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Analysis of jump processes with nondegenerate jumping kernels

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Abstract

We prove regularity estimates for functions which are harmonic with respect to certain jump processes. The aim of this article is to extend the method of Bass–Levin (2002) [3] and Bogdan–Sztonyk (2005) [6] to more general processes. Furthermore, we establish a new version of the Harnack inequality that implies regularity estimates for corresponding harmonic functions.

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1. Introduction

Let $\alpha \in (0, 2)$. We define a non-local operator $\mathcal{L}$ by

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d \setminus \{0\}} (f(x+h) - f(x) - \langle \nabla f(x), h \rangle \mathbb{1}_{\{|h|\le 1\}}) n(x,h)\, dh, \tag{1.1}$$

for $f \in C_b^2(\mathbb{R}^d)$. Assume for a moment, that $n: \mathbb{R}^d \times \left(\mathbb{R}^d \setminus \{0\}\right) \to [0, \infty)$ is a measurable function with

$$c_1 |h|^{-d-\alpha} \le n(x,h) \le c_2 |h|^{-d-\alpha} \tag{1.2}$$

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for every $h \in \mathbb{R}^d \setminus \{0\}$, any $x \in \mathbb{R}^d$ and fixed positive reals $c_1 < c_2$. Note that $n(x,h) = |h|^{-d-\alpha}$ for every h implies $\mathcal{L} f = -c(\alpha)(-\Delta)^{\alpha/2} f$ with some appropriate constant $c(\alpha)$.

In [3] it is shown that harmonic functions with respect to $\mathcal{L}$ satisfy a Harnack inequality in the following sense: there is a constant $c_3 \geq 1$ such that for every ball B_R the following implication holds:

$$f \geq 0 \quad \text{in } \mathbb{R}^d, \qquad f \text{ harmonic in } B_R \Rightarrow \forall x, y \in B_{R/2} : f(x) \leq c_3 f(y).$$

In [3] it is also shown that harmonic functions with respect to $\mathcal{L}$ satisfy the following a-priori estimate: There are constants $\beta \in (0,1)$, $c_4 \geq 1$ such that for every ball B_R the following implication holds:

$$f \text{ harmonic in } B_R \Rightarrow \|f\|_{C^\beta(\overline{B_{R/2}})} \leq c_4 \|f\|_\infty.$$

This result and its proof recently generated several research activities; see the short discussion below. Our aim is to prove similar results under weaker assumptions on the kernel n.

Let us be more precise. We consider kernels $n: \mathbb{R}^d \times \left(\mathbb{R}^d \setminus \{0\}\right) \to [0,\infty)$ that satisfy for every $x, h \in \mathbb{R}^d$, $h \neq 0$

$$n(x,h) = n(x,-h) \tag{1.3}$$

and

$$k\left(\frac{h}{|h|}\right) j(|h|) \leq n(x,h) \leq K_0\, k\left(\frac{h}{|h|}\right) j(|h|) \tag{1.4}$$

where $K_0 \geq 1$ and $k: S^{d-1} \to [0,\infty)$ is a measurable bounded symmetric function on the unit sphere satisfying the following conditions: there are $N \in \mathbb{N}$, $\varepsilon_1, \ldots, \varepsilon_N > 0$ and $\eta_1, \ldots, \eta_N \in S^{d-1}$ such that for $S_i = S^{d-1} \cap (B(\eta_i, \varepsilon_i) \cup B(-\eta_i, \varepsilon_i))$

$$k(\xi) \geq 1 \quad \text{if } \xi \in \bigcup_{i=1}^N S_i. \tag{1.5}$$

Let $j : (0,\infty) \to [0,\infty)$ be a function such that $\int_{\mathbb{R}^d} (|z|^2 \wedge 1)\, j(|z|)\, dz$ is finite. We further assume the following.

(J1) There exist $\alpha \in (0,2)$ and a function $\ell: (0,2) \to (0,\infty)$ which is slowly varying at 0 (i.e. $\lim_{r\to 0+} \frac{\ell(\lambda r)}{\ell(r)} = 1$ for any $\lambda > 0$) and bounded away from 0 on every compact interval such that

$$j(t) = \frac{\ell(t)}{t^{d+\alpha}} \quad \text{for every } 0 < t \leq 1.$$

(J2) There is a constant $\kappa \geq 1$ such that

$$j(t) \leq \kappa j(s) \quad \text{whenever } 1 \leq s \leq t.$$

In order to establish regularity estimates we need an additional weak assumption.

(J3) There is $\sigma > 0$ such that

$$\limsup_{R\to\infty} R^\sigma \int_{|z|>R} j(|z|)\, dz \leq 1.$$

If this condition holds, then one can always choose $\sigma \in (0,\alpha)$.

Remark. The symmetry assumption (1.3) is used only in Proposition 2.3 and can be dispensed with if $\alpha \in (0, 1)$.

