vanish when $N \to \infty$. However, the time-evolution of δh_t^∞ is a priori governed by the Jacobian of (1.10) around h_t^∞, whose characteristics are obtained by linearizing those of the transport operator $\mathcal{H}_{transport}$. Alas, the linearization of the already singular characteristics of $\mathcal{H}_0$, see Proposition 1.3, leads to an exponential factor of the type $\exp\Big(c\int dt\, ((M_t^N)''(Z_t))+(M_t''(Z_t))\Big)$, which is exponentially large for small $|b|$ near the points x of the real axis at which $M_t(x\pm \mathrm{i}0)$ is not differentiable, e.g. near the end points of the support for a standard density of the semi-circle type $\frac{1}{\pi}\sqrt{2-x^2}$ ($a>0$), with associated Stieltjes transform $M_t(z)=-z+\sqrt{z^2-2}$, see (1.29).

5. Large deviation bound for the support of the measure

As a key technical argument required for the convergence of our scheme, we prove in this section the following bound for the probability that the support of the measure is large. Since the number N of eigenvalues varies in this section, we emphasize the N-dependence of the process when we judge it necessary by writing $\lambda_t^{N,i}$ instead of λ_t^i.

Lemma 5.1 (*Large Deviation Bound*)**.** *Assume the large deviation estimate (1.37) for the initial support holds, namely,* $\mathbb{P}[\max_{i=1,\dots,N}|\lambda_0^{N,i}|>R_0]\le C_0 e^{-c_0 N}$ *for some constants* $R_0, c_0, C_0>0$. *Let* $T>0$. *There exist some radius* $R=R(T)$ *and constant* c, *depending on* V *and* R_0, c_0 *but uniform in* N, *such that*

$$\mathbb{P}\left[\sup_{0\le t\le T}\ \sup_{i=1,\dots,N}|\lambda_t^{N,i}|>R\right]\le Ce^{-cN}. \tag{5.1}$$

The principle of the proof was obligingly provided by a referee. It relies on uniform-in-time moment bounds for the empirical measure, and on a comparison principle for sde's.

First, we use as an input moment bounds proved *in the case* $V=0$ by induction on $p=1,2,\dots,\varepsilon N$ in Anderson–Guionnet–Zeitouni [1]. Let $(\tilde\lambda_t^{N,i})_{t\ge 0}, i=1,\dots,N$ be the solution of the modified system of coupled stochastic differential equations with *zero potential*,

$$d\tilde\lambda_t^{N,i}=\frac{1}{\sqrt{N}}dW_t^i+\frac{\beta}{2N}\sum_{j\ne i}\frac{dt}{\tilde\lambda_t^{N,i}-\tilde\lambda_t^{N,j}},\qquad i=1,\dots,N \tag{5.2}$$

with initial condition $\tilde\lambda_0^{N,i}\equiv\lambda_0^{N,i}$ coinciding with that of (1.1), and $\tilde X_t^N:=\frac{1}{N}\sum_{i=1}^N\delta_{\tilde\lambda_t^{N,i}}$ be the corresponding random point process. Under $\Omega_0:\Big(\max_{i=1,\dots,N}|\lambda_0^{N,i}|\le R_0\Big)$, an event of probability $1-C_0e^{-c_0N}$, eq. (4.3.45) in [1] holds, namely,

$$\mathbb{E}\left[\mathbf{1}_{\Omega_0}\sup_{0\le t\le T}\int\tilde X_t^N(dx)\,|x|^p\right]\le R_1(T)^p. \tag{5.3}$$

(An explicit expression for the constants ε and $R_1(T)$, depending on R_0, can be obtained by following computations on p. 274, as a consequence of Lemma 4.3.17.) The above bound implies $\mathbb{E}\Big[\mathbf{1}_{\Omega_0}\sup_{0\le t\le T}\sup_{i=1,\dots,N}|\tilde\lambda_t^{N,i}|^p\Big]\le NR_1(T)^p$ and then (by Markov's inequality), letting $R_2(t):=eR_1(t)$,

$$\mathbb{P}[\mathbf{1}_{\Omega_0}\sup_{0\le t\le T}\ \sup_{i=1,\dots,N}|\tilde\lambda_t^{N,i}|>R_2(T)]\le Ne^{-\varepsilon N}. \tag{5.4}$$

