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Martingale solutions and Markov selections for stochastic partial differential equations

Benjamin Goldys [a], Michael Röckner [b,c,*], Xicheng Zhang [a,b,d]

[a] *School of Mathematics and Statistics, The University of New South Wales, Sydney, 2052, Australia*
[b] *Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany*
[c] *Departments of Mathematics and Statistics, Purdue University, W. Laffayette, IN, 47907, USA*
[d] *Department of Mathematics, Huazhong University of Science and Technology, Wuhan, Hubei 430074, PR China*

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Abstract

We present a general framework for solving stochastic porous medium equations and stochastic Navier–Stokes equations in the sense of martingale solutions. Following Krylov [N.V. Krylov, The selection of a Markov process from a Markov system of processes, and the construction of quasidiffusion processes, Izv. Akad. Nauk SSSR Ser. Mat. 37 (1973) 691–708] and Flandoli–Romito [F. Flandoli, N. Romito, Markov selections for the 3D stochastic Navier–Stokes equations, Probab. Theory Related Fields 140 (2008) 407–458], we also study the existence of Markov selections for stochastic evolution equations in the absence of uniqueness.

Keywords: Markov selection; Martingale solution; Stochastic porous medium equation; Stochastic Navier–Stokes equation

1. Introduction

The purpose of this paper is twofold. First, we prove a general existence result of solutions for a large class of stochastic partial differential equations (SPDE) of evolutionary type in the

* Corresponding author at: Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany.
E-mail addresses: B.Goldys@unsw.edu.au (B. Goldys), roeckner@math.uni-bielefeld.de (M. Röckner), XichengZhang@gmail.com (X. Zhang).

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sense of Stroock and Varadhan's martingale problem (see [17]). Second, because of the lack of uniqueness in general, we construct almost sure (with respect to the time parameter) Markov selections.

Recently, there has been a lot of interest in stochastic porous medium equations. Strong solutions have been constructed for various classes of such equations e.g. in [9,6,15,3,4,18, 16] (see also [14]). Weak solutions, unique in an L^2-sense, were constructed in [5,2]. In all these papers, however, weak monotonicity conditions were imposed on the coefficients. One aim of this paper is to modify and extend the classical work by Mikulevicius and Rozovskii [13] on weak or martingale solutions for SPDE in such a way so as to include stochastic porous medium equations without monotonicity conditions, but merely growth restrictions on the coefficients and quite weak continuity assumptions (i.e. merely demi-continuity). To this end, we suggest a general framework (cf. Section 4) which also comprises the stochastic Navier–Stokes equations over a bounded domain in all dimensions with multiplicative noise (which was, however, already covered in [13] under similar assumptions on the coefficients).

On the other hand, without any at least local weak monotonicity conditions on the coefficients one cannot expect to be able to prove uniqueness of martingale solutions. The least, however, what one can expect is to prove the existence of Markov selections or so-called almost sure Markov selections recently introduced by Flandoli and Romito in [8], generalizing the classical work of Krylov [10], beautifully implemented in finite dimensions in [17]. In this celebrated paper [8], the authors prove the existence of almost sure Markov selections in the case of stochastic 3D Navier–Stokes equations (also showing that the "almost sure" can be dropped for sufficiently regular additive noise). The second aim of our paper is to prove the existence of such almost sure Markov selections in our general framework in Section 4 (cf. Theorem 4.7). As applications, we recover the corresponding results in [8] for the stochastic 3D Navier–Stokes equations (cf. Section 6), but also prove the existence of such selections for non-monotone stochastic porous medium equations for the first time (see Section 5).

Our construction of almost sure Markov selections differs from that in [8] in the following ways: Our abstract Markov selection theorem is stated in a Polish space so that it can be used to deal with more general stochastic equations. Another main difference about the notion of martingale solutions is that we avoid using the notion of "a.s. super martingale" from [8], which would cause some unnecessary difficulties (as e.g. the lack of measurability of $s \mapsto \mathbb{E}(\cdot|\mathcal{F}_s)(\omega)$ for the natural, not right-continuous filtration $(\mathcal{F}_s)_{s\geqslant 0}$).

This paper is organized as follows: In Section 2, we state the abstract Markov selection theorem in a Polish space, whose proof is given in Appendix A. In Section 3, under the assumptions of existence and weak compactness of martingale solutions, we prove a theorem about the existence of Markov selections for abstract SPDE of evolutionary type. In Section 4, we give some concrete conditions for the coefficients of such SPDE so that the assumptions in Section 3 are satisfied. In the next two sections, we apply our general results to non-monotone stochastic generalized porous medium equations and stochastic 3D Navier–Stokes equations with multiplicative noise. In Appendices A–C, for the reader's convenience and completeness, we include some proofs of theorems and lemmas used in the main text.

2. Abstract Markov selections

Let $(\mathbb{X}, \rho_{\mathbb{X}})$ be a Polish space. For fixed $t \geqslant 0$, let $\Omega^t := C([t, \infty); \mathbb{X})$ be the space of all continuous functions from $[t, \infty)$ to $\mathbb{X}$ with the metric

$$\rho^t(x,y) := \sum_{m=[t]+1}^{\infty} \frac{1}{2^m}\left(\sup_{s\in[t,m]} \rho_{\mathbb{X}}(x(s),y(s)) \wedge 1\right).$$

Then (Ω^t, ρ^t) is a Polish space. For $s \geqslant t$, define the σ-algebra $\mathcal{F}_s^t$ on Ω^t by

$$\mathcal{F}_s^t := \sigma\{x(r) : t \leqslant r \leqslant s\},$$

and write

$$\mathcal{F}^t := \vee_{s\geqslant t} \mathcal{F}_s^t.$$

Thus, we have a measurable space with filtration

$$(\Omega^t, \mathcal{F}^t, (\mathcal{F}_s^t)_{s\geqslant t}).$$

If $t = 0$, we simply write

$$(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geqslant 0}).$$

We remark Ω^t can be regarded as a closed subset of Ω by setting

$$x(r) := x(t), \quad r \in [0,t],\ x \in \Omega^t.$$

In this way, for any $s \geqslant t \geqslant 0$, $\Omega^t \in \mathcal{F}_s$ and

$$\mathcal{F}_s^t = \Omega^t \cap \mathcal{F}_s \subset \mathcal{F}_s.$$

The shift operator $\Phi_t : \Omega \to \Omega^t$ defined by

$$\Phi_t(x)(s) := x(s-t), \quad s \geqslant t, \tag{2.1}$$

establishes a measurable isomorphism between $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geqslant 0})$ and $(\Omega^t, \mathcal{F}^t, (\mathcal{F}_s^t)_{s\geqslant t})$.

For a Polish space $\mathbb{V}$ let $\mathcal{B}(\mathbb{V})$ denote its Borel σ-algebra, and $\mathscr{P}(\mathbb{V})$ the set of all probability measures on $(\mathbb{V}, \mathcal{B}(\mathbb{V}))$. It is a classical fact that $\mathcal{F}^t = \mathcal{B}(\Omega^t)$, and $\mathscr{P}(\mathbb{V})$ is still a Polish space with respect to the weak topology. The corresponding metric is denoted by $\mathbf{d}_{\mathbb{V}}$. For each $P \in \mathscr{P}(\Omega^t)$, we may extend P to Ω by putting $P(A) = P(\Omega^t \cap A)$ for $A \in \mathcal{F}$. In this way, $\mathscr{P}(\Omega^t)$ can be thought of as a subset of $\mathscr{P}(\Omega)$. The shift operator Φ_t also establishes an isomorphism between $\mathscr{P}(\Omega)$ and $\mathscr{P}(\Omega^t)$, i.e., if $P \in \mathscr{P}(\Omega)$, then $P \circ \Phi_t^{-1} \in \mathscr{P}(\Omega^t)$; if $P \in \mathscr{P}(\Omega^t)$, then $P \circ \Phi_t \in \mathscr{P}(\Omega)$.

The following lemma is easy.

Lemma 2.1. *For $t \geqslant 0$, let $P \in \mathscr{P}(\Omega^t) \subset \mathscr{P}(\Omega)$. Then for any non-negative $\mathcal{F}$-measurable random variable ξ*

$$\mathbb{E}^P(\xi|\mathcal{F}_s) = \mathbb{E}^P(\xi|\mathcal{F}_s^t), \quad s \geqslant t.$$

Proof. Since $\mathcal{F}_s^t \subset \mathcal{F}_s$, we only need to prove that for any $A \in \mathcal{F}_s$

$$\mathbb{E}^P(1_A \cdot \xi) = \mathbb{E}^P(1_A \cdot \mathbb{E}^P(\xi|\mathcal{F}_s^t)).$$

However this is true because P is concentrated on Ω^t. $\square$

Given $P \in \mathscr{P}(\Omega)$ and $t > 0$, we shall denote by $P(\cdot|\mathcal{F}_t)(x)$ a regular conditional probability distribution (abbreviated as r.c.p.d.) of P with respect to $\mathcal{F}_t$. In particular, $P(\cdot|\mathcal{F}_t)(x)$ is a probability measure on $(\Omega, \mathcal{F})$ and for any bounded $\mathcal{F}$-measurable function f on Ω

$$\mathbb{E}^P(f|\mathcal{F}_t) = \int_\Omega f(y)P(\mathrm{d}y|\mathcal{F}_t), \quad P\text{-a.s.}, \tag{2.2}$$

and there exists a P-null set $N \in \mathcal{F}_t$ such that for any $x \notin N$ (cf. [17, Theorem 1.1.8])

$$P(\{y : y(s) = x(s), s \in [0,t]\}|\mathcal{F}_t)(x) = 1. \tag{2.3}$$

In particular, by (2.3) we can also consider $P(\cdot|\mathcal{F}_t)(x)$ as a measure on $(\Omega^t, \mathcal{F}^t)$, i.e.,

$$P(\cdot|\mathcal{F}_t)(x) \in \mathscr{P}(\Omega^t). \tag{2.4}$$

Below we shall do this without further comments.

Let us recall the following result (cf. [17, Theorem 6.1.2]).

Theorem 2.2. *Fix $t > 0$. Let $x \mapsto Q_x$ be a mapping from Ω to $\mathscr{P}(\Omega^t)$ such that for any $A \in \mathcal{F}^t$, $x \mapsto Q_x(A)$ is $\mathcal{F}_t$-measurable, and for any $x \in \Omega$*

$$Q_x(y \in \Omega^t : y(t) = x(t)) = 1.$$

Then for any $P \in \mathscr{P}(\Omega)$, there exists a unique $P \otimes_t Q \in \mathscr{P}(\Omega)$ such that

$$(P \otimes_t Q)(A) = P(A), \quad \forall A \in \mathcal{F}_t, \tag{2.5}$$

and for $P \otimes_t Q$-almost all $x \in \Omega$

$$Q_x = (P \otimes_t Q)(\cdot|\mathcal{F}_t)(x). \tag{2.6}$$

Let $\mathbb{B}$ be another Polish space, which is continuously and densely injected into $\mathbb{X}$. By Kuratowski's theorem, $\mathbb{B}$ is a Borel subset of $(\mathbb{X}, \rho_{\mathbb{X}})$ and $\mathcal{B}(\mathbb{B}) = \mathbb{B} \cap \mathcal{B}(\mathbb{X})$.

Definition 2.3. We say $P \in \mathscr{P}_{\mathbb{B}}(\Omega) \subset \mathscr{P}(\Omega)$ is concentrated on the paths with values in $\mathbb{B}$, if there exists an $A \in \mathcal{F}$ with $P(A) = 1$ such that $A \subset \{x \in \Omega : x(t) \in \mathbb{B}, \forall t \geqslant 0\}$.

Remark 2.4. As a subset of $(\mathscr{P}(\Omega), \mathbf{d}_\Omega)$, $(\mathscr{P}_{\mathbb{B}}(\Omega), \mathbf{d}_\Omega)$ is a separable metric space, but, may be not complete. It is clear that $\mathcal{B}(\mathscr{P}_{\mathbb{B}}(\Omega)) = \mathscr{P}_{\mathbb{B}}(\Omega) \cap \mathcal{B}(\mathscr{P}(\Omega))$.

Following [8, Definitions 2.4, 2.5], we introduce the following notions.

Definition 2.5. A family $(P_b)_{b\in\mathbb{B}}$ of probability measures in $\mathscr{P}_{\mathbb{B}}(\Omega)$, is called an almost sure Markov family (resp. Markov family) if for any $A \in \mathcal{F}$, $b \mapsto P_b(A)$ is $\mathcal{B}(\mathbb{B})/\mathcal{B}([0,1])$-measurable, and for each $b \in \mathbb{B}$ there exists a Lebesgue null set $\mathbb{T}_{P_b} \subset (0,\infty)$ (resp. $\mathbb{T}_{P_b} = \emptyset$) such that for all $t \notin \mathbb{T}_{P_b}$ and P_b-almost all $x \in \Omega$

$$P_b(\cdot|\mathcal{F}_t)(x) = P_{x(t)} \circ \Phi_t^{-1}.$$

In other words, for any bounded $\mathcal{F}^t$-measurable function f on Ω^t

$$\int_{\Omega^t} f(y)P_b(\mathrm{d}y|\mathcal{F}_t)(x) = \mathbb{E}^{P_{x(t)}}(f \circ \Phi_t), \quad P_b\text{-a.s. } x \in \Omega.$$

By $\mathrm{Comp}(\mathscr{P}_{\mathbb{B}}(\Omega))$ denote the space of all compact subsets of $\mathscr{P}_{\mathbb{B}}(\Omega)$. Define a metric $\tilde{\mathbf{d}}_C(K_1, K_2)$ between two points $K_1, K_2 \in \mathrm{Comp}(\mathscr{P}_{\mathbb{B}}(\Omega))$ by

$$\tilde{\mathbf{d}}_C(K_1, K_2) := \inf\{\epsilon > 0 : K_1 \subset K_2^\epsilon, K_2 \subset K_1^\epsilon\},$$

where for any set $K \in \mathrm{Comp}(\mathscr{P}_{\mathbb{B}}(\Omega))$, $K^\epsilon := \{y : \mathbf{d}_\Omega(x, y) < \epsilon, \exists x \in K\}$. We remark that for any $x, y \in \mathscr{P}_{\mathbb{B}}(\Omega)$

$$\tilde{\mathbf{d}}_C(\{x\}, \{y\}) = \mathbf{d}_\Omega(x, y).$$

It is easy to see that $(\mathrm{Comp}(\mathscr{P}_{\mathbb{B}}(\Omega)), \tilde{\mathbf{d}}_C)$ is a separable metric space, which will be endowed with the Borel sigma algebra.

Definition 2.6. Let $\mathbb{B} \ni b \mapsto \mathscr{C}(b) \in \mathrm{Comp}(\mathscr{P}_{\mathbb{B}}(\Omega))$ be a measurable mapping. We say $(\mathscr{C}(b))_{b\in\mathbb{B}}$ forms an almost sure pre-Markov family (resp. pre-Markov family) if for each $b \in \mathbb{B}$ and $P \in \mathscr{C}(b)$, there exists a Lebesgue null set $\mathbb{T}_P \subset (0, \infty)$ (resp. $\mathbb{T}_P = \emptyset$) such that for all $0 \leqslant t \notin \mathbb{T}_P$,

1. (Disintegration) there is a P-null set $N \in \mathcal{F}_t$ such that for $x \notin N$,

$$x(t) \in \mathbb{B}, \quad P(\Phi_t(\cdot)|\mathcal{F}_t)(x) \in \mathscr{C}(x(t));$$

2. (Reconstruction) for each mapping $\Omega \ni x \mapsto Q_x \in \mathscr{P}_{\mathbb{B}}(\Omega^t)$ satisfying the assumptions in Theorem 2.2 such that there is a P-null set $N \in \mathcal{F}_t$ such that for all $x \notin N$

$$x(t) \in \mathbb{B}, \quad Q_x \circ \Phi_t \in \mathscr{C}(x(t));$$

then $P \otimes_t Q \in \mathscr{C}(b)$.

We are now in a position to state the following abstract Markov selection theorem (cf. [8, Theorems 2.8, 2.12]).

Theorem 2.7. *Let $(\mathscr{C}(b))_{b\in\mathbb{B}}$ be an almost sure pre-Markov family (resp. pre-Markov family). Suppose that for each $b \in \mathbb{B}$, $\mathscr{C}(b)$ is non-empty and convex. Then there exists a measurable selection $\mathbb{B} \ni b \mapsto P_b \in \mathscr{P}_{\mathbb{B}}(\Omega)$ such that $P_b \in \mathscr{C}(b)$ for each $b \in \mathbb{B}$, and $(P_b)_{b\in\mathbb{B}}$ is an almost sure Markov family (resp. Markov family). We call $(P_b)_{b\in\mathbb{B}}$ an almost sure Markov selection (resp. Markov selection) of $(\mathscr{C}(b))_{b\in\mathbb{B}}$.*

Although the proof of this theorem is almost the same as that given in [8, Theorem 2.8] (see also [17, Theorem 12.2.3]), for the reader's convenience, the proof will be provided in Appendix A.

3. Markov property for stochastic evolution equations

Let $\mathbb{H}$ be a separable Hilbert space, with inner product $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ and norm $\|\cdot\|_{\mathbb{H}}$. Let $\mathbb{X}, \mathbb{Y}$ be two separable and reflexive Banach spaces with norms $\|\cdot\|_{\mathbb{X}}$ and $\|\cdot\|_{\mathbb{Y}}$, such that

$$\mathbb{Y} \subset \mathbb{H} \subset \mathbb{X}$$

continuously and densely. By Kuratowski's theorem we have that $\mathbb{Y} \in \mathcal{B}(\mathbb{H})$, $\mathbb{H} \in \mathcal{B}(\mathbb{X})$ and $\mathcal{B}(\mathbb{Y}) = \mathcal{B}(\mathbb{H}) \cap \mathbb{Y}$, $\mathcal{B}(\mathbb{H}) = \mathcal{B}(\mathbb{Y}) \cap \mathbb{H}$. If we identify the dual of $\mathbb{H}$ with itself, then we get

$$\mathbb{X}^* \subset \mathbb{H}^* \simeq \mathbb{H} \subset \mathbb{X}.$$

In applications, $\mathbb{X}^*$ is usually embedded in $\mathbb{Y}$. The dual pair between $\mathbb{X}$ and $\mathbb{X}^*$ is denoted by

$${}_{\mathbb{X}}\langle x, y\rangle_{\mathbb{X}^*}, \quad x \in \mathbb{X}, \ y \in \mathbb{X}^*.$$

We remark that if $x \in \mathbb{H}$, then

$${}_{\mathbb{X}}\langle x, y\rangle_{\mathbb{X}^*} = \langle x, y\rangle_{\mathbb{H}}.$$

Let $\mathscr{E}$ be a fixed countable dense subset of $\mathbb{X}^*$ which will be chosen in each case.

Let $(W(t))_{t\geqslant 0}$ be a cylindrical Brownian motion in another separable Hilbert space $\mathbb{U}$ with identity covariance. Let $L_2(\mathbb{U};\mathbb{H})$ be the space of all Hilbert–Schmidt operators from $\mathbb{U}$ to $\mathbb{H}$ with inner product $\langle\cdot,\cdot\rangle_{L_2(\mathbb{U},\mathbb{H})}$ and norm $\|\cdot\|_{L_2(\mathbb{U},\mathbb{H})}$.

Consider the following stochastic evolution equation:

$$\mathrm{d}x(t) = A(x(t))\mathrm{d}t + B(x(t))\mathrm{d}W_t, \quad t \geqslant 0, x(0) = x_0 \in \mathbb{H}, \tag{3.1}$$

where $A : \mathbb{Y} \to \mathbb{X}$ and $B : \mathbb{Y} \to L_2(\mathbb{U};\mathbb{H})$ are $\mathcal{B}(\mathbb{Y})/\mathcal{B}(\mathbb{X})$ and $\mathcal{B}(\mathbb{Y})/\mathcal{B}(L_2(\mathbb{U};\mathbb{H}))$-measurable respectively.

