

On an extension of Lévy's stochastic area process to higher dimensions

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Received 22 November 1990

Revised 22 November 1991

For $n \geq 2$ an $(n-1)$ -parameter real process V_n , called stochastic volume, is defined. This process is an extension to higher dimensions of Lévy's stochastic area which is obtained from V_n by setting $n = 2$. For V_3 , a Strassen-type functional law of the iterated logarithm is proved by making use of large deviations techniques.

AMS 1991 Subject Classifications: Primary 60F17, 60F10; Secondary 60H05, 60J65.

Brownian motion * Lévy's stochastic area * functional law of the iterated logarithm * large deviations

1. Introduction

Let $\mathbb{R}_+ = [0, +\infty)$. Paul Lévy (1940) initiated the study of the so-called area process

$$L(t) = \frac{1}{2} \left(\int_0^t W_1(s) W_2(ds) - \int_0^t W_2(s) W_1(ds) \right), \quad t \in \mathbb{R}_+,$$

where $W = \{W(s) = (W_1(s), W_2(s)); s \in \mathbb{R}_+\}$ is a Wiener process on a complete probability space (Ω, \mathcal{F}, P) , as a stochastic analogue of the formula yielding the area contained in a lens-shaped domain. Several extensions of this process to higher dimensions already exist in the literature (e.g., Bethuet, 1986, and Helmes and Schwane, 1983). Our extension is based on the simple remark that, in \mathbb{R}^2 , such a lens-shaped domain can be viewed as a cone. Indeed, in \mathbb{R}^n , let $V_n(t)$ be the stochastic analogue of the volume of a cone, where W is now a $(n-1)$ -parameter Wiener process; $V_n = \{V_n(t); t \in \mathbb{R}_+^{n-1}\}$ is our extension to higher dimensions of Lévy's stochastic area process $\{L(t); t \in \mathbb{R}_+\}$ which we shall examine subsequently. It is to be expected that proofs for several-parameter processes will be technically difficult and will use complicated notation.

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Work supported by the Natural Sciences and Engineering Research Council of Canada and by the Fonds F.C.A.R. of the Province of Quebec.

In Section 2 we construct, in two different ways, the stochastic volume V_n associated with the $(n-1)$ -parameter Wiener process W in \mathbb{R}^n . Our main results are given in Sections 3 and 4, where we use large deviations techniques, similar to those in Baldi (1986), to prove a Strassen-type functional law of the iterated logarithm for V_3 .

The following notations and conventions are adopted throughout: (Ω, \mathcal{F}, P) is a complete probability space on which are defined all random variables considered. Let $k \in \mathbb{N} = \{1, 2, \dots\}$. For every $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$, $\langle x, y \rangle$ is the scalar product in \mathbb{R}^k and $|x|$ stands for Euclidean norm; further, $x + y = (x_1 + y_1, \dots, x_k + y_k)$, $xy = (x_1 y_1, \dots, x_k y_k)$, $x^y = (x_1^{y_1}, \dots, x_k^{y_k})$, and $x/y = (x_1/y_1, \dots, x_k/y_k)$. \mathbb{R}^k is partially ordered by setting $x \leq y$ if, and only if, $x_i \leq y_i$ for every $i \in \{1, \dots, k\}$; if $x \leq y$ then $[x, y]$ stands for $\prod_{i=1}^k [x_i, y_i]$. Next, if $C \subset \{1, \dots, k\}$ and $t = (t_1, \dots, t_k) \in \mathbb{R}_+^k$, then t_C stands for $(t_i : i \in C)$. $\bar{C} = \{1, \dots, k\} \setminus C$. For every $t \in \mathbb{R}_+$, \underline{t} is the point whose coordinates are all equal to t . Let now $t^0 = (t_1^0, \dots, t_k^0) \in \mathbb{R}_+^k$ and let $X = \{X(t) : t \in \mathbb{R}_+^k\}$ be a k -parameter process. The restriction of X to $\{t \in \mathbb{R}_+^k : t_i = t_i^0, i \in C\}$ is denoted by ${}^{t_C^0}X$. $X \stackrel{\mathcal{L}}{=} Y$ means that the random variables X and Y have the same probability distribution. Finally, every symbol with a hat is omitted.

2. Construction of the stochastic volume

In this section we construct, by two different methods, the stochastic volume V_n . For $n=3$, we show that either construction leads to a version of the same process.

2.1. Preliminaries

Let $n \geq 2$ be a positive integer and let $V_n(t)$, $t \in \mathbb{R}_+^{n-1}$, be the volume of a cone $K[t]$ with vertex 0 and basis the portion $I[t] = \{I(s) : s \in [0, t]\}$ of a parametrized surface $s \mapsto I(s) = (I_1(s), \dots, I_n(s)) \in \mathbb{R}^n$, $s \in \mathbb{R}_+^{n-1}$, of class C^{n-1} . Then

$$V_n(t) = \frac{1}{n} \sum_{\sigma \in S_n} \varepsilon_\sigma I_n^\sigma(t), \quad t \in \mathbb{R}_+^{n-1}, \quad n \geq 2, \quad (2.1)$$

where S_n is the set of all permutations $\sigma = (\sigma(1), \dots, \sigma(n))$ of $\{1, \dots, n\}$, ε_σ is the signature of $\sigma \in S_n$, and

$$I_n^\sigma(t) = \int_0^{t_{n-1}} \cdots \int_0^{t_1} I_{\sigma(1)}(u_1, \dots, u_{n-1}) \prod_{i=2}^n \frac{\partial I_{\sigma(i)}}{\partial u_{i-1}}(u_1, \dots, u_{n-1}) du_1 \cdots du_{n-1};$$

in particular

$$V_2(t_1) = \frac{1}{2} \left(\int_0^{t_1} I_1(u_1) I_2(du_1) - \int_0^{t_1} I_2(u_1) I_1(du_1) \right). \quad (2.2)$$

Further, for every $n \geq 3$ and every $u = (u_1, \dots, u_{n-1}) \in \mathbb{R}_+^{n-1}$, set

$$f_n^\sigma(u) = \int_0^{u_{n-2}} \cdots \int_0^{u_1} \Gamma_{\sigma(1)}(a_1, \dots, a_{n-2}, u_{n-1}) \\ \times \prod_{i=2}^{n-1} \frac{\partial \Gamma_{\sigma(i)}}{\partial u_{i-1}}(a_1, \dots, a_{n-2}, u_{n-1}) da_1 \cdots da_{n-2}.$$

Moreover, for every $k \in \{1, \dots, n-2\}$, put $\mathcal{T}_k = \{i = (i_1, \dots, i_k) \in \mathbb{N}^k : 1 \leq i_1 < \dots < i_k \leq n-2\}$. Now, for every $i = (i_1, \dots, i_k) \in \mathcal{T}_k$ and every $t = (t_1, \dots, t_{n-1}) \in \mathbb{R}_+^{n-1}$, put $(i) = \{i_1, \dots, i_k\}$ and $[0, t]^{(i)} = \prod_{m \in (i)} [0, t_m]$, where $(i) = \{1, \dots, n-1\} \setminus \{i\}$. By virtue of the formula of integration by parts in Dozzi (1989, p. 66-67), we have

$$I_n^\sigma(t) = \sum_{k=1}^{n-2} \sum_{i \in \mathcal{T}_k} (-1)^{n-2-k} \int_{[0,t]^{(i)}} {}^{t(i)} f_n^\sigma(u_{\overline{(i)}}) \\ \times \frac{\partial^{n-1-k} \Gamma_{\sigma(n)}}{\partial u_1 \cdots \partial \widehat{u_{i_1}} \cdots \partial \widehat{u_{i_k}} \cdots \partial u_{n-1}}(u_{\overline{(i)}}) \prod_{j \in \overline{(i)}} du_j \\ + (-1)^{n-2} \int_0^{t_{n-1}} \cdots \int_0^{t_1} f_n^\sigma(u_1, \dots, u_{n-1}) \\ \times \frac{\partial^{n-1} \Gamma_{\sigma(n)}}{\partial u_1 \cdots \partial u_{n-1}}(u_1, \dots, u_{n-1}) du_1 \cdots du_{n-1} \quad (2.3)$$

$({}^{t(i)} g(u_{\overline{(i)}}) = g(t_{(i)}, u_{\overline{(i)}}))$; in particular

$$I_3^\sigma(t_1, t_2) = \int_0^{t_2} f_3^\sigma(t_1, u_2) \frac{\partial \Gamma_{\sigma(3)}}{\partial u_2}(t_1, u_2) du_2 \\ - \int_0^{t_2} \int_0^{t_1} f_3^\sigma(u_1, u_2) \frac{\partial^2 \Gamma_{\sigma(3)}}{\partial u_1 \partial u_2}(u_1, u_2) du_1 du_2.$$

By summing over σ in (2.3), we obtain

$$V_n(t) = \frac{n-1}{n} \sum_{k=1}^{n-2} \sum_{i \in \mathcal{T}_k} (-1)^{n-2-k} \int_{[0,t]^{(i)}} \langle {}^{t(i)} \tilde{V}_{n-1}(u_{\overline{(i)}}), {}^{t(i)} \Gamma(du_{\overline{(i)}}) \rangle \\ + \frac{n-1}{n} (-1)^{n-2} \\ \times \int_0^{t_{n-1}} \cdots \int_0^{t_1} \langle \tilde{V}_{n-1}(u_1, \dots, u_{n-1}), \Gamma(du_1, \dots, du_{n-1}) \rangle, \quad n \geq 3, \quad (2.4)$$

where $\tilde{V}_{n-1} = ((-1)^{n-1} V_{n-1}^1, \dots, (-1)^{j(n-j)} V_{n-1}^j, \dots, V_{n-1}^n)$; here $V_{n-1}^j(\cdot, u_{n-1})$, $j \in \{1, \dots, n\}$, $u_{n-1} \in \mathbb{R}_+$, is the analogue of $V_n(\cdot)$ associated with

$$\Gamma^j(\cdot, u_{n-1}) = (\Gamma_{\tau_j(1)}(\cdot, u_{n-1}), \dots, \Gamma_{\tau_j(n-1)}(\cdot, u_{n-1})),$$

with $\tau_j = (\tau_j(1), \dots, \tau_j(n-1))$ defined by $\tau_1 = (2, \dots, n)$, $\tau_j = (j+1, \dots, n, 1, \dots, j-1)$ for $2 \leq j \leq n-1$, and $\tau_n = (1, \dots, n-1)$. In particular

$$V_3(t_1, t_2) = \frac{2}{3} \left(\int_0^{t_2} \langle \tilde{V}_2(t_1, u_2), \Gamma(t_1, du_2) \rangle - \int_0^{t_2} \int_0^{t_1} \langle \tilde{V}_2(u_1, u_2), \Gamma(du_1, du_2) \rangle \right),$$

where $\tilde{V}_2 = (V_2^1, V_2^2, V_2^3)$.

