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Normal approximation by Stein's method under sublinear expectations

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Abstract

Peng (2008) proved the Central Limit Theorem under a sublinear expectation:

Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d random variables under a sublinear expectation $\hat{\mathbf{E}}$ with $\hat{\mathbf{E}}[X_1] = \hat{\mathbf{E}}[-X_1] = 0$ and $\hat{\mathbf{E}}[|X_1|^3] < \infty$. Setting $W_n := \frac{X_1+\cdots+X_n}{\sqrt{n}}$, we have, for each bounded Lipschitz function φ,

$$\lim_{n\to\infty} \left|\hat{\mathbf{E}}[\varphi(W_n)] - \mathcal{N}_G(\varphi)\right| = 0,$$

where $\mathcal{N}_G$ is the G-normal distribution with $G(a) = \frac{1}{2}\hat{\mathbf{E}}[aX_1^2]$, $a \in \mathbb{R}$

In this paper, we shall give an estimate of the convergence rate of this CLT by Stein's method under sublinear expectations:

Under the same conditions as above, there exists a constant $\alpha \in (0, 1)$ depending on $\underline{\sigma}$ and $\overline{\sigma}$, and a positive constant $C_{\alpha,G}$ depending on $\alpha, \underline{\sigma}$ and $\overline{\sigma}$ such that

$$\sup_{|\varphi|_{Lip}\leq 1} \left|\hat{\mathbf{E}}[\varphi(W_n)] - \mathcal{N}_G(\varphi)\right| \leq C_{\alpha,G} \frac{\hat{\mathbf{E}}[|X_1|^{2+\alpha}]}{n^{\frac{\alpha}{2}}},$$

where $\overline{\sigma}^2 = \hat{\mathbf{E}}[X_1^2]$, $\underline{\sigma}^2 = -\hat{\mathbf{E}}[-X_1^2] > 0$ and $\mathcal{N}_G$ is the G-normal distribution with

$$G(a) = \frac{1}{2}\hat{\mathbf{E}}[aX_1^2] = \frac{1}{2}(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-), \quad a \in \mathbb{R}.$$

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1. Introduction

The Central Limit Theorem is one of the most striking and useful results in probability and statistics, and explains why the normal distribution appears in areas as diverse as gambling, measurement error, sampling, and statistical mechanics. In essence, the Central Limit Theorem in its classical form states that a normal approximation applies to the distribution of quantities that can be modeled as the sum of many independent contributions, all of which are roughly the same size.

Motivated by problems of model uncertainty in statistics, measures of risk and superhedging in finance, Peng [10] introduced the notion of sublinear expectations. A random variable X in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbf{E}})$ with $\hat{\mathbf{E}}[|X|^3] < \infty$ is called G-normally distributed if for any independent copy X' of X and $\alpha, \beta \in \mathbb{R}$,

$$\alpha X + \beta X' \overset{d}{=} \sqrt{\alpha^2 + \beta^2} X.$$

As known, if $\hat{\mathbf{E}}$ is a linear expectation generated by a probability, a random variable X with the above property is normally distributed. Suppose X is G-normally distributed under $\hat{\mathbf{E}}$. For $\varphi \in C_{b,Lip}(\mathbb{R})$, the collection of bounded Lipschitz functions on $\mathbb{R}$, set $\mathcal{N}_G[\varphi] = \hat{\mathbf{E}}[\varphi(X)]$. We call $\mathcal{N}_G$, a sublinear expectation on $C_{b,Lip}(\mathbb{R})$, a G-normal distribution. Here, the function G, defined by $G(a) = \frac{1}{2}\hat{\mathbf{E}}[aX^2]$, $a \in \mathbb{R}$, characterizes the variances of X.

In the seminal paper [7], Peng S. proved the Central Limit Theorem under a sublinear expectation, which is a milestone in the theory of sublinear expectations.

Theorem 1.1. *Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d random variables under a sublinear expectation $\hat{\mathbf{E}}$ with $\hat{\mathbf{E}}[X_1] = \hat{\mathbf{E}}[-X_1] = 0$ and $\hat{\mathbf{E}}[|X_1|^3] < \infty$. Setting $W_n := \frac{X_1+\cdots+X_n}{\sqrt{n}}$, we have, for each $\varphi \in C_{b,Lip}(\mathbb{R})$,*

$$\lim_{n\to\infty} \left| \hat{\mathbf{E}}[\varphi(W_n)] - \mathcal{N}_G(\varphi) \right| = 0,$$

where $\mathcal{N}_G$ is the G-normal distribution with $G(a) = \frac{1}{2}\hat{\mathbf{E}}[aX_1^2]$, $a \in \mathbb{R}$.

Just like the linear case, this theorem mathematically justifies, at least asymptotically, the G-normal distribution would be may be used to approximate quantities which can be formulated as the sums of independent and identically distributed random variables under a sublinear expectation. However, even though in practice sample sizes may be large, or may appear to be sufficient for the purposes to handle, depending on that and other factors, the normal approximation may or may not be accurate. It is here that the need for the evaluation of the quality of the normal approximation arises.

For the linear case, Stein's method, which made its first appearance in the ground breaking work of Stein [13], is a powerful tool to estimate the error of normal approximation. The cornerstone of Stein's method is the Stein equation (refer to [1] for more details): For a standard normally distributed random variable Z and given φ, solve the following equation for f,

$$f'(x) - xf(x) = \varphi(x) - E[\varphi(Z)]. \tag{1.1}$$

Then, for any random variable W, evaluate the left hand side of the Stein equation at W and take the expectation, obtaining $E[\varphi(W)] - E[\varphi(Z)]$.

The objective of this paper is to introduce the ideas of Stein's method to the nonlinear case. The expected Stein equation for G-normal distribution would be

$$G(f''(x)) - \frac{x}{2} f'(x) = \varphi(x) - \mathcal{N}_G[\varphi]. \tag{1.2}$$

Unfortunately, for $\varphi \in C_{b,Lip}(\mathbb{R})$, Eq. (1.2) generally does not have a solution. Therefore, the first step is to find a substitute of the Stein equation.