Let $f : \mathbb{Y} \to \mathbb{R}$ be a $\mathcal{B}(\mathbb{Y})/\mathcal{B}(\mathbb{R})$-measurable real function. Setting $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, we may extend f to a $\mathcal{B}(\mathbb{X})/\mathcal{B}(\bar{\mathbb{R}})$-measurable function on $\mathbb{X}$ by setting

$$f(x) = \infty, \quad x \in \mathbb{X} \setminus \mathbb{Y}. \tag{3.2}$$

In the following, we shall always use this extension if it is necessary, and keep the same notations as in Section 2 such as $\Omega, \mathcal{F}$ and $\mathcal{F}_t$. We now introduce the following notion of martingale solution to Eq. (3.1).

Definition 3.1. Let $x_0 \in \mathbb{H}$. A probability measure $P \in \mathscr{P}(\Omega)$ is called a martingale solution of Eq. (3.1) with initial value x_0 if:

(M1) $P(x(0) = x_0) = 1$ and for any $n \in \mathbb{N}$

$$P\left\{x \in \Omega : \int_0^n \|A(x(s))\|_{\mathbb{X}}\mathrm{d}s + \int_0^n \|B(x(s))\|^2_{L_2(\mathbb{U};\mathbb{H})}\mathrm{d}s < +\infty\right\} = 1;$$

(M2) for every $\ell \in \mathscr{E}$, the process

$$M_\ell(t,x) := {}_{\mathbb{X}}\langle x(t), \ell\rangle_{\mathbb{X}^*} - \int_0^t {}_{\mathbb{X}}\langle A(x(s)), \ell\rangle_{\mathbb{X}^*}\mathrm{d}s$$

is a continuous square integrable $\mathcal{F}_t$-martingale with respect to P, whose quadratic variation process is given by

$$\langle M_\ell\rangle(t,x) := \int_0^t \|B^*(x(s))(\ell)\|^2_{\mathbb{U}}\mathrm{d}s,$$

where the asterisk denotes the adjoint operator of $B(x(s))$;

(M3) for any $p \in \mathbb{N}$, there exist a continuous positive real function $t \mapsto C_{t,p}$ (only depending on p and A, B), a lower semi-continuous functional $\mathcal{N}_p : \mathbb{Y} \to [0,\infty]$, and a Lebesgue null set $\mathbb{T}_P \subset (0,\infty)$ such that for all $0 \leqslant s \notin \mathbb{T}_P$ and all $t \geqslant s$

$$\mathbb{E}^P\left(\sup_{r\in[s,t]} \|x(r)\|^{2p}_{\mathbb{H}} + \int_s^t \mathcal{N}_p(x(r))\mathrm{d}r \,\middle|\, \mathcal{F}_s\right) \leqslant C_{t-s,p}\cdot(\|x(s)\|^{2p}_{\mathbb{H}} + 1). \tag{3.3}$$

Remark 3.2. If a martingale solution $P \in \mathscr{P}(\Omega)$ is concentrated on the paths that are right continuous in $\mathbb{H}$, then the exceptional set $\mathbb{T}_P$ in **(M3)** is empty. In fact, letting $t > s \geqslant 0$, we choose $s_n \notin \mathbb{T}_P$ with $s_n \downarrow s$. Then

$$\mathbb{E}^P\left(\sup_{r\in[s_n,t]} \|x(r)\|^{2p}_{\mathbb{H}} + \int_{s_n}^t \mathcal{N}_p(x(r))\mathrm{d}r \,\middle|\, \mathcal{F}_{s_n}\right) \leqslant C_{t-s_n,p}\cdot\left(\|x(s_n)\|^{2p}_{\mathbb{H}} + 1\right).$$

Taking conditional expectations with respect to $\mathcal{F}_s$ gives

$$\mathbb{E}^P\left(\sup_{r\in[s_n,t]}\|x(r)\|_{\mathbb{H}}^{2p}+\int_{s_n}^t \mathcal{N}_p(x(r))\mathrm{d}r\middle|\mathcal{F}_s\right)\leqslant C_{t-s_n,p}\cdot\left(\mathbb{E}^P\left(\|x(s_n)\|_{\mathbb{H}}^{2p}|\mathcal{F}_s\right)+1\right).$$

Letting $n\to\infty$ and using the dominated convergence theorem and the right continuity of $s\mapsto x(s)$, we obtain

$$\mathbb{E}^P\left(\sup_{r\in[s,t]}\|x(r)\|_{\mathbb{H}}^{2p}+\int_{s}^t \mathcal{N}_p(x(r))\mathrm{d}r\middle|\mathcal{F}_s\right)\leqslant C_{t-s,p}\cdot\left(\|x(s)\|_{\mathbb{H}}^{2p}+1\right).$$

We make the following assumptions:

(H1) For each $x_0\in\mathbb{H}$, there exists a martingale solution $P\in\mathscr{P}(\Omega)$ starting from x_0 to Eq. (3.1) in the sense of Definition 3.1. The set of all such martingale solutions is denoted by $\mathscr{C}(x_0)$.

(H2) Let $x_n\to x_0$ as $n\to\infty$ in $\mathbb{H}$ and $P_n\in\mathscr{C}(x_n)$. Then for some subsequence n_k, P_{n_k} weakly converges to some $P\in\mathscr{C}(x_0)$.

Lemma 3.3. *Under* **(H1)** *and* **(H2)***, the family* $(\mathscr{C}(x_0))_{x_0\in\mathbb{H}}$ *satisfies the disintegration property in Definition 2.6.*

Proof. Fix $x_0\in\mathbb{H}$ and $P\in\mathscr{C}(x_0)$. Let $\mathbb{T}_P$ be the exceptional set in **(M2)**. We also fix $0\leqslant r\notin\mathbb{T}_P$. Let $Q_x^r:=P(\cdot|\mathcal{F}_r)(x)$ be an r.c.p.d. of P with respect to $\mathcal{F}_r$. We want to show that there is a P-null set $N\in\mathcal{F}_r$ such that for all $x\notin N$

$$Q_x^r\circ\Phi_r\in\mathscr{C}(x(r)).$$

That is, we need to check $Q_x^r\circ\Phi_r$ satisfies **(M1)–(M3)**.

(M1). Setting

$$\Omega_n:=\left\{x\in\Omega:\int_0^n\|A(x(s))\|_{\mathbb{X}}\mathrm{d}s+\int_0^n\|B(x(s))\|^2_{L_2(\mathbb{U};\mathbb{H})}\mathrm{d}s<+\infty\right\}$$

and $\Omega':=\cap_{n\in\mathbb{N}}\Omega_n$, by (2.2) and (2.4) we have

$$1=P(\Omega')=\int_\Omega Q_x^r(\Omega'\cap\Omega^r)P(\mathrm{d}x)=\int_\Omega Q_x^r(\Phi_r\Omega')P(\mathrm{d}x),$$

which together with (2.3) implies that for some P-null set $N_1\in\mathcal{F}_r$ and all $x\notin N_1$

$$Q_x^r\circ\Phi_r(\Omega')=1,$$

and

$$Q_x^r\circ\Phi_r(y:y(0)=x(r))=Q_x^r(y:y(r)=x(r))=1.$$

(M2). Since $\mathscr{E}$ is countable, by (III) of Lemma B.3 there exists a P-null set $N_2\in\mathcal{F}_r$ such that for all $x\notin N_2$, $Q_x^r\circ\Phi_r$ satisfies **(M2)**.

(M3). We choose ξ and η in Lemma B.2 as follows:

$$\xi(t,s):=\sup_{s'\in[s,t]}\|x(s')\|_{\mathbb{H}}^{2p}+\int_s^t\mathcal{N}_p(x(s'))\mathrm{d}s'$$

and

$$\eta(t,s):=\text{the right-hand side of (3.3)}.$$

It is clear that for each $0 \leqslant s \leqslant t$, $\eta(t,s)$ is $\mathcal{F}_s$-measurable, and $t \mapsto \eta(t,s)$ is continuous, $t \mapsto \xi(t,s)$ is increasing, and (iii) in Lemma B.2 holds. The integrability conditions on ξ and η in Lemma B.2 follow from **(M3)**, i.e.,

$$\mathbb{E}^P(\xi(t,0)) \leqslant C_{t,p} \cdot (\|x_0\|_{\mathbb{H}}^{2p} + 1), \quad \forall t \geqslant 0.$$

Thus, by (III) of Lemma B.2 there exists a P-null set $N_3 \in \mathcal{F}_r$ such that for all $x \notin N_3$, $Q_x^r \circ \Phi_r$ satisfies **(M3)**.

Finally, letting $N := N_1 \cup N_2 \cup N_3$, we obtain the desired result. □

Lemma 3.4. *Under* **(H1)** *and* **(H2)**, *the family* $(\mathscr{C}(x_0))_{x_0 \in \mathbb{H}}$ *satisfies the reconstruction property in Definition 2.6.*

Proof. Fix $x_0 \in \mathbb{H}$ and $P \in \mathscr{C}(x_0)$. Let $\mathbb{T}_P$ be the exceptional set in **(M2)**. We also fix $0 \leqslant r \notin \mathbb{T}_P$. Let $\Omega \ni x \mapsto Q_x \in \mathscr{P}_{\mathbb{H}}(\Omega^r)$ satisfying the assumptions in Theorem 2.2. Suppose also that for some P-null set $N \in \mathcal{F}_r$ and all $x \notin N$

$$x(r) \in \mathbb{H}, \qquad Q_x \circ \Phi_r \in \mathscr{C}(x(r)).$$

Our aim is to show

$$P \otimes_r Q \in \mathscr{C}(x_0).$$

(M1). Since P agrees with $P \otimes_r Q$ on $\mathcal{F}_r$,

$$(P \otimes_r Q)(y : y(0) = x_0) = 1.$$

Let Ω' be as in Lemma 3.3. By (2.6) we have

$$(P \otimes_r Q)(\Omega') = \int_{\Omega} Q_x(\Phi_r \Omega') P(\mathrm{d}x) = \int_N Q_x \circ \Phi_r(\Omega') P(\mathrm{d}x) = 1.$$

(M2) and **(M3)** for $P \otimes_r Q$ are direct consequences of Lemmas B.2 and B.3 and the fact that P agrees with $P \otimes_r Q$ on $\mathcal{F}_t$. □

We can now give our main result in this section.

Theorem 3.5. *Under* **(H1)** *and* **(H2)**, $(\mathscr{C}(x_0))_{x_0 \in \mathbb{H}}$ *defined above admits a measurable almost sure Markov selection. In this sense, there exists an almost sure Markov family* $(P_{x_0})_{x_0 \in \mathbb{H}}$ *for Eq.* (3.1).

Proof. By **(H1)** and Definition 3.1, it is clear that $\mathscr{C}(x_0)$ is non-empty and convex for each $x_0 \in \mathbb{H}$. Note that

$$L^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{H}) \cap \Omega \in \mathcal{F},$$

and by **(M3)**

$$P(L^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{H}) \cap \Omega) = 1.$$

Since $t \mapsto x(t)$ is weakly continuous in $\mathbb{H}$ for any $x \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{H}) \cap \Omega$, P is concentrated on the paths with values in $\mathbb{H}$, i.e, $P \in \mathscr{P}_{\mathbb{H}}(\mathcal{F})$.

In **(H2)**, taking $x_n = x_0$ gives that $\mathscr{C}(x_0) \in \mathrm{Comp}(\mathscr{P}_{\mathbb{H}}(\mathcal{F}))$. By **(H2)** again, Remark 2.4 and [17, Lemma 12.1.8], $x_0 \mapsto \mathscr{C}(x_0)$ is a Borel measurable map of $\mathbb{H}$ into $\mathrm{Comp}(\mathscr{P}_{\mathbb{H}}(\mathcal{F}))$. Thus, by Lemmas 3.3 and 3.4, Theorem 2.7 implies the assertion. □

4. Martingale solutions for stochastic evolution equations

In this section, we shall give conditions on A and B such that **(H1)** and **(H2)** hold. For this purpose, we first introduce the following function class $\mathfrak{A}^q$, $q \geqslant 1$: A lower semi-continuous function $\mathcal{N} : \mathbb{Y} \to [0, \infty]$ belongs to $\mathfrak{A}^q$ if $\mathcal{N}(x) = 0$ implies $x = 0$, and

$$\mathcal{N}(cy) \leqslant c^q \mathcal{N}(y), \quad \forall c \geqslant 0, \ y \in \mathbb{Y}, \tag{4.1}$$

and

$$\{y \in \mathbb{Y} : \mathcal{N}(y) \leqslant 1\} \text{ is relatively compact in } \mathbb{Y}. \tag{4.2}$$

Remark 4.1. We extend $\mathcal{N}$ to a $\mathcal{B}(\mathbb{X})/\mathcal{B}([0, \infty])$-measurable function on $\mathbb{X}$ as done in (3.2) so that $\int_0^t \mathcal{N}(x(s))\mathrm{d}s$ is well defined for all $x \in C([0, \infty), \mathbb{X})$.

In this section we assume

$$\mathbb{X} \text{ is a Hilbert space and } \mathbb{X}^* \subset \mathbb{Y} \text{ compactly.} \tag{4.3}$$

The assumptions on A and B are given as follows:

(C1) (Demi-Continuity) For any $x \in \mathbb{X}^*$, if y_n strongly converges to y in $\mathbb{Y}$, then

$$\lim_{n\to\infty} {}_{\mathbb{X}}\langle A(y_n), x\rangle_{\mathbb{X}^*} = {}_{\mathbb{X}}\langle A(y), x\rangle_{\mathbb{X}^*},$$

and

$$\lim_{n\to\infty} \|B^*(y_n)(x) - B^*(y)(x)\|_{\mathbb{U}} = 0.$$

(C2) (Coercivity Condition) There exist $\lambda_1 \geqslant 0$ and $\mathcal{N}_1 \in \mathfrak{A}^q$ for some $q \geqslant 2$ such that for all $x \in \mathbb{X}^*$

$${}_{\mathbb{X}}\langle A(x), x\rangle_{\mathbb{X}^*} \leqslant -\mathcal{N}_1(x) + \lambda_1(1 + \|x\|_{\mathbb{H}}^2).$$

(C3) (Growth Condition) There exist $\lambda_2, \lambda_3, \lambda_4 > 0$ and $\gamma' \geqslant \gamma > 1$ such that for all $x \in \mathbb{Y}$

$$\|A(x)\|_{\mathbb{X}}^{\gamma} \leqslant \lambda_2 \mathcal{N}_1(x) + \lambda_3(1 + \|x\|_{\mathbb{H}}^{\gamma'}),$$
$$\|B(x)\|_{L_2(\mathbb{U};\mathbb{H})}^2 \leqslant \lambda_4(1 + \|x\|_{\mathbb{H}}^2),$$

where $\mathcal{N}_1$ is as in **(C2)**.

Remark 4.2. We note that because no monotonicity conditions are imposed, **(C1)–(C3)** above are considerably weaker than the usual conditions to get strong solutions to Eq. (3.1) (cf. [14]). We recall that demi-continuity is implied by hemi-continuity for weakly monotone maps (cf. [14, Remark 4.1]).

Below we set

$$\mathbb{S} := C([0, \infty), \mathbb{X}) \cap L^q_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{Y})$$

and for $p \geqslant 1$

$$\mathcal{N}_p(x) := \|x\|_{\mathbb{H}}^{2(p-1)} \cdot \mathcal{N}_1(x), \quad x \in \mathbb{Y}.$$

It is clear that $x \mapsto \mathcal{N}_p(x)$ is still a lower semi-continuous function on $\mathbb{Y}$.

The following two lemmas are well known (cf. [7,12]). For the reader's convenience, the proofs are provided in Appendix C.

Lemma 4.3. *Let $(P_n)_{n\in\mathbb{N}}$ be a family of probability measures on $\Omega = C([0,\infty),\mathbb{X})$. Assume that $\mathbb{X}^*$ is compactly embedded into $\mathbb{H}$, and for some $\beta > 0$ and any $T > 0$*

$$\sup_{n\in\mathbb{N}} \mathbb{E}^{P_n}\left(\sup_{t\in[0,T]} \|x(t)\|_{\mathbb{H}} + \sup_{s\neq t\in[0,T]} \frac{\|x(t)-x(s)\|_{\mathbb{X}}}{|t-s|^{\beta}} + \int_0^T \mathcal{N}_1(x(s))\mathrm{d}s\right) < \infty. \tag{4.4}$$

Then $(P_n)_{n\in\mathbb{N}}$ is tight in $\mathbb{S}$.

Lemma 4.4. *Under (4.3), there exists an orthonormal basis $\mathscr{E} := \{\ell_i, i \in \mathbb{N}\} \subset \mathbb{X}^*$ of $\mathbb{H}$ such that for some $\kappa > 0$*

$$\|\Pi_n x\|_{\mathbb{X}} \leqslant \kappa\|x\|_{\mathbb{X}}, \quad \forall n \in \mathbb{N},\ x \in \mathbb{X}, \tag{4.5}$$

where Π_n is the projection operator defined by

$$\Pi_n x := \sum_{i=1}^{n} {}_{\mathbb{X}}\langle x, \ell_i\rangle_{\mathbb{X}^*}\ell_i, \quad x \in \mathbb{X}.$$

Below we shall fix this orthonormal basis $\mathscr{E} = \{\ell_i, i \in \mathbb{N}\}$ of $\mathbb{H}$. Let us first verify assumption **(H2)**.

Theorem 4.5. *Under* **(C1)–(C3)***, assume $x_n \to x_0$ in $\mathbb{H}$ as $n \to \infty$ and let $P_n \in \mathscr{C}(x_n)$. Then there exists a subsequence n_k, and $P \in \mathscr{C}(x_0)$ such that P_{n_k} weakly converges to P.*

Proof. We divide the proof into four steps.

(*Step* 1): In this step we prove that $(P_n)_{n\in\mathbb{N}}$ is tight in $\mathbb{S}$. Recall that each P_n satisfies **(M1)–(M3)**. Define for each $n \in \mathbb{N}$

$$M_n(t,x) := \sum_{j=1}^{\infty} M_{\ell_j}(t,x)\ell_j - x_n,$$

where $M_{\ell_j}(t,x)$ is given in **(M2)**. The process $(t,x) \mapsto M_n(t,x)$ is then a continuous $\mathbb{H}$-valued $\mathcal{F}_t$-martingale with respect to P_n with initial value (due to **(M1)** and **(M2)**)

$$M_n(0,x) = 0, \quad P_n\text{-a.s.},$$

and whose covariation operator process in $\mathbb{H}$ is given by

$$\langle\!\langle M_n\rangle\!\rangle(t,x) := \int_0^t B(x(s))B^*(x(s))\mathrm{d}s. \tag{4.6}$$

In fact, set $M_n^{(k)}(t,x) := \sum_{j=1}^{k} M_{\ell_j}(t,x)\ell_j - \Pi_k x_n$. By **(M2)**, the process $(t,x) \mapsto M_n^{(k)}(t,x)$ is a continuous $\mathbb{H}$-valued $\mathcal{F}_t$-martingale with respect to P_n. Note that by **(C3)** and **(M3)**

$$\int_0^T \mathbb{E}^{P_n}\|B(x(s))\|^2_{L_2(\mathbb{U},\mathbb{H})}\mathrm{d}s \leqslant \lambda_4 \int_0^T \mathbb{E}^{P_n}\left(\|x(s)\|^2_{\mathbb{H}} + 1\right)\mathrm{d}s < +\infty.$$

By Burkholder's inequality and **(M2)**, we have for any $l \geqslant k$

$$\mathbb{E}^{P_n}\left(\sup_{t\in[0,T]} \|M_n^{(k)}(t,x) - M_n^{(l)}(t,x)\|^2_{\mathbb{H}}\right) \leqslant C\sum_{j=k}^{l}\int_0^T \mathbb{E}^{P_n}\|B^*(x(s))(\ell_j)\|^2_{\mathbb{U}}\mathrm{d}s,$$

which tends to zero as l, k go to infinity. Therefore, $M_n(t,x)$ is a continuous $\mathbb{H}$-valued $\mathcal{F}_t$-martingale, and by polarization

$$\begin{aligned}\langle\langle M_n\rangle\rangle(t,x) &= \sum_{i,j}\langle M_n^{(i)}, M_n^{(j)}\rangle(t,x)\ell_i\otimes\ell_j\\ &= \sum_{i,j}\int_0^t \langle B^*(x(s))(\ell_i), B^*(x(s))(\ell_j)\rangle_{\mathbb{U}}\mathrm{d}s\,\ell_i\otimes\ell_j,\end{aligned}$$

which gives (4.6).