2.2. First method of construction

Let $W = \{W(t) = (W_1(t), \dots, W_n(t)): t \in \mathbb{R}_+^{n-1}\}$ be an $(n-1)$ -parameter Wiener process in \mathbb{R}^n . In the sequel \mathcal{F}_t denotes the completion of the σ -field $\sigma(W(s): s \in [0, t])$. Further, let Γ be replaced by W in (2.2) and (2.3). $\{V_n(t): t \in \mathbb{R}_+^{n-1}\}$ so obtained is a $(n-1)$ -parameter process. It remains to give a precise sense to the integrals occurring in the expression of $V_n(t)$. For this purpose, we need a result from Cairoli and Walsh (1975, p. 173) which is stated here in a general form:

Lemma 2.1. *Let $t^0 \in \mathbb{R}_+^k$, let $X = \{X(t): t \in \mathbb{R}_+^k\}$ be a measurable, \mathcal{F}_t -adapted k -parameter process, and let $C \subset \{1, \dots, k\}$ be such that*

$$\int_{[0, t_C^0]} E[({}^t_C X(u_C))^2] \prod_{i \in C} du_i < +\infty.$$

Then, for every $t \leq t^0$,

$$\begin{aligned} E \left(\int_{[0, t_C^0]} {}^t_C X(u_C) {}^t_C W(du_C) \middle| \mathcal{F}_t \right) \\ = \int_{[0, t_C^0]} E({}^t_C X(u_C) | \mathcal{F}_{(u_C, t_C)}) {}^t_C W(du_C). \quad \square \end{aligned}$$

We begin by showing that the expressions

$$V_2(t_1) = \frac{1}{2} \left(\int_0^{t_1} W_1(u_1) W_2(du_1) - \int_0^{t_1} W_2(u_1) W_1(du_1) \right), \quad t_1 \in \mathbb{R}_+, \quad (2.5)$$

and

$$\begin{aligned} V_3(t_1, t_2) = \frac{2}{3} \left(\int_0^{t_2} \langle \tilde{V}_2(t_1, u_2), W(t_1, du_2) \rangle \right. \\ \left. - \int_0^{t_2} \int_0^{t_1} \langle \tilde{V}_2(u_1, u_2), W(du_1, du_2) \rangle \right), \quad (t_1, t_2) \in \mathbb{R}_+^2, \quad (2.6) \end{aligned}$$

are well-defined, where $\tilde{V}_2 = (V_2^1, V_2^2, V_2^3)$ and

$$\begin{aligned} V_2^1(u_1, u_2) = \frac{1}{2} \left(\int_0^{u_1} W_{\tau_j(1)}(a_1, u_2) W_{\tau_j(2)}(da_1, u_2) \right. \\ \left. - \int_0^{u_1} W_{\tau_j(2)}(a_1, u_2) W_{\tau_j(1)}(da_1, u_2) \right), \end{aligned}$$

$j \in \{1, 2, 3\}$, $(u_1, u_2) \in \mathbb{R}_+^2$. Indeed, the integrals in $V_2(t_1)$ are defined as Itô integrals and therefore (2.5) is Lévy's stochastic area process associated with the Wiener process $W = (W_1, W_2)$. For every $u_2 \in \mathbb{R}_+$ and every $j \in \{1, 2, 3\}$, $V_2^j(\cdot, u_2)$ is an Itô integral as well. By virtue of Lemma 2.1, $\{V_2^j(u), \mathcal{F}_u : u \in \mathbb{R}_+^2\}$ is a martingale. Since V_2^j is square integrable, the corollary of the theorem in Bakry (1979, p. 173) implies the existence of a right-continuous version of V_2^j . Hence the integrals in V_3 can be defined in the sense of Cairoli and Walsh (1975).

Let us note that for every $t_2 \in \mathbb{R}_+ \setminus \{0\}$ and every $j \in \{1, 2, 3\}$, $V_2^j(\cdot, t_2)/t_2$ is Lévy's stochastic area process associated with the Wiener process

$$(W_{\tau_j(1)}(\cdot, t_2)/\sqrt{t_2}, W_{\tau_j(2)}(\cdot, t_2)/\sqrt{t_2}).$$

V_3 , given by (2.6), represents the stochastic volume process and is taken as the analogue of Lévy's area process (2.5) in dimension three.

Finally, for $n \geq 4$, we define the stochastic volume by the recurrence formula

$$\begin{aligned} V_n(t) = & \frac{n-1}{n} \sum_{k=1}^{n-2} \sum_{i \in \mathcal{I}_k} (-1)^{n-2-k} \int_{[0,t]^{(i)}} \langle {}^{(i)}\tilde{V}_{n-1}(u_{(\bar{i})}), {}^{(i)}W(du_{(\bar{i})}) \rangle \\ & + \frac{n-1}{n} (-1)^{n-2} \\ & \times \int_0^{t_{n-1}} \cdots \int_0^{t_1} \langle \tilde{V}_{n-1}(u_1, \dots, u_{n-1}), W(du_1, \dots, du_{n-1}) \rangle, \end{aligned}$$

where

$$\tilde{V}_{n-1} = ((-1)^{n-1} V_{n-1}^1, \dots, (-1)^{j(n-j)} V_{n-j}^j, \dots, V_{n-1}^n);$$

here $V_{n-1}^j(\cdot, t_{n-1})/(t_{n-1})^{(n-1)/2}$, $j \in \{1, \dots, n\}$, $t_{n-1} \in \mathbb{R}_+ \setminus \{0\}$, is the stochastic volume process associated with the $(n-2)$ -parameter Wiener process

$$(W_{\tau_j(1)}(\cdot, t_{n-1})/\sqrt{t_{n-1}}, \dots, W_{\tau_j(n-1)}(\cdot, t_{n-1})/\sqrt{t_{n-1}})$$

in \mathbb{R}^{n-1} . The stochastic integrals occurring in the expression of $V_n(t)$ are defined in the sense of Cairoli and Walsh (1975).

The following simple result concerns time change.

Lemma 2.2. *Let $\{M(t), \mathcal{G}_t : t \in \mathbb{R}_+^k\}$ be a continuous strong L_2 -martingale and $\{X(t) : t \in \mathbb{R}_+^k\}$ a measurable and \mathcal{G}_t -adapted k -parameter process satisfying*

$$\int_0^{t_k} \cdots \int_0^{t_1} E(X(u_1, \dots, u_k))^2 \langle M \rangle(du_1, \dots, du_k) < +\infty,$$

for every $t = (t_1, \dots, t_k) \in \mathbb{R}_+^k$, where $\langle M \rangle$ denotes the quadratic variation of M . Then, for every $t = (t_1, \dots, t_k) \in \mathbb{R}_+^k$,

$$\begin{aligned} & \int_0^{t_k} \cdots \int_0^{t_1} X(u_1, \dots, u_k) M(du_1, \dots, du_k) \\ & = \int_0^1 \cdots \int_0^1 X^{(t)}(u_1, \dots, u_k) M^{(t)}(du_1, \dots, du_k), \end{aligned}$$

where $Y^{(t)}(u_1, \dots, u_k) = Y(t_1 u_1, \dots, t_k u_k)$ for every process Y . \square

Next, we need a result from Karatzas and Shreve (1988, p. 55 and 118).

Lemma 2.3. Let $X = \{X(t): t \in \mathbb{R}_+^k\}$ be a k -parameter process and assume that there exist three positive constants α , β , and C such that, for every $(s, t) \in \mathbb{R}_+^k \times \mathbb{R}_+^k$,

$$E(|X(s) - X(t)|^\alpha) \leq C|s - t|^{k+\beta}.$$

Then X admits a continuous version. \square

For $V_n(t)$ we obtain:

Proposition 2.4. The following hold:

- (i) (symmetry). For every $t \in \mathbb{R}_+^{n-1}$, $-V_n(t) \stackrel{\mathcal{L}}{=} V_n(t)$, $n \geq 2$.
- (ii) (scaling). For every $t \in \mathbb{R}_+^{n-1}$, $n \geq 2$, and $c = (c_1, \dots, c_{n-1}) \in (0, +\infty)^{n-1}$,

$$V_n(ct) \stackrel{\mathcal{L}}{=} \left(\prod_{i=1}^{n-1} c_i \right)^{n/2} V_n(t).$$

- (iii) $\{V_n(t), \mathcal{F}_t: t \in \mathbb{R}_+^{n-1}\}$, $n \geq 2$, is a continuous L_2 -martingale.