For $\varphi \in C_{b,Lip}(R)$, the function $u(x,t) := \mathcal{N}_G[\varphi(x + \sqrt{t}\cdot)]$ is the unique viscosity solution of the G-heat equation below

$$\begin{aligned} &\partial_t u - G(D_x^2 u) = 0, \ (x,t) \in \mathbb{R} \times (0,\infty), \\ &\qquad u(x,0) = \varphi(x), \end{aligned}$$

where $G(a) = \frac{1}{2}\mathcal{N}_G[ax^2]$, $a \in \mathbb{R}$, is determined by the variances $\overline{\sigma}^2 := \mathcal{N}_G[x^2]$ and $\underline{\sigma}^2 := -\mathcal{N}_G[-x^2]$. So, if $\overline{\sigma} = \underline{\sigma} = \sigma$, $\mathcal{N}_G$ is nothing but the classical normal distribution $N(0,\sigma^2)$.

Let Θ be a weakly compact subset of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. For the sublinear expectation $\mathcal{N}[\varphi] = \sup_{\mu\in\Theta} \mu[\varphi]$ on $C_{b,Lip}(\mathbb{R})$ and a function $\phi \in C_{b,Lip}(\mathbb{R})$, set

$$w(t) = \mathcal{N}[v(\sqrt{1-t}\cdot, t)],$$

where v is the solution to the G-heat equation with initial value ϕ. Then $w(1) = \mathcal{N}_G[\phi]$, $w(0) = \mathcal{N}[\phi]$, and it can be shown that, for a.e. $s \in (0,1)$,

$$w'(s) = \frac{1}{1-s}\mu_s[G(\phi_s''(x)) - \frac{1}{2}x\phi_s'(x)], \tag{1.3}$$

where $\phi_s(x) = v(\sqrt{1-s}x, s)$ and $\mu_s \in \Theta$ with $\mu_s[\phi_s] = \mathcal{N}[\phi_s]$. From this, we get a substitute of the Stein equation.

Step 1. $\mathcal{N}_G[\phi] - \mathcal{N}[\phi] = \int_0^1 \frac{1}{1-s}\mu_s[G(\phi_s''(x)) - \frac{1}{2}x\phi_s'(x)]\mathrm{ds}$.

Return to the linear case, i.e., $\underline{\sigma} = \overline{\sigma}$ and $\Theta = \{\mu\}$ is a singleton, the above formula will reduce to the classical Stein equation (see Remark 4.2 for details).

Now the next task is to calculate the expectation on the right side of the equality (1.3).

Let $\alpha \in (0,1)$. Suppose $\mathcal{N}[x] = \mathcal{N}[-x] = 0$ and $\mathcal{N}[|x|^{2+\alpha}] < \infty$. For $\phi \in C_b^{2,\alpha}(\mathbb{R})$ and $\mu \in \Theta$ with $\mu[\phi] = \mathcal{N}[\phi]$, we have

Step 2. $\left|\mu[G(\phi''(x)) - \frac{1}{2}x\phi'(x)]\right| \leq 2[\phi'']_\alpha \mathcal{N}[|x|^{2+\alpha}]$, where $G(a) = \frac{1}{2}\mathcal{N}[a|x|^2]$, $a \in \mathbb{R}$.

It merits to emphasize that the function G in Step 1 is determined by the variances of $\mathcal{N}_G$ and that the function G in Step 2 is determined by the variances of $\mathcal{N}$. In other words, to estimate $\mathcal{N}_G[\phi] - \mathcal{N}[\phi]$ applying Step 1 and Step 2 requires that $\mathcal{N}_G$ and $\mathcal{N}$ have the same variances.

Besides, note that $\phi_s(x)$ in the equality (1.3) is the solution to the G-heat equation. Therefore, to apply the estimate in Step 2, we need the regularity properties of the G-heat equation, which can be found in the literature on the partial differential equations (see Section 3 for details).

Step 3. $[D_x^2 v(\cdot,t)]_\alpha \leq c_{\alpha,G}\frac{1}{t^{\frac{1}{2}+\frac{\alpha}{2}}}|\phi|_{Lip}$ for some $\alpha \in (0,1)$ and $c_{\alpha,G} > 0$.

Following these three steps, we give an estimate of the convergence rate of Peng's Central Limit Theorem.

Under the same conditions as those in Theorem 1.1, there exists a constant $\alpha \in (0,1)$ depending on $\underline{\sigma}$ and $\overline{\sigma}$, and a positive constant $C_{\alpha,G}$ depending on $\alpha, \underline{\sigma}$ and $\overline{\sigma}$ such that

$$\sup_{|\varphi|_{Lip}\leq 1}\left|\hat{\mathbf{E}}[\varphi(\frac{X_1+\cdots+X_n}{\sqrt{n}})]-\mathcal{N}_G(\varphi)\right|\leq C_{\alpha,G}\frac{\hat{\mathbf{E}}[|X_1|^{2+\alpha}]}{n^{\frac{\alpha}{2}}},$$

where $\overline{\sigma}^2=\hat{\mathbf{E}}[X_1^2]$, $\underline{\sigma}^2=-\hat{\mathbf{E}}[-X_1^2]>0$ and $\mathcal{N}_G$ is the G-normal distribution with

$$G(a)=\frac{1}{2}\hat{\mathbf{E}}[aX_1^2]=\frac{1}{2}(\overline{\sigma}^2a^+-\underline{\sigma}^2a^-),\ a\in\mathbb{R}.$$

Here α is the Hölder exponent in Step 3, and $C_{\alpha,G}$ can be chosen as $\frac{4}{1-\alpha}c_{\alpha,G}$ with $c_{\alpha,G}$ the α-Hölder constant in Step 3.

In Section 2, we review the basic notions and results of sublinear expectations. In Section 3, we introduce the regularity properties of the G-heat equation that will be used in this paper. In Section 4, we shall generalize the idea of Stein's method to the sublinear expectation space, based on which we get the rate of convergence of Peng's Central Limit Theorem. In Section 5, we consider the CLT under sublinear expectations of a sequence of independent random variables which may not be identically distributed.

2. Basic notions of sublinear expectations

Here we review basic notions and results of sublinear expectations. The readers may refer to [7–12] for more details.

Let Ω be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on Ω such that for any $X\in\mathcal{H}$ and $\varphi\in C_{b,Lip}(\mathbb{R})$, we have $\varphi(X)\in\mathcal{H}$. The space $\mathcal{H}$ is considered as our space of random variables.