Thus, the following equality holds in $\mathbb{X}$

$$x(t) = x_n + \int_0^t A(x(s))\mathrm{d}s + M_n(t,x), \quad P_n\text{-a.s.} \tag{4.7}$$

By Hölder's inequality, **(C3)** and **(M3)**, **(M1)** for P_n we have

$$\begin{aligned}&\mathbb{E}^{P_n}\left[\sup_{s\neq t\in[0,T]}\left(\left\|\int_s^t A(x(r))\mathrm{d}r\right\|_{\mathbb{X}}^{\gamma}\Big/|t-s|^{\gamma-1}\right)\right] \leqslant \mathbb{E}^{P_n}\left[\int_0^T \|A(x(r))\|_{\mathbb{X}}^{\gamma}\mathrm{d}r\right]\\ &\leqslant \mathbb{E}^{P_n}\left[\int_0^T\left(\lambda_2\mathcal{N}_1(x(r)) + \lambda_3(1+\|x(r)\|_{\mathbb{H}}^{\gamma'})\right)\mathrm{d}r\right]\\ &\leqslant C_{T,\gamma'}(\mathbb{E}^{P_n}\|x(0)\|_{\mathbb{H}}^{\gamma'}+1) = C_{T,\gamma'}(\|x_n\|_{\mathbb{H}}^{\gamma'}+1),\end{aligned} \tag{4.8}$$

where $C_{T,\gamma'}$ is independent of n.

Moreover, by **(C3)** and **(M3)**, **(M1)** for P_n again, for any $T \geqslant t > s \geqslant 0$ and $p \in \mathbb{N}$ we have

$$\begin{aligned}\mathbb{E}^{P_n}\|M_n(t,x) - M_n(s,x)\|_{\mathbb{H}}^{2p} &\leqslant C_p\mathbb{E}^{P_n}\left(\int_s^t \|B(x(r))\|^2_{L_2(\mathbb{U};\mathbb{H})}\mathrm{d}r\right)^p\\ &\leqslant C_p|t-s|^{p-1}\int_s^t \mathbb{E}^{P_n}\|B(x(r))\|^{2p}_{L_2(\mathbb{U};\mathbb{H})}\mathrm{d}r\\ &\leqslant C_p|t-s|^{p-1}\int_s^t \mathbb{E}^{P_n}(\|x(r)\|_{\mathbb{H}}^{2p}+1)\mathrm{d}r\\ &\leqslant C_{p,T}|t-s|^p(\|x_n\|_{\mathbb{H}}^{2p}+1).\end{aligned}$$

By Kolmogorov's criterion, for any $\alpha \in (0, \frac{p-1}{2p})$ we get

$$\mathbb{E}^{P_n}\left(\sup_{s\neq t\in[0,T]}\frac{\|M_n(t,x)-M_n(s,x)\|_{\mathbb{H}}^{2p}}{|t-s|^{p\alpha}}\right) \leqslant C_{p,T}(\|x_n\|_{\mathbb{H}}^{2p}+1). \tag{4.9}$$

Combining (4.7)–(4.9) gives for $\beta = 1 - \frac{1}{\gamma}$

$$\sup_{n\in\mathbb{N}}\mathbb{E}^{P_n}\left(\sup_{s\neq t\in[0,T]}\frac{\|x(t)-x(s)\|_{\mathbb{X}}}{|t-s|^{\beta}}\right) < \infty. \tag{4.10}$$

Thus, by **(M3)** for P_n and Lemma 4.3, $(P_n)_{n\in\mathbb{N}}$ is tight in $\mathbb{S}$.

Without loss of generality, we assume that P_n weakly converges to some probability measure P in $\mathbb{S}$. We need to show $P \in \mathscr{C}(x_0)$, i.e, P satisfies **(M1)–(M3)**.

(*Step* 2): In this step we verify **(M1)** for P.

By Skorohod's representation theorem, there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and $\mathbb{S}$-valued random variables $\tilde{x}_n$ and $\tilde{x}$ such that:

(i) $\tilde{x}_n$ has the law P_n for each $n \in \mathbb{N}$;
(ii) $\tilde{x}_n \to \tilde{x}$ in $\mathbb{S}$, $\tilde{P}$-a.e., and $\tilde{x}$ has the law P.

First of all, noting that $x_n \to x_0$ in $\mathbb{H}$, by **(M1)** for P_n we have

$$P(x(0) = x_0) = \tilde{P}(\tilde{x}(0) = x_0) = \lim_{n\to\infty} \tilde{P}(\tilde{x}_n(0) = x_n) = \lim_{n\to\infty} P_n(x(0) = x_n) = 1.$$

For $p \in \mathbb{N}$ and $0 \leqslant s < t$, set

$$\xi_p(t, s, x) := \sup_{r\in[s,t]} \|x(r)\|_{\mathbb{H}}^{2p} + \int_s^t \|x(r)\|_{\mathbb{H}}^{2(p-1)} \mathcal{N}_1(x(r))\mathrm{d}r. \tag{4.11}$$

Since $\mathcal{N}_1$ is lower semi-continuous on $\mathbb{Y}$, it is easy to see that $x \mapsto \xi_p(t, s, x)$ is also lower semi-continuous on $\mathbb{S}$. By Fatou's lemma, from **(M3)** and **(M1)** for P_n we have

$$\begin{aligned}\mathbb{E}^P\left(\xi_p(t, 0, x)\right) &= \mathbb{E}^{\tilde{P}}\left[\xi_p(t, 0, \tilde{x})\right] \leqslant \varliminf_{n\to\infty} \mathbb{E}^{\tilde{P}}\left(\xi_p(t, 0, \tilde{x}_n)\right) \\ &= \varliminf_{n\to\infty} \mathbb{E}^{P_n}\left(\xi_p(t, 0, x)\right) \\ &\leqslant C \varliminf_{n\to\infty} \mathbb{E}^{P_n}\left(\|x(0)\|_{\mathbb{H}}^{2p} + 1\right) \\ &\leqslant C \varliminf_{n\to\infty} (\|x_n\|_{\mathbb{H}}^{2p} + 1) < +\infty. \end{aligned} \tag{4.12}$$

Thus, **(M1)** follows from **(C3)**.

(*Step* 3): In this step we verify **(M2)** for P.

Fixing $\ell \in \mathscr{E}$, we want to show $M_\ell(t, x)$ in **(M2)** is a continuous $\mathcal{F}_t$-martingale with respect to P, whose square variation process is given by

$$\langle M_\ell\rangle(t, x) = \int_0^t \|B^*(x(s))(\ell)\|_{\mathbb{U}}^2 \mathrm{d}s. \tag{4.13}$$

Set for $R > 0$

$$G_R^{(1)}(t, x) := {}_{\mathbb{X}}\langle x(t), \ell\rangle_{\mathbb{X}^*} \cdot \chi_R({}_{\mathbb{X}}\langle x(t), \ell\rangle_{\mathbb{X}^*})$$
$$G_R^{(2)}(t, x) := \int_0^t {}_{\mathbb{X}}\langle A(x(s)), \ell\rangle_{\mathbb{X}^*} \cdot \chi_R({}_{\mathbb{X}}\langle A(x(s)), \ell\rangle_{\mathbb{X}^*})\mathrm{d}s,$$

where $\chi_R \in C_0^\infty(\mathbb{R})$ is a cutoff function with

$$\chi_R(r) = \begin{cases} 1, & |r| \leqslant R \\ 0, & |r| > 2R. \end{cases} \tag{4.14}$$

Then for any $t > 0$, $x \mapsto G_R^{(i)}(t, x)$, $i = 1, 2$ are bounded continuous functions on $\mathbb{S}$. In fact, let $x_n \to x$ in $\mathbb{S}$, then for every $t > 0$, $x_n(t) \to x(t)$ in $\mathbb{X}$ and

$$\int_0^t \|x_n(s) - x(s)\|_{\mathbb{Y}}^q \mathrm{d}s \to 0. \tag{4.15}$$

Clearly,

$$\lim_{n\to\infty} G_R^{(1)}(t, x_n) = G_R^{(1)}(t, x).$$

Since $y \mapsto {}_{\mathbb{X}}\langle A(y), \ell\rangle_{\mathbb{X}^*} \cdot \chi_R({}_{\mathbb{X}}\langle A(y), \ell\rangle_{\mathbb{X}^*})$ is a bounded continuous function on $\mathbb{Y}$ by **(C1)**, by (4.15) we have

$$\lim_{n\to\infty} G_R^{(2)}(t, x_n) = G_R^{(2)}(t, x).$$

Thus, by the dominated convergence theorem

$$\lim_{n\to\infty} \mathbb{E}^{\tilde{P}}|G_R^{(i)}(t, \tilde{x}_n) - G_R^{(i)}(t, \tilde{x})| = 0, \quad i = 1, 2. \tag{4.16}$$

On the other hand, setting

$$G^{(1)}(t, x) := {}_{\mathbb{X}}\langle x(t), \ell\rangle_{\mathbb{X}^*},$$
$$G^{(2)}(t, x) := \int_0^t {}_{\mathbb{X}}\langle A(x(s)), \ell\rangle_{\mathbb{X}^*} \mathrm{d}s,$$

by (4.8) we then have

$$\begin{aligned}
&\lim_{R\to\infty} \sup_n \mathbb{E}^{\tilde{P}} \left|G_R^{(2)}(t, \tilde{x}_n) - G^{(2)}(t, \tilde{x}_n)\right| \\
&\leqslant \lim_{R\to\infty} \sup_n \mathbb{E}^{\tilde{P}} \left(\int_0^t |{}_{\mathbb{X}}\langle A(\tilde{x}_n(s)), \ell\rangle_{\mathbb{X}^*}| \cdot 1_{\{|{}_{\mathbb{X}}\langle A(\tilde{x}_n(s)),\ell\rangle_{\mathbb{X}^*}|\geqslant R\}} \mathrm{d}s\right) \\
&\leqslant \|\ell\|_{\mathbb{X}^*} \lim_{R\to\infty} \sup_n \left[\left(\int_0^t \mathbb{E}^{\tilde{P}} \|A(\tilde{x}_n(s))\|_{\mathbb{X}}^{\gamma} \mathrm{d}s\right)^{1/\gamma}\right. \\
&\quad \left. \times \left(\int_0^t \tilde{P}(|{}_{\mathbb{X}}\langle A(\tilde{x}_n(s)), \ell\rangle_{\mathbb{X}^*}| \geqslant R) \mathrm{d}s\right)^{(\gamma-1)/\gamma}\right] \\
&\leqslant \|\ell\|_{\mathbb{X}^*} \lim_{R\to\infty} \sup_n \left(\int_0^t \mathbb{E}^{\tilde{P}} \|A(\tilde{x}_n(s))\|_{\mathbb{X}}^{\gamma} \mathrm{d}s\right) \Big/ R^{\gamma-1} \\
&= \|\ell\|_{\mathbb{X}^*} \lim_{R\to\infty} \sup_n \left(\int_0^t \mathbb{E}^{P_n} \|A(x(s))\|_{\mathbb{X}}^{\gamma} \mathrm{d}s\right) \Big/ R^{\gamma-1} = 0,
\end{aligned} \tag{4.17}$$

and by **(M3)**

$$\lim_{R\to\infty} \sup_n \mathbb{E}^{\tilde{P}} \left|G_R^{(1)}(t, \tilde{x}_n) - G^{(1)}(t, \tilde{x}_n)\right| = 0. \tag{4.18}$$

Combining (4.12) and (4.16)–(4.18), we obtain

$$\lim_{n\to\infty} \mathbb{E}^{\tilde{P}} \left|G^{(i)}(t, \tilde{x}_n) - G^{(i)}(t, \tilde{x})\right| = 0, \quad i = 1, 2,$$

which due to the definition of M_ℓ in **(M2)** implies

$$\lim_{n\to\infty} \mathbb{E}^{\tilde{P}}|M_\ell(t, \tilde{x}_n) - M_\ell(t, \tilde{x})| = 0. \tag{4.19}$$

Let $t > s$ and g be any bounded and real-valued $\mathcal{F}_s$-measurable continuous function on $\mathbb{S}$. Using (4.19) we have

$$\begin{aligned}
\mathbb{E}^{P}((M_\ell(t, x) - M_\ell(s, x))g(x)) &= \mathbb{E}^{\tilde{P}}((M_\ell(t, \tilde{x}) - M_\ell(s, \tilde{x}))g(\tilde{x})) \\
&= \lim_{n\to\infty} \mathbb{E}^{\tilde{P}}((M_\ell(t, \tilde{x}_n) - M_\ell(s, \tilde{x}_n))g(\tilde{x}_n)) \\
&= \lim_{n\to\infty} \mathbb{E}^{P_n}((M_\ell(t, x) - M_\ell(s, x))g(x)) = 0,
\end{aligned}$$

where the last step is due to **(M2)** for P_n. The arbitrariness of g yields

$$\mathbb{E}^P(M_\ell(t,x)|\mathcal{F}_s) = M_\ell(s,x). \tag{4.20}$$

On the other hand, by BDG's inequality and **(C3)**, **(M3)** for P_n we have

$$\begin{aligned}\sup_n \mathbb{E}^{\tilde{P}}|M_\ell(t,\tilde{x}_n)|^{2p} &\leqslant C\sup_n \mathbb{E}^{\tilde{P}}\left(\int_0^t \|B^*(\tilde{x}_n(s))(\ell)\|_{\mathbb{U}}^2 \mathrm{d}s\right)^p \\ &\leqslant C\sup_n \int_0^t \mathbb{E}^{\tilde{P}}\left(\|B^*(\tilde{x}_n(s))(\ell)\|_{\mathbb{U}}^{2p}\right)\mathrm{d}s < +\infty.\end{aligned}$$

Since $p > 1$, by (4.19) we obtain

$$\lim_{n\to\infty} \mathbb{E}^{\tilde{P}}|M_\ell(t,\tilde{x}_n) - M_\ell(t,\tilde{x})|^2 = 0,$$

and by **(C1)**

$$\lim_{n\to\infty} \mathbb{E}^{\tilde{P}}\left|\int_0^t \|B^*(\tilde{x}_n(s))(\ell) - B^*(\tilde{x}(s))(\ell)\|_{\mathbb{U}}^2 \mathrm{d}r\right| = 0.$$

Thus, using the same method used for proving (4.20), we obtain

$$\mathbb{E}^P\left(M_\ell^2(t,x) - \int_0^t \|B^*(x(s))(\ell)\|_{\mathbb{U}}^2\mathrm{d}r\,\middle|\,\mathcal{F}_s\right) = M_\ell^2(s,x) - \int_0^s \|B^*(x(s))(\ell)\|_{\mathbb{U}}^2\mathrm{d}r,$$

which means that (4.13) holds.

(*Step* 4): In this step we verify **(M3)** for P.

Fix a $p \in \mathbb{N}$. Since $\tilde{x}_n$ converges to $\tilde{x}$ in $\mathbb{S}$, $\tilde{P}$-a.s., and $\mathbb{Y} \subset \mathbb{H}$, we also have this convergence in $L^2_{\mathrm{loc}}(0,\infty;\mathbb{H})$. By **(M3)** for P_n, for any $T > 0$ we have

$$\lim_{n\to\infty} \int_0^T \mathbb{E}^{\tilde{P}}\|\tilde{x}_n(s) - \tilde{x}(s)\|_{\mathbb{H}}^{2p}\mathrm{d}s = 0.$$

So, by choosing a subsequence if necessary, there exists a Lebesgue null set $\mathbb{T}_0 \subset (0,\infty)$ such that for all $s \notin \mathbb{T}_0$

$$\lim_{n\to\infty} \mathbb{E}^{\tilde{P}}\|\tilde{x}_n(s) - \tilde{x}(s)\|_{\mathbb{H}}^{2p} = 0. \tag{4.21}$$

Let $\mathbb{T}_{P_n}$ be the exceptional set in **(M3)** for P_n. Set $\mathbb{T}_P := \cup_{n=0}^\infty \mathbb{T}_{P_n} \cup \mathbb{T}_0$. For any $s \notin \mathbb{T}_P$ and $t \geqslant s$, we need to prove

$$\mathbb{E}^P\left(\xi_p(t,s,x)|\mathcal{F}_s\right) \leqslant C_{t-s,p}\cdot(\|x(s)\|_{\mathbb{H}}^{2p}+1), \quad P\text{-a.s.},$$

where ξ_p is defined by (4.11), which is equivalent to proving that for any $\mathcal{F}_s$-measurable and bounded continuous function g on $\Omega = C([0,\infty),\mathbb{X})$

$$\mathbb{E}^P\left[\left(\xi_p(t,s,x)\right)g(x)\right] \leqslant C_{t-s,p}\cdot\mathbb{E}^P\left[(\|x(s)\|_{\mathbb{H}}^{2p}+1)g(x)\right].$$

By the lower semi-continuity of $x \mapsto \xi_p(t,s,x)$ we have

$$\begin{aligned}
\mathbb{E}^{P}\left[\left(\xi_p(t,s,x)\right)g(x)\right] &= \mathbb{E}^{\tilde{P}}\left[\left(\xi_p(t,s,\tilde{x})\right)g(\tilde{x})\right]\\
&\leqslant \varliminf_{n\to\infty} \mathbb{E}^{\tilde{P}}\left[\left(\xi_p(t,s,\tilde{x}_n)\right)g(\tilde{x}_n)\right]\\
&= \varliminf_{n\to\infty} \mathbb{E}^{P_n}\left[\left(\xi_p(t,s,x)\right)g(x)\right]\\
(\text{by } \textbf{(M3)} \text{ for } P_n) &\leqslant C_{t-s,p}\cdot \varliminf_{n\to\infty} \mathbb{E}^{P_n}\left[(\|x(s)\|_{\mathbb{H}}^{2p}+1)g(x)\right]\\
&= C_{t-s,p}\cdot \varliminf_{n\to\infty} \cdot\mathbb{E}^{\tilde{P}}\left[(\|\tilde{x}_n(s)\|_{\mathbb{H}}^{2p}+1)g(\tilde{x}_n)\right]\\
(\text{by } (4.21)) &= C_{t-s,p}\cdot \mathbb{E}^{\tilde{P}}\left[(\|\tilde{x}(s)\|_{\mathbb{H}}^{2p}+1)g(\tilde{x})\right]\\
&= C_{t-s,p}\cdot \mathbb{E}^{P}\left[(\|x(s)\|_{\mathbb{H}}^{2p}+1)g(x)\right],
\end{aligned}$$

which means that **(M3)** holds for P. □

In the following result, we prove the existence of martingale solutions to Eq. (3.1) under **(C1)–(C3)**.

Theorem 4.6. *Assume* **(C1)–(C3)**. *For each $x_0 \in \mathbb{H}$, there exists a martingale solution $P \in \mathscr{P}(\Omega)$ starting from x_0 to Eq.* (3.1) *in the sense of Definition* 3.1.

Proof. We shall use Galerkin's approximation to prove this theorem, and divide the proof into three steps.