Proof. (i) is a consequence of the fact that $-V_n$ is the stochastic volume associated with the $(n-1)$ -parameter Wiener process $\tilde{W} = (-W_1, W_2, \dots, W_n)$. (ii) follows from Lemma 2.2 and the scaling property of the multiparameter Wiener process. (iii) follows from Lemmas 2.1 and 2.3. \square

2.3. Second method of construction

Let us now turn our attention to another method of constructing the stochastic volume. For this purpose, let us write (2.1) under the form

$$V_n(t) = \frac{1}{n} \sum_{\sigma \in S_n} \varepsilon_\sigma \int_0^{t_{n-1}} \cdots \int_0^{t_1} \Gamma_{\sigma(1)}(u_1, \dots, u_{n-1}) \\ \times \prod_{i=2}^n \Gamma_{\sigma(i)}(u_1, \dots, u_{i-2}, du_{i-1}, u_i, \dots, u_{n-1}).$$

Replacing Γ by W , we obtain the formal expression

$$V_n(t) = \frac{1}{n} \sum_{\sigma \in S_n} \varepsilon_\sigma \int_0^{t_{n-1}} \cdots \int_0^{t_1} W_{\sigma(1)}(u_1, \dots, u_{n-1}) \\ \times \prod_{i=2}^n W_{\sigma(i)}(u_1, \dots, u_{i-2}, du_{i-1}, u_i, \dots, u_{n-1}). \quad (2.7)$$

Combining the constructions in Yor (1976, pp. 125-127) and in Guyon and Prum (1980, p. 2.9) one can construct a continuous L_2 -martingale $J_{W_k \dots W_1}$ with respect to a k -parameter Wiener process in \mathbb{R}^k , $W = (W_1, \dots, W_k)$, such that

$$J_{W_k \dots W_1}(du_1, \dots, du_k) = \prod_{i=1}^k W_i(u_1, \dots, u_{i-1}, du_i, u_{i+1}, \dots, u_k).$$

Therefore the expression (2.7) is well defined; in particular

$$V_3(t_1, t_2) = \frac{1}{3} \sum_{\sigma \in S_3} \varepsilon_\sigma \int_0^{t_2} \int_0^{t_1} W_{\sigma(1)}(u_1, u_2) J_{W_{\sigma(3)} W_{\sigma(2)}}(du_1, du_2). \quad (2.8)$$

One may ask whether the two methods of construction of the stochastic volume lead to the same process. We have:

Proposition 2.5. *The two-parameter processes defined by (2.6) and (2.8) are versions of the same process.*

Proof. It is an immediate consequence of the so-called Green's stochastic formula given in Cairoli and Walsh (1975, Theorem 6.1, p. 150). \square

Remarks. (i) We conjecture that we have the same result as in Proposition 2.5 for $n \geq 4$. To prove this conjecture one has to generalize the so-called Green's stochastic formula. (ii) The second method leads to a process with a simpler expression. But the first one has the advantage of expressing V_n in terms of V_{n-1} ; for example, V_3 is expressed in terms of Lévy's stochastic area process V_2 .

3. Large deviations estimates

This section deals with Ventsel and Freidlin's estimates (Ventsel and Freidlin, 1970) for the stochastic volume V_3 .

Let (S, δ) be an arbitrary metric space. In the sequel $\mathcal{C}^{m,k}$ (resp. $\mathcal{C}^{S,k}$) stands for the set of all continuous functions from $[0, 1]^k$ to \mathbb{R}^m (resp. S) endowed with the topology induced by the uniform convergence norm $\|\cdot\|$ (resp. distance d). Further, let $\mathcal{C}_0^{m,k} = \{f \in \mathcal{C}^{m,k} : f \text{ vanishes on the axes}\}$,

$$H^{m,k} = \left\{ f \in \mathcal{C}_0^{m,k} : f \text{ absolutely continuous and } \frac{\partial^k f}{\partial u_1 \cdots \partial u_k} \in L^2([0, 1]^k, \mathbb{R}^m) \right\}.$$

For every $f \in H^{m,k}$, we put

$$\|f\|_{H^{m,k}}^2 = \int_0^1 \cdots \int_0^1 \left| \frac{\partial^k f}{\partial u_1 \cdots \partial u_k}(u_1, \dots, u_k) \right|^2 du_1 \cdots du_k.$$

Let us consider the map $\tilde{\lambda} : \mathcal{C}^{m,k} \rightarrow [0, +\infty]$ defined by

$$\tilde{\lambda}(f) = \begin{cases} \frac{1}{2} \|f\|_{H^{m,k}}^2 & \text{for } f \in H^{m,k}, \\ +\infty & \text{otherwise,} \end{cases}$$

and an arbitrary map $F : H^{m,k} \rightarrow \mathcal{C}^{S,k}$. Let also $\lambda : \mathcal{C}^{S,k} \rightarrow [0, +\infty]$ be the map defined by

$$\lambda(g) = \begin{cases} \inf\{\tilde{\lambda}(f) : F(f) = g\} & \text{for } F^{-1}(\{g\}) \neq \emptyset, \\ +\infty & \text{otherwise.} \end{cases} \quad (3.1)$$

For every Borel set $A \subset \mathcal{C}^{m,k}$ (resp. $\mathcal{C}^{S,k}$) we put

$$\tilde{\Lambda}(A) = \inf_{f \in A} \tilde{\lambda}(f) \quad \left(\text{resp. } \Lambda(A) = \inf_{g \in A} \lambda(g) \right). \quad (3.2)$$

Now, let $W = \{W(t) = (W_1(t), \dots, W_m(t)): t \in \mathbb{R}_+^k\}$ be a k -parameter Wiener process in \mathbb{R}^m . Further, let $\{X^\varepsilon = \{X^\varepsilon(t): t \in \mathbb{R}_+^k\}: \varepsilon \in \mathbb{R}_+ \setminus \{0\}\}$ be an arbitrary family of k -parameter processes in S . In order to prove large deviations properties for this family we need the following regularity conditions:

(H1) For every $a \in \mathbb{R}_+$, the restriction of F to $\tilde{K}(a) = \{f \in H^{m,k}: \tilde{\lambda}(f) \leq a\}$ is continuous.

(H2) For every $(R, \rho, a) \in \mathbb{R}_+^3$, there exists $(\varepsilon_0, \alpha) \in (\mathbb{R}_+ \setminus \{0\})^2$ such that, for every $f \in \tilde{K}(a)$ and $\varepsilon \in (0, \varepsilon_0]$,

$$P(d(X^\varepsilon, F(f)) \geq \rho, \|\varepsilon W - f\| < \alpha) \leq \exp(-R/\varepsilon^2).$$

Theorem 3.1. Assume (H1) and (H2). Then the following hold:

- (i) λ is lower semicontinuous.
- (ii) For every $a \in \mathbb{R}_+$, $K(a) = \{g \in \mathcal{C}^{S,k}: \lambda(g) \leq a\}$ is compact in $\mathcal{C}^{S,k}$.
- (iii) For every open set $A \subset \mathcal{C}^{S,k}$,

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log P(X^\varepsilon \in A) \geq -\Lambda(A),$$

and, for every closed set $B \subset \mathcal{C}^{S,k}$,

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log P(X^\varepsilon \in B) \leq -\Lambda(B).$$

Proof. Similar to that given in Doss and Priouret (1983, p. 363) for $k = 1$. \square

From now on $S = \mathbb{R}$. For our purpose we consider V_3 viewed as a Wiener functional $F(W)$ (i.e., F is a measurable function from $\mathcal{C}^{3,2}$ to $\mathcal{C}^{1,2}$) and, for $f = (f_1, f_2, f_3) \in H^{3,2}$ and every $t = (t_1, t_2) \in \mathbb{R}_+^2$,

$$F(f)(t) = \frac{1}{3} \sum_{\sigma \in S_3} \varepsilon_\sigma \int_0^{t_2} \int_0^{t_1} f_{\sigma(1)}(u_1, u_2) \frac{\partial f_{\sigma(2)}}{\partial u_1}(u_1, u_2) \frac{\partial f_{\sigma(3)}}{\partial u_2}(u_1, u_2) du_1 du_2.$$

Let us remark that $\varepsilon^3 V_3 = F(\varepsilon W)$. To apply Theorem 3.1 to $\{\varepsilon^3 V_3: \varepsilon \in \mathbb{R}_+ \setminus \{0\}\}$, it suffices to verify conditions (H1) and (H2).

Lemma 3.2. (H1) is satisfied.

Proof. Let $a \in \mathbb{R}_+$, $f \in \tilde{K}(a)$, and let $\{f^n: n \in \mathbb{N}\}$ be a sequence in $\tilde{K}(a)$ such that $\lim_{n \rightarrow +\infty} \|f^n - f\| = 0$. We have to show that $\lim_{n \rightarrow +\infty} \|F(f^n) - F(f)\| = 0$. It is easy to see that, for every $t = (t_1, t_2) \in [0, 1]^2$,

$$|F(f^n)(t) - F(f)(t)| \leq \frac{1}{3} \sum_{\sigma \in S_3} (|J_{1,\sigma}^n(t)| + |J_{2,\sigma}^n(t)| + |J_{3,\sigma}^n(t)|),$$

where

$$J_{1,\sigma}^n(t) = \int_0^{t_2} \int_0^{t_1} (f_{\sigma(1)}^n - f_{\sigma(1)})(u_1, u_2) \frac{\partial f_{\sigma(2)}^n}{\partial u_1}(u_1, u_2) \frac{\partial f_{\sigma(3)}^n}{\partial u_2}(u_1, u_2) du_1 du_2,$$

$$J_{2,\sigma}^n(t) = \int_0^{t_2} \int_0^{t_1} f_{\sigma(1)}(u_1, u_2) \frac{\partial (f_{\sigma(2)}^n - f_{\sigma(2)})}{\partial u_1}(u_1, u_2) \frac{\partial f_{\sigma(3)}^n}{\partial u_2}(u_1, u_2) du_1 du_2,$$

$$J_{3,\sigma}^n(t) = \int_0^{t_2} \int_0^{t_1} f_{\sigma(1)}(u_1, u_2) \frac{\partial f_{\sigma(2)}^n}{\partial u_1}(u_1, u_2) \frac{\partial (f_{\sigma(3)}^n - f_{\sigma(3)})}{\partial u_2}(u_1, u_2) du_1 du_2.$$

(a) *Proof of $\lim_{n \rightarrow +\infty} \|J_{1,\sigma}^n\| = 0$.* We have

$$\|J_{1,\sigma}^n\| \leq \|f_{\sigma(1)}^n - f_{\sigma(1)}\| \int_0^1 \int_0^1 \left| \frac{\partial f_{\sigma(2)}^n}{\partial u_1}(u_1, u_2) \frac{\partial f_{\sigma(3)}^n}{\partial u_2}(u_1, u_2) \right| du_1 du_2.$$

By using the inequality of Cauchy-Schwarz, we see that the double integral is $\leq 2a$.

