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Multivariate CARMA processes

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Abstract

A multivariate Lévy-driven continuous time autoregressive moving average (CARMA) model of order (p, q), $q < p$, is introduced. It extends the well-known univariate CARMA and multivariate discrete time ARMA models. We give an explicit construction using a state space representation and a spectral representation of the driving Lévy process. Furthermore, various probabilistic properties of the state space model and the multivariate CARMA process itself are discussed in detail.
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1. Introduction

Being the continuous time analogue of the well-known ARMA processes (see e.g. [1]), continuous time ARMA (CARMA) processes, dating back to [2], have been extensively studied over the recent years (see e.g. [3–5] and references therein) and widely used in various areas of application like engineering, finance and the natural sciences (e.g. [6,7,5]). The advantage of continuous time modelling is that it allows handling irregularly spaced time series and in particular high frequency data often appearing in finance. Originally, driving processes of CARMA models were restricted to Brownian motion; however, [4] allowed for Lévy processes which have a finite r-th moment for some $r > 0$.

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As CARMA processes are short memory moving average processes, [8] developed fractionally integrated CARMA (FICARMA) processes, which exhibit long range dependence. So far only univariate CARMA processes have been defined and investigated. However, in order to model the joint behaviour of several time series (e.g. prices of various stocks) multivariate models are required. Thus, we develop multivariate CARMA processes and study their probabilistic properties in this paper.

Unfortunately, it is not straightforward to define multivariate CARMA processes analogously to the univariate ones, as the state space representation (see Section 3.1) relies on the ability to exchange the autoregressive and moving average operators, which is only possible in one dimension. Simply taking this approach would lead to a spectral representation which does not reflect the autoregressive moving average structure. Our approach leads to a model which can be interpreted as a solution to the formal differential equation $P(D)Y(t) = Q(D)DL(t)$, where D denotes the differential operator with respect to t, L a Lévy process and P and Q the autoregressive and moving average polynomial, respectively. Moreover, it is the continuous time analogue of the multivariate ARMA model.

The paper is organized as follows. In Section 2 we review elementary properties of multidimensional Lévy processes and the stochastic integration theory for deterministic functions with respect to them. A brief summary of univariate Lévy-driven CARMA processes forms the first part of the third section and is followed by the development of what will turn out to be the state space representation of multivariate CARMA (MCARMA) processes. We start by constructing a random orthogonal measure allowing for a spectral representation of the driving Lévy process and continue by studying a stochastic differential equation. Analysing the spectral representation of its solution shows that it can be used to define multivariate CARMA processes. After taking a closer look at the probabilistic properties of this SDE (second moments, Markov property, stationary and limiting distributions and path behaviour), we state the definition of MCARMA processes in Section 3.3. Furthermore, we establish a kernel representation, which enables us to derive some further probabilistic properties of MCARMA models. In particular, we characterize the stationary distribution and path behaviour and give conditions for the existence of moments, the existence of a C_b^∞ density as well as for strong mixing.

Throughout this paper we use the following notation. We call the space of all real or complex $m \times m$ matrices $M_m(\mathbb{R})$ or $M_m(\mathbb{C})$, respectively, and the space of all complex invertible $m \times m$ matrices $\mathcal{G}l_m(\mathbb{C})$. Furthermore, A^* denotes the adjoint of the matrix A and $\operatorname{Ker} A$ its kernel. $I_m \in M_m(\mathbb{C})$ is the identity matrix and $\|A\|$ is the operator norm corresponding to the norm $\|x\|$ for $x \in \mathbb{C}^m$. Finally, $I_B(\cdot)$ is the indicator function of the set B and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2. Multivariate Lévy processes

2.1. Basic facts on multivariate Lévy processes

We state some elementary properties of multivariate Lévy processes that will be needed. For a more general treatment and proofs we refer the reader to [9–11].

We consider a Lévy process $L = \{L(t)\}_{t\geq 0}$ (where $L(0) = 0$ a.s.) in $\mathbb{R}^m$ without a Brownian component determined by its characteristic function in the Lévy–Khintchine form $E\left[\mathrm{e}^{\mathrm{i}\langle u, L(t)\rangle}\right] = \exp\{t\psi_L(u)\}$, $t \geq 0$, where

$$\psi_L(u) = \mathrm{i}\langle \gamma, u\rangle + \int_{\mathbb{R}^m} (\mathrm{e}^{\mathrm{i}\langle u,x\rangle} - 1 - \mathrm{i}\langle u, x\rangle I_{\{\|x\|\leq 1\}})\, \nu(\mathrm{d}x), \quad u \in \mathbb{R}^m, \tag{2.1}$$

where $\gamma \in \mathbb{R}^m$ and ν is a measure on $\mathbb{R}^m$ that satisfies $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^m} (\|x\|^2 \wedge 1)\, \nu(\mathrm{d}x) < \infty$. The measure ν is referred to as the Lévy measure of L. It is a well-known fact that with every càdlàg Lévy process L on $\mathbb{R}^m$ one can associate a random measure J on $\mathbb{R} \times \mathbb{R}^m \setminus \{0\}$ describing the jumps of L. For any measurable set $B \subset \mathbb{R} \times \mathbb{R}^m \setminus \{0\}$,

$$J(B) = \sharp\{s \geq 0 :\ (s, L_s - L_{s-}) \in B\}.$$

The jump measure J is a Poisson random measure on $\mathbb{R} \times \mathbb{R}^m \setminus \{0\}$ (see e.g. Definition 2.18 in [12]) with intensity measure $n(\mathrm{d}s, \mathrm{d}x) = \mathrm{d}s\, \nu(\mathrm{d}x)$. By the Lévy–Itô decomposition we can rewrite L almost surely as

$$L(t) = \gamma t + \int_{\|x\|\geq 1, s\in[0,t]} x\, J(\mathrm{d}s, \mathrm{d}x) + \lim_{\varepsilon \downarrow 0} \int_{\varepsilon \leq \|x\| \leq 1,\, s\in[0,t]} x\, \tilde{J}(\mathrm{d}s, \mathrm{d}x), \quad t \geq 0. \tag{2.2}$$

Here $\tilde{J}(\mathrm{d}s, \mathrm{d}x) = J(\mathrm{d}s, \mathrm{d}x) - \mathrm{d}s\,\nu(\mathrm{d}x)$ is the compensated jump measure, the terms in (2.2) are independent and the convergence in the last term is a.s. and locally uniform in $t \geq 0$.

In the sequel we will work with a two-sided Lévy process $L = \{L(t)\}_{t\in\mathbb{R}}$, constructed by taking two independent copies $\{L_1(t)\}_{t\geq 0}$, $\{L_2(t)\}_{t\geq 0}$ of a one-sided Lévy process and setting

$$L(t) = \begin{cases} L_1(t) & \text{if } t \geq 0 \\ -L_2(-t-) & \text{if } t < 0. \end{cases} \tag{2.3}$$

Assuming that ν satisfies additionally

$$\int_{\|x\|>1} \|x\|^2\, \nu(\mathrm{d}x) < \infty, \tag{2.4}$$

L has finite mean and covariance matrix Σ_L given by

$$\Sigma_L = \int_{\mathbb{R}^m} x x^*\, \nu(\mathrm{d}x). \tag{2.5}$$

Furthermore, if we suppose that $E[L(1)] = \gamma + \int_{\|x\|>1} x\, \nu(\mathrm{d}x) = 0$, then it follows that (2.1) can be written in the form

$$\psi_L(u) = \int_{\mathbb{R}^m} (\mathrm{e}^{\mathrm{i}\langle u,x\rangle} - 1 - \mathrm{i}\langle u, x\rangle)\, \nu(\mathrm{d}x), \quad u \in \mathbb{R}^m, \tag{2.6}$$

and (2.2) simplifies to

$$L(t) = \int_{x\in\mathbb{R}^m\setminus\{0\},\, s\in[0,t]} x\, \tilde{J}(\mathrm{d}s, \mathrm{d}x), \quad t \in \mathbb{R}. \tag{2.7}$$

In this case $L = \{L(t)\}_{t\geq 0}$ is a martingale.

2.2. *Stochastic integrals with respect to Lévy processes*

In this section we consider the stochastic process $X = \{X(t)\}_{\in\mathbb{R}}$ given by

$$X(t) = \int_{\mathbb{R}} f(t, s)\, L(\mathrm{d}s), \quad t \in \mathbb{R}, \tag{2.8}$$

where $f : \mathbb{R} \times \mathbb{R} \to M_m(\mathbb{R})$ is a measurable function and $L = \{L(t)\}_{t\in\mathbb{R}}$ is an m-dimensional Lévy process without a Brownian component. For integration with respect to Brownian motion we refer the reader to any of the standard books.

We first assume that the process L in (2.8) is an m-dimensional Lévy process without a Gaussian component satisfying $E[L(1)] = 0$ and $E[L(1)L(1)^*] < \infty$, i.e., L can be represented as in (2.7).

In this case it follows from (2.7) that the process X can be represented by

$$X(t) = \int_{\mathbb{R}\times\mathbb{R}^m} f(t,s)x\,\tilde{J}(\mathrm{d}s,\mathrm{d}x), \quad t \in \mathbb{R}, \tag{2.9}$$

where $\tilde{J}(\mathrm{d}s,\mathrm{d}x) = J(\mathrm{d}s,\mathrm{d}x) - \mathrm{d}s\,\nu(\mathrm{d}x)$ is the compensated jump measure of L. A necessary and sufficient condition for the existence of the stochastic integral (2.9) in $L^2(\Omega, P)$ (see e.g. [13] or [14]) is that

$$\int_{\mathbb{R}}\int_{\mathbb{R}^m} (\|f(t,s)x\|^2 \wedge \|f(t,s)x\|)\,\nu(\mathrm{d}x)\,\mathrm{d}s < \infty, \quad \forall\, t \in \mathbb{R}.$$

Then the law of $X(t)$ is for all $t \in \mathbb{R}$ infinitely divisible with characteristic function

$$E\left[\exp\{\mathrm{i}\langle u, X(t)\rangle\}\right] = \exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}^m} \left(\mathrm{e}^{\mathrm{i}\langle u, f(t,s)x\rangle} - 1 - \mathrm{i}\langle u, f(t,s)x\rangle\right)\nu(\mathrm{d}x)\,\mathrm{d}s\right\}.$$

Furthermore, if $f(t,\cdot) \in L^2(\mathbb{R}; M_m(\mathbb{R}))$, the integral (2.9) exists in $L^2(\Omega, P)$ and

$$E\left[X(t)X(t)^*\right] = \int_{\mathbb{R}} f(t,s)\Sigma_L f^*(t,s)\,\mathrm{d}s. \tag{2.10}$$

If

$$\int_{\mathbb{R}}\int_{\mathbb{R}^m} (\|f(t,s)x\| \wedge 1)\,\nu(\mathrm{d}x)\,\mathrm{d}s < \infty, \quad \forall\, t \in \mathbb{R},$$

the stochastic integral (2.8) exists without a compensator and we can write

$$X(t) = \int_{\mathbb{R}\times\mathbb{R}^m} f(t,s)x\,J(\mathrm{d}s,\mathrm{d}x), \quad t \in \mathbb{R}. \tag{2.11}$$