(*Step* 1): Let $\{\ell_i, i \in \mathbb{N}\}$ be the orthonormal basis of $\mathbb{H}$ in Lemma 4.4. Let

$$\mathbb{H}_n := \mathrm{span}\{\ell_1,\ldots,\ell_n\} \subset \mathbb{X}^* \subset \mathbb{Y} \subset \mathbb{H} \subset \mathbb{X}.$$

Define the operators $A_n : \mathbb{H}_n \to \mathbb{H}_n$ and $B_n : \mathbb{H}_n \to L_2(\mathbb{U},\mathbb{H}_n)$ as follows:

$$A_n(x) := \Pi_n A(x), \qquad B_n(x) := \Pi_n B(x).$$

Then we have by **(C2)**

$$\langle A_n(x), x\rangle_{\mathbb{H}_n} \leqslant -\mathcal{N}_1(x) + \lambda_1(1+\|x\|_{\mathbb{H}_n}^2), \quad x \in \mathbb{H}_n, \tag{4.22}$$

and by **(C3)**

$$\|B_n(x)\|_{L_2(\mathbb{U},\mathbb{H}_n)}^2 \leqslant \lambda_4(1+\|x\|_{\mathbb{H}_n}^2), \quad x \in \mathbb{H}_n. \tag{4.23}$$

Consider the following finite-dimensional SDE in $\mathbb{H}_n$

$$\mathrm{d}x_n(t) = A_n(x_n(t))\mathrm{d}t + B_n(x_n(t))\mathrm{d}W(t), \qquad x_n(0) = \Pi_n x_0. \tag{4.24}$$

Set

$$\Omega^{(n)} := C([0,\infty),\mathbb{H}_n)$$

and

$$\mathcal{F}_t^{(n)} := \mathcal{B}(C([0,t],\mathbb{H}_n)), \qquad \mathcal{F}^{(n)} := \vee_{t\geqslant 0}\mathcal{F}_t^{(n)}.$$

By Theorem C.3 in Appendix C, there exists a probability measure $P_n \in \mathscr{P}(\Omega^{(n)})$ such that **(M1)** and **(M2)** hold. The generic point in $\Omega^{(n)}$ is denoted by x_n.

(*Step* 2): We now prove that **(M3)** holds for each P_n with $\mathbb{T}_{P_n} = \emptyset$. Fixing $p \in \mathbb{N}$ and $t > s \geqslant 0$, we need to prove

$$\mathbb{E}^{P_n}\left(\xi_p(t,s,x_n)|\mathcal{F}_s^{(n)}\right) \leqslant C_{t-s,p}\cdot(\|x_n(s)\|_{\mathbb{H}}^{2p}+1), \quad P_n\text{-a.s.}, \tag{4.25}$$

where ξ_p is defined by (4.11), and $t \mapsto C_{t,p}$ is some positive continuous real function independent of n.

First of all, by **(M2)** the following equality holds in $\mathbb{H}_n$

$$x_n(t) = \Pi_n x_0 + \int_0^t A_n(x_n(r))\mathrm{d}r + M_n(t,x_n), \tag{4.26}$$

where $M_n(t,x_n)$ is a continuous $\mathbb{H}_n$-valued $\mathcal{F}_t^{(n)}$-martingale with respect to P_n, whose covariation operator process in $\mathbb{H}^n$ is given by

$$\langle\langle M_n\rangle\rangle(t,x_n) := \int_0^t B_n(x_n(s))B_n^*(x_n(s))\mathrm{d}s.$$

Using Itô's formula twice we have

$$\begin{aligned}\|x_n(t)\|_{\mathbb{H}_n}^2 &= \|\Pi_n x_0\|_{\mathbb{H}}^2 + 2\int_0^t \langle A_n(x_n(s)), x_n(s)\rangle_{\mathbb{H}_n}\mathrm{d}s \\ &\quad + \int_0^t \|B_n(x_n(s))\|_{L_2(\mathbb{U};\mathbb{H}_n)}^2\mathrm{d}s + \int_0^t \langle x_n(s), \mathrm{d}M_n(t)\rangle_{\mathbb{H}_n},\end{aligned}$$

and

$$\begin{aligned}\|x_n(t)\|_{\mathbb{H}}^{2p} &= \|x_n(s)\|_{\mathbb{H}}^{2p} + 2p\int_s^t \|x_n(r)\|_{\mathbb{H}}^{2(p-1)}\langle A_n(x_n(r)), x_n(r)\rangle_{\mathbb{H}_n}\mathrm{d}r \\ &\quad + p\int_s^t \|x_n(r)\|_{\mathbb{H}}^{2(p-1)}\|B_n(x_n(r))\|_{L_2(\mathbb{U};\mathbb{H}_n)}^2\mathrm{d}r \\ &\quad + 2p(p-1)\int_s^t \|x_n(r)\|_{\mathbb{H}}^{2(p-2)}\|B^*(x_n(r))(x_n(r))\|_{\mathbb{U}}^2\mathrm{d}r \\ &\quad + M_n^{(p)}(t,x_n) - M_n^{(p)}(s,x_n),\end{aligned}$$

where $M_n^{(p)}(t,x_n)$ is a continuous real-valued $\mathcal{F}_t^{(n)}$-martingale with respect to P_n, whose quadratic variation process is given by

$$\langle M_n^{(p)}\rangle(t,x_n) := 4p^2\int_0^t \|x_n(r)\|_{\mathbb{H}}^{4(p-1)}\|B_n^*(x_n(r))(x_n(r))\|_{\mathbb{U}}^2\mathrm{d}s.$$

By (4.22) and (4.23) we have

$$\begin{aligned}\|x_n(t)\|_{\mathbb{H}}^{2p} &\leqslant \|x_n(s)\|_{\mathbb{H}}^{2p} + 2p\int_s^t \|x_n(r)\|_{\mathbb{H}}^{2(p-1)}(-\mathcal{N}_1(x_n(r)) + \lambda_1(1+\|x_n(r)\|_{\mathbb{H}}^2))\mathrm{d}r \\ &\quad + p(2p-1)\lambda_4\cdot\int_s^t \|x_n(r)\|_{\mathbb{H}}^{2(p-1)}(1+\|x_n(r)\|_{\mathbb{H}}^2)\mathrm{d}s \\ &\quad + M_n^{(p)}(t,x_n) - M_n^{(p)}(s,x_n) \\ &\leqslant \|x_n(s)\|_{\mathbb{H}}^{2p} - 2p\int_s^t \mathcal{N}_p(x_n(r))\mathrm{d}r + C_p\int_s^t(\|x_n(r)\|_{\mathbb{H}}^{2p}+1)\mathrm{d}s \\ &\quad + M_n^{(p)}(t,x_n) - M_n^{(p)}(s,x_n).\end{aligned} \tag{4.27}$$

Put

$$g(t) := \mathbb{E}^{P_n}\left(\sup_{r\in[s,t]} \|x_n(r)\|_{\mathbb{H}}^{2p} | \mathcal{F}_s^{(n)}\right).$$

By Corollary B.4 and Young's inequality we have

$$\begin{aligned}
&\mathbb{E}^{P_n}\left(\sup_{r\in[s,t]} |M_n^{(p)}(r,x_n) - M_n^{(p)}(s,x_n)||\mathcal{F}_s^{(n)}\right)\\
&\leqslant C_p\mathbb{E}^{P_n}\left[\left(\int_s^t \|x_n(r)\|_{\mathbb{H}}^{4(p-1)}\|B_n^*(x_n(r))(x_n(r))\|_{\mathbb{U}}^2 \mathrm{d}r\right)^{1/2}\middle| \mathcal{F}_s^{(n)}\right]\\
&\leqslant C_p\mathbb{E}^{P_n}\left[\sup_{r\in[s,t]} \|x_n(r)\|_{\mathbb{H}}^{p}\left(\int_s^t (\|x_n(r)\|_{\mathbb{H}}^{2p}+1)\mathrm{d}r\right)^{1/2}\middle| \mathcal{F}_s^{(n)}\right]\\
&\leqslant \frac{1}{2}g(t) + C_p\mathbb{E}^{P_n}\left(\int_s^t (\|x_n(r)\|_{\mathbb{H}}^{2p}+1)\mathrm{d}r|\mathcal{F}_s^{(n)}\right]\\
&\leqslant \frac{1}{2}g(t) + C_p\int_s^t (g(r)+1)\mathrm{d}r.
\end{aligned}$$

Thus, taking first supremums and then conditional expectations with respect to $\mathcal{F}_s^{(n)}$ for both sides of (4.27), we obtain

$$\begin{aligned}
&g(t) + 2p\mathbb{E}^{P_n}\left(\int_s^t \mathcal{N}_p(x_n(r))\mathrm{d}r\middle| \mathcal{F}_s^{(n)}\right)\\
&\leqslant \|x_n(s)\|_{\mathbb{H}}^{2p} + \frac{1}{2}g(t) + C_p\int_s^t (g(r)+1)\mathrm{d}r,
\end{aligned} \tag{4.28}$$

and furthermore

$$g(t) \leqslant 2\|x_n(s)\|_{\mathbb{H}}^{2p} + 2C_p\int_s^t (g(r)+1)\mathrm{d}r.$$

By Gronwall's inequality, we obtain

$$g(t) \leqslant g(t)+1 \leqslant \mathrm{e}^{2C_p(t-s)}(2\|x_n(s)\|_{\mathbb{H}}^{2p}+1),$$

which gives the desired estimate (4.25) due to (4.28).

(*Step* 3): We remark that $\Omega_n = C([0,\infty),\mathbb{H}_n)$ is a closed subset of Ω. We extend P_n to a probability measure $\hat{P}_n$ on $(\Omega,\mathcal{F})$ by setting

$$\hat{P}_n(A) := P_n(A\cap\Omega_n), \quad A\in\mathcal{F}.$$

In particular, by (4.25) we also have

$$\mathbb{E}^{\hat{P}_n}\left(\xi_p(t,s,x)|\mathcal{F}_s\right) \leqslant C_{t-s,p}\cdot(\|x(s)\|_{\mathbb{H}}^{2p}+1).$$

We now show that $(\hat{P}_n)_{n\in\mathbb{N}}$ is tight in $\mathbb{S}$. As in the proof of Step 1 in Theorem 4.5, we only need to prove that for some $\beta > 0$ and any $T > 0$

$$\sup_{n\in\mathbb{N}} \mathbb{E}^{\hat{P}_n}\left(\sup_{s\neq t\in[0,T]} \frac{\|x(t)-x(s)\|_{\mathbb{X}}}{|t-s|^{\beta}}\right)$$
$$= \sup_{n\in\mathbb{N}} \mathbb{E}^{P_n}\left(\sup_{s\neq t\in[0,T]} \frac{\|x_n(t)-x_n(s)\|_{\mathbb{X}}}{|t-s|^{\beta}}\right) < \infty. \tag{4.29}$$

By **(C3)**, (4.5) and (4.25) we have

$$\mathbb{E}^{P_n}\left[\sup_{s\neq t\in[0,T]}\left(\left\|\int_s^t A_n(x_n(r))\mathrm{d}r\right\|_{\mathbb{X}}^{\gamma}\Big/|t-s|^{\gamma-1}\right)\right]$$
$$\leqslant \mathbb{E}^{P_n}\left[\int_0^T \|A_n(x_n(r))\|_{\mathbb{X}}^{\gamma}\mathrm{d}r\right] \leqslant \kappa^{\gamma}\mathbb{E}^{P_n}\left[\int_0^T \|A(x_n(r))\|_{\mathbb{X}}^{\gamma}\mathrm{d}r\right]$$
$$\leqslant C_{\gamma}\mathbb{E}^{P_n}\left[\int_0^T \left[\mathcal{N}_1(x_n(r)) + (\|x_n(r)\|_{\mathbb{H}}^{\gamma'}+1)\right]\mathrm{d}r\right] \leqslant C_{T,\gamma'},$$

and similar to (4.9), for any $p \in \mathbb{N}$ and $\alpha \in (0, \frac{p-1}{2p})$

$$\sup_{n\in\mathbb{N}} \mathbb{E}^{P_n}\left(\sup_{r\neq t\in[0,T]} \frac{\|M_n(t,x_n)-M_n(s,x_n)\|_{\mathbb{H}}^{2p}}{|t-r|^{p\alpha}}\right) < \infty. \tag{4.30}$$

Observing (4.26) we obtain (4.29).

Without loss of generality, we may assume that $\hat{P}_n$ weakly converges to some probability measure $\hat{P}$ in $\mathbb{S}$. As in the proof of Theorem 4.5 we then show $\hat{P}$ satisfies **(M1)–(M3)**. □

Thus we obtain the following main result in this section.

Theorem 4.7. *Under* **(C1)–(C3)**, *there exists an almost sure Markov family* $(P_{x_0})_{x_0\in\mathbb{H}}$ *to Eq.* (3.1).

5. Stochastic generalized porous medium equations

Let $\mathcal{O}$ be a bounded open subset of $\mathbb{R}^d$ with smooth boundary. For $k \geqslant 0$ and $p > 1$, let $W_0^{k,p}(\mathcal{O})$ be the usual Sobolev space on $\mathcal{O}$ with Dirichlet boundary conditions. The norm in $W_0^{k,p}(\mathcal{O})$ is denoted by $\|\cdot\|_{k,p}$. The dual space of $W_0^{k,p}(\mathcal{O})$ is given by $W^{-k,p'}(\mathcal{O})$, where $p' = \frac{p}{p-1}$. The following Sobolev embeddings hold (cf. [1]):

$$W_0^{k,p}(\mathcal{O}) \subset C^m(\overline{\mathcal{O}}), \quad 0 \leqslant m < k - d/p. \tag{5.1}$$

By Poincarè's inequality, one has for $x \in W_0^{1,2}(\mathcal{O})$

$$\int_{\mathcal{O}} |x(u)|^2\mathrm{d}u \leqslant \rho_{\mathcal{O}} \int_{\mathcal{O}} |\nabla x(u)|^2\mathrm{d}u. \tag{5.2}$$

An equivalent norm in $W_0^{1,2}(\mathcal{O})$ is thus given by

$$\|x\|_{1,2} = \left(\int_{\mathcal{O}} |\nabla x(u)|^2\mathrm{d}u\right)^{1/2}. \tag{5.3}$$

We shall use this norm below, as well as the notations

$$\partial_i := \frac{\partial}{\partial u_i}, \qquad \partial_{ij}^2 = \frac{\partial^2}{\partial u_i \partial u_j},$$

and the usual Einstein summation convention.

Let $\{W^k(t); t \geqslant 0, k \in \mathbb{N}\}$ be a sequence of independent standard Brownian motions, and l^2 the Hilbert space of all square summable real number sequences. Consider the following quasi-linear SPDE with Dirichlet boundary condition:

$$\begin{cases} \mathrm{d}x(t) = \left[\partial_{ij}^2 a^{ij}(u, x(t)) + \partial_i b^i(u, x(t)) + c(u, x(t))\right] \mathrm{d}t + \sigma_i(u, x(t))\mathrm{d}W^i(t), \\ x(t, u) = 0, \quad (t, u) \in \mathbb{R}_+ \times \partial\mathcal{O}, \\ x(0) = x_0 \in L^2(\mathcal{O}), \end{cases} \tag{5.4}$$

where a, b, c and σ are continuous functions from $\mathcal{O} \times \mathbb{R}$ to $\mathbb{R}^{2d}, \mathbb{R}^d, \mathbb{R}$ and l^2 respectively with respect to the second variable, and satisfy for some fixed $q \geqslant 2$ and all $u \in \mathcal{O}, r \in \mathbb{R}$:

$$\partial_r a^{ij}(u, r)\xi_i\xi_j \geqslant -\kappa_{a,0} \cdot |r|^{q-2}|\xi|^2, \quad \xi \in \mathbb{R}^d, \tag{5.5}$$

$$\|a^{\cdot\cdot}(u, r)\|_{\mathbb{R}^{2d}} \leqslant \kappa_{a,1} \cdot (|r|^{q-1} + 1), \tag{5.6}$$

$$\|\partial_j a^{\cdot j}(u, r)\|_{\mathbb{R}^d} + \|b^{\cdot}(u, r)\|_{\mathbb{R}^d} \leqslant \kappa_{a,b} \cdot |r|^{q-1} + \kappa'_{a,b} \cdot |r|^{\frac{q}{2}}, \tag{5.7}$$

$$|c(u, r)| \leqslant \kappa_{c,1} \cdot |r|^{q-1} + \kappa_{c,2} \cdot (|r| + 1), \tag{5.8}$$

$$\|\sigma(u, r)\|_{l^2} \leqslant \kappa_\sigma \cdot (|r| + 1), \tag{5.9}$$

where all κ with subscripts are strictly positive constants, and

$$\frac{\kappa_{a,b}}{2}\left(1 + \frac{q^2\rho_{\mathcal{O}}}{4}\right) + \frac{\kappa_{c,1} \cdot q^2\rho_{\mathcal{O}}}{4} \leqslant \frac{\kappa_{a,0}}{2}. \tag{5.10}$$

In the following we take

$$\mathbb{Y} := L^q(\mathcal{O}), \qquad \mathbb{H} := L^2(\mathcal{O})$$

and

$$\mathbb{X}^* := W_0^{d+2,2}(\mathcal{O}), \qquad \mathbb{X} := W^{-d-2,2}(\mathcal{O}).$$

Then (4.3) holds.

Define the functional $\mathcal{N}_1$ on $\mathbb{Y}$ as follows:

$$\mathcal{N}_1(y) := \begin{cases} \dfrac{\kappa_0}{q^2}\displaystyle\int_{\mathcal{O}} |\nabla(|y(u)|^{\frac{q}{2}-1}\, y(u))|^2 \mathrm{d}u & \text{if } |y|^{\frac{q}{2}-1} y \in W_0^{1,2}(\mathcal{O}), \\ +\infty, & \text{otherwise.} \end{cases}$$

Then

Lemma 5.1. *$\mathcal{N}_1$ is lower semi-continuous on $\mathbb{Y}$, and the set $\{y \in \mathbb{Y} : \mathcal{N}_1(y) \leqslant 1\}$ is relatively compact in $\mathbb{Y}$. In particular, $\mathcal{N}_1 \in \mathfrak{A}^q$.*

Proof. Let y_n converge to y in $\mathbb{Y} = L^q(\mathcal{O})$. For the lower semi-continuity of $\mathcal{N}_1$, we need to prove

$$\mathcal{N}_1(y) \leqslant \varliminf_{n\to\infty} \mathcal{N}_1(y_n).$$

Without loss of generality, we assume $\sup_{n\in\mathbb{N}}\mathcal{N}_1(y_n) < +\infty$. Noticing that as $n \to \infty$

$$\begin{aligned}
&\int_{\mathcal{O}} \left| |y_n(u)|^{\frac{q}{2}-1} y_n(u) - |y(u)|^{\frac{q}{2}-1} y(u) \right|^2 \mathrm{d}u \\
&= \frac{q^2}{4} \int_{\mathcal{O}} |y(u) - y_n(u)|^2 \left| \int_0^1 |y(u) + s(y_n(u) - y(u))|^{\frac{q}{2}-1} \mathrm{d}s \right|^2 \mathrm{d}u \\
&\leqslant C_q \int_{\mathcal{O}} \left(|y(u) - y_n(u)|^2 |y(u)|^{q-2} + |y_n(u) - y(u)|^q \right) \mathrm{d}u \to 0,
\end{aligned}$$

we get

$$|y|^{(q-2)/2} y \in W_0^{1,2}(\mathcal{O}) \quad \text{and} \quad \mathcal{N}_1(y) < +\infty,$$

as well as by (5.3)

$$\begin{aligned}
\int_{\mathcal{O}} \left| \nabla (|y(u)|^{\frac{q}{2}-1} y(u)) \right|^2 \mathrm{d}u &= \sup_{x\in\mathbb{X}^*, \|x\|_{1,2}\leqslant 1} \left| \int_{\mathcal{O}} |y(u)|^{\frac{q}{2}-1} y(u) \cdot \Delta x(u) \mathrm{d}u \right| \\
&= \sup_{x\in\mathbb{X}^*, \|x\|_{1,2}\leqslant 1} \lim_{n\to\infty} \left| \int_{\mathcal{O}} |y_n(u)|^{\frac{q}{2}-1} y_n(u) \cdot \Delta x(u) \mathrm{d}u \right| \\
&\leqslant \varliminf_{n\to\infty} \sup_{x\in\mathbb{X}^*, \|x\|_{1,2}\leqslant 1} \left| \int_{\mathcal{O}} |y_n(u)|^{\frac{q}{2}-1} y_n(u) \cdot \Delta x(u) \mathrm{d}u \right| \\
&\leqslant \varliminf_{n\to\infty} \int_{\mathcal{O}} \left| \nabla (|y_n(u)|^{\frac{q}{2}-1} y_n(u)) \right|^2 \mathrm{d}u,
\end{aligned}$$

which gives the lower semi-continuity of $\mathcal{N}_1$.