Then, one compares the two eigenvalue processes $(\lambda_t^i)_{0\le t\le T}$ and $(\bar\lambda_t^i)_{0\le t\le T}$, adapting the argument given in [1], Lemma 4.3.6. Let $E_t^{N,i}:=\lambda_t^{N,i}-\bar\lambda_t^{N,i}-\alpha t$ ($\alpha>0$). Subtracting the sde's for the two processes, one gets

$$\frac{dE_t^{N,i}}{dt} = -\frac{\beta}{2N}\sum_{j\neq i}\frac{E_t^{N,i}-E_t^{N,j}}{(\lambda_t^{N,i}-\lambda_t^{N,j})(\bar\lambda_t^{N,i}-\bar\lambda_t^{N,j})} \\ -(V'(\lambda_t^{N,i})-V'(\bar\lambda_t^{N,i}))-V'(\bar\lambda_t^{N,i})-\alpha. \tag{5.5}$$

Whatever the ordering chosen for the eigenvalues, the denominator in (5.5) is always >0 because eigenvalues never cross. We assume that the event $\Omega : \left(\max_{i=1,\dots,N}|\lambda_0^{N,i}|\le R_0\right)\cap\left(\sup_{0\le t\le T}\sup_{i=1,\dots,N}|\bar\lambda_t^{N,i}|\le R_2(T)\right)$ is realized, an event of probability $1-e^{-c_1 N}$; then $|V'(\bar\lambda_t^{N,i})|$ is bounded uniformly in N, i, and $t\le T$ by some constant C_2 depending on V; we assume $\alpha > C_2$. Initially, $E_0^{N,i}\le R_0+R_2(T)$, $i=1,\dots,N$ by construction. *Assume* that there exists some $t<T$ and i such that $E_t^{N,i}\ge R_3(T) := R_0+R_2(T)+1$, and let $t_{min}>0$ be the first time at which one such inequality holds, so that $E_{t_{min}}^{N,i} = R_3(T)$ for some i, while $E_t^{N,j} < R_3(T)$ for $t<t_{min}$ and $j=1,\dots,N$. But then $E_{t_{min}}^{N,i}-E_{t_{min}}^{N,j}\ge 0$ for all $j\ne i$, and (by convexity of V) $V'(\lambda_{t_{min}}^{N,i})-V'(\bar\lambda_{t_{min}}^{N,i})\ge 0$. Hence $\frac{dE_{t_{min}}^{N,i}}{dt}<0$: a contradiction. Reversing the signs of the inequalities, one proves similarly that $\tilde\lambda_t^{N,i}-\lambda_t^{N,i}-\alpha t\le R_3(T)$. Concluding: with high probability, $\sup_{0\le t\le T}\sup_{i=1,\dots,N}|\lambda_t^{N,i}|\le R_0+2R_2(T)+C_2T+1$. □

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Appendix A. Generalized transport operators

Many operators in this article are of the following type,

$$\mathcal{H}_t f(x) = \sum_k v_k(t,x)\partial_{x_k} f(x) + \tau(t,x) f(x) \tag{A.1}$$

with $f:\Omega\to\mathbb{R}$, where Ω is a domain in $\mathbb{R}^d$ (in practice, we need only consider $\Omega = \Pi^{\pm}$), and $\boldsymbol{v}(t,\cdot)$ a vector field, resp. $\tau(t,\cdot)$ a function, on Ω. Let us call such operators *generalized transport operators*.