Therefore $\|J_{1,\sigma}^n\| \leq 2a \|f_{\sigma(1)}^n - f_{\sigma(1)}\| \leq 2a \|f^n - f\|$. So $\lim_{n \rightarrow +\infty} \|J_{1,\sigma}^n\| = 0$.

(b) *Proof of $\lim_{n \rightarrow +\infty} \|J_{2,\sigma}^n\| = 0$.* We have

$$\begin{aligned} & |J_{2,\sigma}^n(t)| \\ & \leq \left| \int_0^{t_2} f_{\sigma(1)}(t_1, t_2) \frac{\partial f_{\sigma(3)}^n}{\partial u_2}(t_1, u_2) (f_{\sigma(2)}^n - f_{\sigma(2)})(t_1, u_2) du_2 \right| \\ & + \left| \int_0^{t_2} \int_0^{t_1} (f_{\sigma(2)}^n - f_{\sigma(2)})(u_1, u_2) f_{\sigma(1)}(u_1, u_2) \frac{\partial^2 f_{\sigma(3)}^n}{\partial u_1 \partial u_2}(u_1, u_2) du_1 du_2 \right| \\ & + \left| \int_0^{t_2} \int_0^{t_1} (f_{\sigma(2)}^n - f_{\sigma(2)})(u_1, u_2) \frac{\partial f_{\sigma(1)}^n}{\partial u_1}(u_1, u_2) \frac{\partial f_{\sigma(3)}^n}{\partial u_2}(u_1, u_2) du_1 du_2 \right|. \end{aligned}$$

So,

$$\begin{aligned} \|J_{2,\sigma}^n\| & \leq \sqrt{2a} \|f_{\sigma(1)}\| \|f_{\sigma(2)}^n - f_{\sigma(2)}\| + \sqrt{2a} \|f_{\sigma(1)}\| \|f_{\sigma(2)}^n - f_{\sigma(2)}\| \\ & + 2a \|f_{\sigma(2)}^n - f_{\sigma(2)}\| \leq 6a \|f^n - f\|, \end{aligned}$$

for $\|f_{\sigma(1)}\| \leq \sqrt{2a}$. Therefore $\lim_{n \rightarrow +\infty} \|J_{2,\sigma}^n\| = 0$.

(c) *Proof of $\lim_{n \rightarrow +\infty} \|J_{3,\sigma}^n\| = 0$.* The proof is similar to that of part (b) and is omitted.

Hence $\lim_{n \rightarrow +\infty} \|F(f^n) - F(f)\| = 0$. \square

Let us turn now to condition (H2). We need two auxiliary lemmas.

Lemma 3.3. Let $W = \{W(t) = (W_1(t), W_2(t), W_3(t)): t \in \mathbb{R}_+^2\}$ be a two-parameter Wiener process in \mathbb{R}^3 . Then, for every $(R, a, \rho) \in \mathbb{R}_+^3$, there exists $\alpha \in \mathbb{R}_+ \setminus \{0\}$ such that

$$P(\|F(\varepsilon W + f) - F(f)\| \geq \rho, \|\varepsilon W\| < \alpha) \leq \exp(-R/\varepsilon^2).$$

Proof. For every $t = (t_1, t_2) \in \mathbb{R}_+^2$, we have

$$\begin{aligned} F(\varepsilon W + f)(t) - F(f)(t) \\ = \varepsilon^3 F(W) + \frac{1}{3} \sum_{\sigma \in S_3} \varepsilon_\sigma \left(\varepsilon^2 \sum_{i=1}^3 J_i^\sigma(W, f)(t) + \varepsilon \sum_{i=4}^6 J_i^\sigma(W, f)(t) \right), \end{aligned}$$

where

$$\begin{aligned} J_1^\sigma(W, f)(t) &= \int_0^{t_1} \int_0^{t_2} W_{\sigma(1)}(u_1, u_2) \frac{\partial f_{\sigma(2)}}{\partial u_1}(u_1, u_2) W_{\sigma(3)}(u_1, du_2) du_1, \\ J_2^\sigma(W, f)(t) &= \int_0^{t_2} \int_0^{t_1} W_{\sigma(1)}(u_1, u_2) \frac{\partial f_{\sigma(3)}}{\partial u_2}(u_1, u_2) W_{\sigma(2)}(du_1, u_2) du_2, \\ J_3^\sigma(W, f)(t) &= \int_0^{t_2} \int_0^{t_1} f_{\sigma(1)}(u_1, u_2) J_{W_{\sigma(3)} W_{\sigma(2)}}(du_1, du_2), \\ J_4^\sigma(W, f)(t) &= \int_0^{t_2} \int_0^{t_1} W_{\sigma(1)}(u_1, u_2) \frac{\partial f_{\sigma(2)}}{\partial u_1}(u_1, u_2) \frac{\partial f_{\sigma(3)}}{\partial u_2}(u_1, u_2) du_1 du_2, \\ J_5^\sigma(W, f)(t) &= \int_0^{t_1} \int_0^{t_2} f_{\sigma(1)}(u_1, u_2) \frac{\partial f_{\sigma(2)}}{\partial u_1}(u_1, u_2) W_{\sigma(3)}(u_1, du_2) du_1, \\ J_6^\sigma(W, f)(t) &= \int_0^{t_2} \int_0^{t_1} f_{\sigma(1)}(u_1, u_2) \frac{\partial f_{\sigma(3)}}{\partial u_2}(u_1, u_2) W_{\sigma(2)}(du_1, u_2) du_2. \end{aligned}$$

It follows that

$$\begin{aligned} P(\|F(\varepsilon W + f) - F(f)\| \geq \rho, \|\varepsilon W\| < \alpha) \\ \leq P(\varepsilon^3 \|F(W)\| \geq \frac{1}{7}\rho, \|\varepsilon W\| < \alpha) \\ + \sum_{i=1}^3 P\left(\frac{1}{3} \left\| \varepsilon^2 \sum_{\sigma \in S_3} J_i^\sigma(W, f) \right\| \geq \frac{1}{7}\rho, \|\varepsilon W\| < \alpha\right) \\ + \sum_{i=4}^6 P\left(\frac{1}{3} \left\| \varepsilon \sum_{\sigma \in S_3} J_i^\sigma(W, f) \right\| \geq \frac{1}{7}\rho, \|\varepsilon W\| < \alpha\right) \\ \leq P(\varepsilon^3 \|F(W)\| \geq \frac{1}{7}\rho, \|\varepsilon W\| < \alpha) \\ + \sum_{i=1}^3 \sum_{\sigma \in S_3} P(\varepsilon^2 \|J_i^\sigma(W, f)\| \geq \frac{1}{14}\rho, \|\varepsilon W\| < \alpha) \\ + \sum_{i=4}^6 \sum_{\sigma \in S_3} P(\varepsilon \|J_i^\sigma(W, f)\| \geq \frac{1}{14}\rho, \|\varepsilon W\| < \alpha) \\ = A_1 + A_2 + A_3. \end{aligned}$$

(a) *Majoration of A_1 .* We have

$$A_1 \leq \sum_{\sigma \in S_3} P \left(\varepsilon^3 \left\| \int_0^{t_1} \int_0^{t_2} W_{\sigma(1)}(u_1, u_2) J_{W_{\sigma(3)} W_{\sigma(2)}}(du_1, du_2) \right\| \geq \frac{1}{14} \rho, \|\varepsilon W\| < \alpha \right).$$

In view of the so-called Green stochastic formula (Cairolì and Walsh, 1975, Theorem 6.1, p. 150), for every $t = (t_1, t_2) \in [0, 1]^2$, we have

$$\begin{aligned} & \int_0^{t_2} \int_0^{t_1} W_{\sigma(1)}(u_1, u_2) J_{W_{\sigma(3)} W_{\sigma(2)}}(du_1, du_2) \\ &= \int_0^{t_2} f_3^\sigma(t_1, u_2) W_{\sigma(3)}(t_1, du_2) - \int_0^{t_2} \int_0^{t_1} f_3^\sigma(u_1, u_2) W_{\sigma(3)}(du_1, du_2), \quad P\text{-a.s.}, \end{aligned}$$

where $f_3^\sigma(t_1, u_2) = \int_0^{t_1} W_{\sigma(1)}(a_1, u_2) W_{\sigma(2)}(da_1, u_2)$. Therefore it suffices to bound from above

$$\begin{aligned} & P \left(\varepsilon^3 \left\| \int_0^{t_1} f_3^\sigma(\cdot, u_2) W_{\sigma(3)}(\cdot, du_2) \right\| \geq \frac{1}{28} \rho, \|\varepsilon W\| < \alpha \right) \\ &+ P \left(\varepsilon^3 \left\| \int_0^{t_2} \int_0^{t_1} f_3^\sigma(u_1, u_2) W_{\sigma(3)}(du_1, du_2) \right\| \geq \frac{1}{28} \rho, \|\varepsilon W\| < \alpha \right) \\ &= B_1 + B_2. \end{aligned}$$

(a₁) *Majoration of B_1 .* Let $\delta \in \mathbb{R}_+ \setminus \{0\}$ be arbitrary. For every $t_2 \in [0, 1]$, put

$$D^\sigma(t_2, \delta, \varepsilon) = \left\{ t_1 \in [0, 1] : \varepsilon^2 \sup_{0 \leq u_2 \leq t_2} |f_3^\sigma(t_1, u_2)| > \delta \right\}$$

and

$$\tau_{t_2} = \begin{cases} \inf D^\sigma(t_2, \delta, \varepsilon) & \text{for } D^\sigma(t_2, \delta, \varepsilon) \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