Definition 2.1. A sublinear expectation is a functional $\hat{\mathbf{E}}:\mathcal{H}\to\mathbb{R}$ satisfying

E1. $\hat{\mathbf{E}}[X]\geq\hat{\mathbf{E}}[Y]$, if $X\geq Y$;

E2. $\hat{\mathbf{E}}[\lambda X]=\lambda\hat{\mathbf{E}}[X]$, for $\lambda\geq 0$;

E3. $\hat{\mathbf{E}}[c]=c$, for $c\in\mathbb{R}$;

E4. $\hat{\mathbf{E}}[X+Y]\leq\hat{\mathbf{E}}[X]+\hat{\mathbf{E}}[Y]$, for $X,Y\in\mathcal{H}$;

E5. $\hat{\mathbf{E}}[\varphi_n(X)]\downarrow 0$, for $X\in\mathcal{H}$ and $\varphi_n\in C_{b,Lip}(\mathbb{R})$, $\varphi_n\downarrow 0$.

The triple $(\Omega,\mathcal{H},\hat{\mathbf{E}})$ is called a sublinear expectation space. For $X\in\mathcal{H}$, set

$$\mathcal{N}^X[\varphi]=\hat{\mathbf{E}}[\varphi(X)],\ \varphi\in C_{b,Lip}(\mathbb{R}),$$

which is a sublinear expectation on $C_{b,Lip}(\mathbb{R})$. X follows the distribution $\mathcal{N}^X$, and we write $X\sim\mathcal{N}^X$. A functional $\mathcal{N}$ is a sublinear expectation on $C_{b,Lip}(\mathbb{R})$ if and only if it can be represented as the supremum expectation of a weakly compact subset Θ of probability measures on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ (see [2]),

$$\mathcal{N}[\varphi]=\sup_{\mu\in\Theta}\mu[\varphi],\ \text{for all }\varphi\in C_{b,Lip}(\mathbb{R}).\tag{2.1}$$

Definition 2.2. Let $(\Omega,\mathcal{H},\hat{\mathbf{E}})$ be a sublinear expectation space. We say a random vector $\mathbf{X}=(X_1,\ldots,X_m)\in\mathcal{H}^m$ is independent of $\mathbf{Y}=(Y_1,\ldots,Y_n)\in\mathcal{H}^n$ if for any $\varphi\in C_{b,Lip}(\mathbb{R}^{n+m})$

$$\hat{\mathbf{E}}[\varphi(\mathbf{Y},\mathbf{X})]=\hat{\mathbf{E}}[\hat{\mathbf{E}}[\varphi(\mathbf{y},\mathbf{X})]|_{\mathbf{y}=\mathbf{Y}}].$$

In a sublinear expectation space, the fact that **X** is independent of **Y** does not imply that **Y** is independent of **X**. We say $(X_i)_{i\geq 1}$ is a sequence of independent random variables, if X_{i+1} is independent of $(X_1, \dots, X_i)$ for all $i \in \mathbb{N}$.

Definition 2.3. Let $(\Omega, \mathcal{H}, \hat{\mathbf{E}})$ and $(\widetilde{\Omega}, \widetilde{\mathcal{H}}, \widetilde{\mathbf{E}})$ be two sublinear expectations. A random vector $\mathbf{X}$ in $(\Omega, \mathcal{H}, \hat{\mathbf{E}})$ is said to be identically distributed with another random vector $\mathbf{Y}$ in $(\widetilde{\Omega}, \widetilde{\mathcal{H}}, \widetilde{\mathbf{E}})$ (write $\mathbf{X} \overset{d}{=} \mathbf{Y}$), if for any bounded Lipschitz function φ,

$$\hat{\mathbf{E}}[\varphi(X)] = \widetilde{\mathbf{E}}[\varphi(Y)].$$

3. Regularity estimates for the G-heat equation

In this section, we shall introduce a regularity result for the G-heat equation, which is crucial to obtain the convergence rate in Peng's Central Limit Theorem.

$$\partial_t u(x,t) - G(D_x^2 u(x,t)) = 0, \ (x,t) \in \mathbb{R} \times (0,\infty), \tag{3.1}$$

$$u(x,0) = \varphi(x), \tag{3.2}$$

where $G(a) = \frac{1}{2}(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-)$ for some $\overline{\sigma} \geq \underline{\sigma} > 0$.

Throughout this paper, we shall always suppose that $\underline{\sigma} > 0$.

For regularity estimates of (more general) fully nonlinear partial differential equations, we refer the readers to the papers by Kruzhkov [4], Krylov [5], Wang [14], and the book by Lieberman [6] and the references therein. Here we only introduce a result that will be used in this paper.

First of all, for any initial value $\varphi \in C_{b,Lip}(\mathbb{R})$, the collection of bounded Lipschitz functions on $\mathbb{R}$, the G-heat equation has a unique classical solution. Furthermore, we have the following interior regularity estimate:

There exists a constant $\alpha \in (0,1)$ depending on $\underline{\sigma}$ and $\overline{\sigma}$, and a positive constant $c_{\alpha,G}$ depending on $\alpha, \underline{\sigma}$ and $\overline{\sigma}$ such that if $u \in C^{2,1}(\mathbb{R} \times (0,+\infty))$ is a solution to the G-heat equation, we have

$$[D_x^2 u(\cdot,1)]_\alpha \leq c_{\alpha,G} \|Du\|_{\infty,\mathbb{R}\times[0,1]}. \tag{3.3}$$

Here, $[f]_\alpha = \sup_{x,y\in\mathbb{R},x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$ and $\|v\|_{\infty,\mathbb{R}\times[0,1]} = \sup_{(x,t)\in\mathbb{R}\times(0,1]} |v(x,t)|$.

Set $v_\varepsilon(x,t) = \frac{1}{\varepsilon^2} u(\varepsilon x, \varepsilon^2 t)$ for $\varepsilon \in (0,1)$. Then v_ε is also a solution to the G-heat equation. So we have

$$[D_x^2 v_\varepsilon(\cdot,1)]_\alpha \leq c_{\alpha,G} \|Dv_\varepsilon\|_{\infty,\mathbb{R}\times[0,1]}.$$

Noting that

$$[D_x^2 v_\varepsilon(\cdot,1)]_\alpha = \varepsilon^\alpha [D_x^2 u(\cdot,\varepsilon^2)]_\alpha$$

and

$$\|D_x v_\varepsilon\|_{\infty,\mathbb{R}\times[0,1]} \leq \varepsilon^{-1} \|D_x u\|_{\infty,\mathbb{R}\times[0,1]},$$

we get

$$\varepsilon^{1+\alpha} [D_x^2 u(\cdot,\varepsilon^2)]_\alpha \leq c_{\alpha,G} \|D_x u\|_{\infty,\mathbb{R}\times[0,1]}.$$

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We summarize the above arguments in the following theorem.