Let $\{y_n, n \in \mathbb{N}\}$ be a sequence in $\mathbb{Y}$ such that $\mathcal{N}_1(y_n) \leqslant 1$. Since $W_0^{1,2}(\mathcal{O})$ is compactly embedded into $L^2(\mathcal{O})$ (cf. [1]), there exists a subsequence n_k such that

$$\lim_{k,l\to\infty} \| |y_{n_k}|^{\frac{q}{2}-1} y_{n_k} - |y_{n_l}|^{\frac{q}{2}-1} y_{n_l} \|_{L^2(\mathcal{O})} = 0.$$

Noting the following elementary inequality: for some $C_q > 0$

$$|r - r'|^{\frac{q}{2}} \leqslant C_q \big| |r|^{\frac{q}{2}-1} r - |r'|^{\frac{q}{2}-1} r' \big|, \quad r, r' \in \mathbb{R},$$

we obtain

$$\lim_{k,l\to\infty} \|y_{n_k} - y_{n_l}\|_{\mathbb{Y}} = \lim_{k,l\to\infty} \|y_{n_k} - y_{n_l}\|_{L^q(\mathcal{O})} = 0. \quad \square$$

Define for $x \in \mathbb{Y} = L^q(\mathcal{O})$

$$A(x) := \partial_{ij}^2 a^{ij}(\cdot, x(\cdot)) + \partial_i b^i(\cdot, x(\cdot)) + c(\cdot, x(\cdot)) \in \mathbb{X}$$

and

$$B(x) := \sigma(\cdot, x(\cdot)) \in L_2(l^2; \mathbb{H}).$$

Lemma 5.2. *Assume that (5.5)–(5.10) hold. Then* **(C1)**–**(C3)** *hold for* A *and* B *defined above.*

Proof. We divide the proof into three steps.

(*Step* 1): Let $x \in \mathbb{X}^* = W_0^{d+2,2}(\mathcal{O})$ and y_n converge to y in $\mathbb{Y} = L^q(\mathcal{O})$. Note that by (5.1)

$$\begin{aligned}
{}_{\mathbb{X}}\langle A(y_n) - A(y), x\rangle_{\mathbb{X}^*} = & \int_{\mathcal{O}} (a^{ij}(u, y_n(u)) - a^{ij}(u, y(u))) \cdot \partial_{ij}^2 x(u)\mathrm{d}u, \\
& - \int_{\mathcal{O}} (b^i(u, y_n(u)) - b^i(u, y(u))) \cdot \partial_i x(u)\mathrm{d}u, \\
& + \int_{\mathcal{O}} (c(u, y_n(u)) - c(u, y(u))) \cdot x(u)\mathrm{d}u \\
\leqslant & \ C\|x\|_{C^2(\bar{\mathcal{O}})} \int_{\mathcal{O}} |a(u, y_n(u)) - a(u, y(u))|\mathrm{d}u, \\
& + C\|x\|_{C^1(\bar{\mathcal{O}})} \int_{\mathcal{O}} |b(u, y_n(u)) - b(u, y(u))|\mathrm{d}u, \\
& + C\|x\|_{C(\bar{\mathcal{O}})} \int_{\mathcal{O}} |c(u, y_n(u)) - c(u, y(u))|\mathrm{d}u \\
=: & \ I_1^{(n)} + I_2^{(n)} + I_3^{(n)}.
\end{aligned}$$

For $I_1^{(n)}$, by (5.6) we have for all $n \in \mathbb{N}$

$$\begin{aligned}
\int_{\mathcal{O}} |a(u, y_n(u)) - a(u, y(u))|^{\frac{q}{q-1}} \mathrm{d}x & \leqslant C \int_{\mathcal{O}} (|y_n(u)|^{q-1} + |y(u)|^{q-1} + 1)^{\frac{q}{q-1}} \mathrm{d}x \\
& \leqslant C \int_{\mathcal{O}} (|y_n(u)|^q + |y(u)|^q + 1)\mathrm{d}u \leqslant C,
\end{aligned}$$

which implies that $\{|a(\cdot, y_n(\cdot)) - a(\cdot, y(\cdot))|, n \in \mathbb{N}\}$ is uniformly integrable, and so by the continuity of a in r

$$I_1^{(n)} \to 0.$$

Similarly, by (5.7) and (5.8)

$$I_2^{(n)} + I_3^{(n)} \to 0,$$

and by (5.9)

$$\|B^*(y_n)(x) - B^*(y)(x)\|_{l^2} \leqslant \int_{\mathcal{O}} \|\sigma(u, y_n(u)) - \sigma(u, y(u))\|_{l^2} \cdot |x(u)|\mathrm{d}u \to 0.$$

Thus, **(C1)** holds.

(*Step* 2): For $x \in \mathbb{X}^*$, noting

$$\partial_j a^{ij}(u, x(u)) = (\partial_r a^{ij})(u, x(u))\partial_j x(u) + (\partial_j a^{ij})(u, x(u)),$$

by (5.5), (5.7) and (5.8) we get

$$
\begin{aligned}
{}_{\mathbb{X}}\langle A(x), x\rangle_{\mathbb{X}^*} &= -\int_{\mathcal{O}} \Big[(\partial_r a^{ij})(u, x(u))\partial_j x(u) + (\partial_j a^{ij})(u, x(u))\Big] \cdot \partial_i x(u)\mathrm{d}u \\
&\quad - \int_{\mathcal{O}} b^i(u, x(u)) \cdot \partial_i x(u)\mathrm{d}u + \int_{\mathcal{O}} c(u, x(u)) \cdot x(u)\mathrm{d}u \\
&\leqslant -\kappa_{a,0} \int_{\mathcal{O}} |x(u)|^{q-2}|\nabla x(u)|^2\mathrm{d}u + \kappa_{a,b} \int_{\mathcal{O}} |x(u)|^{q-1} \cdot |\nabla x(u)|\mathrm{d}u \\
&\quad + \kappa'_{a,b} \int_{\mathcal{O}} |x(u)|^{\frac{q}{2}} \cdot |\nabla x(u)|\mathrm{d}u + \int_{\mathcal{O}} \Big(\kappa_{c,1}|x(u)|^q + \kappa_{c,2}(|x(u)|^2 + |x(u)|)\Big)\,\mathrm{d}u.
\end{aligned}
$$

By Poincare's inequality (5.2) we have

$$
\begin{aligned}
\int_{\mathcal{O}} |x(u)|^q\mathrm{d}u &\leqslant \rho_{\mathcal{O}} \int_{\mathcal{O}} |\nabla(|x(u)|^{\frac{q}{2}-1} x(u))|^2\mathrm{d}u \\
&= \frac{q^2\rho_{\mathcal{O}}}{4} \int_{\mathcal{O}} |x(u)|^{q-2}|\nabla x(u)|^2\mathrm{d}u.
\end{aligned} \tag{5.11}
$$

By Young's inequality we have

$$
\begin{aligned}
\kappa_{a,b} \int_{\mathcal{O}} |x(u)|^{q-1} \cdot |\nabla x(u)|\mathrm{d}u &\leqslant \frac{\kappa_{a,b}}{2} \int_{\mathcal{O}} \Big[|x(u)|^{q-1} \cdot |\nabla x(u)|^2 + |x(u)|^q\Big]\,\mathrm{d}u \\
&\leqslant \frac{\kappa_{a,b}}{2}\left(1 + \frac{q^2\rho_{\mathcal{O}}}{4}\right) \int_{\mathcal{O}} |x(u)|^{q-1} \cdot |\nabla x(u)|^2\mathrm{d}u.
\end{aligned}
$$

Hence, by relations (5.10) and (5.11), for some $C > 0$ we get

$$
\begin{aligned}
{}_{\mathbb{X}}\langle A(x), x\rangle_{\mathbb{X}^*} &\leqslant -\frac{\kappa_{a,0}}{4} \int_{\mathcal{O}} |x(u)|^{q-2}|\nabla x(u)|^2\mathrm{d}u + C \int_{\mathcal{O}} (|x(u)|^2 + 1)\mathrm{d}u \\
&= -\mathcal{N}_1(x) + C(\|x\|_{\mathbb{H}}^2 + 1).
\end{aligned}
$$

Thus, **(C2)** holds.

(*Step* 3): Since ∂_{ij}^2 and ∂_i are bounded linear operators from $W^{-d,2}(\mathcal{O})$ to $\mathbb{X} = W^{-2-d,2}(\mathcal{O})$, by (5.1), we have for $\gamma = \frac{q}{q-1}$ and $x \in \mathbb{Y}$

$$
\begin{aligned}
\|A(x)\|_{\mathbb{X}}^{\gamma} &\leqslant C(\|\partial_{ij}^2 a^{ij}(\cdot, x)\|_{\mathbb{X}}^{\gamma} + \|\partial_i b^i(\cdot, x)\|_{\mathbb{X}}^{\gamma} + \|c(\cdot, x)\|_{\mathbb{X}}^{\gamma}) \\
&\leqslant C(\|a(\cdot, x)\|_{-d,2}^{\gamma} + \|b(\cdot, x)\|_{-d,2}^{\gamma} + \|c(\cdot, x)\|_{-d,2}^{\gamma}) \\
&\leqslant C(\|a(\cdot, x)\|_{L^1(\mathcal{O})}^{\gamma} + \|b(\cdot, x)\|_{L^1(\mathcal{O})}^{\gamma} + \|c(\cdot, x)\|_{L^1(\mathcal{O})}^{\gamma}) \\
&\leqslant C\||x|^{q-1} + 1\|_{L^1(\mathcal{O})}^{\gamma} \leqslant C(\|x\|_{L^{q-1}(\mathcal{O})}^q + 1) \\
&\leqslant C(\|x\|_{L^q(\mathcal{O})}^q + 1) \leqslant C \cdot (\mathcal{N}_1(x) + 1),
\end{aligned}
$$

where the last step is due to (5.11).

On the other hand, we have

$$
\begin{aligned}
\|B(x)\|_{L_2(l^2;\mathbb{H})}^2 &= \int_{\mathcal{O}} \|\sigma(u, x(u))\|_{l^2}^2\mathrm{d}u \\
&\leqslant C \int_{\mathcal{O}} (|x(u)|^2 + 1)\mathrm{d}u \\
&= C(\|x\|_{\mathbb{H}}^2 + \mathrm{Vol}(\mathcal{O})).
\end{aligned}
$$

Thus, **(C3)** holds and the proof is complete. $\square$

Combining Lemmas 5.1 and 5.2 and Theorem 4.7, we obtain:

Theorem 5.3. *Assume (5.5)–(5.10). For each $x_0 \in L^2(\mathcal{O})$, there exists a martingale solution $P_{x_0} \in \mathscr{P}(\Omega)$ to Eq. (5.4) in the sense of Definition 3.1 such that for any $T > 0$*

$$\mathbb{E}^{P_{x_0}}\left(\sup_{t\in[0,T]} \|x(t)\|^2_{L^2(\mathcal{O})} + \int_0^T \mathcal{N}_1(x(s))\mathrm{d}s\right) < +\infty.$$

Moreover, there exists an almost sure Markov selection $(P_{x_0})_{x_0\in L^2(\mathcal{O})}$ for Eq. (5.4).

Below we discuss the existence of Markov selections in two situations. Let us first see the simple case of $q = 2$.

Theorem 5.4. *Assume that (5.5)–(5.9) hold with $q = 2$, as well as that*

$$|\partial_r a(x, r)| \leqslant \kappa_0, \quad \forall (x, r) \in \mathcal{O} \times \mathbb{R}.$$

Then for each $x_0 \in L^2(\mathcal{O})$, there exists a martingale solution $P_{x_0} \in \mathscr{P}(\Omega)$ to Eq. (5.4) in the sense of Definition 3.1 such that $t \mapsto x(t) \in L^2(\mathcal{O})$ is continuous and for any $T > 0$

$$\mathbb{E}^{P_{x_0}}\left(\sup_{t\in[0,T]} \|x(t)\|^2_{L^2(\mathcal{O})} + \int_0^T \|\nabla x(s)\|^2_{L^2(\mathcal{O})}\mathrm{d}s\right) < +\infty.$$

Moreover, there exists a Markov selection $(P_{x_0})_{x_0\in L^2(\mathcal{O})}$.

Proof. By Remark 3.2, we only need to prove that for $q = 2$, every martingale solution is path continuous in $\mathbb{H} = L^2(\mathcal{O})$. By Itô's formula due to Krylov and Rozovskii [11], it is enough to show that the operator A maps $W^{1,2}_0(\mathcal{O})$ into $W^{-1,2}(\mathcal{O})$, because then, the following equality holds in $W^{-1,2}(\mathcal{O})$

$$x(t) = x_0 + \int_0^t A(x(s))\mathrm{d}s + M(t, x), \quad P_{x_0}\text{-a.s.},$$

where $M(t, x)$ is a continuous $L^2(\mathcal{O})$-valued square integrable martingale as defined in (Step 1) of Theorem 4.5.

For any $x, y \in W^{1,2}_0(\mathcal{O})$, we have

$$\begin{aligned}
|{}_{\mathbb{X}}\langle A(x), y\rangle_{\mathbb{X}^*}| &= \Bigg| -\int_{\mathcal{O}} \Big[(\partial_r a^{ij})(u, x(u))\partial_j x(u) + (\partial_j a^{ij})(u, x(u))\Big] \cdot \partial_i y(u)\mathrm{d}u \\
&\quad - \int_{\mathcal{O}} b^i(u, x(u)) \cdot \partial_i y(u)\mathrm{d}u + \int_{\mathcal{O}} c(u, x(u)) \cdot y(u)\mathrm{d}u \Bigg| \\
&\leqslant C\int_{\mathcal{O}} (|\nabla x(u)| + (|x(u)| + 1)) \cdot |\nabla y(u)|\mathrm{d}u \\
&\quad + C\int_{\mathcal{O}} (|x(u)| + 1) \cdot |\nabla y(u)|\mathrm{d}u + C\int_{\mathcal{O}} (|x(u)| + 1) \cdot y(u)\mathrm{d}u \\
&\leqslant C(\|x\|_{1,2} + 1) \cdot \|y\|_{1,2}.
\end{aligned}$$

Hence, for each $x \in W^{1,2}_0(\mathcal{O})$, $A(x)$ is a bounded linear functional on $W^{1,2}_0(\mathcal{O})$, i.e., $A(x) \in W^{1,2}_0(\mathcal{O})^* = W^{-1,2}(\mathcal{O})$. The proof is complete. $\square$

Next for the case $q > 2$, we consider the following type of equation

$$\mathrm{d}x(t) = \left[\Delta(|x(t)|^{q-2}x(t)) + c(u, x(t))\right]\mathrm{d}t + \sigma_i(u, x(t))\mathrm{d}W^i(t).$$

We shall follow the method in [16, Theorem 2.8] to prove that any martingale solution of this equation is path right continuous. Thus, we obtain a Markov selection by Remark 3.2.

Theorem 5.5. *Consider the above equation, and assume that (5.8) and (5.9) hold with $q > 2$. For each $x_0 \in L^2(\mathcal{O})$ and each martingale solution $P_{x_0} \in \mathscr{P}(\Omega)$ to Eq. (5.4) in the sense of Definition 3.1, P is concentrated on the paths that are right continuous in $L^2(\mathcal{O})$. In particular, there exists a Markov selection $(P_{x_0})_{x_0 \in L^2(\mathcal{O})}$.*

Proof. Below we choose a special triple, namely:

$$\mathbb{V} = L^q(\mathcal{O}) \subset W^{-1,2}(\mathcal{O}) =: \mathbb{H}_0 \simeq \mathbb{H}_0^* =: W_0^{1,2}(\mathcal{O}) \subset (L^q(\mathcal{O}))^* = \mathbb{V}^*,$$

and consider a family of equivalent norms in $\mathbb{H}_0$

$$\|x\|_\epsilon := \int_{\mathcal{O}} |(I - \epsilon\Delta)^{-1/2}x(u)|^2\mathrm{d}u, \quad \epsilon > 0,$$

where $\simeq$ is understood via the Riesz map $\mathcal{R} := I - \epsilon\Delta$.

The Hilbert space $(\mathbb{H}_0, \|\cdot\|_\epsilon)$ will be written as $\mathbb{H}_0^\epsilon$. It is clear that for any $x \in \mathbb{H}$

$$\|x\|_\epsilon \leqslant \|x\|_{\mathbb{H}} \quad \text{and} \quad \lim_{\epsilon\downarrow 0} \|(I - \epsilon\Delta)^{-1}x - x\|_{\mathbb{H}} = \|x\|_{\mathbb{H}}.$$

It is well known that $I - \epsilon\Delta : W_0^{1,2}(\mathcal{O}) \to (L^q(\mathcal{O}))^*$ can be extended to a linear isometry from $L^{\frac{q}{q-1}}(\mathcal{O})$ to $(L^q(\mathcal{O}))^*$ such that for any $y \in L^{\frac{q}{q-1}}(\mathcal{O})$ and $x \in L^q(\mathcal{O})$ (cf. [14])

$${}_{\mathbb{V}}\langle x, (I - \epsilon\Delta)y\rangle_{\mathbb{V}^*} = \int_{\mathcal{O}} x(u)\cdot y(u)\mathrm{d}u.$$

Thus, we have

$$\begin{aligned}
{}_{\mathbb{V}}\langle x, A(x)\rangle_{\mathbb{V}^*} &= -\frac{1}{\epsilon}\int_{\mathcal{O}} |x(u)|^q\mathrm{d}u + \frac{1}{\epsilon}{}_{\mathbb{V}}[x, |x(\cdot)|^{q-2}x(\cdot)]_{\mathbb{V}^*} + {}_{\mathbb{V}}[x, c(\cdot, x(\cdot))]_{\mathbb{V}^*} \\
&\leqslant -\frac{1}{\epsilon}\|x\|_{L^q}^q + \frac{1}{\epsilon}\int_{\mathcal{O}} |(I - \epsilon\Delta)^{-1}x(u)|\cdot|x(u)|^{q-1}\mathrm{d}u \\
&\quad + \int_{\mathcal{O}} |(I - \epsilon\Delta)^{-1}x(u)|\cdot|c(u, x(u))|\mathrm{d}u \\
&\leqslant -\frac{1}{\epsilon}\|x\|_{L^q}^q + \frac{1}{\epsilon}\|(I - \epsilon\Delta)^{-1}x\|_{L^q}\cdot\||x(u)|^{q-1}\|_{L^{q^*}} \\
&\quad + C\|(I - \epsilon\Delta)^{-1}x\|_{L^q}\cdot\|1 + |x(u)|^{q-1}\|_{L^{q^*}} \\
&\leqslant -\frac{1}{\epsilon}\|x\|_{L^q}^q + \frac{1}{\epsilon}\|x\|_{L^q}\cdot\||x(u)|^{q-1}\|_{L^{q^*}} \\
&\quad + C\|x\|_{L^q}\cdot(1 + \||x(u)|^{q-1}\|_{L^{q^*}}) \\
&\leqslant C(1 + \|x\|_{L^q}^q).
\end{aligned} \tag{5.12}$$

As in (Step 1) of Theorem 4.5, the following equality holds in $\mathbb{V}^*$

$$x(t) = x_0 + \int_0^t A(x(s))\mathrm{d}s + M(t, x), \quad P_{x_0}\text{-a.s.}, \tag{5.13}$$

where $M(t,x)$ is a continuous $\mathbb{H}$-valued square integrable $\mathcal{F}_t$-martingale with respect to P_{x_0}, whose square variation process is given by

$$\langle M\rangle(t,x) := \int_0^t \|B(x(s))\|^2_{L_2(l^2;\mathbb{H})}\mathrm{d}s.$$

By Itô's formula and (5.12) we get for any $t > r$

$$\begin{aligned}\|x(t)\|^2_\epsilon &= \|x(r)\|^2_\epsilon + 2\int_r^t {}_{\mathbb{V}}[x(s), A(x(s))]_{\mathbb{V}^*}\mathrm{d}s \\ &\quad + 2\int_r^t \langle x(s), \mathrm{d}M(s,x)\rangle_{\mathbb{H}^\epsilon} + \int_r^t \|B(x(s))\|^2_{L_2(l^2;\mathbb{H}_0^\epsilon)}\mathrm{d}s \\ &\leqslant \|x(r)\|^2_\epsilon + C\int_r^t (1+\|x(s)\|^q_{\mathbb{V}})\mathrm{d}s \\ &\quad + 2\int_r^t \langle x(s), \mathrm{d}M(s,x)\rangle_{\mathbb{H}^\epsilon} + \int_r^t \|B(x(s))\|^2_{L_2(l^2;\mathbb{H}_0)}\mathrm{d}s. \end{aligned} \tag{5.14}$$