For every fixed $t_2 \in [0, 1]$, τ_{t_2} is a stopping time with respect to the filtration $\{\mathcal{F}_{(t_1, t_2)} : t_1 \in [0, 1]\}$. We have

$$\begin{aligned} B_1 &\leq P \left(\varepsilon^3 \sup_{0 \leq u \leq 1} \left| \int_0^{t_2} f_3^\sigma(t_1, u_2) W_{\sigma(3)}(t_1, du_2) \right| \geq \frac{1}{28} \rho, \inf_{0 \leq t_2 \leq 1} \tau_{t_2} = 1 \right) \\ &+ P \left(\inf_{0 \leq t_2 \leq 1} \tau_{t_2} \neq 1, \|\varepsilon W\| < \alpha \right) \\ &\leq P \left(\varepsilon^3 \sup_{0 \leq t \leq 1} \left| \int_0^{t_2} f_3^\sigma(\min(t_1, \tau_{u_2}), u_2) W_{\sigma(3)}(t_1, du_2) \right| \geq \frac{1}{28} \rho \right) \\ &+ P(\varepsilon^2 \|f_3^\sigma\| \geq \delta, \|\varepsilon W\| < \alpha) = C_1 + C_2. \end{aligned}$$

By virtue of Lemmas 2.1 and 2.3,

$$\left\{ \varepsilon^3 \int_0^{t_2} f_3^\sigma(\min(t_1, \tau_{u_2}), u_2) W_{\sigma(3)}(t_1, du_2), \mathcal{F}_t : t \in [0, 1]^2 \right\}$$

is a continuous L_2 -martingale with 2-quadratic variation bounded by $\varepsilon^2 \delta^2 t_1 t_2$, for every $t = (t_1, t_2) \in [0, 1]^2$. The exponential estimate in Mishura (1987, Theorem 1, p. 276) implies

$$C_1 \leq \frac{2e}{e-1} \left[\exp\left(\frac{-4e\gamma^2}{3\varepsilon^2 \delta^2 (2e+1)}\right) + \exp\left(\frac{-4e\gamma^2}{9\varepsilon^2 \delta^2 (2e+1)}\right) \right], \quad (3.3)$$

where $\gamma = \frac{1}{28}\rho$.

Further, there exists $\delta = \delta(R, \rho)$ such that the right term of (3.3) is less than $\frac{1}{48} \exp(-R/\varepsilon^2)$ for every $\varepsilon \in [0, 1]$. For δ so chosen, let $\beta \in \mathbb{R}_+ \setminus \{0\}$ be arbitrary and put, for every $t_1 \in [0, 1]$,

$$G^\sigma(t_1, \beta, \varepsilon) = \left\{ t_2 \in [0, 1]: \varepsilon \sup_{0 \leq u_1 \leq t_1} |W_{\sigma(1)}(u_1, t_2)| > \beta \right\}$$

and

$$\tau_{t_1} = \begin{cases} \inf G^\sigma(t_1, \beta, \varepsilon) & \text{for } G^\sigma(t_1, \beta, \varepsilon) \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

By the same techniques as above, we have

$$C_2 \leq P\left(\varepsilon^2 \sup_{0 \leq t \leq 1} \left| \int_0^{t_1} W_{\sigma(1)}(u_1, \min(t_2, \tau_{u_1})) W_{\sigma(2)}(du_1, t_2) \right| \geq \delta \right) \\ + P(\|\varepsilon W\| \geq \beta, \|\varepsilon W\| < \alpha) = D_1 + D_2.$$

In view of Theorem 1 in Mishura (1987, p. 276), and by reasoning as above, we obtain

$$D_1 \leq \frac{2e}{e-1} \left[\exp\left(\frac{-4e\delta^2}{3\varepsilon^2 \beta^2 (2e+1)}\right) + \exp\left(\frac{-4e\delta^2}{9\varepsilon^2 \beta^2 (2e+1)}\right) \right]. \quad (3.4)$$

Choose now $\beta = \beta(R, \delta)$ such that the right term of (3.4) is less than $\frac{1}{48} \exp(-R/\varepsilon^2)$ for every $\varepsilon \in (0, 1]$. Next, for $\alpha < \beta$, we have $D_2 = 0$. Therefore, there exists $\alpha = \alpha'_1 < \beta$ such that, for every $\varepsilon \in (0, 1]$,

$$B_1 \leq \frac{1}{24} \exp(-R/\varepsilon^2). \quad (3.5)$$

(a₂) *Majoration of B_2 .* Let $\delta \in \mathbb{R}_+ \setminus \{0\}$ be arbitrary. We have

$$B_2 \leq P\left(\varepsilon^3 \left\| \int_0^\cdot \int_0^\cdot f_3^\sigma(u_1, u_2) W_{\sigma(3)}(du_1, du_2) \right\| \geq \frac{1}{28}\rho, \varepsilon^2 \|f_3^\sigma\| < \delta \right) \\ + P(\varepsilon^2 \|f_3^\sigma\| \geq \delta, \|\varepsilon W\| < \alpha) = E_1 + E_2.$$

In view of the exponential estimates in Dozzi (1989, Proposition 7, p. 114) we have

$$E_1 \leq 4 \exp\left(\frac{-\gamma^2}{18\varepsilon^2 \delta^2}\right), \quad (3.6)$$

where $\gamma = \frac{1}{28}\rho$.

Choose now $\delta = \delta(R, \rho)$ such that the right member of (3.6) is less than $\frac{1}{48} \exp(-R/\varepsilon^2)$, for every $\varepsilon \in (0, 1]$. For δ so chosen, in view of the argument used to bound from above C_2 , there exists $\alpha = \alpha_1'' < \delta$ such that $E_2 \leq \frac{1}{48} \exp(-R/\varepsilon^2)$ for every $\varepsilon \in (0, 1]$. So, for $\alpha = \alpha_1''$, we obtain

$$B_2 \leq \frac{1}{24} \exp(-R/\varepsilon^2). \quad (3.7)$$

Finally, for $\alpha = \alpha_1 = \min(\alpha_1', \alpha_1'')$, (3.5) and (3.7) imply that, for every $\varepsilon \in (0, 1]$,

$$A_1 \leq 6(\frac{1}{12} \exp(-R/\varepsilon^2)) = \frac{1}{2} \exp(-R/\varepsilon^2). \quad (3.8)$$

(b) *Majoration of A_2 .* We have

$$A_2 = \sum_{\sigma \in S_3} \sum_{i=1}^3 F_i^\sigma,$$

where

$$F_i^\sigma = P(\varepsilon^2 \|J_i^\sigma(W, f)\| \geq \frac{1}{14}\rho, \|\varepsilon W\| < \alpha), \quad i = 1, 2, 3.$$

(b₁) *Majoration of F_1^σ .* For every $t = (t_1, t_2) \in [0, 1]^2$,

$$\begin{aligned} J_1^\sigma(W, f)(t) &= \int_0^{t_1} \int_0^{t_2} W_{\sigma(1)}(u_1, u_2) \\ &\quad \times \left(\int_0^{u_2} \frac{\partial^2 f_{\sigma(2)}}{\partial u_1 \partial u_2}(u_1, a) da \right) W_{\sigma(3)}(u_1, du_2) du_1. \end{aligned}$$

The stochastic Fubini's Theorem in Kailath, Segall and Zakai (1978, Theorem 1, p. 139) implies that

$$\begin{aligned} J_1^\sigma(W, f)(t) &= \int_0^{t_1} \int_0^{t_2} \left(\int_\alpha W_{\sigma(1)}(u_1, u_2) W_{\sigma(3)}(u_1, du_2) \right) \\ &\quad \times \frac{\partial^2 f_{\sigma(2)}}{\partial u_1 \partial u_2}(u_1, a) da du_1. \end{aligned}$$

So

$$\|J_1^\sigma(W, f)\| \leq 2\sqrt{2a} \left\| \int_0^\cdot W_{\sigma(1)}(\cdot, u_1), W_{\sigma(3)}(\cdot, du_2) \right\|,$$

therefore we obtain

$$F_1^\sigma \leq P\left(\varepsilon^2 \left\| \int_0^\cdot W_{\sigma(1)}(\cdot, u_2) W_{\sigma(3)}(\cdot, du_2) \right\| \geq \rho/(28\sqrt{2a}), \|\varepsilon W\| < \alpha\right).$$

In view of the argument used to bound from above C_2 , there exists $\alpha = \alpha_2'$ such that, for every $\varepsilon \in (0, 1]$,

$$F_1^\sigma \leq \frac{1}{36} \exp(-R/\varepsilon^2). \quad (3.9)$$

(b₂) *Majoration of F_2^σ .* F_2 can be bounded from above by the same techniques. Therefore, there exists $\alpha = \alpha_2''$ such that, for every $\varepsilon \in (0, 1]$,

$$F_2^\sigma \leq \frac{1}{36} \exp(-R/\varepsilon^2). \quad (3.10)$$

(b₃) *Majoration of F_3^σ* , f being of bounded variation, the integration by parts formula in Dozzi (1989, p. 67) yields, for every $t = (t_1, t_2) \in (0, 1]^2$,

$$\begin{aligned} J_3^\sigma(W, f)(t) &= f_{\sigma(1)}(t) J_{W_{\sigma(3)} W_{\sigma(2)}}(t) - \int_0^{t_2} J_{W_{\sigma(3)} W_{\sigma(2)}}(t_1, u_2) \frac{\partial f_{\sigma(1)}}{\partial u_2}(t_1, u_2) du_2 \\ &\quad - \int_0^{t_1} J_{W_{\sigma(3)} W_{\sigma(2)}}(u_1, t_2) \frac{\partial f_{\sigma(1)}}{\partial u_1}(u_1, t_2) du_1 \\ &\quad + \int_0^{t_2} \int_0^{t_1} J_{W_{\sigma(3)} W_{\sigma(2)}}(u_1, u_2) \frac{\partial^2 f_{\sigma(1)}}{\partial u_1 \partial u_2}(u_1, u_2) du_1 du_2. \end{aligned}$$