Theorem 3.1. *There exists a constant $\alpha \in (0,1)$ depending on $\underline{\sigma}$ and $\overline{\sigma}$, and a positive constant $c_{\alpha,G}$ depending on $\alpha, \underline{\sigma}$ and $\overline{\sigma}$ such that if $u \in C^{2,1}(\mathbb{R}\times(0,+\infty))$ is a solution to the G-heat equation, we have, for $t \in (0,1]$,*

$$[D_x^2 u(\cdot,t)]_\alpha \leq c_{\alpha,G}\frac{1}{t^{\frac{1}{2}+\frac{\alpha}{2}}}\|D_x u\|_{\infty,\mathbb{R}\times[0,1]}. \tag{3.4}$$

For $\varphi \in C_{b,Lip}(\mathbb{R})$, if u is the solution to the G-heat equation with initial value φ, we know that $u(\cdot,t)$ is also uniformly Lipschitz continuous with

$$\|D_x u\|_{\infty,\mathbb{R}\times[0,1]} \leq |\varphi|_{Lip}.$$

Hence, we have the following immediate corollary of Theorem 3.1.

Corollary 3.2. *There exists a constant $\alpha \in (0,1)$ depending on $\underline{\sigma}$ and $\overline{\sigma}$, and a positive constant $c_{\alpha,G}$ depending on $\alpha, \underline{\sigma}$ and $\overline{\sigma}$ such that if $\varphi \in C_{b,Lip}(\mathbb{R})$ with $|\varphi|_{Lip} \leq 1$, and u is the solution to the G-heat equation with initial value φ, then we have*

$$[D_x^2 u(\cdot,t)]_\alpha \leq c_{\alpha,G}\frac{1}{t^{\frac{1}{2}+\frac{\alpha}{2}}}. \tag{3.5}$$

4. Rate of convergence of Peng's CLT

Let $\mathcal{N}[\varphi] = \sup_{\mu\in\Theta}\mu[\varphi]$ be a sublinear expectation on $C_{b,Lip}(\mathbb{R})$, where Θ is a weakly compact subset of probability measures on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$. Sometimes, we also write $\mathcal{N}[\psi]$ for $\sup_{\mu\in\Theta}\mu[\psi]$ when ψ is a Borel measurable function such that $\sup_{\mu\in\Theta}\mu[\psi]$ makes sense. Throughout this article, we suppose the following additional property:

(H) $\lim_{N\to\infty}\mathcal{N}[|x|1_{[|x|>N]}] = 0$.

Note that the condition (H) is naturally satisfied if $\mathcal{N}[|x|^{1+\delta}] < \infty$ for some $\delta > 0$.

Define $\xi : \mathbb{R} \to \mathbb{R}$ by $\xi(x) = x$. Sometimes, we write $\mathcal{N}_G[\varphi]$, $\mathcal{N}[\varphi]$ and $\mu[\varphi]$ by $\mathbb{E}_G[\varphi(\xi)]$, $\mathbb{E}[\varphi(\xi)]$ and $E_\mu[\varphi(\xi)]$, respectively. For $\varphi \in C_{b,Lip}(\mathbb{R})$, set $\Theta_\varphi = \{\mu \in \Theta : E_\mu[\varphi(\xi)] = \mathbb{E}[\varphi(\xi)]\}$.

Lemma 4.1. *For $\phi \in C_{b,Lip}(\mathbb{R})$, let v be the solution to the G-heat equation with initial value ϕ and set $\phi_s(x) := v(\sqrt{1-s}x, s)$. Then*

$$\mathcal{N}_G[\phi] - \mathcal{N}[\phi] = \int_0^1 \frac{1}{1-s}\sup_{\mu_s\in\Theta_s} E_{\mu_s}[\mathcal{L}_G\phi_s(\xi)]\mathrm{d}s = \int_0^1 \frac{1}{1-s}\inf_{\mu_s\in\Theta_s} E_{\mu_s}[\mathcal{L}_G\phi_s(\xi)]\mathrm{d}s, \tag{4.1}$$

where $\mathcal{L}_G\phi_s(x) = G(\phi_s''(x)) - \frac{x}{2}\phi_s'(x)$, $\Theta_s = \Theta_{\phi_s}$. Particularly, we have, for a.e. $s \in (0,1)$,

$$\sup_{\mu_s\in\Theta_s} E_{\mu_s}[\mathcal{L}_G\phi_s(\xi)] = \inf_{\mu_s\in\Theta_s} E_{\mu_s}[\mathcal{L}_G\phi_s(\xi)].$$

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Proof. Set $w(s)=\mathbb{E}[v(\sqrt{1-s}\xi,s)]$. Then $w(1)=\mathcal{N}_G[\phi]$ and $w(0)=\mathcal{N}[\phi]$. By Lemma 2.4 in Hu, Peng and Song [3], we have, for $s\in(0,1)$,

$$\begin{aligned}\partial_s^+ w(s):&=\lim_{\delta\to 0+}\frac{w(s+\delta)-w(s)}{\delta}\\ &=\frac{1}{1-s}\sup_{\mu_s\in\Theta_s}E_{\mu_s}[\mathcal{L}_G\phi_s(\xi)]\end{aligned}$$

and

$$\begin{aligned}\partial_s^- w(s):&=\lim_{\delta\to 0+}\frac{w(s-\delta)-w(s)}{-\delta}\\ &=\frac{1}{1-s}\inf_{\mu_s\in\Theta_s}E_{\mu_s}[\mathcal{L}_G\phi_s(\xi)].\end{aligned}$$

Noting that w is continuous on $[0,1]$ and locally Lipschitz continuous on $(0,1)$ by the regularity properties of the solution v of the G-heat equation, we have $w'(s)=\partial_s^+w(s)=\partial_s^-w(s)$ for a.e. $s\in(0,1)$ and consequently

$$w(1)-w(0)=\int_0^1\partial_s^+w(s)ds=\int_0^1\partial_s^-w(s)ds.\quad\square$$

Remark 4.2. Suppose that $G(a)=\frac{1}{2}\mathcal{N}_G[ax^2]=\frac{1}{2}\sigma^2 a$ is linear, i.e., $\mathcal{N}_G=N(0,\sigma^2)$, and that $\mathcal{N}$ is a linear expectation, i.e., $\Theta=\{\mu\}$ is a singleton. Then (4.1) can be rewritten as

$$E[\phi(Z)]-E_\mu[\phi(\xi)]=E_\mu[\int_0^1\frac{1}{1-s}\left(\frac{\sigma^2}{2}\phi_s''(\xi)-\frac{\xi}{2}\phi_s'(\xi)\right)ds]=E_\mu[\frac{\sigma^2}{2}g''(\xi)-\frac{\xi}{2}g'(\xi)],$$

where $g(x)=\int_0^1\frac{1}{1-s}\phi_s(x)ds$ and $Z\sim N(0,\sigma^2)$ under E. Since this equality holds for any distribution μ, we have, by choosing $\mu=\delta_x$,

$$E[\phi(Z)]-\phi(x)=\frac{\sigma^2}{2}g''(x)-\frac{x}{2}g'(x),\ \ x\in\mathbb{R},$$

which is just the classical Stein Equation. Eq. (4.1) will be used as an analogue of the Stein equation under sublinear expectations.