Finally, in the general case, where condition (2.4) is not satisfied, necessary and sufficient conditions for the integral (2.8) to exist are (see [13,15])

$$\int_{\mathbb{R}}\int_{\mathbb{R}^m} (\|f(t,s)x\|^2 \wedge 1)\,\nu(\mathrm{d}x)\,\mathrm{d}s < \infty, \quad \forall\, t \in \mathbb{R}, \tag{2.12}$$

and

$$\int_{\mathbb{R}} \left\| f(t,s)\gamma + \int_{\mathbb{R}^m} f(t,s)x\left(I_{\{\|f(t,s)x\|\le 1\}} - I_{\{\|x\|\le 1\}}\right)\nu(\mathrm{d}x)\right\| \mathrm{d}s < \infty. \tag{2.13}$$

Then we represent X as

$$\begin{aligned} X(t) = &\int_{\mathbb{R}}\int_{\mathbb{R}^m} f(t,s)x\left[J(\mathrm{d}s,\mathrm{d}x) - (1 \vee \|f(t,s)x\|)^{-1}\,\nu(\mathrm{d}x)\,\mathrm{d}s\right] \\ &+ \int_{\mathbb{R}} f(t,s)\gamma\,\mathrm{d}s, \quad t \in \mathbb{R}. \end{aligned}$$

Moreover, if the integral in (2.8) is well-defined, the distribution of $X(t)$ is infinitely divisible with characteristic triplet $(\gamma_X^t, 0, \nu_X^t)$ given by

$$\gamma_X^t = \int_{\mathbb{R}} f(t,s)\gamma \, \mathrm{d}s + \int_{\mathbb{R}} \int_{\mathbb{R}^m} f(t,s)x[I_{\{\|f(t,s)x\| \leq 1\}} - I_{\{\|x\| \leq 1\}}] \, \nu(\mathrm{d}x) \, \mathrm{d}s, \tag{2.14}$$

$$\nu_X^t(B) = \int_{\mathbb{R}} \int_{\mathbb{R}^m} 1_B(f(t,s)x) \, \nu(\mathrm{d}x) \, \mathrm{d}s. \tag{2.15}$$

It follows that the characteristic function of $X(t)$ can be written as

$$\begin{aligned} E\left[\mathrm{e}^{\mathrm{i}\langle u, X(t)\rangle}\right] &= \exp\left\{\mathrm{i}\langle \gamma_X^t, u\rangle + \int_{\mathbb{R}^m} [\mathrm{e}^{\mathrm{i}\langle u,x\rangle} - 1 - \mathrm{i}\langle u, x\rangle I_{\{\|x\|\leq 1\}}] \, \nu_X^t(\mathrm{d}x)\right\} \\ &= \exp\left\{\int_{\mathbb{R}} \psi_L(f(t,s)^* u) \, \mathrm{d}s\right\}, \end{aligned} \tag{2.16}$$

where ψ_L is given as in (2.1). These facts follow from [15, Theorem 3.1, Proposition 2.17 and Corollary 2.19].

3. Multivariate CARMA processes

In this section we discuss CARMA processes driven by general Lévy processes, i.e., the Lévy processes may have a Brownian component and does not need to have finite variance, if not stated otherwise. We start with a brief review of the well-known one-dimensional case.

3.1. Univariate Lévy-driven CARMA processes

Continuous time ARMA (CARMA) processes constitute a special class of short memory moving average (MA) processes (see, for instance, [9, Section 4.3.5]) and are the continuous time analogues of the well-known autoregressive moving average (ARMA) processes. We give here a short summary of their definition and properties. For further details see [3,4,16].

Definition 3.1 (*CARMA Process*). Let $\{L(t)\}_{t\in\mathbb{R}}$ be a Lévy process satisfying $\int_{|x|\geq 1} \log|x|\nu(\mathrm{d}x) < \infty$, p, q be in $\mathbb{N}_0$ with $p > q$ and $a_1, \ldots, a_p, b_0, \ldots, b_q \in \mathbb{R}$, $a_p, b_0 \neq 0$ such that

$$A := \left[\begin{array}{c|ccc} 0 & & I_{p-1} & \\ \hline -a_p & -a_{p-1} & \ldots & -a_1 \end{array}\right]$$

has only eigenvalues with strictly negative real part. Furthermore, denote by $\{X(t)\}_{t\in\mathbb{R}}$ the stationary solution to

$$\mathrm{d}X(t) = AX(t)\mathrm{d}t + e\,L(\mathrm{d}t), \quad t \in \mathbb{R}, \tag{3.1}$$

where $e^{\mathrm{T}} = [0, \ldots, 0, 1]$. Then the process

$$Y(t) = b^{\mathrm{T}} X(t), \tag{3.2}$$

with $b^{\mathrm{T}} = \left[b_q, b_{q-1}, \ldots, b_{q-p+1}\right]$, is called a Lévy-driven continuous time autoregressive moving average process of order (p, q) (CARMA(p, q), for short). If $q < p - 1$, we set $b_{-1} = \cdots = b_{q-p+1} = 0$.

The CARMA(p, q) process can be interpreted as the stationary solution of the p-th-order linear differential equation,

$$p(D)Y(t) = q(D)DL(t), \quad t \geq 0, \tag{3.3}$$

where D denotes differentiation with respect to t and

$$p(z) := z^p + a_1 z^{p-1} + \cdots + a_p \quad \text{and} \quad q(z) := b_0 z^q + b_1 z^{q-1} + \cdots + b_q$$

are the so-called autoregressive and moving average polynomials, respectively. To see this note first that in the case $q(z) = 1$ (i.e. $q = 0$ and $b^{\mathrm{T}} = (1, 0, \ldots, 0)$) rewriting (3.3) as a system of first-order differential equations in the standard way gives (3.1) and (3.2) with $X_t^{\mathrm{T}} = (Y_t, DY_t, \ldots, D^{p-1}Y_t)$. In the general case we transform (3.3) to $Y(t) = p(D)^{-1}q(D)DL(t) = q(D)p(D)^{-1}DL(t)$ (note that we may commute $p^{-1}(D)$ and $q(D)$, since the real coefficients and the operator D all commute). From the previous case we infer that the process in (3.1) is formed by $p(D)^{-1}DL(t)$ and the first $p-1$ derivatives of this process. Now one can immediately see that $Y_t = b^{\mathrm{T}} X_t = q(D)p(D)^{-1}DL_t$.

Remark 3.2. Observe that the process $\{X(t)\}_{t\in\mathbb{R}}$ can be represented as

$$X(t) = \int_{-\infty}^{t} \mathrm{e}^{A(t-u)} e \, L(\mathrm{d}u), \quad t \in \mathbb{R}, \tag{3.4}$$

and is a multivariate Ornstein–Uhlenbeck-type process [17–19]. Hence, we have

$$Y(t) = \int_{-\infty}^{t} b^{\mathrm{T}} \mathrm{e}^{A(t-u)} e \, L(\mathrm{d}u), \quad t \in \mathbb{R}. \tag{3.5}$$

From (3.5) it is obvious that $\{Y(t)\}_{t\in\mathbb{R}}$ is a causal short memory moving average process, since it has the form

$$Y(t) = \int_{-\infty}^{\infty} g(t-u) \, L(\mathrm{d}u), \quad t \in \mathbb{R}, \tag{3.6}$$

with kernel $g(t) = b^{\mathrm{T}} \mathrm{e}^{At} e I_{[0,\infty)}(t)$. Replacing e^{At} by its spectral representation, the kernel g can be expressed as

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t\lambda} \frac{q(\mathrm{i}\lambda)}{p(\mathrm{i}\lambda)} \mathrm{d}\lambda, \quad t \in \mathbb{R}. \tag{3.7}$$

Note that the representation of $\{Y(t)\}_{t\in\mathbb{R}}$ given by (3.6) together with (3.7) defines a strictly stationary process even if there are eigenvalues of A with strictly positive real part. However, if there are eigenvalues with positive real part, the CARMA process will be no longer causal. Henceforth, we focus on causal CARMA processes.

Proposition 3.3 (*[16, Section 2]*). *If $E[L(1)^2] < \infty$, the spectral density f_Y of $Y = \{Y(t)\}_{t\in\mathbb{R}}$ is given by*

$$f_Y(\lambda) = \frac{\mathrm{var}(L(1))}{2\pi} \frac{|q(\mathrm{i}\lambda)|^2}{|p(\mathrm{i}\lambda)|^2}, \quad \lambda \in \mathbb{R}.$$

Consequently, the autocovariance function γ_Y of the CARMA process Y can be expressed as

$$\gamma_Y(h) = \operatorname{cov}(Y(t+h), Y(t)) = \frac{\operatorname{var}(L(1))}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}h\lambda} \left| \frac{q(\mathrm{i}\lambda)}{p(\mathrm{i}\lambda)} \right|^2 \mathrm{d}\lambda, \quad h \in \mathbb{R}.$$

Moreover, for a causal CARMA process an application of the residue theorem leads to

$$\gamma_Y(h) = \operatorname{var}(L(1)) \sum_{r=1}^{p} \frac{q(\lambda_r) q(-\lambda_r)}{p'(\lambda_r) p(-\lambda_r)} \mathrm{e}^{\lambda_r |h|}, \quad h \in \mathbb{R},$$

provided all eigenvalues $\lambda_1, \ldots, \lambda_p$ of the matrix A are algebraically simple.

3.2. State space representation of multivariate CARMA processes

This section contains the necessary results and insights enabling us to define multivariate CARMA processes in the next section. As we shall heavily make use of spectral representations of stationary processes (see [20–22] for comprehensive treatments), let us briefly recall the notions and results we shall employ.

Definition 3.4. Let $\mathcal{B}(\mathbb{R})$ denote the Borel-σ-algebra over $\mathbb{R}$. A family $\{\zeta(\Delta)\}_{\Delta \in \mathcal{B}(\mathbb{R})}$ of $\mathbb{C}^m$-valued random variables is called an m-dimensional random orthogonal measure if

(a) $\zeta(\Delta) \in L^2$ for all bounded $\Delta \in \mathcal{B}(\mathbb{R})$,
(b) $\zeta(\emptyset) = 0$,
(c) $\zeta(\Delta_1 \cup \Delta_2) = \zeta(\Delta_1) + \zeta(\Delta_2)$ a.s. if $\Delta_1 \cap \Delta_2 = \emptyset$ and
(d) $F : \mathcal{B}(\mathbb{R}) \to M_m(\mathbb{C})$, $\Delta \mapsto E[\zeta(\Delta)\zeta(\Delta)^*]$ defines a σ-additive positive definite matrix measure (i.e., a σ-additive set function that assumes values in the positive semi-definite matrices) and it holds that $E[\zeta(\Delta_1)\zeta(\Delta_2)^*] = F(\Delta_1 \cap \Delta_2)$ for all $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{R})$.

F is referred to as the spectral measure of ζ.

The definition above obviously implies $E[\zeta(\Delta_1)\zeta(\Delta_2)^*] = 0$ for disjoint Borel sets Δ_1, Δ_2.