So,

$$\|J_3^\sigma(W, f)\| \leq 4\sqrt{2a} \|J_{W_{\sigma(3)} W_{\sigma(2)}}\|.$$

But the so-called Green's stochastic formula (Cairolì and Walsh, 1975, Theorem 6.1, p. 150) implies

$$\begin{aligned} J_{W_{\sigma(3)} W_{\sigma(2)}}(t) &= \int_0^{t_2} W_{\sigma(2)}(t_1, u_2) W_{\sigma(3)}(t_1, du_2) \\ &\quad - \int_0^{t_2} \int_0^{t_1} W_{\sigma(2)}(u_1, u_2) W_{\sigma(3)}(du_1, du_2) \quad P\text{-a.s.} \end{aligned}$$

So, by the arguments used to bound from above C_2 and E_1 , we obtain that there exists $\alpha = \alpha_2'''$ such that, for every $\varepsilon \in (0, 1]$,

$$F_3^\sigma \leq \frac{1}{36} \exp(-R/\varepsilon^2). \quad (3.11)$$

Next, by taking $\alpha = \alpha_2' = \min(\alpha_2', \alpha_2'', \alpha_2''')$ and by combining (3.9), (3.10), and (3.11), we have, for every $\varepsilon \in (0, 1]$,

$$A_2 \leq 6(\frac{1}{12} \exp(-R/\varepsilon^2)) = \frac{1}{2} \exp(-R/\varepsilon^2). \quad (3.12)$$

(c) *Majoration of A_3* . We have

$$A_3 = \sum_{\sigma \in S_3} \sum_{i=4}^6 G_i^\sigma,$$

where

$$G_i^\sigma = P(\varepsilon \|J_i^\sigma(W, f)\| \geq \frac{1}{14}\rho, \|\varepsilon W\| < \alpha), \quad i = 4, 5, 6.$$

(c₁) *Majoration of G_4^σ* . We have

$$\|J_4^\sigma(W, f)\| \leq 2a \|W_{\sigma(1)}\| \leq 2a \|W\|.$$

So,

$$G_4^\sigma \leq P(\|\varepsilon W\| \geq \rho/(28a), \|\varepsilon W\| < \alpha).$$

By taking $\alpha = \alpha_3' < \rho/(28a)$, we obtain $G_4^\sigma = 0$ for every $\varepsilon \in (0, 1]$.

(c₂) *Majoration of G_5^σ .* For every $t = (t_1, t_2) \in (0, 1]^2$, we have

$$\begin{aligned} J_5^\sigma(W, f)(t) &= \int_0^{t_1} \int_0^{t_2} \int_0^{u_2} f_{\sigma(1)}(u_1, u_2) \frac{\partial^2 f_{\sigma(2)}}{\partial u_1 \partial u_2}(u_1, a) da W_{\alpha(3)}(u_1, du_2) du_1 \\ &= \int_0^{t_1} \int_0^{t_2} \left(\int_a^{t_2} f_{\sigma(1)}(u_1, u_2) W_{\sigma(3)}(u_1, du_2) \right) \frac{\partial^2 f_{\sigma(2)}}{\partial u_1 \partial u_2}(u_1, a) da du_1. \end{aligned}$$

Let us point out that the interchange of the deterministic integral and the stochastic integral can be justified by Theorem 1 in Kailath, Segall and Zakai (1978, p. 139). Therefore we obtain

$$\|J_5^\sigma(W, f)\| \leq 2\sqrt{2a} \left\| \int_0^\cdot f_{\sigma(1)}(\cdot, u_2) W_{\sigma(3)}(\cdot, du_2) \right\|.$$

It follows that there exists $\alpha = \alpha_3''$ such that, for every $\varepsilon \in (0, 1]$, $G_5^\sigma = 0$.

(c₃) *Majoration of G_6^σ .* G_6^σ can be examined by the same techniques used in (c₂). So, there exists $\alpha = \alpha_3'''$ such that, for every $\varepsilon \in (0, 1]$, $G_6^\sigma = 0$.

Next, by taking $\alpha = \alpha_3 = \min(\alpha_3', \alpha_3'', \alpha_3''')$, we have, for every $\varepsilon \in (0, 1]$,

$$A_3 = 0. \quad (3.13)$$

Finally, by taking $\alpha = \min(\alpha_1, \alpha_2, \alpha_3)$, (3.8), (3.12), and (3.13) imply that, for every $\varepsilon \in (0, 1]$,

$$P(\|F(\varepsilon W + f) - F(f)\| \geq \rho, \|\varepsilon W\| < \alpha) \leq \exp(-R/\varepsilon^2). \quad \square$$

We are now ready to verify condition (H2).

Lemma 3.4. (H2) is satisfied.

Proof. Let $f \in \tilde{K}(a)$ and put $A^\varepsilon = (\|F(\varepsilon W) - F(f)\| \geq \rho, \|\varepsilon W - f\| < \alpha)$,

$$V^\varepsilon = \exp\left(-\frac{1}{\varepsilon} \int_0^1 \int_0^1 \left\langle \frac{\partial^2 f}{\partial u_1 \partial u_2}(u_1, u_2), W(du_1, du_2) \right\rangle\right),$$

$$W^\varepsilon = W - \frac{1}{\varepsilon} f.$$

Then the Girsanov formula (Dozzi, 1989, p. 89) asserts that W^ε is a two-parameter Wiener process under the probability measure Q^ε defined by

$$\frac{dQ^\varepsilon}{dP} = \exp\left(\frac{1}{\varepsilon} \int_0^1 \int_0^1 \left\langle \frac{\partial^2 f}{\partial u_1 \partial u_2}(u_1, u_2), W(du_1, du_2) \right\rangle - \frac{\tilde{\lambda}(f)}{\varepsilon^2}\right).$$

Therefore,

$$P(A^\varepsilon) = \int_{A^\varepsilon} \exp\left(-\frac{1}{\varepsilon} \int_0^1 \int_0^1 \left\langle \frac{\partial^2 f}{\partial u_1 \partial u_2}(u_1, u_2), W(du_1, du_2) \right\rangle + \frac{\tilde{\lambda}(f)}{\varepsilon^2}\right) dQ^\varepsilon.$$

Let $\gamma \in \mathbb{R}$ be arbitrary. We have

$$\begin{aligned} P(A^\varepsilon) &\leq \int_{A^\varepsilon \cap (V^\varepsilon \leq \exp(\gamma/\varepsilon^2))} \exp\left(-\frac{1}{\varepsilon} \int_0^1 \int_0^1 \left\langle \frac{\partial^2 f}{\partial u_1 \partial u_2}(u_1, u_2), W(du_1, du_2) \right\rangle \right. \\ &\quad \left. + \frac{\tilde{\lambda}(f)}{\varepsilon^2} \right) dQ^\varepsilon \\ &\quad + P(V^\varepsilon \geq \exp(\gamma/\varepsilon^2)) \\ &\leq \exp((\alpha + \gamma)/\varepsilon^2) Q^\varepsilon(A^\varepsilon) + \exp((\alpha - \gamma)/\varepsilon^2). \end{aligned} \quad (3.14)$$

Choose now $\gamma = \gamma(R, \rho, a)$ such that $\exp((\alpha - \gamma)/\varepsilon^2) \leq \frac{1}{2} \exp(-R/\varepsilon^2)$ for every $\varepsilon \in (0, 1]$. Let us note that

$$Q^\varepsilon(A^\varepsilon) = Q^\varepsilon(\|F(\varepsilon W^\varepsilon + f) - F(f)\| \geq \rho, \|\varepsilon W^\varepsilon\| < \alpha).$$

For the chosen γ , in view of Lemma 3.3, there exists α such that, for every $\varepsilon \in (0, 1]$,

$$\exp((\alpha + \gamma)/\varepsilon^2) Q^\varepsilon(A^\varepsilon) \leq \frac{1}{2} \exp(-R/\varepsilon^2).$$

By taking $\varepsilon_0 = 1$, we obtain (3.14). \square

We are now in a position to state the large deviations properties for the stochastic volume V_3 . Lemmas 3.2 and 3.4, and Theorem 3.1 lead to:

Theorem 3.5. *The following hold:*

- (i) λ is lower semicontinuous.
- (ii) For every open set $A \subset \mathcal{C}^{1,2}$,

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log P(\varepsilon^3 V_3 \in A) \geq -\Lambda(A)$$

and, for every closed set $B \subset \mathcal{C}^{1,2}$,

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log P(\varepsilon^3 V_3 \in B) \leq -\Lambda(B). \quad \square$$

4. Functional law of the iterated logarithm

Let us apply the above large deviations estimates to prove a Strassen-type functional law of the iterated logarithm for V_3 . We shall adopt the methods used in Baldi (1986) and we shall begin by establishing this result for an arbitrary Wiener functional.

Let $W = \{(W_1(t), \dots, W_m(t)) : t \in \mathbb{R}_+^k\}$ be a k -parameter Wiener process in \mathbb{R}^m . For every $u = (u_1, \dots, u_k) \in \mathbb{R}_+^k$, put

$$\phi(u) = \begin{cases} \log \log \left(\prod_{i=1}^k u_i \right) & \text{for } \prod_{i=1}^k u_i \geq 3, \\ 1 & \text{otherwise,} \end{cases}$$

and $\Phi(u) = (\prod_{i=1}^k u_i) \phi(u)$. Further, let $\xi = \{\xi_u : u \in \mathbb{R}_+^k\}$ denote the k -parameter

process in $\mathcal{C}^{m,k}$ defined, for every $t \in [0, 1]^k$, by

$$\xi_u(t) = W(ut)/(\Phi(u))^{1/2}.$$

For every separable Banach space S and every k -parameter process $\{X_u : u \in \mathbb{R}_+^k\}$ in S , $C(X_u : u \in \mathbb{R}_+^k)$ stands for the set of all limit points of $\{X_u : u \in \mathbb{R}_+^k\}$ as all coordinates of u converge to infinity. With the notations of Section 3 we have (Park, 1974) the following functional law of the iterated logarithm for the Wiener process:

Theorem 4.1. *The k -parameter process $\xi = \{\xi_u : u \in \mathbb{R}_+^k\}$ is almost surely relatively compact in $\mathcal{C}^{m,k}$ and $P(C(\xi_u : u \in \mathbb{R}_+^k) = \tilde{K}(k)) = 1$. \square*

Let $F(W)$ be a Wiener functional, with values in $\mathcal{C}_0^{l,k}$. We want to find sufficient conditions for F which guarantee the transfer of the result of Theorem 4.1 to the k -parameter process $Z = \{Z_u = F(\xi_u) : u \in \mathbb{R}_+^k\}$. Let $X^\varepsilon = F(\varepsilon W)$.