The next Lemma gives an estimate of the expectations on the right hand of Eq. (4.1).

Lemma 4.3. *Let $\alpha\in(0,1]$. Suppose $\mathbb{E}[\xi]=\mathbb{E}[-\xi]=0$ and $\mathbb{E}[|\xi|^{2+\alpha}]<\infty$. For $\phi\in C_b^{2,\alpha}(\mathbb{R})$ and $\mu\in\Theta_\phi$, we have*

$$\left|E_\mu[\frac{\xi}{2}\phi'(\xi)-G(\phi''(\xi))]\right|\le 2[\phi'']_\alpha\mathbb{E}[|\xi|^{2+\alpha}],$$

where $G(a)=\frac{1}{2}\mathbb{E}[a|\xi|^2]$, $a\in\mathbb{R}$.

Proof. Taylor's formula gives

$$\phi(\xi)=\phi(0)+\phi'(0)\xi+\frac{1}{2}\phi''(0)|\xi|^2+R_\xi,\tag{4.2}$$

$$\phi'(\xi)=\phi'(0)+\phi''(0)\xi+R'_\xi,\tag{4.3}$$

$$\phi''(\xi)=\phi''(0)+R''_\xi,\tag{4.4}$$

with $|R_\xi|\le\frac{1}{2}[\phi'']_\alpha|\xi|^{2+\alpha}$, $|R'_\xi|\le[\phi'']_\alpha|\xi|^{1+\alpha}$ and $|R''_\xi|\le[\phi'']_\alpha|\xi|^\alpha$.

Set $A := \mathbb{E}[\phi(\xi)] = E_\mu[\phi(\xi)]$. Then

$$\begin{aligned} A = \mathbb{E}[\phi(\xi)] &= \mathbb{E}[\phi(0) + \phi'(0)\xi + \frac{1}{2}\phi''(0)|\xi|^2 + R_\xi] \\ &\le \phi(0) + \mathbb{E}[\frac{1}{2}\phi''(0)|\xi|^2] + \mathbb{E}[R_\xi] \\ &\le \phi(0) + G(\phi''(0)) + \frac{1}{2}[\phi'']_\alpha \mathbb{E}[|\xi|^{2+\alpha}], \end{aligned}$$

and

$$\begin{aligned} A = \mathbb{E}[\phi(\xi)] &= \mathbb{E}[\phi(0) + \phi'(0)\xi + \frac{1}{2}\phi''(0)|\xi|^2 + R_\xi] \\ &\ge \phi(0) + \mathbb{E}[\frac{1}{2}\phi''(0)|\xi|^2] - \mathbb{E}[-R_\xi] \\ &\ge \phi(0) + G(\phi''(0)) - \frac{1}{2}[\phi'']_\alpha \mathbb{E}[|\xi|^{2+\alpha}]. \end{aligned}$$

Therefore,

$$\left| A - \phi(0) - G(\phi''(0)) \right| \le \frac{1}{2}[\phi'']_\alpha \mathbb{E}[|\xi|^{2+\alpha}].$$

Noting that $A = E_\mu[\phi(\xi)] = \phi(0) + \frac{1}{2}\phi''(0)E_\mu[|\xi|^2] + E_\mu[R_\xi]$, we have

$$\left| \frac{1}{2}\phi''(0)E_\mu[|\xi|^2] - G(\phi''(0)) \right| = \left| A - \phi(0) - E_\mu[R_\xi] - G(\phi''(0)) \right| \le [\phi'']_\alpha \mathbb{E}[|\xi|^{2+\alpha}]. \quad (4.5)$$

Now let us compute the expectation $E_\mu[\frac{\xi}{2}\phi'(\xi) - G(\phi''(\xi))]$. By (4.3) and (4.4), we have

$$\begin{aligned} &\frac{\xi}{2}\phi'(\xi) - G(\phi''(\xi)) \\ &= \frac{\xi}{2}(\phi'(0) + \phi''(0)\xi + R'_\xi) - G(\phi''(0) + R''_\xi) \\ &= \frac{\xi}{2}\phi'(0) + [\frac{1}{2}\phi''(0)|\xi|^2 - G(\phi''(0))] + [G(\phi''(0)) - G(\phi''(0) + R''_\xi)] + \frac{\xi}{2}R'_\xi. \end{aligned}$$

So, by (4.5),

$$\begin{aligned} &\left| E_\mu[\frac{\xi}{2}\phi'(\xi) - G(\phi''(\xi))] \right| \\ &= \left| E_\mu[\frac{1}{2}\phi''(0)|\xi|^2 - G(\phi''(0))] + E_\mu[G(\phi''(0)) - G(\phi''(0) + R''_\xi)] + \frac{1}{2}E_\mu[\xi R'_\xi] \right| \\ &\le [\phi'']_\alpha \mathbb{E}[|\xi|^{2+\alpha}] + \frac{1}{2}[\phi'']_\alpha \overline{\sigma}^2 \mathbb{E}[|\xi|^\alpha] + \frac{1}{2}[\phi'']_\alpha \mathbb{E}[|\xi|^{2+\alpha}] \\ &\le 2[\phi'']_\alpha \mathbb{E}[|\xi|^{2+\alpha}]. \end{aligned}$$

The last inequality holds since

$$\mathbb{E}[|\xi|^2]\mathbb{E}[|\xi|^\alpha] \le \left(\mathbb{E}[|\xi|^{2\times\frac{2+\alpha}{2}}]\right)^{\frac{2}{2+\alpha}} \left(\mathbb{E}[|\xi|^{\alpha\times\frac{2+\alpha}{\alpha}}]\right)^{\frac{\alpha}{2+\alpha}} = \mathbb{E}[|\xi|^{2+\alpha}]. \qquad \square$$

Remark 4.4. We emphasize that the function G in Lemma 4.1 is determined by the variances of $\mathcal{N}_G$ and that the function G in Lemma 4.3 is determined by the variances of $\mathcal{N}$. In other words, to estimate $\mathcal{N}_G[\phi] - \mathcal{N}[\phi]$ applying these two lemmas requires that $\mathcal{N}_G$ and $\mathcal{N}$ have the same variances.