Stochastic integrals $\int_\Delta f(t)\zeta(\mathrm{d}t)$ of deterministic Lebesgue-measurable functions $f : \mathbb{R} \to M_m(\mathbb{C})$ with respect to a random orthogonal measure ζ are now as usually defined in an L^2-sense (see, in particular, [22, Ch. 1] for details). Note that the integration can be understood componentwise: denoting the coordinates of ζ by ζ_i, i.e. $\zeta = (\zeta_1, \ldots, \zeta_m)^*$, the i-th element $\left(\int_\Delta f(t)\zeta(\mathrm{d}t)\right)_i$ of $\int_\Delta f(t)\zeta(\mathrm{d}t)$ is given by $\sum_{k=1}^m \int_\Delta f_{ik}(t)\zeta_k(\mathrm{d}t)$, where the integrals are standard one-dimensional stochastic integrals in an L^2-sense and $f_{ik}(t)$ denotes the element in the i-th row and k-th column of $f(t)$. The above integral is defined whenever the integral

$$\int_\Delta f(t)F(\mathrm{d}t)f(t)^* := \left(\sum_{k,l=1}^{m} \int_{\mathbb{R}} f_{ik}(t)\overline{f}_{jl}(t)F_{kl}(\mathrm{d}t) \right)_{1 \le i,j \le m}$$

exists. Functions satisfying this condition are said to be in $L^2(F)$. For two functions $f, g \in L^2(F)$ we have

$$E\left[\int_\Delta f(t)\zeta(\mathrm{d}t) \left(\int_\Delta g(t)\zeta(\mathrm{d}t)\right)^*\right] = \int_\Delta f(t)F(\mathrm{d}t)g(t)^*. \tag{3.8}$$

In the following we will only encounter random orthogonal measures, whose associated spectral measures have constant density with respect to the Lebesgue measure λ on $\mathbb{R}$, i.e. $F(\mathrm{d}t) =$

$C\lambda(\mathrm{d}t) := C\,\mathrm{d}t$ for some positive definite $C \in M_m(\mathbb{C})$, which simplifies the integration theory considerably. In this case it is easy to see that it is sufficient for $\int_\Delta f(t)F(\mathrm{d}t)f(t)^*$ to exist that $\int_\Delta \|f(t)\|^2\,\mathrm{d}t$ is finite, where $\|\cdot\|$ is some norm on $M_m(\mathbb{C})$. To ease notation we define the space of square integrable matrix-valued functions

$$L^2(\mathbb{R}; M_m(\mathbb{C})) := \left\{ f : \mathbb{R} \to M_m(\mathbb{C}),\ \int_{\mathbb{R}} \|f(t)\|^2 \mathrm{d}t < \infty \right\}. \tag{3.9}$$

In the following we abbreviate $L^2(\mathbb{R}; M_m(\mathbb{C}))$ by $L^2(M_m(\mathbb{C}))$. This space is independent of the norm $\|\cdot\|$ on $M_m(\mathbb{C})$ used in the definition and is equal to the space of functions $f = (f_{ij}) : \mathbb{R} \to M_m(\mathbb{C})$ where all components f_{ij} are in the usual space $L^2(\mathbb{R}; \mathbb{C})$.

$$\|f\|_{L^2(M_m(\mathbb{C}))} = \left(\int_{\mathbb{R}} \|f(t)\|^2 \mathrm{d}t \right)^{1/2} \tag{3.10}$$

defines a norm on $L^2(M_m(\mathbb{C}))$ and again it is immaterial which norm we use, as all norms $\|\cdot\|$ on $M_m(\mathbb{C})$ lead to equivalent norms $\|\cdot\|_{L^2(M_m(\mathbb{C}))}$. With this norm $L^2(M_m(\mathbb{C}))$ is a Banach space and even a Hilbert space, provided the original norm $\|\cdot\|$ on $M_m(\mathbb{C})$ is induced by a scalar product. Observe that as usual we do not distinguish between functions and equivalence classes in $L^2(\cdot)$. The integrals $\int_\Delta f(t)\zeta(\mathrm{d}t)$ and $\int_\Delta g(t)\zeta(\mathrm{d}t)$ agree (in L^2) if f and g are identical in $L^2(M_m(\mathbb{C}))$, and a sequence of integrals $\int_\Delta \|f_n(t)\|^2\,\mathrm{d}t$ converges (in L^2) to $\int_\Delta \|f(t)\|^2\,\mathrm{d}t$ for $n \to \infty$ if $\|f_n(t) - f(t)\|_{L^2(M_m(\mathbb{C}))} \to 0$ as $n \to \infty$. Moreover,

$$E\left[\int_\Delta f(t)\zeta(\mathrm{d}t) \left(\int_\Delta g(t)\zeta(\mathrm{d}t) \right)^* \right] = \int_\Delta f(t)Cg(t)^* \mathrm{d}t. \tag{3.11}$$

Our first step in the construction of multivariate CARMA processes is the following theorem extending the well-known fact that

$$W(t) = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\mu t} - 1}{\mathrm{i}\mu}\, \phi(\mathrm{d}\mu), \quad t \in \mathbb{R},$$

is an m-dimensional standard Wiener process if ϕ is an m-dimensional Gaussian random orthogonal measure satisfying $E[\phi(A)] = 0$ and $E[\phi(A)\phi(A)^*] = \frac{I_m}{2\pi}\lambda(A)$ for all $A \in \mathcal{B}(\mathbb{R})$ (see e.g. [23, Section 2.1, Lemma 5]).

Theorem 3.5. *Let $L = \{L(t)\}_{t\in\mathbb{R}}$ be a two-sided square integrable m-dimensional Lévy process with $E[L(1)] = 0$ and $E[L(1)L(1)^*] = \Sigma_L$. Then there exists an m-dimensional random orthogonal measure Φ_L with spectral measure F_L such that $E[\Phi_L(\Delta)] = 0$ for any bounded Borel set Δ,*

$$F_L(\mathrm{d}t) = \frac{\Sigma_L}{2\pi}\mathrm{d}t \tag{3.12}$$

and

$$L(t) = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\mu t} - 1}{\mathrm{i}\mu}\, \Phi_L(\mathrm{d}\mu). \tag{3.13}$$

The random measure Φ_L is uniquely determined by

$$\Phi_L([a,b)) = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\mu a} - \mathrm{e}^{-\mathrm{i}\mu b}}{2\pi\mathrm{i}\mu}\, L(\mathrm{d}\mu) \tag{3.14}$$

for all $-\infty < a < b < \infty$.

Proof. Observe that setting $\tilde{\Phi}([a,b)) = L(b) - L(a)$ defines a random orthogonal measure on the semi-ring of intervals $[a;b)$, with $-\infty < a < b < \infty$. Using an obvious multidimensional extension of [22, Theorem 2.1], we extend $\tilde{\Phi}_L$ to a random orthogonal measure on the Borel sets. It is immediate that the associated spectral measure $\tilde{F}_L$ satisfies $\tilde{F}_L(\mathrm{d}t) = \Sigma_L\,\mathrm{d}t$ and that integrating with respect to $\tilde{\Phi}_L$ is the same as integrating with respect to the Lévy process L.

Now define $\Phi_L([a,b))$ for $-\infty < a < b < \infty$ by (3.14) which is equivalent to

$$\Phi_L([a,b)) = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\mu a} - \mathrm{e}^{-\mathrm{i}\mu b}}{2\pi\mathrm{i}\mu}\,\tilde{\Phi}_L(\mathrm{d}\mu). \tag{3.15}$$

Using (3.11) we obtain for any two intervals $[a,b)$ and $[a',b')$

$$\begin{aligned} E[\Phi_L([a,b))\Phi_L([a',b'))^*] &= \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\mu a} - \mathrm{e}^{-\mathrm{i}\mu b}}{2\pi\mathrm{i}\mu}\Sigma_L \overline{\left(\frac{\mathrm{e}^{-\mathrm{i}\mu a'} - \mathrm{e}^{-\mathrm{i}\mu b'}}{2\pi\mathrm{i}\mu}\right)}\mathrm{d}\mu \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\mu a} - \mathrm{e}^{-\mathrm{i}\mu b}}{2\pi\mathrm{i}\mu}\Sigma_L^{1/2}\left(\frac{\mathrm{e}^{-\mathrm{i}\mu a'} - \mathrm{e}^{-\mathrm{i}\mu b'}}{2\pi\mathrm{i}\mu}\Sigma_L^{1/2}\right)^* \mathrm{d}\mu, \end{aligned} \tag{3.16}$$

where $\Sigma_L^{1/2}$ denotes the unique square root of Σ_L defined by spectral calculus. The crucial point is now to observe that the function $\hat{\phi}_{a,b}(\mu) = \frac{\mathrm{e}^{-\mathrm{i}\mu a} - \mathrm{e}^{-\mathrm{i}\mu b}}{\sqrt{2\pi}\mathrm{i}\mu}\Sigma_L^{1/2}$ is the Fourier transform of the function $I_{[a,b)}(t)\Sigma_L^{1/2}$, i.e.,

$$\hat{\phi}_{a,b}(\mu) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\mu t} I_{[a,b)}(t)\Sigma_L^{1/2}\mathrm{d}t.$$

The standard theory of Fourier–Plancherel transforms $\mathcal{F}$ (see e.g. [24, Chapter II] or [25, Chapter 6]) extends immediately to the space $L^2(M_m(\mathbb{C}))$ on setting

$$\mathcal{F}_m : L^2(M_m(\mathbb{C})) \to L^2(M_m(\mathbb{C})), \qquad f(t) \mapsto \hat{f}(\mu) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\mu t} f(t)\mathrm{d}t$$

where $\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\mu t} f(t)\mathrm{d}t$ is the limit in $L^2(M_m(\mathbb{C}))$ of $\int_{-R}^{R} \mathrm{e}^{-\mathrm{i}\mu t} f(t)\mathrm{d}t$ as $R \to \infty$, because this can be interpreted as a componentwise Fourier–Plancherel transformation and, as stated before, a function f is in $L^2(M_m(\mathbb{C}))$ if and only if all components f_{ij} are in $L^2(\mathbb{R};\mathbb{C})$. In particular, $\mathcal{F}_m$ is an invertible continuous linear operator on $L^2(M_m(\mathbb{C}))$ with

$$\mathcal{F}_m^{-1} : L^2(M_m(\mathbb{C})) \to L^2(M_m(\mathbb{C})), \qquad \hat{f}(\mu) \mapsto f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\mu t} \hat{f}(\mu)\mathrm{d}\mu,$$

and Plancherel's identity generalizes to:

$$\int_{\mathbb{R}} f(t)g(t)^*\mathrm{d}t = \int_{\mathbb{R}} \hat{f}(\mu)\hat{g}(\mu)^*\mathrm{d}\mu. \tag{3.17}$$

Combining (3.16) with (3.17) gives

$$\begin{aligned} E[\Phi_L([a,b))\Phi_L([a',b'))^*] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\phi}_{a,b}(\mu)\left(\hat{\phi}_{a',b'}(\mu)\right)^* \mathrm{d}\mu \\ &= \frac{\Sigma_L}{2\pi}\int_{-\infty}^{\infty} I_{[a,b)}(t)I_{[a',b')}(t)\,\mathrm{d}t. \end{aligned}$$

This implies immediately that $E[\Phi_L([a,b))\Phi_L([a',b'))^*]=0$ if $[a,b)$ and $[a',b')$ are disjoint, $E[\Phi_L([a,b))\Phi_L([a,b))^*]=\frac{\Sigma_L\lambda([a,b))}{2\pi}$ and that Φ_L is a random orthogonal measure on the semi-ring of intervals $[a,b)$, which we extend to one on all Borel sets. Therefore, (3.15) extends to

$$\int_{-\infty}^{\infty} I_\Delta(t)\,\Phi_L(\mathrm{d}t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\phi}_\Delta(\mu)\,\tilde{\Phi}_L(\mathrm{d}\mu) \tag{3.18}$$

for all Borel sets Δ, where $\hat{\phi}_\Delta=\mathcal{F}_m(I_\Delta)$ is the Fourier transform of I_Δ.