Theorem 4.2. *Assume (H1) and (H2). Then the following hold:*

(i) *For every open set $A \subset \mathcal{C}^{l,k}$,*

$$\liminf_{u \rightarrow +\infty} \frac{1}{\phi(u)} \log P(Z_u \in A) \geq -\Lambda(A).$$

(ii) *For every closed subset $B \subset \mathcal{C}^{l,k}$,*

$$\limsup_{u \rightarrow +\infty} \frac{1}{\phi(u)} \log P(Z_u \in B) \leq -\Lambda(B).$$

here $u \rightarrow +\infty$ stands for $\min_{1 \leq i \leq k} u_i \rightarrow +\infty$.

Proof. It suffices to take into account that Z_u and $F(W/\sqrt{\phi(u)})$ have the same law and to apply Theorem 3.1(iii) to the family $\{F(\varepsilon W) : \varepsilon > 0\}$. \square

To prove the functional law of the iterated logarithm, we shall use Theorem 4.2 and the following lemmas.

Lemma 4.3. *Assume (H1) and (H2). Then, for every $c \in (1, +\infty)$ and $\varepsilon \in \mathbb{R}_+ \setminus \{0\}$, there exists P -a.s. $j^0 = (j_1^0, \dots, j_k^0) \in \mathbb{N}^k$ such that, if $j \geq j^0$, then $d(Z_{\varepsilon j}, K(k)) < \varepsilon$, where $d(f, A) = \inf_{g \in A} d(f, g)$ for every $A \subset \mathcal{C}^{l,k}$.*

Proof. Let $K_\varepsilon = \{g \in \mathcal{C}^{l,k} : d(g, K(k)) \geq \varepsilon\}$. In view of the definition of $\Lambda(K_\varepsilon)$, there exists a sequence $\{g_n : n \in \mathbb{N}\}$ in K_ε such that $\lim_{n \rightarrow +\infty} \lambda(g_n) = \Lambda(K_\varepsilon)$. Therefore the sequence $\{\lambda(g_n) : n \in \mathbb{N}\}$ is bounded. Since Theorem 3.1(ii) implies that the sets $K(a)$, $a \in \mathbb{R}_+$, are compact, it follows that $\{g_n : n \in \mathbb{N}\}$ is relatively compact. So, there exist a subsequence $\{g_{k_n} : n \in \mathbb{N}\}$ of $\{g_n : n \in \mathbb{N}\}$ and $g \in K_\varepsilon$ such that $\lim_{n \rightarrow +\infty} g_{k_n} = g$. Since λ is lower semicontinuous, we have $\Lambda(K_\varepsilon) \leq \lambda(g) \leq \lim_{n \rightarrow +\infty} \lambda(g_{k_n}) \leq \Lambda(K_\varepsilon)$. Therefore $\lambda(g) = \Lambda(K_\varepsilon)$. Since $g \notin K(k)$, we have $\Lambda(K_\varepsilon) > k$. Let $\delta \in \mathbb{R}_+ \setminus \{0\}$ such that $\Lambda(K_\varepsilon) > k + 2\delta$. By virtue of Theorem 4.2(ii), for j sufficiently large, we have $P(Z_{\varepsilon j} \in K_\varepsilon \text{ i.o.}) = 0$. \square

In addition to (H1) and (H2), we need two more conditions.

(H3) There exists $\alpha \in \mathbb{R}_+ \setminus \{0\}$ such that, for every $\varepsilon \in \mathbb{R}_+ \setminus \{0\}$ and $u \in \mathbb{R}_+^k$, $F(\varepsilon W(u \cdot)) = \varepsilon^\alpha F(W(u \cdot))$ and $F(W(u \cdot))(t) = F(W)(ut)$.

Let λ and Λ be the functions defined by (3.1) and (3.2), respectively, with $S = \mathbb{R}^l$.

(H4) For every $\varepsilon \in \mathbb{R}_+ \setminus \{0\}$, there exists $c_\varepsilon \in (1, +\infty)$ such that, for every $c \in (1, c_\varepsilon]$, $\Lambda(A_{\varepsilon, c}) \geq k+2$, where

$$A_{\varepsilon, c} = \left\{ g \in \mathcal{C}^{l, k}: \sup_{\substack{0 \leq s \leq 1 \\ s/\varepsilon \leq t \leq s}} |g(s) - g(t)| \geq \frac{1}{2}\varepsilon \right\}.$$

Lemma 4.4. Assume (H1)–(H4). Then, for every $\varepsilon \in \mathbb{R}_+ \setminus \{0\}$, there exists $c_\varepsilon \in (1, +\infty)$ such that, if $c \in (1, c_\varepsilon]$, then

$$P\left(\sup_{u \in I_j(c)} \left\| Z_u - \left(\frac{\Phi(\xi^j)}{\Phi(u)} \right)^{\alpha/2} Z_{\xi^j} \right\| \geq \varepsilon \text{ i.o.} \right) = 0,$$

where

$$I_j(c) = [\xi^{j-1}, \xi^j], \quad j = (j_1, \dots, j_k) \in \mathbb{N}^k.$$

Proof. For every $j \in \mathbb{N}^k$, put

$$Y_j = \sup_{u \in I_j(c)} \left\| \frac{1}{(\Phi(u))^{\alpha/2}} (F(W(u \cdot)) - F(W(\xi^j \cdot))) \right\|.$$

The following inclusions are easy to establish:

$$\begin{aligned} \{Y_j \geq \varepsilon\} &\subset \left\{ \sup_{u \in I_j(c)} \left\| \frac{1}{(\Phi(\xi^{j-1}))^{\alpha/2}} (F(W(u \cdot)) - F(W(\xi^j \cdot))) \right\| \geq \varepsilon \right\} \\ &\subset \left\{ \sup_{\substack{0 \leq s \leq 1 \\ s/\varepsilon \leq t \leq s}} \frac{1}{(\Phi(\xi^{j-1}))^{\alpha/2}} |F(W)(\xi^j t) - F(W)(\xi^j s)| \geq \varepsilon \right\} \quad (\text{by H3}) \\ &\subset \left\{ \sup_{\substack{0 \leq s \leq 1 \\ s/\varepsilon \leq t \leq s}} \left(\frac{\Phi(\xi^j)}{\Phi(\xi^{j-1})} \right)^{\alpha/2} |Z_{\xi^j}(t) - Z_{\xi^j}(s)| \geq \varepsilon \right\} \quad (\text{by H3}) \end{aligned}$$

Now, let $c_\varepsilon \in (1, +\infty)$ (from (H4)) and choose $c \in (1, +\infty)$ such that $c^{\alpha k/2} \in (1, \min(\frac{3}{2}, c_\varepsilon^{\alpha k/2}))$. Since $\lim_{j \rightarrow +\infty} (\Phi(\xi^j)/\Phi(\xi^{j-1}))^{\alpha/2} = c^{\alpha k/2}$, there exists $j^0 \in \mathbb{N}^k$ such that, if $j \geq j^0$, then $(\Phi(\xi^j)/\Phi(\xi^{j-1}))^{\alpha/2} \leq 2$. Therefore, for $j \geq j^0$, we have $\{Y_j \geq \varepsilon\} \subset \{Z_{\xi^j} \in A_{c_\varepsilon}\}$. Now, in view of Theorem 4.2(ii), there exists $j^1 \in \mathbb{N}^k$ such that, if $j \geq j^1$, then $P(Z_{\xi^j} \in A_{c_\varepsilon}) \leq \exp[(-\Lambda(A_{c_\varepsilon}) + 1)\phi(\xi^j)]$. So, for $j \geq \max(j^0, j^1)$, we have

$$P(Y_j \geq \varepsilon) \leq P(Z_{\xi^j} \in A_{c_\varepsilon}) \leq \exp[(-\Lambda(A_{c_\varepsilon}) + 1)\phi(\xi^j)]. \quad (4.1)$$

Further, since $c \in (1, c_\varepsilon]$, (H4) implies that $\Lambda(A_{c_\varepsilon}) > k+2$. So, in view of (4.1), we obtain for $j \geq \max(j^0, j^1)$,

$$P(Y_j \geq \varepsilon) \leq \exp[-(k+1)\phi(\xi^j)]. \quad (4.2)$$

The right member of (4.2) being summable, the Borel–Cantelli Lemma yields the proof. \square

Lemma 4.5. Assume (H1)–(H4). Then, for every $\varepsilon \in \mathbb{R}_+ \setminus \{0\}$, there exists P-a.s. $u^0 \in \mathbb{R}_+^k$ such that, if $u > u^0$, we have $d(Z_u, K(k)) < \varepsilon$.