With these preparations, we are now ready to prove the convergence rate of Peng's Central Limit Theorem under sublinear expectations.

Theorem 4.5. *Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d random variables under a sublinear expectation $\hat{\mathbf{E}}$ with $\hat{\mathbf{E}}[X_1] = \hat{\mathbf{E}}[-X_1] = 0$ and $\hat{\mathbf{E}}[X_1^2] = \overline{\sigma}^2 \geq -\hat{\mathbf{E}}[-X_1^2] = \underline{\sigma}^2 > 0$. For $\varphi \in C_{b,Lip}(\mathbb{R})$, let u be the solution to the G-heat equation with initial value φ. Setting $W_n := \frac{X_1+\cdots+X_n}{\sqrt{n}}$, we have, for $\alpha \in (0,1]$,*

$$\left|\hat{\mathbf{E}}[\varphi(W_n)] - \mathcal{N}_G(\varphi)\right| \leq 2\int_0^1 [D_x^2 u(\cdot,s)]_\alpha \mathrm{d}s \frac{\hat{\mathbf{E}}[|X_1|^{2+\alpha}]}{n^{\frac{\alpha}{2}}},$$

where $\mathcal{N}_G$ is the G-normal distribution with $G(a) = \frac{1}{2}\hat{\mathbf{E}}[aX_1^2]$.

Proof. Let $u(x,t)$ be the solution to the G-heat equation with $u(x,0) = \varphi(x)$. Assume $\int_0^1 [D_x^2 u(\cdot,s)]_\alpha \mathrm{d}s < \infty$ and $\hat{\mathbf{E}}[|X_1|^{2+\alpha}] < \infty$. For the other case, the result is trivial.

Fix $n \in \mathbb{N}$. Set, for $1 \leq i \leq n$,

$$\xi_{i,n} = \frac{X_i}{\sqrt{n}}, \quad W_{0,n} = 0, \quad W_{i,n} = \sum_{k=1}^{i} \xi_{k,n},$$

and, for $0 \leq i \leq n$,

$$A_{i,n} = \hat{\mathbf{E}}[u(W_{i,n}, 1 - \frac{i}{n})].$$

Then $A_{n,n} = \hat{\mathbf{E}}[\varphi(W_n)]$, $A_{0,n} = \mathcal{N}_G[\varphi]$, and

$$\left|\hat{\mathbf{E}}[\varphi(W_n)] - \mathcal{N}_G[\varphi]\right| \leq \sum_{i=1}^{n} \left|A_{i,n} - A_{i-1,n}\right| \tag{4.6}$$

$$= \sum_{i=1}^{n} \left|\hat{\mathbf{E}}[b_{i,n}(W_{i-1,n})] - \hat{\mathbf{E}}[c_{i,n}(W_{i-1,n})]\right|, \tag{4.7}$$

$$\leq \sum_{i=1}^{n} \sup_{x\in\mathbb{R}} \left|b_{i,n}(x) - c_{i,n}(x)\right| \tag{4.8}$$

where $b_{i,n}(x) = \hat{\mathbf{E}}[u(x + \frac{X_i}{\sqrt{n}}, 1 - \frac{i}{n})]$ and $c_{i,n}(x) = \mathbb{E}_G[u(x + \frac{\xi}{\sqrt{n}}, 1 - \frac{i}{n})]$. Here and below we write $\mathbb{E}_G[\phi(\xi)]$ for $\mathcal{N}_G[\phi]$.

Let us now compute $b_{i,n}(x) - c_{i,n}(x)$.

Set $\phi(y) := \phi_{x,i,n}(y) = u(x + \frac{y}{\sqrt{n}}, 1 - \frac{i}{n})$. Then $c_{i,n}(x) = \mathcal{N}_G[\phi]$ and $b_{i,n}(x) = \hat{\mathbf{E}}[\phi(X_1)]$. The latter, as a sublinear expectation on $C_{b,Lip}(\mathbb{R})$, can be represented as

$$\hat{\mathbf{E}}[\phi(X_1)] = \sup_{\mu\in\Theta} \mu[\phi],$$

where Θ is a weakly compact subset of probabilities on $\mathbb{R}$. In the sequel, we employ the notations in Lemma 4.1. By this lemma, we have

$$c_{i,n}(x) - b_{i,n}(x) = \int_0^1 \frac{1}{1-s} \sup_{\mu_s\in\Theta_s} \mu_s[\mathcal{L}_G\phi_s]\mathrm{d}s = \int_0^1 \frac{1}{1-s} \inf_{\mu_s\in\Theta_s} \mu_s[\mathcal{L}_G\phi_s]\mathrm{d}s,$$

where

$$\begin{aligned}\phi_s(y) &= \mathbb{E}_G[\phi(\sqrt{1-s}y+\sqrt{s}\xi)]\\ &= \mathbb{E}_G[u(x+\sqrt{\frac{1-s}{n}}y+\sqrt{\frac{s}{n}}\xi, 1-\frac{i}{n})]\\ &= u(x+\sqrt{\frac{1-s}{n}}y, 1-\frac{i}{n}+\frac{s}{n}).\end{aligned}$$

Therefore

$$[D_y^2\phi_s]_\alpha = (\frac{1-s}{n})^{1+\frac{\alpha}{2}}[D_x^2u(\cdot, 1-\frac{i}{n}+\frac{s}{n})]_\alpha.$$

Now Lemma 4.3 gives

$$\begin{aligned}\Big|b_{i,n}(x)-c_{i,n}(x)\Big| &\le \int_0^1 \frac{2}{1-s}[D_y^2\phi_s]_\alpha \mathrm{ds}\times\hat{\mathbf{E}}[|X_1|^{2+\alpha}]\\ &= \int_0^1 \frac{2(1-s)^{\frac{\alpha}{2}}}{n^{1+\frac{\alpha}{2}}}[D_x^2u(\cdot, 1-\frac{i}{n}+\frac{s}{n})]_\alpha \mathrm{ds}\times\hat{\mathbf{E}}[|X_1|^{2+\alpha}]\\ &\le \int_0^1 \frac{2}{n^{1+\frac{\alpha}{2}}}[D_x^2u(\cdot, 1-\frac{i}{n}+\frac{s}{n})]_\alpha \mathrm{ds}\times\hat{\mathbf{E}}[|X_1|^{2+\alpha}]\\ &= \frac{2}{n^{\frac{\alpha}{2}}}\int_{1-\frac{i}{n}}^{1-\frac{i-1}{n}}[D_x^2u(\cdot, s)]_\alpha \mathrm{ds}\times\hat{\mathbf{E}}[|X_1|^{2+\alpha}].\end{aligned}$$