For any function $\varphi\in L^2(M_m(\mathbb{C}))$ there is a sequence of elementary functions $\varphi_k(t)$, $k\in\mathbb{N}$ (i.e., matrix-valued functions of the form $\sum_{i=1}^N C_i I_{\Delta_i}(t)$ with appropriate $N\in\mathbb{N}$, $C_i\in M_m(\mathbb{C})$ and Borel sets Δ_i), which converges to φ in $L^2(M_m(\mathbb{C}))$. As the Fourier–Plancherel transform is a topological isomorphism that maps $L^2(M_m(\mathbb{C}))$ onto itself, the Fourier–Plancherel transforms $\hat{\varphi}_k(t)$ converge to the Fourier–Plancherel transform $\hat{\varphi}(t)$ in $L^2(M_m(\mathbb{C}))$, which allows us to extend (3.18), exchanging the roles of μ and t, to

$$\int_{-\infty}^{\infty}\varphi(\mu)\,\Phi_L(\mathrm{d}\mu)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\varphi}(t)\,\tilde{\Phi}_L(\mathrm{d}t) \tag{3.19}$$

for all functions φ in $L^2(M_m(\mathbb{C}))$ and their Fourier–Plancherel transforms $\hat{\varphi}$. Now choose $\varphi(\mu)=\frac{\mathrm{e}^{\mathrm{i}\mu b}-\mathrm{e}^{\mathrm{i}\mu a}}{\mathrm{i}\mu}$; then $\hat{\varphi}(t)=\sqrt{2\pi}I_{[a,b)}(t)$. This shows that

$$\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\mu b}-\mathrm{e}^{\mathrm{i}\mu a}}{\mathrm{i}\mu}\,\Phi_L(\mathrm{d}\mu)=L(b)-L(a)$$

and thus (3.13) is shown.

The uniqueness of Φ_L follows easily, as (3.13) implies (3.19) using arguments analogous to the above ones. □

Note that for one-dimensional random orthogonal measures such results can already be found in [20, Section IX.4].

Remark 3.6. If we formally differentiate (3.13), we obtain

$$\frac{\mathrm{d}L(t)}{\mathrm{d}t}=\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\mu t}\,\Phi_L(\mathrm{d}\mu),$$

as in the spectral representation differentiation is the transform given by

$$\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\mu t}\,\Phi(\mathrm{d}\mu)\mapsto\int_{-\infty}^{\infty}\mathrm{i}\mu\mathrm{e}^{\mathrm{i}\mu t}\,\Phi(\mathrm{d}\mu).$$

Thus, a univariate CARMA process should have the representation

$$Y(t)=\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\mu t}\frac{q(\mathrm{i}\mu)}{p(\mathrm{i}\mu)}\,\Phi_L(\mathrm{d}\mu), \tag{3.20}$$

as this reflects the differential equation (3.3). Later, in Theorem 3.22, we will see that this is indeed the case. The square integrability necessary for (3.20) to be defined explains why one can only consider CARMA processes with $q<p$ (cf. Lemma 3.11).

The next lemma deals with the spectral representation of integrals of processes.

Lemma 3.7. *Let Φ be an m-dimensional random orthogonal measure with spectral measure $F(\mathrm{d}t) = C\,\mathrm{d}t$ for some positive definite $C \in M_m(\mathbb{C})$ and $g \in L^2(M_m(\mathbb{C}))$. Define the m-dimensional random process $G = \{G(t)\}_{t\in\mathbb{R}}$ by*

$$G(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\mu t} g(\mathrm{i}\mu)\,\Phi(\mathrm{d}\mu).$$

Then G is weakly stationary,

$$\int_0^t G(s)\,\mathrm{d}s < \infty \quad \text{a.s. for every } t > 0 \quad \text{and}$$
$$\int_0^t G(s)\,\mathrm{d}s = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\mu t} - 1}{\mathrm{i}\mu} g(\mathrm{i}\mu)\,\Phi(\mathrm{d}\mu), \quad t > 0.$$

Proof. Weak stationarity follows immediately from (3.11), which implies

$$E[G(t)G(s)^*] = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\mu(t-s)} g(\mathrm{i}\mu) C g(\mathrm{i}\mu)^* \mathrm{d}\mu.$$

The weak stationarity implies that $\|G(s)\|_{L_2} := E[\|G(s)\|_2^2]^{1/2} = E[G(s)^*G(s)]^{1/2}$ is finite and constant, where $\|\cdot\|_2$ denotes the Euclidean norm. Thus an elementary Fubini argument and using $\|\cdot\|_{L^1} \le \|\cdot\|_{L^2}$ gives:

$$E\left\|\int_0^t G(s)\mathrm{d}s\right\|_2 \le E\left[\int_0^t \|G(s)\|_2 \mathrm{d}s\right] = \int_0^t E\left[\|G(s)\|_2\right]\mathrm{d}s \le \int_0^t \|G(s)\|_{L_2}\mathrm{d}s < \infty.$$

In particular, $\int_0^t G(s)\mathrm{d}s$ is almost surely finite. Finally, we obtain

$$\begin{aligned}\int_0^t G(s)\mathrm{d}s &= \int_0^t \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\mu s} g(\mathrm{i}\mu)\,\Phi(\mathrm{d}\mu)\mathrm{d}s = \int_{-\infty}^{\infty} \int_0^t \mathrm{e}^{\mathrm{i}\mu s}\mathrm{d}s\, g(\mathrm{i}\mu)\,\Phi(\mathrm{d}\mu) \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\mu t} - 1}{\mathrm{i}\mu} g(\mathrm{i}\mu)\,\Phi(\mathrm{d}\mu),\end{aligned}$$

using a stochastic version of Fubini's theorem (e.g. the obvious multidimensional extension of [21, Section IV.4, Lemma 4]). □

Before turning to a theorem enabling us to define MCARMA processes we establish three lemmata and one corollary which contain necessary technical results relating the zeros of what is to become the autoregressive polynomial to the spectrum of a particular matrix A. The first lemma contains furthermore some additional insight into the eigenvectors of A.

Lemma 3.8. *Let $A_1, \ldots, A_p \in M_m(\mathbb{C})$, $p \in \mathbb{N}$, define $P : \mathbb{C} \to M_m(\mathbb{C})$, $z \mapsto I_m z^p + A_1 z^{p-1} + A_2 z^{p-2} + \cdots + A_p$ and set*

$$\mathcal{N}(P) = \{z \in \mathbb{C} : \det(P(z)) = 0\}, \tag{3.21}$$

i.e., $\mathcal{N}(P)$ is the set of all $z \in \mathbb{C}$ such that $P(z) \notin \mathcal{G}l_m(\mathbb{C})$. Furthermore, set

$$A = \begin{pmatrix} 0 & I_m & 0 & \dots & 0 \\ 0 & 0 & I_m & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & I_m \\ -A_p & -A_{p-1} & \dots & \dots & -A_1 \end{pmatrix} \in M_{mp}(\mathbb{C}) \tag{3.22}$$

and denote the spectrum of A by $\sigma(A)$. Then $\mathcal{N}(P) = \sigma(A)$ and $\bar{x} \in \mathbb{C}^{mp} \setminus \{0\}$ is an eigenvector of A with corresponding eigenvalue λ if and only if there is an $\tilde{x} \in \operatorname{Ker} P(\lambda) \setminus \{0\}$, such that $\bar{x} = (\tilde{x}^*, (\lambda\tilde{x})^*, \dots, (\lambda^{p-1}\tilde{x})^*)^*$. Moreover, $0 \in \sigma(A)$ if and only if $0 \in \sigma(A_p)$.

Proof. It is immediate from the structure of A that A is of full rank if and only if A_p is of full rank.

Let λ be an eigenvalue of A and $\bar{x} = (x_1^*, \dots, x_p^*)^* \in \mathbb{R}^{mp}$, $x_i \in \mathbb{R}^m$, a corresponding eigenvector, i.e., $A\bar{x} - \lambda\bar{x} = 0$ from which $\lambda x_1 = x_2, \lambda x_2 = x_3, \dots, \lambda x_{p-1} = x_p$, $\lambda x_p + A_1 x_p + A_2 x_{p-1} + \dots + A_p x_1 = 0$ follows. Hence, $x_i = \lambda^{i-1} x_1$, $i = 1, 2, \dots, p$ and

$$\begin{aligned} &\lambda^p x_1 + A_1 \lambda^{p-1} x_1 + A_2 \lambda^{p-2} x_1 + \dots + A_p x_1 \\ &\quad = (I_m \lambda^p + A_1 \lambda^{p-1} + \dots + A_p) x_1 = 0. \end{aligned} \tag{3.23}$$

As $\bar{x} \neq 0$, we have $x_1 \neq 0$ and (3.23) gives $x_1 \in \operatorname{Ker} P(\lambda)$. Hence, we can set $\tilde{x} = x_1$. Furthermore the non-triviality of the kernel of $P(\lambda)$ implies $\det(P(\lambda)) = 0$. Thus $\mathcal{N}(P) \supseteq \sigma(A)$ has been established.

Now we turn to the converse implication. Let $\lambda \in \mathcal{N}(P)$; then $P(\lambda)$ has a non-trivial kernel. Take any $\tilde{x} \in \operatorname{Ker} P(\lambda) \setminus \{0\}$ and set $\bar{x} = (\tilde{x}^*, (\lambda\tilde{x})^*, \dots, (\lambda^{p-1}\tilde{x})^*)^*$. Then (3.23) shows that $A\bar{x} = \lambda\bar{x}$ and thus $\lambda \in \sigma(A)$. Therefore $\mathcal{N}(P) \subseteq \sigma(A)$ and $\bar{x}$ is an eigenvector of A to the eigenvalue λ. □

Corollary 3.9. $\sigma(A) \subseteq (-\infty, 0) + \mathrm{i}\mathbb{R}$ *if and only if* $\mathcal{N}(P) \subseteq (-\infty, 0) + \mathrm{i}\mathbb{R}$.

Lemma 3.10. *If $\mathcal{N}(P) \subseteq \mathbb{R} \setminus \{0\} + \mathrm{i}\mathbb{R}$, then $P(\mathrm{i}z) \in \mathcal{G}l_m(\mathbb{C})$ for all $z \in \mathbb{R}$.*

Proof. As all zeros of $\det(P(z))$ have non-vanishing real part, all zeros of $\det(P(\mathrm{i}z))$ must have non-vanishing imaginary part and thus $P(\mathrm{i}z)$ is invertible for all $z \in \mathbb{R}$. □

Lemma 3.11. *Let $C_0, C_1, \dots, C_{p-1} \in M_m(\mathbb{C})$ and $R(z) = \sum_{i=0}^{p-1} C_i z^i$. Assume that $\mathcal{N}(P) \subseteq \mathbb{R} \setminus \{0\} + \mathrm{i}\mathbb{R}$; then*

$$\int_{-\infty}^{\infty} \|P(\mathrm{i}z)^{-1} R(\mathrm{i}z)\|^2 \,\mathrm{d}z < \infty,$$

where $P(z) = I_m z^p + A_1 z^{p-1} + \dots + A_p$.