Note that by BDG's inequality and the dominated convergence theorem

$$\begin{aligned}&\mathbb{E}^{P_{x_0}}\left|\sup_{t\in[0,T]}\int_0^t \langle x(s), \mathrm{d}M(s,x)\rangle_{\mathbb{H}^\epsilon} - \int_0^t \langle x(s), \mathrm{d}M(s,x)\rangle_{\mathbb{H}}\right| \\ &= \mathbb{E}^{P_{x_0}}\left|\sup_{t\in[0,T]}\int_0^t \langle (I-\epsilon\Delta)^{-1}x(s) - x(s), \mathrm{d}M(s,x)\rangle_{\mathbb{H}}\right| \\ &\leqslant 3\mathbb{E}^{P_{x_0}}\left(\int_0^T \|(I-\epsilon\Delta)^{-1}x(s)-x(s)\|^2_{\mathbb{H}}\cdot\|B(x(s))\|^2_{L_2(l^2;\mathbb{H})}\mathrm{d}s\right)^{1/2} \\ &\to 0 \quad \text{as } \epsilon\downarrow 0.\end{aligned}$$

Hence, there exist a P_{x_0}-null set N and some subsequence ϵ_k such that for all $x \notin N$

$$\lim_{\epsilon_k\to 0}\int_0^t \langle x(s), \mathrm{d}M(s,x)\rangle_{\mathbb{H}^{\epsilon_k}} = \int_0^t \langle x(s), \mathrm{d}M(s,x)\rangle_{\mathbb{H}}, \quad \forall t \geqslant 0.$$

Taking firstly limits $\epsilon_k \downarrow 0$ for both sides of (5.14), and then $t \downarrow r$, we get for P_{x_0}-almost all $x \in \Omega$ and any $t \geqslant r$

$$\overline{\lim_{t\downarrow r}}\, \|x(t)\|_{\mathbb{H}} \leqslant \|x(r)\|_{\mathbb{H}}.$$

On the other hand, by the weak continuity of $x(t)$ in $\mathbb{H}$ we have

$$\underline{\lim_{t\downarrow r}}\, \|x(t)\|_{\mathbb{H}} \geqslant \|x(r)\|_{\mathbb{H}}.$$

Hence, $t \mapsto \|x(t)\|_{\mathbb{H}}$ is right continuous, and therefore also $t \mapsto x(t)$ in $\mathbb{H}$. □

6. Stochastic Navier–Stokes equations

In this section, we want to apply Theorem 4.7 to the following d-dimensional stochastic Navier–Stokes equation in a bounded domain $\mathcal{O} \subset \mathbb{R}^d$ with smooth boundary:

$$\begin{aligned}\mathrm{d}\mathbf{u}(t) &= [\Delta\mathbf{u}(t) - (\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t) + \nabla p(t) + \mathbf{f}(x,\mathbf{u}(t))]\,\mathrm{d}t \\ &\quad + \left[\nabla\tilde{p}_i(t) + \mathbf{h}_i(x,\mathbf{u}(t))\right]\mathrm{d}W^i_t,\end{aligned} \tag{6.1}$$

subject to the incompressibility condition

$$\operatorname{div}\mathbf{u}(t) = 0, \tag{6.2}$$

Dirichlet boundary condition

$$\mathbf{u}(t,x) = 0, \quad (t,x) \in \mathbb{R}_+ \times \partial\mathcal{O}, \tag{6.3}$$

and with the initial condition

$$\mathbf{u}(0) = \mathbf{u}_0, \tag{6.4}$$

where $p(t,x)$ and $\tilde{p}_i(t,x)$ are unknown scalar functions, $\mathbf{u}$ is the velocity vector, $\mathbf{f}$ and $\mathbf{h}$ are respectively functions from $\mathcal{O} \times \mathbb{R}^d$ to $\mathbb{R}^d$ and $\mathbb{R}^d \times l^2$, continuous with respect to the second variable, and satisfy for some $\kappa_0 > 0$ and $g \in L^2(\mathcal{O})$

$$|\mathbf{f}(x,\mathbf{u})| + \|\mathbf{h}(x,\mathbf{u})\|_{l^2} \leqslant \kappa_0 \cdot |\mathbf{u}| + g(x), \quad \forall (x,\mathbf{u}) \in \mathcal{O} \times \mathbb{R}^d. \tag{6.5}$$

Let $C^\infty_{0,\sigma}(\mathcal{O})^d$ be the space of all smooth d-dimensional vector fields on $\mathcal{O}$ with compact supports in $\mathcal{O}$ and divergence free. The completion of $C^\infty_{0,\sigma}(\mathcal{O})^d$ in $W^{k,p}_0(\mathcal{O})^d$ will be denoted by $\mathbf{W}^{k,p}_{0,\sigma}(\mathcal{O})$.

Below we choose

$$\mathbb{Y} = \mathbb{H} = \mathbf{W}^{0,2}_{0,\sigma}(\mathcal{O})$$

and

$$\mathbb{X} = (\mathbf{W}^{2+d,2}_{0,\sigma}(\mathcal{O}))^*, \qquad \mathbb{X}^* = \mathbf{W}^{2+d,2}_{0,\sigma}(\mathcal{O}).$$

Then (4.3) holds.

Let $\mathcal{P}$ be the orthogonal projection operator from $L^2(\mathcal{O})^d$ onto $\mathbb{H}$. We define the operators A and B as follows: for $\mathbf{u} \in C^\infty_{0,\sigma}(\mathcal{O})$

$$A(\mathbf{u}) := \mathcal{P}[\Delta\mathbf{u}] - \mathcal{P}[(\mathbf{u}\cdot\nabla)\mathbf{u}] + \mathcal{P}[\mathbf{f}(\cdot,\mathbf{u})]$$

and

$$B(\mathbf{u}) := \mathcal{P}[\mathbf{h}(\cdot,\mathbf{u})].$$

Then

Lemma 6.1. *For any* $\mathbf{u},\mathbf{v} \in C^\infty_{0,\sigma}(\mathcal{O})$*, we have*

$$\|\mathcal{P}[\Delta\mathbf{u}] - \mathcal{P}[\Delta\mathbf{v}]\|_{\mathbb{X}} \leqslant C\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}},$$
$$\|\mathcal{P}[(\mathbf{u}\cdot\nabla)\mathbf{u}] - \mathcal{P}[(\mathbf{v}\cdot\nabla)\mathbf{v}]\|_{\mathbb{X}} \leqslant C(\|\mathbf{u}\|_{\mathbb{H}} + \|\mathbf{v}\|_{\mathbb{H}}) \cdot \|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}}.$$

In particular, we can extend the operators A and B to $\mathbb{H}$ such that for $\mathbf{u} \in \mathbb{H}$, $A(\mathbf{u}) \in \mathbb{X}$ and $B(\mathbf{u}) \in L_2(l^2;\mathbb{H})$.

Proof. We only prove the second assertion, the first can be proved analogously. By the Sobolev embedding theorem we have

$$\begin{aligned}
&\|\mathcal{P}[(\mathbf{u}\cdot\nabla)\mathbf{u}] - \mathcal{P}[(\mathbf{v}\cdot\nabla)\mathbf{v}]\|_{\mathbb{X}} \\
&\quad = \sup_{\mathbf{w}\in C^\infty_{0,\sigma}(\mathcal{O}):\|\mathbf{w}\|_{2+d,2}\leqslant 1} |\langle \mathcal{P}[(\mathbf{u}\cdot\nabla)\mathbf{u}] - \mathcal{P}[(\mathbf{v}\cdot\nabla)\mathbf{v}], \mathbf{w}\rangle_{\mathbb{H}}| \\
&\quad = \sup_{\mathbf{w}\in C^\infty_{0,\sigma}(\mathcal{O}):\|\mathbf{w}\|_{2+d,2}\leqslant 1} |\langle \nabla(\mathbf{u}\otimes\mathbf{u} - \mathbf{v}\otimes\mathbf{v}), \mathbf{w}\rangle_{\mathbb{H}}| \\
&\quad = \sup_{\mathbf{w}\in C^\infty_{0,\sigma}(\mathcal{O}):\|\mathbf{w}\|_{2+d,2}\leqslant 1} |\langle \mathbf{u}\otimes\mathbf{u} - \mathbf{v}\otimes\mathbf{v}, \nabla\mathbf{w}\rangle_{\mathbb{H}}| \\
&\quad \leqslant C\left[\sup_{\mathbf{w}\in C^\infty_{0,\sigma}(\mathcal{O}):\|\mathbf{w}\|_{2+d,2}\leqslant 1} \|\nabla\mathbf{w}\|_{C(\bar{\mathcal{O}})}\right]\cdot \||\mathbf{u}\otimes\mathbf{u} - \mathbf{v}\otimes\mathbf{v}|\|_{L^1(\mathcal{O})} \\
&\quad \leqslant C(\|\mathbf{u}\|_{\mathbb{H}} + \|\mathbf{v}\|_{\mathbb{H}})\cdot\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}}. \quad \square
\end{aligned}$$

Thus, we can write the system (6.1)–(6.4) in the following abstract form:

$$\mathrm{d}\mathbf{u}(t) = A(\mathbf{u}(t))\mathrm{d}t + B(\mathbf{u}(t))\mathrm{d}W(t), \qquad \mathbf{u}(0) = \mathbf{u}_0. \tag{6.6}$$

In order to use Theorem 4.7, we define the functional $\mathcal{N}_1$ on $\mathbb{Y}$ as follows:

$$\mathcal{N}_1(\mathbf{u}) := \begin{cases} \|\nabla\mathbf{u}\|^2_{L^2(\mathcal{O})}, & \text{if } \mathbf{u}\in\mathbf{W}^{1,2}_{0,\sigma}(\mathcal{O}), \\ +\infty, & \text{otherwise.}\end{cases}$$

As in the proof of Lemma 5.1, we can prove that $\mathcal{N}_1 \in \mathfrak{A}^2$. The following is the main result in this section.

Theorem 6.2. *Assume (6.5). Then there exists an almost sure Markov family* $(P_{\mathbf{u}_0})_{\mathbf{u}_0\in\mathbb{H}}$ *for Eq. (6.6).*

Proof. By Theorem 4.7, it suffices to check **(C1)–(C3)** for the above A and B.

For **(C1)**, using Lemma 6.1, as in the proof of (Step 1) in Lemma 5.2, we can prove the demi-continuity of A and B.

For **(C2)**, noting that for $\mathbf{u}\in\mathbb{X}^*$

$${}_{\mathbb{X}}\langle \mathcal{P}[(\mathbf{u}\cdot\nabla)\mathbf{u}], \mathbf{u}\rangle_{\mathbb{X}^*} = \langle \mathcal{P}[(\mathbf{u}\cdot\nabla)\mathbf{u}], \mathbf{u}\rangle_{\mathbb{H}} = \langle (\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{u}\rangle_{L^2} = 0,$$

by (6.5) we have

$${}_{\mathbb{X}}\langle A(\mathbf{u}), \mathbf{u}\rangle_{\mathbb{X}^*} = -\mathcal{N}_1(\mathbf{u}) + \langle \mathbf{f}(\cdot,\mathbf{u}), \mathbf{u}\rangle_{L^2} \leqslant -\mathcal{N}_1(\mathbf{u}) + C(\|\mathbf{u}\|^2_{\mathbb{H}} + 1).$$

For **(C3)**, it is clear that by Lemma 6.1

$$\|A(\mathbf{u})\|_{\mathbb{X}} \leqslant C(\|\mathbf{u}\|^2_{\mathbb{H}} + 1)$$

and

$$\|B(\mathbf{u})\|_{L_2(l^2;\mathbb{H})} \leqslant C(\|\mathbf{u}\|_{\mathbb{H}} + 1).$$

This completes the proof. $\square$

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Appendix A. Section 2

A.1. *Proof of* Theorem 2.7

For each $f \in C_b(\mathbb{X})$ and $\lambda > 0$, we define

$$L_f^\lambda(t,x) := \int_t^\infty \mathrm{e}^{-\lambda s} f(x(s))\mathrm{d}s, \quad t \geqslant 0,\ x \in \Omega^t$$

and

$$J_f^\lambda(P) := \mathbb{E}^P(L_f^\lambda(0,\cdot)), \quad P \in \mathscr{P}(\Omega).$$

It is clear that

$$L_f^\lambda(t,x) = \mathrm{e}^{-\lambda t} L_f^\lambda(0, \Phi_t^{-1}(x)), \quad x \in \Omega^t. \tag{A.1}$$

For each $b \in \mathbb{B}$, we define

$$M_f^\lambda(b) := \sup_{P \in \mathscr{C}(b)} J_f^\lambda(P)$$

and

$$\mathscr{C}_f^\lambda(b) := \{P \in \mathscr{C}(b) : J_f^\lambda(P) = M_f^\lambda(b)\}. \tag{A.2}$$

Lemma A.1. *Let $(\mathscr{C}(b))_{b\in\mathbb{B}}$ be an almost sure pre-Markov family (resp. pre-Markov family) with $\mathscr{C}(b)$ non-empty and convex. Then for any $f \in C_b(\mathbb{X})$ and $\lambda > 0$, $(\mathscr{C}_f^\lambda(b))_{b\in\mathbb{B}}$ is still an almost sure pre-Markov family (resp. pre-Markov family) with $\mathscr{C}_f^\lambda(b)$ non-empty and convex.*

Proof. Since $L_f^\lambda(0,\cdot) \in C_b(\Omega)$, the map $P \mapsto \mathbb{E}^P(L_f^\lambda(0,\cdot)) = J_f^\lambda(P)$ is linear and continuous with respect to weak convergence. By the compactness and convexity of $\mathscr{C}(b)$, we know $\mathscr{C}_f^\lambda(b) \in \mathrm{Comp}(\mathscr{P}_{\mathbb{B}}(\Omega))$ is non-empty and convex. By [17, Lemma 12.1.7], the map $b \mapsto \mathscr{C}_f^\lambda(b)$ is measurable.

We now prove the disintegration and reconstruction properties for $(\mathscr{C}_f^\lambda(b))_{b\in\mathbb{B}}$. Fix a $b \in \mathbb{B}$ and $P \in \mathscr{C}_f^\lambda(b) \subset \mathscr{C}(b)$. Let $\mathbb{T}_P \subset (0,\infty)$ be the corresponding exceptional set of P. We also fix a $t \notin \mathbb{T}_P$.

Let $P(\cdot|\mathcal{F}_t)(x)$ be an r.c.p.d. of P with respect to $\mathcal{F}_t$. Define

$$N_1 := \{x : x(t) \notin \mathbb{B}\} \cup \{x : x(t) \in \mathbb{B},\ P(\Phi_t(\cdot)|\mathcal{F}_t)(x) \notin \mathscr{C}(x(t))\},$$

and

$$N_2 := \{x \in N_1^c : P(\Phi_t(\cdot)|\mathcal{F}_t)(x) \notin \mathscr{C}_f^\lambda(x(t))\}.$$

By the disintegration property for $(\mathscr{C}(b))_{b\in\mathbb{B}}$, one knows that $N_1 \in \mathcal{F}_t$ and $P(N_1) = 0$. So, by [17, Lemma 12.1.9], $N_2 \in \mathcal{F}_t$. We want to show that $P(N_2) = 0$.

By the measurable selection theorem (cf. [17, Theorem 12.1.10]), there is a measurable map $b \mapsto R_b$ so that $R_b \in \mathscr{C}_f^\lambda(b)$ for all $b \in \mathbb{B}$. Define

$$Q_x := \begin{cases} P(\cdot|\mathcal{F}_t)(x), & x \notin N_1 \cup N_2, \\ R_{x(t)} \circ \Phi_t^{-1}, & x \in N_2, \\ \delta_x, & x \in N_1, \end{cases}$$

where δ_x is the Dirac measure concentrated on x.

By the reconstruction property for $(\mathscr{C}(b))_{b\in\mathbb{B}}$, we have $P \otimes_t Q \in \mathscr{C}(b)$. Hence,

$$\begin{aligned} J_f^\lambda(P) &= M_f^\lambda(b) \geqslant J_f^\lambda(P \otimes_t Q) \\ \text{(by (2.5))} &= \mathbb{E}^P\left(\int_0^t \mathrm{e}^{-\lambda s} f(x(s))\mathrm{d}s\right) + \mathbb{E}^{P\otimes_t Q}\left(L_f^\lambda(t,\cdot)\right) \\ \text{(by (2.2))} &= J_f^\lambda(P) - \int_\Omega\int_\Omega L_f^\lambda(t,y)P(\mathrm{d}y|\mathcal{F}_t)(x)P(\mathrm{d}x) \\ \text{(by (2.2), (2.5) and (2.6))} &+ \int_\Omega\int_\Omega L_f^\lambda(t,y)Q_x(\mathrm{d}y)P(\mathrm{d}x). \end{aligned}$$

Thus, by the definition of Q_x and $P(N_1) = 0$ we have

$$\begin{aligned} 0 &\geqslant \int_{N_2}\int_\Omega L_f^\lambda(t,\Phi_t y)R_{y(t)}(\mathrm{d}y)P(\mathrm{d}x) - \int_{N_1\cup N_2}\int_\Omega L_f^\lambda(t,y)P(\mathrm{d}y|\mathcal{F}_t)(x)P(\mathrm{d}x) \\ &= \mathrm{e}^{-\lambda t}\int_{N_2}\left[\int_\Omega L_f^\lambda(0,y)R_{x(t)}(\mathrm{d}y) - \int_\Omega L_f^\lambda(0,\Phi_t^{-1}y)P(\mathrm{d}y|\mathcal{F}_t)(x)\right]P(\mathrm{d}x), \end{aligned}$$

that is,

$$\int_{N_2}\left[J_f^\lambda(R_{x(t)}) - J_f^\lambda(P(\Phi_t(\cdot)|\mathcal{F}_t)(x))\right]P(\mathrm{d}x) \leqslant 0. \tag{A.3}$$

On the other hand, for each $x \in N_2$, by the definition of N_2 we have $P(\Phi_t(\cdot)|\mathcal{F}_t)(x) \notin \mathscr{C}_f^\lambda(x(t))$, and in view of $R_{x(t)} \in \mathscr{C}_f^\lambda(x(t))$

$$J_f^\lambda(P(\Phi_t(\cdot)|\mathcal{F}_t)(x)) < M_f^\lambda(x(t)) = J_f^\lambda(R_{x(t)}),$$

which together with (A.3) gives

$$P(N_2) = 0.$$

The disintegration property for $(\mathscr{C}_f^\lambda(b))_{b\in\mathbb{B}}$ now follows.