Hence,

$$\begin{aligned}\Big|\hat{\mathbf{E}}[\varphi(W_n)]-\mathcal{N}_G[\varphi]\Big| &\le \sum_{i=1}^n \sup_{x\in\mathbb{R}}\big|b_{i,n}(x)-c_{i,n}(x)\big|\\ &\le \frac{2}{n^{\frac{\alpha}{2}}}\int_0^1[D_x^2u(\cdot, s)]_\alpha \mathrm{ds}\times\hat{\mathbf{E}}[|X_1|^{2+\alpha}]. \quad \square\end{aligned}$$

Corollary 4.6. *Let $(X_i)_{i\ge1}$ be a sequence of i.i.d random variables under a sublinear expectation $\hat{\mathbf{E}}$ with $\hat{\mathbf{E}}[X_1]=\hat{\mathbf{E}}[-X_1]=0$ and $\hat{\mathbf{E}}[X_1^2]=\overline{\sigma}^2\ge-\hat{\mathbf{E}}[-X_1^2]=\underline{\sigma}^2>0$. Setting $W_n:=\frac{X_1+\cdots+X_n}{\sqrt{n}}$, then there exists a constant $\alpha\in(0,1)$ depending on $\underline{\sigma}$ and $\overline{\sigma}$, and a positive constant $C_{\alpha,G}$ depending on $\alpha,\underline{\sigma}$ and $\overline{\sigma}$ such that*

$$\sup_{|\varphi|_{Lip}\le1}\Big|\hat{\mathbf{E}}[\varphi(W_n)]-\mathcal{N}_G(\varphi)\Big|\le C_{\alpha,G}\frac{\hat{\mathbf{E}}[|X_1|^{2+\alpha}]}{n^{\frac{\alpha}{2}}},$$

where $\mathcal{N}_G$ is the G-normal distribution with $G(a)=\frac{1}{2}\hat{\mathbf{E}}[aX_1^2]$.

Here α is the Hölder exponent in Theorem 3.1, and $C_{\alpha,G}$ can be chosen as $\frac{4}{1-\alpha}c_{\alpha,G}$ with $c_{\alpha,G}$ the α-Hölder constant in the same theorem.

Proof. The conclusion follows immediately from Theorem 4.5 and Corollary 3.2. $\square$

5. Non-identically distributed Case

In this section, we consider the normal approximation for an independent but not necessarily identically distributed sequence of random variables. To do so, we first introduce some

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notations. For a random variable X in a sublinear expectation space with $\overline{\sigma}^2 := \hat{\mathbf{E}}[X^2] \geq -\hat{\mathbf{E}}[-X^2] =: \underline{\sigma}^2 > 0$, set $\beta := \frac{\overline{\sigma}}{\underline{\sigma}}$ and $\sigma := \frac{\overline{\sigma}+\underline{\sigma}}{2}$. Now we can use β, σ to characterize the variances of a random variable X in a sublinear expectation space. For example, we shall write $\mathcal{N}_\beta(0, \sigma^2)$ for the G-normal distribution $\mathcal{N}_G$, and write $\mathcal{N}_\beta$ for $\mathcal{N}_\beta(0, 1)$. Clearly, $\mathcal{N}_1(0, \sigma^2) = N(0, \sigma^2)$, the classical normal distribution.

In this section, we shall fix the ratio $\beta \geq 1$ of variances as a constant and call σ^2 the variance. We write G_β for the function G with $\underline{\sigma} = \frac{2}{1+\beta}$ and $\overline{\sigma} = \frac{2\beta}{1+\beta}$. So the G_β-normal distribution is $\mathcal{N}_\beta$.

Theorem 5.1. *Let $(\xi_i)_{1\leq i\leq n}$ be a sequence of independent random variables under a sublinear expectation $\hat{\mathbf{E}}$. We suppose further that, for each $1 \leq i \leq n$, ξ_i has finite variance σ_i^2 and mean 0, i.e., $\hat{\mathbf{E}}[\xi_i] = \hat{\mathbf{E}}[-\xi_i] = 0$. Setting $W := \xi_1 + \cdots + \xi_n$ and $\sigma^2 := \Sigma_{i=1}^n \sigma_i^2$, then there exists a constant $\alpha \in (0, 1)$ depending on β, and a positive constant $C_{\alpha,\beta}$ depending on α, β such that*

$$\sup_{|\varphi|_{Lip}\leq 1} \left| \hat{\mathbf{E}}[\varphi(\frac{W}{\sigma})] - \mathcal{N}_\beta(\varphi) \right| \leq C_{\alpha,\beta} \sup_{1\leq i\leq n} \left\{ \frac{\hat{\mathbf{E}}[|\xi_i|^{2+\alpha}]}{\sigma_i^{2+\alpha}} (\frac{\sigma_i}{\sigma})^\alpha \right\}.$$

Here α is the Hölder exponent in Theorem 3.1, and $C_{\alpha,\beta}$ can be chosen as $\frac{4}{1-\alpha} c_{\alpha,G_\beta}$ with c_{α,G_β} the α-Hölder constant in the same theorem.

Proof. The proof is adapted from that of Theorem 4.5.

Set, for $1 \leq i \leq n$,

$$t_0 = 0, \ \ t_i = \frac{\Sigma_{k=1}^i \sigma_k^2}{\sigma^2}, \ \ W_0 = 0, \ \ W_i = \sum_{k=1}^i \frac{\xi_k}{\sigma}.$$

and, for $0 \leq i \leq n$,

$$A_i = \hat{\mathbf{E}}[u(W_i, 1 - t_i)],$$

where $u(x, t)$ is the solution to the standard G_β-heat equation with $u(x, 0) = \varphi(x)$.