It is well-known how to solve PDEs generated by generalized transport operators, i.e. of the type

$$\frac{\partial f_t}{\partial t}(x) = \mathcal{H}_t f_t(x) \tag{A.2}$$

with terminal condition $f_T \equiv f$. Namely, let $y_t \equiv \Phi_t^T(y)$ (called: *characteristics of* (A.2)) be the solution of the ode $\frac{dy_t}{dt} = \boldsymbol{v}(t,y_t)$ with terminal condition $y_T = y$. One checks immediately that

$$f_t(y) = c_t f(\Phi_t^T(y)), \qquad c_t := \exp\left(-\int_t^T \tau(y_s)\,ds\right) \tag{A.3}$$

is a solution. In particular, $\mathrm{supp}(f_t)$, $t\le T$ is the inverse image of $\mathrm{supp}(f)$ by Φ_t^T; so, if $\boldsymbol{v}\big|_{\partial\Omega}$ is *inward on the boundary of some domain* Ω *containing the support of* f_T, *then* $\mathrm{supp}(f_T)\subset\Omega$

for all $t \le T$. In the article we actually refer either to the basis trajectory $y_{\cdot} = a_{\cdot} + \mathrm{i}b_{\cdot}$ or to the "extended" trajectory $(a_{\cdot} + \mathrm{i}b_{\cdot}, c_{\cdot})$ as characteristics.

The *Jacobian* of the ode, $J_t := \frac{dy_t}{dy}$, solves the linearized ode $\frac{dJ_t}{dt} = \nabla \boldsymbol{v}(t, y_t) J_t$ with terminal condition $J_T = \mathrm{Id}$. In particular (letting $|\cdot|$ denote the determinant), $\frac{d}{dt}\big|_{t=T} |J_t| = \mathrm{Tr}(\nabla \boldsymbol{v}(T, y)) = \nabla \cdot \boldsymbol{v}(T, y)$. The time-variation of the L^1-norm of f_t is

$$\frac{d}{dt}\Big|_{t=T} \int dy\, |f_t(y)| = \int dy \left(\mathrm{Re}\, \frac{d}{dt} c_t(y) \right) |f(y)| - \int dy \left(\frac{d}{dt}\Big|_{t=T} |J_t| \right) |f(y)|$$
$$= \int dy\, (\mathrm{Re}\, \tau(T, y) - \mathrm{Tr}(\nabla \boldsymbol{v}(T, y)))\, |f(y)|; \tag{A.4}$$

it vanishes when $\mathrm{Re}\, c = \mathrm{Tr}(\nabla \boldsymbol{v})$, in particular when

$$\mathcal{H}_t = \left(\sum_k v_k(t, x) \partial_{x_k} \right)^{\dagger} = -\sum_k v_k(t, x) \partial_{x_k} - \nabla \cdot \boldsymbol{v}(t, x) \tag{A.5}$$

is in *divergence form*, i.e. is the *adjoint* of a transport operator. Thus $\mathcal{H}_t$ is the generator of a strongly continuous semi-group of contractions of $L^1(\mathbb{R})$, see e.g. [19], chapter 1. The latter observation extends to the case when $\mathcal{H}_t = \left(\sum_k v_k(t, x) \partial_{x_k} \right)^{\dagger} - \tau(t, x)$ with $\tau(t, \cdot) \le 0$, in the sense that $\int dy\, |f_t(y)| \le \int dy\, |f_T(y)|$ for $0 \le t \le T$.

Appendix B. Stieltjes transforms

We collect in this section some definitions and elementary properties concerning Stieltjes transforms. We make use of the Fourier transform normalized as follows,

$$\mathcal{F}(f)(s) = \int_{-\infty}^{+\infty} f(x) e^{-\mathrm{i}xs}\, dx \tag{B.1}$$

with inverse $\mathcal{F}^{-1}(g)(x) = \frac{1}{2\pi} \int g(s) e^{\mathrm{i}xs}\, ds$.