Proof. As $\det(P(\mathrm{i}z))$, $z \in \mathbb{R}$, has no zeros, $\|P(\mathrm{i}z)^{-1} R(\mathrm{i}z)\|$ is finite for all $z \in \mathbb{R}$, continuous and thus bounded on any compact set. Hence, $\int_{-K}^{K} \|P(\mathrm{i}z)^{-1} R(\mathrm{i}z)\|^2 \,\mathrm{d}z$ exists for all $K \in \mathbb{R}$. For any $x \in \mathbb{R}^m$ we have

$$\|P(z)x\| = \left\| \left(I_m z^p + \sum_{k=0}^{p-1} A_{p-k} z^k \right) x \right\| \geq \|z^p x\| - \left\| \sum_{k=0}^{p-1} A_{p-k} z^k x \right\|$$

$$\geq \left(|z|^p - \sum_{k=0}^{p-1} \|A_{p-k}\||z^k|\right) \|x\|.$$

Thus, there is $K > 0$ such that $\|P(z)x\| \geq |z|^p \|x\|/2$ for all z such that $|z| \geq K$, $x \in \mathbb{R}^m$. This implies $\|P(z)^{-1}\| \leq 2|z|^{-p} \forall |z| \geq K$ and thus for all $z \in \mathbb{R}$, $|z| \geq K$,

$$\|P(\mathrm{i}z)^{-1}R(\mathrm{i}z)\|^2 \leq \|P(\mathrm{i}z)^{-1}\|^2 \|R(\mathrm{i}z)\|^2 \leq \frac{4}{|z|^{2p}} \left(\sum_{i=0}^{p-1} \|C_i\||z|^i\right)^2,$$

which gives the finiteness of $\int_{-\infty}^{-K} \|P(\mathrm{i}z)^{-1}R(\mathrm{i}z)\|^2 \,\mathrm{d}z$ and $\int_K^\infty \|P(\mathrm{i}z)^{-1}R(\mathrm{i}z)\|^2 \,\mathrm{d}z$. $\square$

The following result provides the key to being able to define multivariate CARMA processes.

Theorem 3.12. *Let $L = \{L(t)\}_{t\in\mathbb{R}}$ be an m-dimensional square integrable Lévy process with zero mean and corresponding m-dimensional random orthogonal measure Φ as in Theorem 3.5 and $p, q \in \mathbb{N}_0$, $q < p$ (i.e., $p \geq 1$). Let further $A_1, A_2, \dots, A_p, B_0, B_1, \dots, B_q \in M_m(\mathbb{R})$, where $B_0 \neq 0$ and define $\beta_1 = \beta_2 = \cdots = \beta_{p-q-1} = 0$ (if $p > q+1$) and $\beta_{p-j} = -\sum_{i=1}^{p-j-1} A_i \beta_{p-j-i} + B_{q-j}$ for $j = 0, 1, 2, \dots, q$. (Alternatively, $\beta_{p-j} = -\sum_{i=1}^{p-j-1} A_i \beta_{p-j-i} + B_{q-j}$ for $j = 0, 1, \dots, p-1$, setting $B_i = 0$ for $i < 0$.) Assume that A as defined in (3.22) satisfies $\sigma(A) \subseteq (-\infty, 0) + \mathrm{i}\mathbb{R}$, which implies $A_p \in \mathcal{G}l_m(\mathbb{R})$.*

Denote by $G = (G_1^*(t), \dots, G_p^*(t))^*$ an mp-dimensional process and set $\beta^* = \left(\beta_1^*, \dots, \beta_p^*\right)$. Then the stochastic differential equation

$$\mathrm{d}G(t) = AG(t)\mathrm{d}t + \beta \mathrm{d}L(t) \tag{3.24}$$

is uniquely solved by the process G given by

$$\begin{aligned} G_j(t) &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda t} w_j(\mathrm{i}\lambda)\, \Phi(\mathrm{d}\lambda), \quad j = 1, 2, \dots, p, \quad t \in \mathbb{R}, \quad \textit{where} \\ w_j(z) &= \frac{1}{z}(w_{j+1}(z) + \beta_j), \quad j = 1, 2, \dots, p-1 \quad \textit{and} \\ w_p(z) &= \frac{1}{z}\left(-\sum_{k=0}^{p-1} A_{p-k} w_{k+1}(z) + \beta_p\right). \end{aligned} \tag{3.25}$$

The strictly stationary process G can also be represented as

$$G(t) = \int_{-\infty}^{t} \mathrm{e}^{A(t-s)} \beta \, L(\mathrm{d}s), \quad t \in \mathbb{R}. \tag{3.26}$$

Moreover, $G(0)$ and $\{L(t)\}_{t\geq 0}$ are independent, in particular,

$$E[G_j(0)L(t)^*] = 0 \quad \textit{for all } t \geq 0,\ j = 1, 2, \dots, p.$$

Finally, it holds that

$$w_p(z) = P(z)\left(\beta_p z^{p-1} - \sum_{j=0}^{p-2} \sum_{k=0}^{j} A_{p-k} \beta_{p+k-j-1} z^j\right), \tag{3.27}$$

$$w_1(z) = (P(z))^{-1} Q(z), \tag{3.28}$$

where

$$P(z) = I_m z^p + A_1 z^{p-1} + \cdots + A_p \text{ (“\textit{autoregressive polynomial}”)},$$
$$Q(z) = B_0 z^q + B_1 z^{q-1} + \cdots + B_q \text{ (“\textit{moving average polynomial}”)}$$

and $\int_{-\infty}^{\infty} \|w_j(\mathrm{i}\lambda)\|^2 \,\mathrm{d}\lambda < \infty$ *for all* $j \in \{1, 2, \ldots, p\}$.

Proof. $A_p \in \mathcal{G}l_m(\mathbb{R})$ follows from Lemma 3.8. That (3.26) is the strictly stationary solution of (3.24) is a standard result, since all elements of $\sigma(A)$ have strictly negative real part, and a simple application of Gronwall's Lemma shows that the solution of (3.24) is a.s. unique for all $t \in \mathbb{R}$ (see e.g. [26], Theorem 3.1). Since $G(0) = \int_{-\infty}^{0} \mathrm{e}^{-As}\beta\, L(\mathrm{d}s)$ and the processes $\{L(t)\}_{t<0}$ and $\{L(t)\}_{t\geq 0}$ are independent according to our definition (2.3) of L, $G(0)$ and $\{L(t)\}_{t\geq 0}$ are independent.

To prove (3.27) and (3.28) we first show

$$w_j(z) = \frac{1}{z^{p-j}}\left(w_p(z) + \sum_{i=1}^{p-j} \beta_{p-i} z^{i-1}\right) \quad \text{for } j = 1, \ldots, p-1. \tag{3.29}$$

In fact, for $p - j = 1$ (3.29) becomes $w_{p-1} = \frac{1}{z}(w_p(z) + \beta_{p-1})$ which proves the identity for $j = p - 1$ immediately. Assume the identity holds for $j + 1 \in \{2, 3, \ldots, p - 1\}$, then

$$\begin{aligned} w_j(z) &= \frac{1}{z}(w_{j+1}(z) + \beta_j) = \frac{1}{z}\left[\frac{1}{z^{p-j-1}}\left(w_p(z) + \sum_{i=1}^{p-j-1} \beta_{p-i} z^{i-1}\right) + \beta_j\right] \\ &= \frac{1}{z^{p-j}}\left(w_p(z) + \sum_{i=1}^{p-j} \beta_{p-i} z^{i-1}\right), \end{aligned}$$

which proves (3.29). Now we turn to (3.27):

$$\begin{aligned} w_p(z) &= \frac{1}{z}\left(-\sum_{k=0}^{p-1} A_{p-k} w_{k+1}(z) + \beta_p\right) \\ &\overset{(3.29)}{=} \frac{1}{z}\left[-\sum_{k=0}^{p-1} A_{p-k}\left(\frac{1}{z^{p-k-1}}\left(w_p(z) + \sum_{i=1}^{p-k-1} \beta_{p-i} z^{i-1}\right)\right)\right] + \frac{\beta_p}{z}. \end{aligned}$$

It follows that

$$\left(I_m z^p + \sum_{k=0}^{p-1} A_{p-k} z^k\right) w_p(z) = \beta_p z^{p-1} - \sum_{k=0}^{p-1}\sum_{i=1}^{p-k-1} A_{p-k}\beta_{p-i} z^{k+i-1}.$$

Set $j = k + i - 1$; then

$$\begin{aligned} w_p(z) &= (P(z))^{-1}\left(\beta_p z^{p-1} - \sum_{k=0}^{p-2}\sum_{j=k}^{p-2} A_{p-k}\beta_{p+k-j-1} z^j\right) \\ &= (P(z))^{-1}\left(\beta_p z^{p-1} - \sum_{j=0}^{p-2}\sum_{k=0}^{j} A_{p-k}\beta_{p+k-j-1} z^j\right), \end{aligned}$$

which proves (3.27).

Let now $l \in \{1, 2, \ldots, p-1\}$. Then setting $A_0 = I_m$,

$$\begin{aligned}
w_l(z) &= \frac{1}{z^{p-l}} \left(w_p(z) + \sum_{i=1}^{p-l} \beta_{p-i} z^{i-1} \right) \\
&\overset{(3.27)}{=} \frac{1}{z^{p-l}} \left[(P(z))^{-1} \left(\beta_p z^{p-1} - \sum_{j=0}^{p-2} \sum_{k=0}^{j} A_{p-k} \beta_{p+k-j-1} z^j \right) + \sum_{i=1}^{p-l} \beta_{p-i} z^{i-1} \right] \\
&= \frac{(P(z))^{-1}}{z^{p-l}} \left[\beta_p z^{p-1} - \sum_{j=0}^{p-2} \sum_{k=0}^{j} A_{p-k} \beta_{p+k-j-1} z^j + \left(\sum_{i=1}^{p-l} \beta_{p-i} z^{i-1} \right) \right] \\
&= \frac{(P(z))^{-1}}{z^{p-l}} \left[\beta_p z^{p-1} - \sum_{j=0}^{p-2} \sum_{k=0}^{j} A_{p-k} \beta_{p+k-j-1} z^j + \sum_{k=0}^{p} \sum_{i=0}^{p-l-1} A_{p-k} \beta_{p-i-1} z^{i+k} \right].
\end{aligned}$$

Setting $j = k + l$ we obtain

$$\begin{aligned}
w_l(z) &= \frac{(P(z))^{-1}}{z^{p-l}} \left[\beta_p z^{p-1} - \sum_{j=0}^{p-2} \sum_{k=0}^{j} A_{p-k} \beta_{p+k-j-1} z^j \right. \\
&\quad \left. + \sum_{k=0}^{p} \sum_{j=k}^{k+p-l-1} A_{p-k} \beta_{p+k-j-1} z^j \right] \\
&= \frac{(P(z))^{-1}}{z^{p-l}} \left[- \sum_{j=0}^{p-2} \sum_{k=0}^{j} A_{p-k} \beta_{p+k-j-1} z^j + \sum_{k=0}^{p-l-1} \sum_{j=k}^{p-l-1} A_{p-k} \beta_{p+k-j-1} z^j \right. \\
&\quad + \beta_p z^{p-1} + \sum_{k=p-l}^{p} \sum_{j=k}^{k+p-l-1} A_{p-k} \beta_{p+k-j-1} z^j \\
&\quad \left. + \sum_{k=1}^{p-l-1} \sum_{j=p-l}^{k+p-l-1} A_{p-k} \beta_{p+k-j-1} z^j \right].
\end{aligned}$$