Then $A_n = \hat{\mathbf{E}}[\varphi(W_n)]$, $A_0 = \mathcal{N}_\beta[\varphi]$, and

$$\left| \hat{\mathbf{E}}[\varphi(W_n)] - \mathcal{N}_\beta[\varphi] \right| \leq \sum_{i=1}^n \left| A_i - A_{i-1} \right| \tag{5.1}$$

$$= \sum_{i=1}^n \left| \hat{\mathbf{E}}[b_i(W_{i-1})] - \hat{\mathbf{E}}[c_i(W_{i-1})] \right|, \tag{5.2}$$

$$\leq \sum_{i=1}^n \sup_{x\in\mathbb{R}} \left| b_i(x) - c_i(x) \right| \tag{5.3}$$

where $b_i(x) = \hat{\mathbf{E}}[u(x + \frac{\xi_i}{\sigma}, 1 - t_i)]$ and $c_i(x) = \mathbb{E}_\beta[u(x + \frac{\sigma_i \xi}{\sigma}, 1 - t_i)]$. Here and below we write $\mathbb{E}_\beta[\phi(\xi)]$ for $\mathcal{N}_\beta[\phi]$.

Let us now compute $b_i(x) - c_i(x)$.

Set $\phi(y) := \phi_{x,i}(y) = u(x + \frac{\sigma_i y}{\sigma}, 1 - t_i)$. Then $c_i(x) = \mathcal{N}_\beta[\phi]$ and $b_i(x) = \hat{\mathbf{E}}[\phi(\frac{\xi_i}{\sigma_i})]$. The latter, as a sublinear expectation on $C_{b,Lip}(\mathbb{R})$, can be represented as

$$\hat{\mathbf{E}}[\phi(\frac{\xi_i}{\sigma_i})] = \sup_{\mu\in\Theta} \mu[\phi],$$

where Θ is a weakly compact subset of probabilities on $\mathbb{R}$. In the sequel, we employ the notations in Lemma 4.1. By this lemma, we have

$$c_i(x) - b_i(x) = \int_0^1 \frac{1}{1-s} \sup_{\mu_s \in \Theta_s} \mu_s[\mathcal{L}_{G_\beta}\phi_s]\mathrm{d}s = \int_0^1 \frac{1}{1-s} \inf_{\mu_s \in \Theta_s} \mu_s[\mathcal{L}_{G_\beta}\phi_s]\mathrm{d}s,$$

where

$$\begin{aligned}\phi_s(y) &= \mathbb{E}_\beta[\phi(\sqrt{1-s}y + \sqrt{s}\xi)] \\ &= \mathbb{E}_\beta[u(x + \sqrt{1-s}\frac{\sigma_i}{\sigma}y + \sqrt{s}\frac{\sigma_i}{\sigma}\xi, 1-t_i)] \\ &= u(x + \sqrt{1-s}\frac{\sigma_i}{\sigma}y, 1 - t_i + s\frac{\sigma_i^2}{\sigma^2}).\end{aligned}$$

Therefore

$$\begin{aligned}[D_y^2\phi_s]_\alpha &= (\frac{\sigma_i^2}{\sigma^2}(1-s))^{1+\frac{\alpha}{2}}[D_x^2 u(\cdot, 1-t_i + s\frac{\sigma_i^2}{\sigma^2})]_\alpha \\ &\le c_{\alpha,G_\beta}(\frac{\sigma_i^2}{\sigma^2}(1-s))^{1+\frac{\alpha}{2}}(1-t_i+s\frac{\sigma_i^2}{\sigma^2})^{-(\frac{1}{2}+\frac{\alpha}{2})}.\end{aligned}$$

Now Lemma 4.3 gives, noting that $G_\beta(a) = \frac{1}{2}\hat{\mathbf{E}}[a(\frac{\xi_i}{\sigma_i})^2]$,

$$\begin{aligned}\left|b_{i,n}(x) - c_{i,n}(x)\right| &\le \int_0^1 \frac{2}{1-s}[D_y^2\phi_s]_\alpha \mathrm{d}s \times \frac{\hat{\mathbf{E}}[|\xi_i|^{2+\alpha}]}{\sigma_i^{2+\alpha}} \\ &\le 2c_{\alpha,G_\beta}\frac{\sigma_i^2}{\sigma^2}\int_0^1 (\frac{\sigma_i^2}{\sigma^2}(1-s))^{\frac{\alpha}{2}}(1-t_i+\frac{\sigma_i^2}{\sigma^2}s)^{-(\frac{1}{2}+\frac{\alpha}{2})}\mathrm{d}s \times \frac{\hat{\mathbf{E}}[|\xi_i|^{2+\alpha}]}{\sigma_i^{2+\alpha}} \\ &\le 2c_{\alpha,G_\beta}(\frac{\sigma_i^2}{\sigma^2})^{1+\frac{\alpha}{2}}\int_0^1 (1-t_i+\frac{\sigma_i^2}{\sigma^2}s)^{-(\frac{1}{2}+\frac{\alpha}{2})}\mathrm{d}s \times \frac{\hat{\mathbf{E}}[|\xi_i|^{2+\alpha}]}{\sigma_i^{2+\alpha}} \\ &= 2c_{\alpha,G_\beta}\frac{\sigma_i^\alpha}{\sigma^\alpha}\int_{1-t_i}^{1-t_{i-1}} s^{-(\frac{1}{2}+\frac{\alpha}{2})}\mathrm{d}s \times \frac{\hat{\mathbf{E}}[|\xi_i|^{2+\alpha}]}{\sigma_i^{2+\alpha}}.\end{aligned}$$

Hence,

$$\begin{aligned}\left|\hat{\mathbf{E}}[\varphi(W_n)] - \mathcal{N}_\beta[\varphi]\right| &\le \sum_{i=1}^n \sup_{x\in\mathbb{R}}\left|b_{i,n}(x) - c_{i,n}(x)\right| \\ &\le 2c_{\alpha,G_\beta}\int_0^1 s^{-(\frac{1}{2}+\frac{\alpha}{2})}\mathrm{d}s \sup_{1\le i\le n}\left\{\frac{\hat{\mathbf{E}}[|\xi_i|^{2+\alpha}]}{\sigma_i^{2+\alpha}}(\frac{\sigma_i}{\sigma})^\alpha\right\} \\ &= \frac{4c_{\alpha,G_\beta}}{1-\alpha}\sup_{1\le i\le n}\left\{\frac{\hat{\mathbf{E}}[|\xi_i|^{2+\alpha}]}{\sigma_i^{2+\alpha}}(\frac{\sigma_i}{\sigma})^\alpha\right\}. \quad \square\end{aligned}$$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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