Let us now look at the reconstruction property for $(\mathscr{C}_f^\lambda(b))_{b\in\mathbb{B}}$. Let $x \mapsto Q_x$ be a mapping from Ω to $\mathscr{P}_{\mathbb{B}}(\Omega)$ and satisfy the assumptions in Theorem 2.2. Assume that there is a P-null set $N \in \mathcal{F}_t$ such that for all $x \notin N$

$$x(t) \in \mathbb{B}, \quad Q_x \in \mathscr{C}_f^\lambda(x(t)) \circ \Phi_t^{-1}.$$

By the reconstruction property for $(\mathscr{C}(b))_{b\in\mathbb{B}}$, we have $P \otimes_t Q \in \mathscr{C}(b)$. Moreover, by the above calculations we have

$$\begin{aligned}
J_f^\lambda(P \otimes_t Q) - \mathbb{E}^P\left(\int_0^t e^{-\lambda s} f(x(s))\mathrm{d}s\right) &= \mathbb{E}^{P\otimes_t Q}(L_f^\lambda(t,\cdot)) \\
&= \int_\Omega \mathbb{E}^{Q_x}(L_f^\lambda(t,\cdot))P(\mathrm{d}x) = \mathrm{e}^{-\lambda t}\int_\Omega L_f^\lambda(Q_x \circ \Phi_t)P(\mathrm{d}x) \\
&\geqslant \mathrm{e}^{-\lambda t}\int_\Omega L_f^\lambda\left(P(\cdot|\mathcal{F}_t)(x) \circ \Phi_t\right) P(\mathrm{d}x) \\
&= \int_\Omega\int_\Omega L_f^\lambda(t,y)P(\mathrm{d}y|\mathcal{F}_t)(x)P(\mathrm{d}x) = \mathbb{E}^P(L_f^\lambda(t,\cdot)),
\end{aligned}$$

that is

$$J_f^\lambda(P \otimes_t Q) \geqslant J_f^\lambda(P).$$

We have therefore shown that $P \otimes_t Q \in \mathscr{C}_f^\lambda(b)$ by $P \in \mathscr{C}_f^\lambda(b)$, which completes the proof. □

Lemma A.2. *Let $\mathscr{C}$ be a convex and closed subset of the Polish space $(\mathscr{P}(\Omega), \mathbf{d}_\Omega)$. Let $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ be another probability space, and $\hat{\Omega} \ni \hat{\omega} \mapsto P_{\hat{\omega}} \in \mathscr{C}$ an $\hat{\mathcal{F}}/(\mathscr{P}(\Omega))$-measurable mapping. Then*

$$\int_{\hat{\Omega}} P_{\hat{\omega}}(\cdot)\hat{P}(\mathrm{d}\hat{\omega}) \in \mathscr{C}.$$

Proof. First of all, it is easy to see that

$$P(\cdot) := \int_{\hat{\Omega}} P_{\hat{\omega}}(\cdot)\hat{P}(\mathrm{d}\hat{\omega}) \in \mathscr{P}(\Omega).$$

By [14, Lemma A.1.4], there exists a sequence of $P_{\hat{\omega}}^n \in \mathscr{C}$ only taking a finite number of values such that for each $\hat{\omega} \in \hat{\Omega}$

$$\mathbf{d}_\Omega(P_{\hat{\omega}}^n, P_{\hat{\omega}}) \downarrow 0 \quad \text{as } n \to \infty. \tag{A.4}$$

In particular, $P_{\hat{\omega}}^n$ has the following form

$$P_{\hat{\omega}}^n = \sum_{j=1}^{N_n} \mu_j \cdot 1_{\Lambda_j}(\hat{\omega}),$$

where $\mu_j \in \mathscr{C}$, and $\Lambda_j \in \hat{\mathcal{F}}$, $\Lambda_j \cap \Lambda_i = \emptyset$, $i \neq j$.

By the convexity of $\mathscr{C}$, one knows that

$$P^n := \int_{\hat{\Omega}} P_{\hat{\omega}}^n(\cdot)\hat{P}(\mathrm{d}\hat{\omega}) = \sum_{j=1}^{N_n} \mu_j \cdot \hat{P}(\Lambda_j) \in \mathscr{C}.$$

On the other hand, by (A.4) and the dominated convergence theorem, we have for any $f \in C_b(\Omega)$

$$\lim_{n\to\infty} \mathbb{E}^{P^n}(f) = \lim_{n\to\infty}\int_{\hat{\Omega}} \mathbb{E}^{P_{\hat{\omega}}^n}(f)\hat{P}(\mathrm{d}\hat{\omega}) = \int_{\hat{\Omega}} \lim_{n\to\infty} \mathbb{E}^{P_{\hat{\omega}}^n}(f)\hat{P}(\mathrm{d}\hat{\omega}) = \mathbb{E}^P(f).$$

Now the assertion follows by the closedness of $\mathscr{C}$. □

Proof of Theorem 2.7. Let $(\sigma_n)_{n\in\mathbb{N}}$ be a dense subset of $(0,\infty)$ and $(\varphi_n)_{n\in\mathbb{N}}$ a dense subset of $U_b(\mathbb{X})$, where $U_b(\mathbb{X})$ is the space of bounded and uniformly continuous functions on $\mathbb{X}$ equipped with the supremum norm. Let $(\lambda_n, f_n)_{n\in\mathbb{N}}$ be an enumeration of $(\sigma_n, \varphi_m)_{n,m\in\mathbb{N}}$. For each $b \in \mathbb{B}$, set $\mathscr{C}_0(b) = \mathscr{C}(b)$, and define inductively

$$\mathscr{C}_{n+1}(b) = \mathscr{C}^{\lambda_n}_{f_n}(b),$$

where $\mathscr{C}^{\lambda_n}_{f_n}(b)$ is defined by (A.2) in terms of $\mathscr{C}_n(b)$.

By Lemma A.1, each $(\mathscr{C}_n(b))_{b\in\mathbb{B}}$ is an almost sure pre-Markov family with non-empty convex values. Since $\mathscr{C}_{n+1}(b) \subset \mathscr{C}_n(b)$, it is clear that for each $b \in \mathbb{B}$,

$$\mathscr{C}_\infty(b) := \cap_n \mathscr{C}_n(b) \in \mathrm{Comp}(\mathscr{P}_{\mathbb{B}}(\Omega)),$$

and $(\mathscr{C}_\infty(b))_{b\in\mathbb{B}}$ is still an almost sure pre-Markov family with non-empty convex values. Thus, if we can show that $\mathscr{C}_\infty(b)$ has only one element for each $b \in \mathbb{B}$, the result then follows.

Claim: For any $b \in \mathbb{B}$, $P, Q \in \mathscr{C}_\infty(b)$ and bounded measurable function f on $\mathbb{X}$,

$$\mathbb{E}^P(f(x(t))) = \mathbb{E}^Q(f(x(t))), \quad \forall t \geqslant 0. \tag{A.5}$$

Suppose that $P, Q \in \mathscr{C}_\infty(b)$. By the definition of $\mathscr{C}_\infty(b)$, we have for all $n, m \in \mathbb{N}$

$$\mathbb{E}^P\left(\int_0^\infty \mathrm{e}^{-\lambda_n t} f_m(x(t))\mathrm{d}t\right) = \mathbb{E}^Q\left(\int_0^\infty \mathrm{e}^{-\lambda_n t} f_m(x(t))\mathrm{d}t\right).$$

Since $(\lambda_n)_{n\in\mathbb{N}}$ is dense in $\mathbb{R}_+$, it follows from the uniqueness of the Laplace transform that

$$\mathbb{E}^P(f_m(x(t))) = \mathbb{E}^Q(f_m(x(t))), \quad \forall t \geqslant 0,\ m \in \mathbb{N}.$$

By a monotone class argument, we obtain (A.5).

In the following, we fix $b \in \mathbb{B}$ and $P^1, P^2 \in \mathscr{C}_\infty(b)$, and prove $P^1 = P^2$. Let $\mathbb{T}_{P^i}$ be the exceptional set corresponding to P^i. We only need to prove that for any $0 \leqslant t_1 < t_2 < \cdots < t_n < \infty$ with $t_i \notin \mathbb{T}_{P^1} \cup \mathbb{T}_{P^2}$ and any bounded measurable functions f_i on $\mathbb{X}$

$$\mathbb{E}^{P^1}(f_1(x(t_1))\cdots f_n(x(t_n))) = \mathbb{E}^{P^2}(f_1(x(t_1))\cdots f_n(x(t_n))).$$

Suppose the above equality holds for n. Let $\mathcal{G}_n = \sigma\{x(t_i) : i = 1, \ldots, n\}$. Then

$$P^1 = P^2 \quad \text{on } \mathcal{G}_n.$$

By the disintegration property for P^i, there are $N_i \in \mathcal{F}_{t_n}$ with $P^i(N_i) = 0$ such that for all $x \notin N_i$,

$$x(t_n) \in \mathbb{B} \quad \text{and} \quad P^i(\Phi_{t_n}(\cdot)|\mathcal{F}_{t_n})(x) \in \mathscr{C}_\infty(x(t_n)).$$

On the other hand, since $\mathcal{G}_n \subset \mathcal{F}_{t_n}$, there are $A_i \in \mathcal{G}_n$ with $P^i(A_i) = 0$ such that for all $x \notin A_i$

$$P^i(\cdot|\mathcal{G}_n)(x) = \int_\Omega P^i(\cdot|\mathcal{F}_{t_n})(y) P^i(\mathrm{d}y|\mathcal{G}_n)(x).$$

Since

$$0 = P^i(N_i) = \int_\Omega P^i(N_i|\mathcal{G}_n)(x) P^i(\mathrm{d}x),$$

there are $B_i \in \mathcal{G}_n$ with $P^i(B_i) = 0$ such that for all $x \notin B_i$

$$P^i(N_i|\mathcal{G}_n)(x) = 0.$$

Set $C_i = A_i \cup B_i \in \mathcal{G}_n$. For $x \notin C_i$, we have

$$P^i(\Phi_{t_n}(\cdot)|\mathcal{G}_n)(x) = \int_{N_i^c} P^i(\Phi_{t_n}(\cdot)|\mathcal{F}_{t_n})(y) P^i(\mathrm{d}y|\mathcal{G}_n)(x)$$

$$\text{by (2.3)} = \int_{N_i^c \cap \{y: y(t_n)=x(t_n)\}} P^i(\Phi_{t_n}(\cdot)|\mathcal{F}_{t_n})(y) P^i(\mathrm{d}y|\mathcal{G}_n)(x).$$

If $x(t_n) \notin \mathbb{B}$, then $N_i^c \cap \{y : y(t_n) = x(t_n)\} = \emptyset$. So, for $x \notin C_i$ and $x(t_n) \notin \mathbb{B}$

$$P^1(\Phi_{t_n}(\cdot)|\mathcal{G}_n)(x) = P^2(\Phi_{t_n}(\cdot)|\mathcal{G}_n)(x) = 0.$$

For $x \notin C_i$ and $x(t_n) \in \mathbb{B}$, noting that for $y \in N_i^c \cap \{y : y(t_n) = x(t_n)\}$

$$P^i(\Phi_{t_n}(\cdot)|\mathcal{F}_{t_n})(y) \in \mathscr{C}_\infty(y(t_n)) = \mathscr{C}_\infty(x(t_n))$$

and by the convexity and compactness of $\mathscr{C}_\infty(b)$ as well as by Lemma A.2, we get

$$P^i(\Phi_{t_n}(\cdot)|\mathcal{G}_n)(x) \in \mathscr{C}_\infty(x(t_n)).$$

Set $\tilde{N} := C_1 \cup C_2 \in \mathcal{G}_n$. By the induction hypothesis we have

$$P^1(\tilde{N}) = P^2(\tilde{N}) \leqslant P^2(C_1) + P^2(C_2) = P^1(C_1) + P^2(C_2) = 0.$$

By the above *Claim*, we have for $x \notin \tilde{N}$

$$\int_\Omega f_{n+1}(y(t_{n+1})) P^1(\mathrm{d}y|\mathcal{G}_n)(x) = \int_\Omega f_{n+1}(y(t_{n+1})) P^2(\mathrm{d}y|\mathcal{G}_n)(x),$$

i.e.,

$$\mathbb{E}^{P_1}(f_{n+1}(x(t_{n+1}))|\mathcal{G}_n) = \mathbb{E}^{P_2}(f_{n+1}(x(t_{n+1}))|\mathcal{G}_n).$$

The proof is thus completed by induction. □

Appendix B. Section 3

We need the following three lemmas in Section 3 about regular conditional probabilities, whose proof idea comes from [17, Theorem 1.2.10] and [8, Proposition B.4].

Lemma B.1. *Let $P \in \mathscr{P}(\Omega)$ and $\xi \in L^1(\Omega, \mathcal{F}, P)$. For $r \geqslant 0$, let $Q_x^r := P(\cdot|\mathcal{F}_r)(x)$ be an r.c.p.d. of P with respect to $\mathcal{F}_r$. Then for $s \geqslant r$, there exists a P-null set $N_{s,\xi} \in \mathcal{F}_r$ such that for all $x \in N_{s,\xi}^c$*

$$\mathbb{E}^P(\xi|\mathcal{F}_s) = \mathbb{E}^{Q_x^r}(\xi|\mathcal{F}_s) = \mathbb{E}^{Q_x^r}(\xi|\mathcal{F}_s^r), \quad Q_x^r\text{-a.s.} \tag{B.1}$$

Proof. For $A \in \mathcal{F}_r$ and $B \in \mathcal{F}_s$, by (2.2) we have

$$\begin{aligned}\int_A \mathbb{E}^{Q_x^r}(1_B \mathbb{E}^P(\xi|\mathcal{F}_s)) P(\mathrm{d}x) &= \int_A \mathbb{E}^P(1_B \mathbb{E}^P(\xi|\mathcal{F}_s)|\mathcal{F}_r) P(\mathrm{d}x) \\ &= \mathbb{E}^P(1_A \cdot 1_B \cdot \mathbb{E}^P(\xi|\mathcal{F}_s)) \\ &= \mathbb{E}^P(1_A \cdot 1_B \cdot \xi) = \int_A \mathbb{E}^{Q_x^r}(1_B \xi) P(\mathrm{d}x) \\ &= \int_A \mathbb{E}^{Q_x^r}(1_B \mathbb{E}^{Q_x^r}(\xi|\mathcal{F}_s)) P(\mathrm{d}x).\end{aligned}$$

Hence, there is a P-null set $N_B \in \mathcal{F}_r$ such that for all $x \notin N_B$

$$\mathbb{E}^{Q_x^r}(1_B \mathbb{E}^P(\xi|\mathcal{F}_s)) = \mathbb{E}^{Q_x^r}(1_B \mathbb{E}^{Q_x^r}(\xi|\mathcal{F}_s)).$$

Since $\mathcal{F}_s$ is countably generated, by a monotone class argument, we may find a common P-null set $N_{s,\xi} \in \mathcal{F}_r$ such that for all $x \notin N_{s,\xi}$ and $B \in \mathcal{F}_s$

$$\mathbb{E}^{Q_x^r}(1_B \mathbb{E}^P(\xi|\mathcal{F}_s)) = \mathbb{E}^{Q_x^r}(1_B \mathbb{E}^{Q_x^r}(\xi|\mathcal{F}_s)).$$

The first equality in (B.1) now follows. The second equality is straightforward by Lemma 2.1 and (2.4). $\square$

Lemma B.2. *Let $\mathbb{D} := \{(t, s) : 0 \leqslant s \leqslant t < \infty\}$. Let $\xi, \eta : \mathbb{D} \to \mathbb{R}_+$ be two measurable processes on $(\Omega, \mathcal{F})$. Given $P \in \mathscr{P}(\Omega)$ and $r \geqslant 0$, let $Q_x^r := P(\cdot|\mathcal{F}_r)(x)$ be an r.c.p.d. of P with respect to $\mathcal{F}_r$. Suppose that:*

(i) *for each $s \geqslant 0$, the map $t \mapsto \xi(t, s)$ is a.s. increasing, and $t \mapsto \eta(t, s)$ is a.s. right continuous, $\eta(t, s)$ is $\mathcal{F}_s$-measurable for any $t \geqslant s$;*

(ii) *for each $(t, s) \in \mathbb{D}$,*

$$\xi(t, s), \eta(t, s) \in L^1(\Omega, \mathcal{F}, P)$$

and

$$\xi(t, \cdot), \eta(t, \cdot) \in L^1(0, t; L^1(\Omega, \mathcal{F}, P));$$

(iii) *for any $x \in \Omega$ and $t \geqslant s \geqslant r$*

$$\xi(t, s, \Phi_r x) = \xi(t - r, s - r, x)$$

and

$$\eta(t, s, \Phi_r x) = \eta(t - r, s - r, x),$$

where Φ is defined by (2.1).

Then the following three statements are equivalent:

(I) *There is a Lebesgue null set $\mathbb{T}_r \subset (r, \infty)$ such that for any $r \leqslant s \notin \mathbb{T}_r$ and $t \geqslant s$*

$$\mathbb{E}^P(\xi(t, s)|\mathcal{F}_s) \leqslant \eta(t, s), \quad P\text{-a.s.}$$

(II) *For some P-null set $N \in \mathcal{F}_r$ and each $x \in N^c$, there is a Lebesgue null set $\mathbb{T}_{r,x} \subset (r, \infty)$ such that for any $r \leqslant s \notin \mathbb{T}_{r,x}$ and any $t \geqslant s$*

$$\mathbb{E}^{Q_x^r}(\xi(t, s)|\mathcal{F}_s^r) \leqslant \eta(t, s), \quad Q_x^r\text{-a.s.}$$

(III) *For some P-null set $N \in \mathcal{F}_r$ and each $x \in N^c$, there is a Lebesgue null set $\mathbb{T}_{r,x} \subset (0, \infty)$ such that for any $0 \leqslant s \notin \mathbb{T}_{r,x}$ and any $t \geqslant s$*

$$\mathbb{E}^{Q_x^r \circ \Phi_r}(\xi(t, s)|\mathcal{F}_s) \leqslant \eta(t, s), \quad Q_x^r \circ \Phi_r\text{-a.s.}$$

Moreover, $\mathbb{T}_r = \emptyset \Leftrightarrow \mathbb{T}_{r,x} = \emptyset$.

Proof. (I) $\Rightarrow$ (II). Fix $t > r$. For $B \in \mathcal{F}_t$, by Lemma B.1 we have

$$\begin{aligned} \zeta(x, s) &:= \mathbb{E}^{Q_x^r}(1_B \cdot \mathbb{E}^{Q_x^r}(\eta(t, s) - \xi(t, s)|\mathcal{F}_s^r)) \\ &= \mathbb{E}^{Q_x^r}(1_B \cdot \mathbb{E}^P(\eta(t, s) - \xi(t, s)|\mathcal{F}_s)). \end{aligned}$$

Recalling that conditional expectations with respect to $\mathcal{F}_s^r$ have cadlag versions in s and by a monotone class argument, one easily sees that

$$(x, s) \mapsto \zeta(x, s) \quad \text{is } \mathcal{F} \times \mathcal{B}([r, t])\text{-measurable.}$$

Hence, for any $T \in \mathcal{B}([r, t])$, $A \in \mathcal{F}_r$ and $B \in \mathcal{F}_t$, we have by Fubini's theorem

$$\begin{aligned}
&\int_A \int_r^t \mathbb{E}^{Q_x^r}(1_T \cdot 1_B \cdot \mathbb{E}^{Q_x^r}(\eta(t, s) - \xi(t, s)|\mathcal{F}_s^r))\mathrm{d}s\, P(\mathrm{d}x) \\
&= \int_A \int_r^t 1_T \cdot \mathbb{E}^{Q_x^r}(1_B \cdot \mathbb{E}^P(\eta(t, s) - \xi(t, s)|\mathcal{F}_s))\mathrm{d}s\, P(\mathrm{d}x) \\
&= \int_r^t 1_T \cdot \mathbb{E}^P \left(1_A \cdot \mathbb{E}^P(1_B \cdot \mathbb{E}^P(\eta(t, s) - \xi(t, s)|\mathcal{F}_s)|\mathcal{F}_r)\right) \mathrm{d}s \\
&= \int_r^t 1_T \cdot \mathbb{E}^P \left(1_{A\cap B} \cdot \left[\eta(t, s) - \mathbb{E}^P(\xi(t, s)|\mathcal{F}_s)\right]\right) \mathrm{d}s \geqslant 0.
\end{aligned}$$

As in the proof of Lemma B.1, since $\mathcal{B}([r, t])$ and $\mathcal{F}_t$ are countably generated, by a monotone class argument we may find a common P-null set $N_t \in \mathcal{F}_r$ such that for all $x \in N_t^c$ and any $T \in \mathcal{B}([r, t])$, $B \in \mathcal{F}_t$

$$\int_r^t \mathbb{E}^{Q_x^r}(1_T \cdot 1_B \cdot \mathbb{E}^{Q_x^r}(\eta(t, s) - \xi(t, s)|\mathcal{F}_s^r))\mathrm{d}s \geqslant 0.$$

Hence, there exists a Lebesgue null set $\mathbb{T}_{t,x} \subset [r, t]$ such that for all $s \notin \mathbb{T}_{t,x}$,

$$\mathbb{E}^{Q_x^r}(\eta(t, s) - \xi(t, s)|\mathcal{F}_s^r) \geqslant 0, \quad Q_x^r\text{-a.s.},$$

i.e.,

$$\mathbb{E}^{Q_x^r}(\xi(t, s)|\mathcal{F}_s^r) \leqslant \eta(t, s), \quad Q_x^r\text{-a.s.}$$

Let $\mathbb{Q}_r$ be the set of all rational points in (r, ∞). Set $N := \cup_{t\in\mathbb{Q}_r} N_t$, then $N \in \mathcal{F}_r$ is a P-null set. For each $x \in N^c$, set $\mathbb{T}_{r,x} := \cup_{t\in\mathbb{Q}_r} \mathbb{T}_{t,x}$. Let $t > s > r$ with $s \notin \mathbb{T}_{r,x}$. Choose a sequence of points t_n in $\mathbb{Q}$ such that $t_n \downarrow t$. By (i) and Fatou's lemma, we then obtain

$$\begin{aligned}
\mathbb{E}^{Q_x^r}(\xi(t, s)|\mathcal{F}_s^r) &\leqslant \mathbb{E}^{Q_x^r}\left(\varliminf_{n\to\infty} \xi(t_n, s)|\mathcal{F}_s^r\right) \\
&\leqslant \varliminf_{n\to\infty} \mathbb{E}^{Q_x^r}(\xi(t_n, s)|\mathcal{F}_s^r) \\
&\leqslant \varliminf_{n\to\infty} \eta(t_n, s) = \eta(t, s), \quad Q_x^r\text{-a.s.}
\end{aligned}$$

Lastly, we need to show $r \notin \mathbb{T}_{r,x}$. This can be done as above by taking $s = r$ and without integrating with respect to s.