It follows that

$$\begin{aligned}
w_l(z) &= (P(z))^{-1} \left[\beta_p z^{l-1} - \sum_{j=p-l}^{p-2} \sum_{k=0}^{j} A_{p-k} \beta_{p+k-j-1} z^{j-p+l} \right. \\
&\quad + \sum_{k=p-l}^{p} \sum_{j=k}^{k+p-l-1} A_{p-k} \beta_{p+k-j-1} z^{j-p+l} \\
&\quad \left. + \sum_{k=1}^{p-l-1} \sum_{j=p-l}^{k+p-l-1} A_{p-k} \beta_{p+k-j-1} z^{j-p+l} \right].
\end{aligned}$$

The last term in the bracket appears only if $p - l - 1 \geq 1$. Therefore, the whole term in the bracket is a polynomial of at most order $p - 1$. Fixing $l = 1$ and setting $i = j - p + 1$ we obtain

$$w_1(z) = P(z)^{-1} \left[\beta_p + \sum_{k=p-1}^{p} \sum_{i=k-p+1}^{k-1} A_{p-k} \beta_{k-i} z^i + \sum_{k=1}^{p-2} \sum_{i=0}^{k-1} A_{p-k} \beta_{k-i} z^i \right]$$

$$= P(z)^{-1}\left[\beta_p + \sum_{k=1}^{p-1}\sum_{i=0}^{k-1} A_{p-k}\beta_{k-i}z^i + A_0\sum_{i=1}^{p-1}\beta_{p-i}z^i\right]$$

$$= P(z)^{-1}\left[\sum_{i=0}^{p-1}\beta_{p-i}z^i + \sum_{i=0}^{p-2}\sum_{k=i+1}^{p-1} A_{p-k}\beta_{k-i}z^i\right].$$

Using the fact that $\beta_1 = B_{q-p+1}$ and setting $j = p - k$, we finally get

$$w_1(z) = (P(z))^{-1}\left[B_{q-p+1}z^{p-1} + \sum_{i=0}^{p-2}\left(\beta_{p-i} + \sum_{j=1}^{p-i-1} A_j\beta_{p-j-i}\right)z^i\right]$$

$$= P(z)^{-1}\left[B_{q-p+1}z^{p-1} + \sum_{i=0}^{p-2} B_{q-i}z^i\right] = P(z)^{-1}\sum_{i=0}^{p-1} B_{q-i}z^i$$

$$= P(z)^{-1}\sum_{i=0}^{q} B_{q-i}z^i = P(z)^{-1}Q(z).$$

The finiteness of $\int_{-\infty}^{\infty}\|w_j(\mathrm{i}\lambda)\|^2\,\mathrm{d}\lambda$ for all $j = 1, 2, \ldots, p$ is now a direct consequence of Lemmata 3.10 and 3.11 and Corollary 3.9.

It remains to show that the process defined in (3.25) solves (3.24): For $j = 1, \ldots, p$ we have as a consequence of (3.25),

$$G_j(t) - G_j(0) = \int_{-\infty}^{\infty}(\mathrm{e}^{\mathrm{i}\lambda t} - 1)w_j(\mathrm{i}\lambda)\,\Phi(\mathrm{d}\lambda). \tag{3.30}$$

For $j = 1, \ldots, p-1$ the recursion for w_j together with Lemma 3.7 gives

$$G_j(t) - G_j(0) = \int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\lambda t} - 1}{\mathrm{i}\lambda}w_{j+1}(\mathrm{i}\lambda)\,\Phi(\mathrm{d}\lambda) + \beta_j\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\lambda t} - 1}{\mathrm{i}\lambda}\,\Phi(\mathrm{d}\lambda)$$

$$= \int_0^t\int_{-\infty}^{\infty} w_{j+1}(\mathrm{i}\lambda)\mathrm{e}^{\mathrm{i}\lambda s}\,\Phi(\mathrm{d}\lambda)\,\mathrm{d}s + \beta_j L(t)$$

$$= \int_0^t G_{j+1}(s)\,\mathrm{d}s + \beta_j L(t).$$

Hence,

$$\mathrm{d}G_j(t) = G_{j+1}(t)\mathrm{d}t + \beta_j\mathrm{d}L(t). \tag{3.31}$$

Analogously we obtain for G_p,

$$G_p(t) - G_p(0) = \int_{-\infty}^{\infty}(\mathrm{e}^{\mathrm{i}\lambda t} - 1)w_p(\mathrm{i}\lambda)\,\Phi(\mathrm{d}\lambda)$$

$$= -\sum_{k=0}^{p-1}\int_0^t\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\lambda s}A_{p-k}w_{k+1}(\mathrm{i}\lambda)\,\Phi(\mathrm{d}\lambda)\,\mathrm{d}s + \beta_p L(t)$$

$$= -\left(\int_0^t A_pG_1(s) + \cdots + A_1G_p(s)\,\mathrm{d}s\right) + \beta_p L(t).$$

Therefore,

$$\mathrm{d}G_p(t) = -(A_pG_1(t) + \cdots + A_1G_p(t))\mathrm{d}t + \beta_p\,\mathrm{d}L(t).$$

Together with (3.31) this gives that the process G defined by (3.25) solves (3.24). $\square$

Obviously, $E[G(t)] = 0$ for the process $G = \{G(t)\}_{t\in\mathbb{R}}$ which solves (3.24). Noting that G is a multivariate Ornstein–Uhlenbeck process, the second-order structure follows immediately.

Proposition 3.13. *Let $G = \{G(t)\}_{t\in\mathbb{R}}$ be the process that solves (3.24). Then its autocovariance matrix function has the form*

$$\Gamma(h) = E[G(t+h)G(t)^*] = \mathrm{e}^{Ah}\Gamma(0), \quad h \geq 0, \tag{3.32}$$

with $\Gamma(0) = \int_0^\infty \mathrm{e}^{Au}\beta\Sigma_L\beta^*\mathrm{e}^{A^*u}\,\mathrm{d}u$ satisfying $A\Gamma(0) + \Gamma(0)A^* = -\beta\Sigma_L\beta^*$.

Proof. (3.32) follows from (3.11) and the last identity is a standard result from matrix theory (see e.g. [27, Theorem VII.2.3]). □

From [28,17–19] we know that (3.26) is the unique stationary solution to (3.24) whenever the Lévy measure ν of the driving process $L(t)$ satisfies $\int_{\|x\|\geq 1}\log\|x\|\nu(\mathrm{d}x) < \infty$. This condition is sufficient (and necessary, provided β is injective) for the stochastic integral in (3.26) to exist, as can be seen from substituting $f(t,s) = \mathrm{e}^{A(t-s)}\beta I_{[0,\infty)}(t-s)$ in (2.12) and (2.13). As we shall use this fact later on to define CARMA processes driven by Lévy processes with infinite second moment, we state the following two results on the process G in a general manner.

Proposition 3.14. *For any driving Lévy process $L(t)$, the process $G = \{G(t)\}_{t\in\mathbb{R}}$ solving (3.24) in Theorem 3.12 is a temporally homogeneous strong Markov process with an infinitely divisible transition probability $P_t(x, \mathrm{d}y)$ having characteristic function*

$$\int_{\mathbb{R}^{mp}} \mathrm{e}^{\mathrm{i}\langle u,y\rangle} P_t(x,\mathrm{d}y) = \exp\left\{\mathrm{i}\langle x, \mathrm{e}^{A^*t}u\rangle + \int_0^t \psi_L((\mathrm{e}^{Av}\beta)^*u)\,\mathrm{d}v\right\}, \quad u \in \mathbb{R}^{mp}. \tag{3.33}$$

Proof. See [18, Th. 3.1] and additionally [10, Theorem V.32] for the strong Markov property. □

Proposition 3.15. *Consider the unique solution $G = \{G(t)\}_{t\geq 0}$ of (3.24) with initial value $G(0)$ independent of $L = \{L(t)\}_{t\geq 0}$, where L is a Lévy process on $\mathbb{R}^m$ satisfying $\int_{\|x\|\geq 1}\log\|x\|\nu(\mathrm{d}x) < \infty$.*

Let $\mathcal{L}(G(t))$ denote the marginal distribution of the process $G = \{G(t)\}_{t\geq 0}$ at time t. Then there exists a limit distribution F such that $\mathcal{L}(G(t)) \to F$ as $t \to \infty$. This F is infinitely divisible with characteristic function

$$E\left[\mathrm{e}^{\mathrm{i}\langle u,F\rangle}\right] = \exp\left\{\int_0^\infty \psi_L((\mathrm{e}^{As}\beta)^*u)\,\mathrm{d}s\right\}, \quad u \in \mathbb{R}^{mp}. \tag{3.34}$$

Proof. See [18, Theorem 4.1]. □

Remark 3.16. Obviously F is also the marginal distribution of the stationary solution considered in Theorem 3.12.

The sample path behaviour of the process $G = \{G(t)\}_{t\in\mathbb{R}}$ is described below.

Proposition 3.17. *If the driving Lévy process $L = \{L(t)\}_{t\in\mathbb{R}}$ of the process $G = \{G(t)\}_{t\in\mathbb{R}}$ in Theorem 3.12 is Brownian motion, the sample paths of G are continuous. Otherwise the process G has a jump, whenever L has one. In particular, $\Delta G(t) = \beta\Delta L(t)$.*

3.3. Multivariate CARMA processes

We are now in a position to define an m-dimensional CARMA (MCARMA) process by using the spectral representation for square integrable driving Lévy processes and extend this definition making use of the insight obtained in Theorem 3.12.

Definition 3.18 (*MCARMA Process*). Let $L = \{L(t)\}_{t\in\mathbb{R}}$ be a two-sided square integrable m-dimensional Lévy process with $E[L(1)] = 0$ and $E[L(1)L(1)^*] = \Sigma_L$. An m-dimensional Lévy-driven continuous time autoregressive moving average process $\{Y(t)\}_{t\in\mathbb{R}}$ of order (p, q), $p > q$ (MCARMA(p, q) process) is defined as

$$\begin{aligned} Y(t) &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda t} P(\mathrm{i}\lambda)^{-1} Q(\mathrm{i}\lambda)\, \Phi(\mathrm{d}\lambda), \quad t \in \mathbb{R}, \quad \text{where} \\ P(z) &:= I_m z^p + A_1 z^{p-1} + \cdots + A_p, \\ Q(z) &:= B_0 z^q + B_1 z^{q-1} + \cdots + B_q \quad \text{and} \end{aligned} \tag{3.35}$$

Φ is the Lévy orthogonal random measure of Theorem 3.5 satisfying $E[\Phi(\mathrm{d}\lambda)] = 0$ and $E[\Phi(\mathrm{d}\lambda)\Phi(\mathrm{d}\lambda)^*] = \frac{\mathrm{d}\lambda}{2\pi}\Sigma_L$. Here $A_j \in M_m(\mathbb{R})$, $j = 1, \ldots, p$, and $B_j \in M_m(\mathbb{R})$ are matrices satisfying $B_q \neq 0$ and $\mathcal{N}(P) := \{z \in \mathbb{C} : \det(P(z)) = 0\} \subset \mathbb{R}\setminus\{0\} + \mathrm{i}\mathbb{R}$.