Proof. For every $c \in (1, +\infty)$, $j \in \mathbb{N}^k$, and $u \in I_j(c)$,

$$\begin{aligned} d(Z_u, K(k)) &\leq \left\| Z_u - \left(\frac{\Phi(\xi^j)}{\Phi(u)} \right)^{\alpha/2} Z_{\xi^j} \right\| \\ &\quad + \left\| \left(\frac{\Phi(\xi^j)}{\Phi(u)} \right)^{\alpha/2} Z_{\xi^j} - Z_{\xi^j} \right\| + d(Z_{\xi^j}, K(k)) \\ &= A_1 + A_2 + A_3. \end{aligned}$$

Lemma 4.3 implies that there exists $j^0 \in \mathbb{N}^k$ such that, if $j \geq j^0$, then $A_3 < \frac{1}{3}\varepsilon$. Now, for every $\delta \in \mathbb{R}_+ \setminus \{0\}$, there exists j^1 (not depending on c) such that, for $j \geq j^1$, we have $1 \leq (\Phi(\xi^j)/\Phi(u))^{\alpha/2} \leq c^{\alpha k/2}(1+\delta)$ and, in view of Lemma 4.3, $\|Z_{\xi^j}\|$ is bounded with respect to j . Therefore, if c is close to 1 and j is sufficiently large, we obtain $A_2 < \frac{1}{3}\varepsilon$. Finally, Lemma 4.4 implies that for c close to 1 and for j sufficiently large, $A_1 < \frac{1}{3}\varepsilon$. Therefore, if u is sufficiently large, we conclude that $d(Z_u, K(k)) < \varepsilon$. \square

Lemma 4.6. Let $g \in K(k)$ and assume (H1)–(H4). Then, for every $\varepsilon \in \mathbb{R}_+ \setminus \{0\}$, there exists $c = c_\varepsilon \in (1, +\infty)$ such that

$$P(\|Z_{\xi^j} - g\| \leq \varepsilon \text{ i.o.}) = 1.$$

Proof. Let $g \in K(k)$. There exists $f \in \tilde{K}(k)$ such that $F(f) = g$ and $\tilde{\lambda}(f) = \lambda(g)$. For every $(\varepsilon, \delta) \in \mathbb{R}_+ \setminus \{0\}$, put

$$A_j = \left\{ \left\| \frac{1}{(\phi(\xi^j))^{1/2}} W^{(\xi^j)} - f \right\| < \delta \right\},$$

and $B_j = \{\|Z_{\xi^j} - g\| < \varepsilon\}$, where

$$W^{(u)} = W(u \cdot) / \left(\prod_{i=1}^k u_i \right)^{1/2}.$$

Let $R > k+1$; in view of (H2), for δ sufficiently small, $P(A_j \cap B_j^c) \leq \exp(-(k+1)\phi(\xi^j))$. We deduce that $\sum_{j \in \mathbb{N}^k} P(A_j \cap B_j^c) < +\infty$. Since there exists $c = c_\varepsilon \in (1, +\infty)$ such that $P(A_j \text{ i.o.}) = 1$ (Park, 1974, p. 484), we obtain $P(B_j \text{ i.o.}) = 1$, for $c = c_\varepsilon$. \square

We are now ready to state the analogue of Theorem 4.1 for the k -parameter process $Z = \{Z_u = F(\xi_u) : u \in \mathbb{R}_+^k\}$.

Theorem 4.7. Assume (H1)–(H4). Then the k -parameter process $\{Z_u = F(\xi_u) : u \in \mathbb{R}_+^k\}$ is P-a.s. relatively compact and $P(C(Z_u : u \in \mathbb{R}_+^k) = K(k)) = 1$.

Proof. First, $K(k)$ being compact, Lemma 4.5 shows that $\{Z_u : u \in \mathbb{R}_+^k\}$ is P-a.s. relatively compact and $P(C(Z_u : u \in \mathbb{R}_+^k) \subset K(k)) = 1$. Next, Lemma 4.6 implies that $P(C(Z_u : u \in \mathbb{R}_+^k) \supset K(k)) = 1$. \square

Corollary 4.8. *Let E be a topological space and let $G: \mathcal{C}^{1,k} \rightarrow E$ be a continuous mapping. Further, assume (H1)–(H4). Then the k -parameter process $\{G(Z_u): u \in \mathbb{R}_+^k\}$ is P -a.s. relatively compact and $P(C(G(Z_u): u \in \mathbb{R}_+^k) = G(K(k))) = 1$. \square*

Let us now turn our attention to the application of Theorem 4.7 to the stochastic volume V_3 . Remember that $V_3 = F(W)$, where $F: \mathcal{C}^{3,2} \rightarrow \mathcal{C}^{1,2}$ and

$$F(f)(t) = \frac{1}{3} \sum_{\sigma \in S_3} \varepsilon_\sigma \int_0^{t_2} \int_0^{t_1} f_{\sigma(1)}(u_1, u_2) \frac{\partial f_{\sigma(2)}}{\partial u_1}(u_1, u_2) du_1 du_2, \quad f \in H.$$

For every $u \in \mathbb{R}_+^2$, $t = (t_1, t_2) \in \mathbb{R}_+^2$, we have, in this special case, $Z_u(t) = F(\xi_u)(t) = V_3(ut)/(\Phi(u))^{3/2}$. We already know that conditions (H1) and (H2) are satisfied (Lemmas 3.2 and 3.4).

Lemma 4.9. (H3) and (H4) are satisfied.

Proof. To verify (H3) take $\alpha = 3$ and use Lemma 2.2. For the verification of (H4), let $c \in (1, +\infty)$. If $\Lambda(A_{c,\varepsilon}) = +\infty$ the assertion follows. Let us assume that $\Lambda(A_{c,\varepsilon}) < +\infty$ and $g \in A_{c,\varepsilon}$ such that $\lambda(g) < +\infty$. Let $f \in H$ with $\tilde{\lambda}(f) = \lambda(g)$ and $F(f) = g$. Since $g \in A_{c,\varepsilon}$, there exists $(s, t) \in [0, 1]^2$ with $s/\varepsilon \leq t \leq s$ such that $|g(s) - g(t)| \geq \frac{1}{4}\varepsilon$. We have

$$|g(s) - g(t)| \leq |g(s_1, s_2) - g(s_1, t_2)| + |g(s_1, t_2) - g(t_1, t_2)|.$$

Next,

$$\begin{aligned} & |g(s_1, s_2) - g(s_1, t_2)| \\ & \leq \frac{1}{3} \sum_{\sigma \in S_3} \int_{t_2}^{s_2} \left| \int_0^{s_1} f_{\sigma(1)}(u_1, u_2) \frac{\partial f_{\sigma(2)}}{\partial u_1}(u_1, u_2) \frac{\partial f_{\sigma(3)}}{\partial u_2}(u_1, u_2) du_1 \right| du_2 \\ & \leq \frac{1}{3} |s_2 - t_2|^{1/2} \sum_{\sigma \in S_3} \left(\int_{t_2}^{s_2} \left| \int_0^{s_1} f_{\sigma(1)}(u_1, u_2) \frac{\partial f_{\sigma(2)}}{\partial u_1}(u_1, u_2) \right. \right. \\ & \quad \left. \left. \times \frac{\partial f_{\sigma(3)}}{\partial u_2}(u_1, u_2) du_1 \right|^2 du_2 \right)^{1/2} \\ & \leq \frac{1}{3} |s_2 - t_2|^{1/2} \sum_{\sigma \in S_2} \left[\int_{t_2}^{s_2} \left(\int_0^{s_1} \left| f_{\sigma(1)}(u_1, u_2) \frac{\partial f_{\sigma(2)}}{\partial u_1}(u_1, u_2) \right. \right. \right. \\ & \quad \left. \left. \times \frac{\partial f_{\sigma(3)}}{\partial u_2}(u_1, u_2) \right| du_1 \right)^2 du_2 \Big]^{1/2} \\ & \leq \frac{1}{3} |s_2 - t_2|^{1/2} \|f_{\sigma(1)}\| \sum_{\sigma \in S_3} \left[\int_0^1 \left(\int_0^1 \left| \frac{\partial f_{\sigma(2)}}{\partial u_1}(u_1, u_2) \right. \right. \right. \\ & \quad \left. \left. \times \frac{\partial f_{\sigma(3)}}{\partial u_2}(u_1, u_2) \right| du_1 \right)^2 du_2 \Big]^{1/2} \\ & \leq \frac{1}{3} |s_2 - t_2|^{1/2} \sqrt{2\tilde{\lambda}(f)} \times 6(2\tilde{\lambda}(f)) \leq 4\sqrt{2}(1 - 1/c)^{1/2} (\tilde{\lambda}(f))^{3/2}. \end{aligned}$$

Applying the same argument to $|g(s_1, t_2) - g(t_1, t_2)|$, we obtain $\frac{1}{4}\varepsilon \leq |g(s) - g(t)| \leq 8\sqrt{2}(1 - 1/c)^{1/2}(\tilde{\lambda}(f))^{3/2}$. So, $\tilde{\lambda}(f) \geq (\varepsilon/(32\sqrt{2}))(c/(c-1))^{1/3}$. Therefore there exists $c_\varepsilon \in (1, +\infty)$ such that, for $c \in (1, c_\varepsilon]$, $\lambda(g) = \tilde{\lambda}(f) \geq 4$, so $\Lambda(A_{c,\varepsilon}) \geq 4$. Thus (H4) is satisfied. \square

Henceforth (Lemma 4.9), we have the following functional law of the iterated logarithm for the stochastic volume V_3 .

Theorem 4.10. *The two-parameter process $\{Z_u = V_3(u \cdot)/(\Phi(u))^{3/2}; u \in \mathbb{R}_+^2\}$ is P -a.s. relatively compact in $\mathcal{C}^{1,2}$ and $P(C(Z_u; u \in \mathbb{R}_+^2) = K(2)) = 1$. \square*

Finally, Theorem 4.10, Corollary 4.8, and the symmetry of the law of V_3 (Proposition 2.4(i)) yield:

Corollary 4.11. *We have*

$$-\liminf_{u \rightarrow +\infty} \frac{V_3(u)}{(\Phi(u))^{3/2}} = \limsup_{u \rightarrow +\infty} \frac{V_3(u)}{(\Phi(u))^{3/2}} = \sup_{f \in K(2)} F(f)(1, 1). \quad \square$$

Remarks. (i) Similar techniques were used by Baldi (1986) to prove a functional law of the iterated logarithm for Lévy's area process V_2 . One may expect that these techniques also apply for $n \geq 4$. For this, one should first prove the equivalence of the two methods of construction of the stochastic volume by extending Green's stochastic formula.

(ii) The functional law of the iterated logarithm for V_2 is proved by Helmes, Rémillard and Theodorescu (1986) by other techniques using the so-called Lévy's formulas.

Acknowledgement

The authors wish to thank the referees for their helpful comments.

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