Let, for $z = a + \mathrm{i}b \in \mathbb{C} \setminus \mathbb{R}$

$$\mathfrak{f}_z(x) = \frac{1}{x - z}, \qquad x \in \mathbb{R}. \tag{B.2}$$

For fixed $b \neq 0$, $\mathfrak{f}_z(x)$ may be seen as a convolution kernel K_b,

$$K_b(x - a) = \frac{1}{(x - a) - \mathrm{i}b}. \tag{B.3}$$

Note that

$$\mathrm{Im}\, (\mathfrak{f}_z)(x) = \frac{b}{|x - z|^2} = \frac{b}{(x - a)^2 + b^2}, \qquad \mathrm{Re}\, (\mathfrak{f}_z)(x) = \frac{x - a}{(x - a)^2 + b^2}. \tag{B.4}$$

In particular,

$$\mathrm{Im}\, (\mathfrak{f}_z)(x) \ge 0 \qquad (b > 0). \tag{B.5}$$

Many estimates are based on the simple remark that

$$\int \frac{b}{(x - a)^2 + b^2}\, dx = \pi \qquad (b > 0) \tag{B.6}$$

is a constant. The Plemelj formula,

$$\frac{1}{x - \mathrm{i}0} = p.v. \left(\frac{1}{x} \right) + \mathrm{i}\pi \delta_0 \tag{B.7}$$

implies the following boundary value equations for K_b,

$$\lim_{b\to 0^+}\int dy\, K_b(x-y)\phi(y) \;-\; \lim_{b\to 0^-}\int dy\, K_b(x-y)\phi(y) = 2\mathrm{i}\pi\phi(x) \tag{B.8}$$

$$\lim_{b\to 0^+}\int dy\, K_b(x-y)\phi(y) \;+\; \lim_{b\to 0^-}\int dy\, K_b(x-y)\phi(y) = 2\ p.v.\int \frac{dy}{x-y}\phi(y). \tag{B.9}$$

Then:

$$\mathcal{F}\mathfrak{f}_z(s) = 2\mathrm{i}\pi e^{-b|s|-\mathrm{i}as}\mathbf{1}_{s<0} \qquad (b>0), \qquad -2\mathrm{i}\pi e^{b|s|-\mathrm{i}as}\mathbf{1}_{s>0} \qquad (b<0) \tag{B.10}$$

hence (for $b>0$)

$$\mathcal{F}(\mathrm{Im}\,(\mathfrak{f}_z))(s) = \pi e^{-b|s|-\mathrm{i}as}, \qquad \mathcal{F}(\mathrm{Re}\,(\mathfrak{f}_z)) = -\mathrm{i}\pi\ \mathrm{sgn}(s)e^{-b|s|-\mathrm{i}as}. \tag{B.11}$$

Properties of the Stieltjes transform of ρ_t.

Let $M_t(z) := \langle X_t, \mathfrak{f}_z\rangle$ ($b := \mathrm{Im}\, z > 0$). Then:

$$\mathrm{Im}\,(M_t(z)) = \langle X_t, \frac{b}{(x-a)^2+b^2}\rangle; \tag{B.12}$$

$$|M_t(z)| = |\langle X_t, \frac{1}{(x-a)-\mathrm{i}b}\rangle| \le 1/b; \tag{B.13}$$

$$|M_t'(z)| = |\langle X_t, \frac{1}{((x-a)-\mathrm{i}b)^2}\rangle| \le \frac{1}{b}\langle X_t, \frac{b}{(x-a)^2+b^2}\rangle = \frac{1}{b}\mathrm{Im}\,(M_t(z)). \tag{B.14}$$

When $|a| \gg R$, we get much better estimates, e.g.

$$|M_t(z)| \le 2/|a|, \qquad |a| \ge 2R. \tag{B.15}$$

On the other hand, if $b \to 0$ and $a \in \mathrm{supp}(X_t)$, then $M_t(z)$ may diverge in general. In particular,

$$|\mathrm{Re}\,(M_t(z))| \le C\|\rho_t\|_\infty \ln(R/b) \qquad (b \le \frac{R}{2}, |a| \le 2R). \tag{B.16}$$

However, if ρ_t is bounded then $\mathrm{Im}\,(M_t(z)) \in [0, \pi\|\rho_t\|_\infty]$; and $\mathrm{Re}\,(M_t(z)) = O(1)$ if the space derivative of the density, ρ_t', is bounded.