The process G defined as in Theorem 3.12 is called the state space representation of the MCARMA process Y.

Remark 3.19. (a) There are several reasons why the name "multivariate continuous time ARMA process" is indeed appropriate. The same arguments as in Remark 3.6 show that an MCARMA process Y can be interpreted as a solution to the p-th-order m-dimensional differential equation

$$P(D)Y(t) = Q(D)DL(t),$$

where D denotes the differentiation operator. Moreover, the upcoming Theorem 3.22 shows that for $m = 1$ the well-known univariate CARMA processes are obtained and, finally, the spectral representation (3.35) is the obvious continuous time analogue of the spectral representation of multivariate discrete time ARMA processes (see, for instance, [1, Section 11.8]).

(b) The well-definedness is ensured by Lemma 3.11. Observe also that, if $\det(P(z))$ has zeros with positive real part, all assertions of Theorem 3.12 except the alternative representation (3.26) and the independence of $G(0)$ and $\{L(t)\}_{t\geq 0}$ remain valid interpreting the stochastic differential equation as an integral equation as in the proof of the theorem. However, in this case the process is no longer causal, i.e. adapted to the natural filtration of the driving Lévy process.

(c) Assuming $E[L(1)] = 0$ is actually no restriction. If $E[L(1)] = \mu_L \neq 0$, one simply observes that $\tilde{L}(t) = L(t) - \mu_L t$ has zero expectation and $P(D)^{-1}Q(D)DL(t) = P(D)^{-1}Q(D)D\tilde{L}(t) + P(D)^{-1}Q(D)\mu_L$. The first term simply is the MCARMA process driven by $\tilde{L}(t)$ and the second an ordinary differential equation having the unique "stationary" solution $-A_p^{-1}B_q\mu_L$, as simple calculations show. Thus, the definition can be immediately extended to $E[L(1)] \neq 0$. Moreover, it is easy to see that the SDE representation given in Theorem 3.12 still holds and one can also extend the spectral representation by adding an atom with mass μ_L to $\Phi_{\tilde{L}}$ at 0.

(d) Furthermore, observe that the representation of MCARMA processes by the stochastic differential equation (3.24) is a continuous time version of state space representations for (multivariate) ARMA processes as given in [1, Example 12.1.5] or [29, p. 387]. For the univariate Gaussian case it can already be found in [23, Lemma 3, Chapter 2.2].

As already noted before, we extend the definition of MCARMA processes to driving Lévy processes L with finite logarithmic moment using Theorem 3.12. As they agree with the above defined MCARMA processes, when L is square integrable, and are always causal, we call them *causal MCARMA processes*.

Definition 3.20 (*Causal MCARMA Process*). Let $L = \{L(t)\}_{t\in\mathbb{R}}$ be an m-dimensional Lévy process satisfying

$$\int_{\|x\|\geq 1} \log \|x\| \, \nu(\mathrm{d}x) < \infty, \tag{3.36}$$

$p, q \in \mathbb{N}_0$ with $q < p$, and further $A_1, A_2, \ldots, A_p, B_0, B_1, \ldots, B_q \in M_m(\mathbb{R})$, where $B_0 \neq 0$. Define the matrices A, β and the polynomial P as in Theorem 3.12 and assume $\sigma(A) = \mathcal{N}(P) \subseteq (-\infty, 0) + \mathrm{i}\mathbb{R}$. Then the m-dimensional process

$$Y(t) = \left(I_m, 0_{M_m(\mathbb{C})}, \ldots, 0_{M_m(\mathbb{C})}\right) G(t) \tag{3.37}$$

where G is the unique stationary solution to $\mathrm{d}G(t) = AG(t)\mathrm{d}t + \beta \mathrm{d}L(t)$ is called a causal MCARMA(p, q) process. Again G is referred to as the state space representation.

Remark 3.21. In the following we will write "MCARMA" when referring to Definition 3.18, "causal MCARMA" when referring to Definition 3.20 and "(causal) MCARMA" when referring to both Definitions 3.18 and 3.20.

Let us now state a result extending the short memory moving average representation of univariate CARMA processes to our MCARMA processes and showing that our definition is in line with univariate CARMA processes.

Theorem 3.22. *Analogously to a one-dimensional CARMA process (see (3.7)), the MCARMA process (3.35) can be represented as a moving average process*

$$Y(t) = \int_{-\infty}^{\infty} g(t-s)\, L(\mathrm{d}s), \quad t \in \mathbb{R}, \tag{3.38}$$

where the kernel matrix function $g : \mathbb{R} \to M_m(\mathbb{R})$ is given by

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\mu t} P(\mathrm{i}\mu)^{-1} Q(\mathrm{i}\mu)\, \mathrm{d}\mu. \tag{3.39}$$

Proof. Using the notation of the proof of Theorem 3.5 we obtain this immediately from (3.19):

$$\begin{aligned} Y(t) &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\mu t} P(\mathrm{i}\mu)^{-1} Q(\mathrm{i}\mu)\, \Phi(\mathrm{d}\mu) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\mu(t-s)} P(\mathrm{i}\mu)^{-1} Q(\mathrm{i}\mu)\, \mathrm{d}\mu\, \tilde{\Phi}_L(\mathrm{d}s) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\mu(t-s)} P(\mathrm{i}\mu)^{-1} Q(\mathrm{i}\mu)\, \mathrm{d}\mu\, L(\mathrm{d}s) \end{aligned}$$

$$= \int_{-\infty}^{\infty} g(t-s)\, L(\mathrm{d}s). \quad \square$$

Remark 3.23. For causal MCARMA processes an analogous result holds with the kernel function g replaced by

$$\tilde{g}(s) = (I_m, 0_{M_m(\mathbb{C})}, \ldots, 0_{M_m(\mathbb{C})})\mathrm{e}^{As}\beta I_{[0,\infty)}(s).$$

Moreover, the function g simplifies in the square integrable causal case as the following extension of a well-known result for univariate CARMA processes shows.

Lemma 3.24. *Assume that $\sigma(A) = \mathcal{N}(P) \subseteq (-\infty, 0) + \mathrm{i}\mathbb{R}$. Then the function g given in (3.39) vanishes on the negative real line.*

Proof. We need the following consequence of the residue theorem from complex analysis (cf., for instance, [30, Section VI.2, Theorem 2.2]):

Let q and $p : \mathbb{C} \mapsto \mathbb{C}$ be polynomials where p is of higher degree than q. Assume that p has no zeros on the real line. Then

$$\int_{-\infty}^{\infty} \frac{q(t)}{p(t)} \exp(\mathrm{i}\alpha t)\mathrm{d}t = 2\pi\mathrm{i} \sum_{z\in\mathbb{C}:\Im(z)>0,\, p(\mathrm{i}z)=0} \mathrm{Res}\,(f, z) \qquad \text{for all } \alpha > 0 \quad \text{and} \tag{3.40}$$

$$\int_{-\infty}^{\infty} \frac{q(t)}{p(t)} \exp(\mathrm{i}\alpha t)\mathrm{d}t = -2\pi\mathrm{i} \sum_{z\in\mathbb{C}:\Im(z)<0,\, p(\mathrm{i}z)=0} \mathrm{Res}\,(f, z) \qquad \text{for all } \alpha < 0 \tag{3.41}$$

with $f : \mathbb{C} \mapsto \mathbb{C}, z \mapsto \frac{q(z)}{p(z)} \exp(\mathrm{i}\alpha z)$ and $\mathrm{Res}(f, a)$ denoting the residual of the function f at point a.

Turning to our function g, we have from elementary matrix theory that

$$P(\mathrm{i}z)^{-1}Q(\mathrm{i}z) = \frac{S(z)}{\det(P(\mathrm{i}z))}$$

where $S : \mathbb{C} \mapsto M_m(\mathbb{C})$ is some matrix-valued polynomial in z. Observe that $\det(P(\mathrm{i}z))$ is a complex-valued polynomial in z and that Lemma 3.11 applied to $R = Q$ implies that $\det(P(\mathrm{i}z))$ is of higher degree than $S(z)$. Thus, we can apply the above stated results from complex function theory componentwise to (3.39). But as all zeros of $\det(P(z))$ are in the left half-plane $(-\infty, 0) + \mathrm{i}\mathbb{R}$, all zeros of $\det(P(\mathrm{i}z))$ are in the upper half-plane $\mathbb{R} + \mathrm{i}(0, \infty)$ and therefore (3.41) shows that

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\mu t} P(\mathrm{i}\mu)^{-1} Q(\mathrm{i}\mu)\, \mathrm{d}\mu = 0 \quad \text{for all } t < 0. \quad \square$$

Remark 3.25. The above result again reflects the causality, i.e., that the MCARMA process $Y(t)$ only depends on the past of the driving Lévy process, i.e., on $\{L(s)\}_{s\le t}$. Similarly g vanishes on the positive half-line if $\mathcal{N}(P) \subset (0, \infty) + \mathrm{i}\mathbb{R}$. In this case the MCARMA process $Y(t)$ depends only on the future of the driving Lévy process, i.e., on $\{L(s)\}_{s\ge t}$. In all other non-causal cases the MCARMA process depends on the driving Lévy process at all times.

Using the kernel representations, strict stationarity of MCARMA processes is obtained by applying [9, Theorem 4.3.16].

Proposition 3.26. *The (causal) MCARMA process is strictly stationary.*

Furthermore, we can characterize the stationary distribution by applying representation (3.38) and the results of [15] mentioned at the end of Section 2.2.

Proposition 3.27. *If the driving Lévy process L has characteristic triplet (γ, σ, ν), then the distribution of the MCARMA process $Y(t)$ is infinitely divisible for $t \in \mathbb{R}$ and the characteristic triplet of the stationary distribution is $(\gamma_Y^\infty, \sigma_Y^\infty, \nu_Y^\infty)$, where*

$$\begin{aligned}
\gamma_Y^\infty &= \int_{\mathbb{R}} g(s)\gamma \,\mathrm{d}s + \int_{\mathbb{R}} \int_{\mathbb{R}^m} g(s)x[I_{\{\|g(s)x\|\le 1\}} - I_{\{\|x\|\le 1\}}]\,\nu(\mathrm{d}x)\,\mathrm{d}s,\\
\sigma_Y^\infty &= \int_{\mathbb{R}} g(s)\sigma g^*(s)\mathrm{d}s\\
\nu_Y^\infty(B) &= \int_{\mathbb{R}} \int_{\mathbb{R}^m} I_B(g(s)x)\,\nu(\mathrm{d}x)\,\mathrm{d}s.
\end{aligned} \tag{3.42}$$

For a causal MCARMA process the same result holds with g replaced by $\tilde{g}$.

3.4. Further properties of MCARMA processes

Having defined multivariate CARMA processes above, we analyse their probabilistic behaviour further in this section. First we turn to the second-order properties.