(II) $\Rightarrow$ (I) is completely the same as (I) $\Rightarrow$ (II) by reversing the arguments.

(II) $\Leftrightarrow$ (III) is direct from (iii). Indeed, for any $A \in \mathcal{F}_{s-r}$

$$\begin{aligned}
\mathbb{E}^{Q_x^r\circ\Phi_r}(1_A \cdot \xi(t - r, s - r)) &= \mathbb{E}^{Q_x^r}(1_{\Phi_r A} \cdot \xi(t - r, s - r), \Phi_r^{-1}(\cdot)) \\
&= \mathbb{E}^{Q_x^r}(1_{\Phi_r A} \cdot \xi(t, s)) \leqslant \mathbb{E}^{Q_x^r}(1_{\Phi_r A} \cdot \eta(t, s)) \\
&= \mathbb{E}^{Q_x^r}(1_{\Phi_r A} \cdot \eta(t - r, s - r), \Phi_r^{-1}(\cdot)) \\
&= \mathbb{E}^{Q_x^r\circ\Phi_r}(1_A \cdot \eta(t - r, s - r)).
\end{aligned}$$

This completes the proof. $\square$

The following lemma can be proved as Lemma B.2 (cf. [17, Theorem 1.2.10]).

Lemma B.3. *Let $(M(t))_{t\geqslant 0}$ and $(K(t))_{t\geqslant 0}$ be $\mathcal{F}_t$-adapted real-valued processes on $(\Omega,\mathcal{F})$ which satisfy for $x\in\Omega$, $t\geqslant r\geqslant 0$*

$$M(t,\Phi_r x)=M(t-r,x),\qquad K(t,\Phi_r x)=K(t-r,x).$$

Given $P\in\mathscr{P}(\Omega)$ and $r\geqslant 0$, let $Q_x^r := P(\cdot|\mathcal{F}_r)(x)$ be an r.c.p.d. of P with respect to $\mathcal{F}_r$. Assume that for each $t\geqslant 0$, $\mathbb{E}^P(K(t))<+\infty$. Then the following statements are equivalent:

(I) *$(M_t,\mathcal{F}_t,P)_{t\geqslant r}$ is a continuous martingale with square variation process $(K(t))_{t\geqslant r}$.*
(II) *There exists a P-null set $N\in\mathcal{F}_r$ such that for all $x\notin N$, $(M_t,\mathcal{F}_t,Q_x^r)_{t\geqslant r}$ is a continuous martingale with square variation process $(K(t))_{t\geqslant r}$.*
(III) *There exists a P-null set $N\in\mathcal{F}_r$ such that for all $x\notin N$, $(M_t,\mathcal{F}_t,Q_x^r\circ\Phi_r)_{t\geqslant 0}$ is a continuous martingale with square variation process $(K(t))_{t\geqslant 0}$.*

As a consequence, we have the following BDG's inequality under conditional expectations:

Corollary B.4. *Let $(M_t,\mathcal{F}_t,P)_{t\geqslant r}$ be a continuous square integrable martingale with $M_r=0$. Then*

$$\mathbb{E}^P\left(\sup_{s\in[r,t]}|M_s||\mathcal{F}_r\right)\leqslant 4\sqrt{2}\cdot\mathbb{E}^P\left(\langle M\rangle_t^{1/2}|\mathcal{F}_r\right),\qquad P\text{-a.s.}$$

Appendix C. Section 4

C.1. *Proof of* Lemma 4.3

Before proving Lemma 4.3, we prepare two useful lemmas.

Lemma C.1. *Let $\mathcal{N}\in\mathfrak{A}^q$ for some $q\geqslant 1$. Then for any $\epsilon>0$, there exists an $R_\epsilon>0$ such that for any $x,y\in\mathbb{Y}$ with $\mathcal{N}(x),\mathcal{N}(y)<+\infty$*

$$\|x-y\|_{\mathbb{Y}}^q\leqslant\epsilon(\mathcal{N}(x)+\mathcal{N}(y))+R_\epsilon\|x-y\|_{\mathbb{X}}^q. \tag{C.1}$$

Proof. Suppose that the assertion is false, then there exists an $\epsilon_0>0$ such that for any $n\in\mathbb{N}$, there are $x_n,y_n\in\mathbb{Y}$ with $\mathcal{N}(x_n),\mathcal{N}(y_n)<+\infty$ such that

$$\|x_n-y_n\|_{\mathbb{Y}}^q>\epsilon_0(\mathcal{N}(x_n)+\mathcal{N}(y_n))+n\|x_n-y_n\|_{\mathbb{X}}^q.$$

Since $\mathcal{N}(x)=0$ implies $x=0$, we have

$$\mathcal{N}(x_n)+\mathcal{N}(y_n)>0.$$

Set

$$\tilde{x}_n := x_n/(\mathcal{N}(x_n)+\mathcal{N}(y_n))^{1/q}$$
$$\tilde{y}_n := y_n/(\mathcal{N}(x_n)+\mathcal{N}(y_n))^{1/q}.$$

Then

$$\|\tilde{x}_n-\tilde{y}_n\|_{\mathbb{Y}}^q>\epsilon_0+n\|\tilde{x}_n-\tilde{y}_n\|_{\mathbb{X}}^q>\epsilon_0. \tag{C.2}$$

By (4.1) and (4.2) we have that $\{\tilde{x}_n, n \in \mathbb{N}\}$ and $\{\tilde{y}_n, n \in \mathbb{N}\}$ are relatively compact in $\mathbb{Y}$. Hence, there exist a subsequence n_k and $\tilde{x}, \tilde{y} \in \mathbb{Y}$ such that

$$\lim_{k\to\infty} \|\tilde{x}_{n_k} - \tilde{x}\|_{\mathbb{Y}} = 0, \qquad \lim_{k\to\infty} \|\tilde{y}_{n_k} - \tilde{y}\|_{\mathbb{Y}} = 0.$$

Thus,

$$\lim_{k\to\infty} \|\tilde{x}_{n_k} - \tilde{y}_{n_k} - \tilde{x} + \tilde{y}\|_{\mathbb{Y}} = 0.$$

On the other hand, dividing both sides of (C.2) by n and then taking limits, we obtain that

$$\lim_{n\to\infty} \|\tilde{x}_n - \tilde{y}_n\|_{\mathbb{X}} = 0.$$

Therefore, $\tilde{x} - \tilde{y} = 0$ and

$$\lim_{k\to\infty} \|\tilde{x}_{n_k} - \tilde{x}_{n_k}\|_{\mathbb{Y}} = 0.$$

From (C.2), we then get the contradiction $0 > \epsilon_0$. $\square$

Lemma C.2. *Let $\mathcal{N} \in \mathfrak{A}^q$ for some $q \geqslant 1$, and K a subset of $\Omega = C([0,\infty), \mathbb{X})$. If for any $n \in \mathbb{N}$, K is equi-continuous in $C([0,n]; \mathbb{X})$ and*

$$\sup_{x\in K} \sup_{t\in[0,n]} \|x(t)\|_{\mathbb{H}} + \sup_{x\in K} \int_0^n \mathcal{N}(x(s))\mathrm{d}s < +\infty. \tag{C.3}$$

Then $K \subset \mathbb{S} = C([0,\infty), \mathbb{X}) \cap L^q_{\mathrm{loc}}(0,\infty; \mathbb{Y})$, and relatively compact in $\mathbb{S}$.

Proof. Let $x \in K$. By Remark 4.1 and (C.3), there exists a Lebesgue null set $\mathbb{T} \subset [0,\infty)$ such that for all $t \notin \mathbb{T}$

$$x(t) \in \mathbb{Y}, \quad \mathcal{N}(x(t)) < +\infty.$$

By (C.1) we have

$$\|x(t)\|^q_{\mathbb{Y}} \leqslant \epsilon\mathcal{N}(x(t)) + R_\epsilon \|x(t)\|^q_{\mathbb{X}}.$$

Hence $x \in L^q_{\mathrm{loc}}(0,\infty; \mathbb{Y})$, and $K \subset \mathbb{S}$.

In order to prove the compactness of K in $\mathbb{S}$, it is enough to prove that K is relatively compact in $\mathbb{S}_n := C([0,n], \mathbb{X}) \cap L^q(0,n; \mathbb{Y})$ for every $n \in \mathbb{N}$. Let $\{x_k, k \in \mathbb{N}\}$ be any sequence in K. By (C.3) we have

$$\sup_{k\in\mathbb{N}} \sup_{t\in[0,n]} \|x_k(t)\|_{\mathbb{H}} < +\infty.$$

Since $\mathbb{X}^*$ is compactly embedded in $\mathbb{Y}$, we also have that $\mathbb{H} \simeq \mathbb{H}^* \subset \mathbb{Y}^*$ is compactly embedded in $\mathbb{X}$. By a diagonalization method, we may extract a subsequence x_{k_l} such that for any rational points $t \in [0,n]$

$$\lim_{l,m\to\infty} \|x_{k_l}(t) - x_{k_m}(t)\|_{\mathbb{X}} = 0.$$

By the equi-continuity of $\{x_{k_l}, l \in \mathbb{N}\}$, we further have

$$\lim_{l,m\to\infty} \sup_{t\in[0,n]} \|x_{k_l}(t) - x_{k_m}(t)\|_{\mathbb{X}} = 0,$$

which together with (C.1) and (C.3) also yields

$$\lim_{l,m\to\infty}\int_0^n \|x_{k_l}(t)-x_{k_m}(t)\|_{\mathbb{Y}}^q \mathrm{d}t = 0.$$

Hence, there exists an $x \in \mathbb{S}_n$ such that

$$\lim_{l\to\infty}\sup_{t\in[0,n]}\|x_{k_l}(t)-x(t)\|_{\mathbb{X}} + \lim_{l\to\infty}\int_0^n \|x_{k_l}(t)-x(t)\|_{\mathbb{Y}}^q \mathrm{d}t = 0,$$

which completes the proof. □

Proof of Lemma 4.3. Fix $\epsilon > 0$. For any $n \in \mathbb{N}$, by (4.4) we may choose R_n sufficiently large such that

$$P_n\left\{x \in \Omega : \sup_{t\in[0,n]}\|x(t)\|_{\mathbb{H}} + \sup_{s\neq t\in[0,n]}\frac{\|x(t)-x(s)\|_{\mathbb{X}}}{|t-s|^\beta} + \int_0^n \mathcal{N}_1(x(s))\mathrm{d}s > R_n\right\} \leqslant \epsilon/2^n.$$

We set

$$K := \bigcap_{n\in\mathbb{N}}\left\{x \in \Omega : \sup_{t\in[0,n]}\|x(t)\|_{\mathbb{H}} + \sup_{s\neq t\in[0,n]}\frac{\|x(t)-x(s)\|_{\mathbb{X}}}{|t-s|^\beta} + \int_0^n \mathcal{N}_1(x(s))\mathrm{d}s \leqslant R_n\right\}.$$

Then K is a compact subset of $\mathbb{S}$ by Lemma C.2. Moreover,

$$\sup_n P_n(K^c) \leqslant \epsilon.$$

Hence $(P_n)_{n\in\mathbb{N}}$ is tight in $\mathbb{S}$.

C.2. *Proof of* Lemma 4.4

It is well known that there exists a self-adjoint operator Λ on $\mathbb{H}$ such that $D(\Lambda) = \mathbb{X}^*$ and

$$\langle \Lambda x, \Lambda x\rangle_{\mathbb{H}} \sim \|x\|_{\mathbb{X}^*}^2.$$

On the other hand, since $\mathbb{X}^* \subset \mathbb{H}$ is compact, the spectrum of Λ is discrete, i.e., there are eigenvalues $0 < \lambda_k \uparrow \infty$ and normalized eigenfunctions $\{\ell_i, i \in \mathbb{N}\} \subset \mathbb{X}^*$ in $\mathbb{H}$ such that

$$\Lambda\ell_i = \lambda_i\ell_i,$$

and $\{\ell_i, i \in \mathbb{N}\}$ is a complete orthonormal basis of $\mathbb{H}$. Thus, the spaces $\mathbb{X}^*$ and $\mathbb{X}$ can be characterized respectively by

$$\mathbb{X}^* = \left\{x = \sum_i a_i\ell_k : \sum_i \lambda_i^2|a_i|^2 < +\infty\right\},$$

and

$$\mathbb{X} = \left\{x = \sum_i a_i\ell_i : \sum_k \frac{|a_i|^2}{\lambda_i^2} < +\infty\right\}.$$

The result now follows.

C.3. Martingale solution for SDE with coercivity drift in finite dimension

Consider the following SDE in $\mathbb{R}^d$:

$$\begin{cases} \mathrm{d}x(t) = b(x(t))\mathrm{d}t + \sigma(x(t))\mathrm{d}W(t), \\ x(0) = x_0 \in \mathbb{R}^d, \end{cases} \tag{C.4}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to L_2(\mathbb{U}, \mathbb{R}^d)$ are continuous functions.

We have:

Theorem C.3. *Assume that there is a constant $C > 0$ such that for any $x \in \mathbb{R}^d$*

$$\langle x, b(x)\rangle_{\mathbb{R}^d} \leqslant C(1 + |x|^2), \qquad \|\sigma(x)\|_{L_2(\mathbb{U},\mathbb{R}^d)} \leqslant C(1 + |x|). \tag{C.5}$$

Then there exists a martingale solution $P \in \mathscr{P}(\Omega)$ to Eq. (C.4) satisfying **(M1)** *and* **(M2)** *of Definition 3.1, where $\Omega := C(\mathbb{R}_+; \mathbb{R}^d)$.*

Proof. Define for $n \in \mathbb{N}$

$$b_n(x) := \chi_n(x)b(x), \qquad \sigma_n(x) := \chi_n(x)\sigma(x),$$

where $0 \leqslant \chi_n \in C_0^\infty(\mathbb{R}^d)$ is a cutoff function with

$$\chi_n(x) = \begin{cases} 1, & |x| \leqslant n \\ 0, & |x| > 2n. \end{cases}$$

Then b_n and σ_n are bounded continuous functions and satisfy

$$\langle x, b_n(x)\rangle_{\mathbb{R}^d} \leqslant C(1 + |x|^2), \qquad \|\sigma_n(x)\|_{L_2(\mathbb{U},\mathbb{R}^d)} \leqslant C(1 + |x|), \tag{C.6}$$

where C is independent of n.

It is well known (cf. [17]) that there exists a probability measure $P_n \in \mathscr{P}(\Omega)$ such that $P_n(x(0) = x_0) = 1$ and

$$M_n(t, x) := x(t) - x_0 - \int_0^t b_n(x(s))\mathrm{d}s, \quad x \in \Omega,$$

is a continuous square integrable $\mathcal{F}_t$-martingale with square variation process

$$\langle M_n\rangle(t, x) = \int_0^t \mathrm{tr}(\sigma_n^*(x(s))\sigma_n(x(s)))\mathrm{d}s.$$

By Itô's formula and (C.6), we have

$$\begin{aligned} |x(t)|^2 &= |x_0|^2 + 2\int_0^t \langle x(s), b_n(x(s))\rangle_{\mathbb{R}^d}\mathrm{d}s \\ &\quad + \int_0^t \mathrm{tr}(\sigma_n^*\sigma_n)(x(s))\mathrm{d}s + 2\int_0^t x(s)\mathrm{d}M_n(s) \\ &\leqslant |x_0|^2 + C\int_0^t (1 + |x(s)|^2)\mathrm{d}s + 2\int_0^t x(s)\mathrm{d}M_n(s). \end{aligned}$$

Taking expectations by Gronwall's inequality, we obtain for any $T > 0$

$$\sup_{t\in[0,T]} \mathbb{E}^{P_n}|x(t)|^2 \leqslant C_{x_0,T}.$$

On the other hand, by BDG's inequality and Young's inequality we have

$$\begin{aligned}
\mathbb{E}^{P_n}\left(\sup_{t\in[0,T]}|x(t)|^2\right) &\leqslant C_{x_0,T}+C\mathbb{E}^{P_n}\left(\sup_{t\in[0,T]}\left|\int_0^t x(s)\mathrm{d}M_n(s)\right|\right)\\
&\leqslant C_{x_0,T}+C\mathbb{E}^{P_n}\left(\int_0^T \|\sigma_n^*(x(s))x(s)\|_{\mathbb{U}}^2\mathrm{d}s\right)\\
&\leqslant C_{x,T}+C\mathbb{E}^{P_n}\left(\sup_{t\in[0,T]}|x(t)|^2\int_0^T \|\sigma_n(x(s))\|_{L_2(\mathbb{U},\mathbb{R}^d)}^2\mathrm{d}s\right)\\
&\leqslant C_{x_0,T}+\frac{1}{2}\mathbb{E}^{P_n}\left(\sup_{t\in[0,T]}|x(t)|^2\right)+C\int_0^T(\mathbb{E}^{P_n}|x(s)|^2+1)\mathrm{d}s.
\end{aligned}$$

Hence

$$\mathbb{E}^{P_n}\left(\sup_{t\in[0,T]}|x(t)|^2\right)\leqslant C_{x_0,T}.$$

Set for any $R>0$

$$\tau_R^n:=\inf\{t:|x(t)|\geqslant R\}.$$

Then

$$\sup_n P_n\{\tau_R^n<T\}\leqslant\frac{C_{x_0,T}}{R^2}.$$

Moreover, as in deriving (4.10), we have for some $\beta\in(0,1)$

$$\sup_n\mathbb{E}^{P_n}\left(\sup_{s\neq t\in[0,T]}\frac{|x(t\wedge\tau_R^n)-x(s\wedge\tau_R^n)|}{|t-s|^\beta}\right)\leqslant C_R.$$

So, for any $\delta,\epsilon>0$ and $R>0$

$$\begin{aligned}
&\sup_n P_n\left\{\sup_{|t-s|\leqslant\delta,s,t\in[0,T]}|x(t)-x(s)|>\epsilon\right\}\\
&\quad\leqslant\sup_n P_n\left\{\sup_{|t-s|\leqslant\delta,s,t\in[0,T]}|x(t)-x(s)|>\epsilon,\tau_R^n\geqslant T\right\}\\
&\qquad+\sup_n P_n\{\tau_R^n<T\}\leqslant\frac{C_R\cdot\delta}{\epsilon^2}+\frac{C_{x_0,T}}{R^2},
\end{aligned}$$

which implies that for any $T>0$

$$\lim_{\delta\to0}\sup_n P_n\left\{\sup_{|t-s|\leqslant\delta,s,t\in[0,T]}|x(t)-x(s)|>\epsilon\right\}=0.$$

Therefore, $(P_n)_{n\in\mathbb{N}}$ is tight. Without loss of generality, we assume that P_n weakly converges to a probability measure P on $C(\mathbb{R}_+;\mathbb{R}^d)$. For example, as in (Step 2) and (Step 3) of Theorem 4.5, one can easily show that P satisfies **(M1)** and **(M2)** of Definition 3.1. □

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