Some distributions.

Let $\phi \in C_c^\infty$ be a smooth function supported on $[-r, r]$, and $b > 0$. Let

$$\langle \mathfrak{f}_{\mathrm{i}b}, \phi\rangle := \int dy\, \frac{\phi(y)}{y-\mathrm{i}b}. \tag{B.17}$$

Then

$$\begin{aligned}\langle \mathfrak{f}_{\mathrm{i}b}, \phi\rangle &= \phi(0)\int_{-r}^{r}\frac{dy}{y-\mathrm{i}b} + \mathrm{i}\int_{-r}^{r} dy\,(\phi(y)-\phi(0))\frac{b}{y^2+|b_T|^2} \\ &\quad + \int_{-r}^{r} dy\, \frac{y(\phi(y)-\phi(0))}{y^2+b^2}\end{aligned} \tag{B.18}$$

is $O(\|\phi\|_\infty + r\|\phi'\|_\infty)$ since: $|\int_{-r}^{r}\frac{dy}{y-\mathrm{i}b}| \le \int dy\, \frac{b}{y^2+b^2} = O(1)$, and $|\frac{y(\phi(y)-\phi(0))}{y^2+b^2}| \le \|\phi'\|_\infty$. Hence (as seen by integration by parts), $y \mapsto (y-\mathrm{i}b)^{-n}$ ($n \ge 1$) is a distribution of order n, namely,

$$\left|\int dy \frac{\phi(y)}{(y-\mathrm{i}b)^n}\right| = O(\|\phi^{(n-1)}\|_\infty + r\|\phi^{(n)}\|_\infty). \tag{B.19}$$

The $\left(\frac{1}{x}\right)$-kernel and its family.

It is known that

$$\mathcal{F}^{-1}(s \mapsto \mathrm{sgn}(s)\mathcal{F}f(s))(x) = \mathrm{i}Hf(x) := \frac{\mathrm{i}}{\pi} p.v. \int_{-\infty}^{+\infty} \frac{1}{x-y} f(y)\, dy \tag{B.20}$$

defined for a compactly supported $f \in C^1$ either as $\frac{\mathrm{i}}{\pi}\lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon}\frac{1}{x-y}f(y)\,dy$ or as $\frac{\mathrm{i}}{\pi}\int\frac{f(y)-f(x)}{x-y}\,dy$, from which by differentiating

$$\mathcal{F}^{-1}(s \mapsto |s|\mathcal{F}f(s))(x) = -\frac{1}{\pi} p.v. \int_{-\infty}^{+\infty} \frac{1}{(x-y)^2} f(y)\, dy. \tag{B.21}$$

For a function f supported on $[-R, R]$, we have the following bounds:

$$\begin{aligned} \left| p.v. \int \frac{1}{x-y} f(y)\, dy \right| &= \mathbf{1}_{|x|\le 2R} \left| \int_{x-3R}^{x+3R} \frac{f(y)-f(x)}{x-y}\, dy \right| + \mathbf{1}_{|x|>2R} \left| \int_{-R}^{R} \frac{f(y)}{x-y}\, dy \right| \\ &= \mathbf{1}_{|x|\le 2R}\ O(R\|f'\|_\infty) + \mathbf{1}_{|x|>2R}\ O(\|f\|_\infty R/|x|), \end{aligned} \tag{B.22}$$

and similarly

$$\begin{aligned} \left| p.v. \int \frac{1}{(x-y)^2} f(y)\, dy \right| &= \mathbf{1}_{|x|\le 2R} \left| \int_{x-3R}^{x+3R} \frac{f'(y)-f'(x)}{x-y}\, dy \right| \\ &\quad + \mathbf{1}_{|x|>2R} \left| \int_{-R}^{R} \frac{f(y)}{(x-y)^2}\, dy \right| \\ &= \mathbf{1}_{|x|\le 2R}\ O(R\|f''\|_\infty) + \mathbf{1}_{|x|>2R}\ O(\|f\|_\infty R/x^2). \end{aligned} \tag{B.23}$$

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