Proposition 3.28. *Let $Y = \{Y(t)\}_{t\in\mathbb{R}}$ be the MCARMA process defined by (3.35). Then its autocovariance matrix function is given by*

$$\Gamma_Y(h) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda h} P(\mathrm{i}\lambda)^{-1} Q(\mathrm{i}\lambda)\Sigma_L Q(\mathrm{i}\lambda)^*(P(\mathrm{i}\lambda)^{-1})^*\,\mathrm{d}\lambda, \quad h \in \mathbb{R}.$$

Proof. It follows directly from the spectral representation (3.35) that the MCARMA process $Y = \{Y(t)\}_{t\in\mathbb{R}}$ has the spectral density

$$f_Y(\lambda) = \frac{1}{2\pi} P(\mathrm{i}\lambda)^{-1} Q(\mathrm{i}\lambda)\Sigma_L Q(\mathrm{i}\lambda)^*(P(\mathrm{i}\lambda)^{-1})^*, \quad \lambda \in \mathbb{R}. \tag{3.43}$$

The autocovariance function is the Fourier transform of (3.43). □

Remark 3.29. Note that in Proposition 3.13 we already obtained an expression for the autocovariance matrix function of the process $\{G(t)\}_{t\in\mathbb{R}}$ of Theorem 3.12. The upper left $m \times m$ block of (3.32) is also equal to Γ_Y.

Regarding the general existence of moments, it is mainly the driving Lévy process that matters.

Proposition 3.30. *Let Y be a causal MCARMA process and assume that the driving Lévy process L is in $L^r(\Omega, P)$ for some $r > 0$. Then Y and its state space representation G are in $L^r(\Omega, P)$. Provided β is injective, the converse is true as well for G.*

Proof. We use the general fact that an infinitely divisible distribution with characteristic triplet (γ, σ, ν) has finite r-th moment if and only if $\int_{\|x\|\ge C} \|x\|^r \nu(\mathrm{d}x) < \infty$ for one and hence all

$C > 0$ (see [11, Corollary 25.8]). Using the kernel representation (3.38) with

$$\tilde{g}(s) = (I_m, 0_{M_m(\mathbb{C})}, \ldots, 0_{M_m(\mathbb{C})})\mathrm{e}^{As}\beta I_{[0,\infty)}(s),$$

(3.42) and the fact that there are $C, c > 0$ such that $\|(I_m, 0_{M_m(\mathbb{C})}, \ldots, 0_{M_m(\mathbb{C})})\mathrm{e}^{As}\beta\| \leq C\mathrm{e}^{-cs}$ we obtain for the stationary distribution of Y

$$\begin{aligned}\int_{\|x\|\geq 1} \|x\|^r \nu_Y^\infty(\mathrm{d}x) &= \int_0^\infty \int_{\mathbb{R}^m} I_{[1,\infty)}\left(\left\|(I_m, 0_{M_m(\mathbb{C})}, \ldots, 0_{M_m(\mathbb{C})})\mathrm{e}^{As}\beta x\right\|\right) \\ &\quad \times \left\|(I_m, 0_{M_m(\mathbb{C})}, \ldots, 0_{M_m(\mathbb{C})})\mathrm{e}^{As}\beta x\right\|^r \nu(\mathrm{d}x)\mathrm{d}s \\ &\leq \int_0^\infty \int_{\mathbb{R}^m} I_{[1,\infty)}\left(C\mathrm{e}^{-cs}\|x\|\right) C^r \mathrm{e}^{-rcs}\|x\|^r \nu(\mathrm{d}x)\mathrm{d}s \\ &= \int_{\|x\|\geq 1/C} \int_0^{\frac{\log(1/(C\|x\|))}{-c}} C^r \mathrm{e}^{-rcs}\|x\|^r \mathrm{d}s\nu(\mathrm{d}x) \\ &= \frac{C^r}{rc}\int_{\|x\|\geq 1/C}\left(\|x\|^r - 1/C^r\right)\nu(\mathrm{d}x),\end{aligned}$$

which is finite if and only if L has a finite r-th moment.

Basically the same arguments apply to $G(t) = \int_{-\infty}^t \mathrm{e}^{A(t-s)}\beta L(\mathrm{d}s)$. Provided β is injective, there are $D, d > 0$ such that $\|\mathrm{e}^{As}\beta\| \geq D\mathrm{e}^{-\mathrm{d}s}$ and calculations analogous to the above one lead to a lower bound which establishes the necessity of $L \in L^r$ for $G \in L^r$. □

Since the characteristic function of $Y(t)$ for each t is explicitly given, we can investigate the existence of a C_b^∞ density, where C_b^∞ denotes the space of bounded continuous, infinitely often differentiable functions whose derivatives are bounded.

Proposition 3.31. *Suppose that there exists an $\alpha \in (0, 2)$ and a constant $C > 0$ such that*

$$\int_{\mathbb{R}}\int_{\mathbb{R}^m} |\langle u, g(t-s)x\rangle|^2 1_{\{|\langle u, g(t-s)x\rangle|\leq 1\}}\nu(\mathrm{d}x)\,\mathrm{d}s \geq C\|u\|^{2-\alpha} \tag{3.44}$$

for any vector u such that $\|u\| \geq 1$. Then the MCARMA process $Y(t)$ has a C_b^∞ density.

The same holds for a causal MCARMA $Y(t)$ process with g replaced by $\tilde{g}$.

Proof. It is sufficient to show that $\int \|u\|^k \|\Phi(u)\|\,\mathrm{d}u < \infty$ for any non-negative integer k, where Φ denotes the characteristic function of $Y(t)$ (see e.g. [31, Proposition 0.2]).

The characteristic function of the (causal) MCARMA process $Y(t)$ is given by

$$\Phi(u) = \exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}^m}\left[\mathrm{e}^{\mathrm{i}\langle u, g(t-s)x\rangle} - 1 - \mathrm{i}\langle u, g(t-s)x\rangle I_{\{|\langle u, g(t-s)x\rangle|\leq 1\}}\right]\nu(\mathrm{d}x)\,\mathrm{d}s\right\},$$

where g stands for either g or $\tilde{g}$. Thus,

$$\begin{aligned}\|\Phi(u)\| &= \left(\exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}^m}\left[\mathrm{e}^{\mathrm{i}\langle u, g(t-s)x\rangle} + \mathrm{e}^{-\mathrm{i}\langle u, g(t-s)x\rangle} - 2\right]\nu(\mathrm{d}x)\,\mathrm{d}s\right\}\right)^{1/2} \\ &= \exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}^m}(\cos\langle u, g(t-s)x\rangle - 1)\;\nu(\mathrm{d}x)\,\mathrm{d}s\right\} \\ &\leq \exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}^m}(\cos\langle u, g(t-s)x\rangle - 1)\,I_{\{|\langle u, g(t-s)x\rangle|\leq 1\}}\,\nu(\mathrm{d}x)\,\mathrm{d}s\right\},\end{aligned}$$

as $\cos\langle u, g(t-s)x\rangle - 1 \leq 0$. Then, using the inequality $1-\cos(z) \geq 2(z/\pi)^2$ for $|z| \leq \pi$ and assumption (3.44) we have

$$\begin{aligned}\|\Phi(u)\| &\leq \exp\left\{-\tilde{C}\int_{\mathbb{R}}\int_{\mathbb{R}^m} |\langle u, g(t-s)x\rangle|^2 I_{\{|\langle u,g(t-s)x\rangle|\leq 1\}}\,\nu(\mathrm{d}x)\,\mathrm{d}s\right\}\\ &\leq \exp\{-C\|u\|^{2-\alpha}\},\end{aligned}$$

where $C, \tilde{C} > 0$ are generic constants and the proof is complete. The inequality $1-\cos(z) \geq 2(z/\pi)^2$ for $|z| \leq \pi$ can be easily shown: Define $f(z) = 1-\cos(z) - 2(z/\pi)^2$. Then $f(0) = f(\pi) = 0$ and there is $y \in (0,\pi)$ such that $f'(z) > 0$, $z \in [0, y)$ and $f'(z) < 0$, $z \in (y, \pi]$. Hence, $f(z) > 0$ for all $z \in (0,\pi)$. □

We summarize the sample path behaviour of the MCARMA(p,q) process $Y = \{Y(t)\}_{t\in\mathbb{R}}$, which is immediate from the state space representation (3.24) and the proof of Theorem 3.12.

Proposition 3.32. *If $p > q+1$, then the (causal) MCARMA(p,q) process $Y = \{Y(t)\}_{t\in\mathbb{R}}$ is $(p-q-1)$-times differentiable. Using the state space representation $G = \{G(t)\}_{t\in\mathbb{R}}$ we have $\frac{\mathrm{d}^i}{\mathrm{d}t^i}Y(t) = G_{i+1}(t)$ for $i = 1, 2, \ldots, p-q-1$.*

If $p = q+1$, then $\Delta Y(t) = \beta_1 \Delta L(t)$, i.e., Y has a jump, whenever L has one.

If the driving Lévy process $L = \{L(t)\}_{t\in\mathbb{R}}$ of the MCARMA(p,q) process is Brownian motion, the sample paths of Y are continuous and $(p-q-1)$-times continuously differentiable, provided $p > q+1$.

Ergodicity and mixing properties (see, for instance, [32] for a comprehensive treatment) have far reaching implications. We thus conclude the analysis of MCARMA processes with a result on their mixing behaviour. Recall the following notions:

Definition 3.33 (*Cf. [33]*). A continuous time stationary stochastic process $X = \{X_t\}_{t\in\mathbb{R}}$ is called strongly (or α-) mixing if

$$\alpha_l := \sup\left\{|P(A\cap B) - P(A)P(B)| : A \in \mathcal{F}_{-\infty}^0, B \in \mathcal{F}_l^\infty\right\} \to 0$$

as $l \to \infty$, where $\mathcal{F}_{-\infty}^0 := \sigma\left(\{X_t\}_{t\leq 0}\right)$ and $\mathcal{F}_l^\infty = \sigma\left(\{X_t\}_{t\geq l}\right)$.

It is said to be β-mixing (or completely regular) if

$$\beta_l := E\left(\sup\left\{\left|P(B|\mathcal{F}_{-\infty}^0) - P(B)\right| : B \in \mathcal{F}_l^\infty\right\}\right) \to 0$$

as $l \to \infty$.

Note that $\alpha_l \leq \beta_l$ and thus any β-mixing process is strongly mixing.

Proposition 3.34. *Let Y be a causal MCARMA process and G be its state space representation. If the driving Lévy process L satisfies*

$$\int_{\|x\|\geq 1} \|x\|^r \nu(\mathrm{d}x) < \infty \tag{3.45}$$

for some $r > 0$, then G is β-mixing with mixing coefficients $\beta_l = O(\mathrm{e}^{-al})$ for some $a > 0$ and Y is strongly mixing. In particular, both G and Y are ergodic.

Proof. As $G(t) = \int_{-\infty}^{t} e^{A(t-s)}\beta L(ds)$ is a multidimensional Ornstein–Uhlenbeck process driven by the Lévy process βL, we may apply [34, Theorem 4.3] noting that (3.45) together with Proposition 3.30 ensure that all conditions are satisfied. Hence, the β-mixing of G with exponentially decaying coefficients is shown. But this implies that $G = (G_1^*, G_2^*, \dots, G_p^*)^*$ is also strongly mixing, which in turn shows the strong mixing property for Y, since Y is equal to G_1 and it is obvious from the definition of strong mixing that strong mixing of a multidimensional process implies strong mixing of its components. Note that we also obtain $\alpha_l \leq \beta_l$ for the mixing coefficients α_l of Y. Using the well-known result that mixing implies ergodicity concludes the proof. □

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