Example 1. If a kernel n satisfies condition (1.2), then it also satisfies (J1)–(J3). Choose $N = 1$, $\varepsilon_1 = 4$, i.e. $S_1 = S^{d-1}$, $k \equiv 1$, $K_0 = c_2/c_1$, $j(s) = c_1 s^{-d-\alpha}$ in (1.4), $\ell \equiv c_1$ in (J1), $\kappa = 1$ in (J2) and $\sigma \in (0, \alpha)$ arbitrarily in (J3). In general, (J1)–(J3) hold for jumping kernels corresponding to stable processes, stable-like processes and truncated versions. Sums of such jumping kernels can be considered, too.

Example 2. Let $N \in \mathbb{N}$, $\eta_1, \dots, \eta_N \in S^{d-1}$ and $\varepsilon_1, \dots, \varepsilon_N$ be positive real numbers such that the sets $S_i = S^{d-1} \cap (B(\eta_i, \varepsilon_i) \cup B(-\eta_i, \varepsilon_i))$ are pairwise disjoint for $i = 1, \dots, N$. Set $B = \bigcup_{i=1}^{N} S_i$. Let $k = \mathbb{1}_B$ and $K_0 = c$ for some $c > 1$. Let $j(s) = s^{-d-\alpha}$ for $s > 0$. Then our assumptions are satisfied if (1.4) and (1.3) hold true. For the particular choice where $x \mapsto n(x, h)$ is constant (case of Lévy process), this class of examples is treated in [6, p. 148], where it is shown that for $N = \infty$ the Harnack inequality fails.

Given a linear operator $\mathcal{L}$ as in (1.1) satisfying (J1) and (J2) we assume that there exists a strong Markov process $X = (X_t, \mathbb{P}^x)$ with paths that are right-continuous with left limits such that the process

$$\left\{ f(X_t) - f(X_0) - \int_0^t \mathcal{L} f(X_s)\, ds \right\}_{t \ge 0}$$

is a $\mathbb{P}^x$-martingale for all $x \in \mathbb{R}^d$ and $f \in C_b^2(\mathbb{R}^d)$. We say that a bounded function $f : \mathbb{R}^d \to \mathbb{R}$ is harmonic with respect to $\mathcal{L}$ in an open set Ω if $\{f\left(X_{\min(t, \tau_{\Omega'})}\right)\}_{t \ge 0}$ is a right-continuous martingale for every open $\Omega' \subset \mathbb{R}^d$ with $\overline{\Omega'} \subset \Omega$, where $\tau_{\Omega'} = \inf\{t > 0 \colon X_t \notin \Omega'\}$ denotes the first exit time from Ω'.

We prove the following version of the Harnack inequality.

Theorem 1.1. *Assume* (J1) *and* (J2). *There exists a constant $c \ge 1$ such that for every $x_0 \in \mathbb{R}^d$, $r \in (0, \frac{1}{4})$ and every bounded function $f \colon \mathbb{R}^d \to \mathbb{R}$ which is non-negative in $B(x_0, 4r)$ and harmonic in $B(x_0, 4r)$ the following estimate holds*

$$f(x) \le c f(y) + c \, \frac{r^\alpha}{\ell(r)} \sup_{v \in B(x_0, 2r)} \int_{B(x_0, 4r)^c} f^-(z) n(v, z - v)\, dz$$

for all $x, y \in B(x_0, r)$.

Remark. If f is, in addition, non-negative in all of $\mathbb{R}^d$, then the classical version of the Harnack inequality follows, i.e. for all $x, y \in B(x_0, r)$:

$$f(x) \le c_1 f(y).$$

As a corollary to the Harnack inequality we obtain the following regularity result.

Theorem 1.2. *Assume* (J1)–(J3). *Then there exist $\beta \in (0, 1)$, $c_3, c_4 \ge 1$ such that for every $x_0 \in \mathbb{R}^d$, every $R \in (0, 1)$, every function $f : \mathbb{R}^d \to \mathbb{R}$ which is harmonic in $B(x_0, R)$ and every $\rho \in (0, R/2)$*

$$\sup_{x, y \in B(x_0, \rho)} |f(x) - f(y)| \le c_3 \|f\|_\infty (\rho/R)^\beta, \tag{1.6}$$

in particular $\|f\|_{C^\beta(\overline{B(x_0, R/2)})} \le c_4 \|f\|_\infty$. (1.7)

Let us comment on the differences between our results and those of [3]:

(1) We can treat kernels $n(x,h)$ for which the quantity

$$\inf_{x\in\mathbb{R}^d} \liminf_{r\to 0+} \frac{|\{h\in B(0,r); n(x,h)=0\}|}{|B(0,r)|}$$

is arbitrarily close to 1, e.g. $n(x,h)$ as in (1.9).

(2) For fixed $x\in\mathbb{R}^d$, upper and lower bounds for $n(x,h)$ may not allow for scaling.

(3) Large jumps of the process might not be comparable, i.e. the quantity

$$\sup\left\{\frac{n(x,h_1)}{n(y,h_2)} : |x-y|\le 1, |h_1-h_2|\le 1, |h_2|+|h_1|\ge 2\right\}$$

might be infinite.

(4) We establish a new version of the Harnack inequality and derive a-priori Hölder regularity estimate as a consequence. In a different setting, this procedure was recently established in [10].

(5) We establish a general tool, Theorem 4.1, that allows to deduce Hölder a-priori estimates from the Harnack inequality.

The constants in the main results of our work and [3] depend on α. It would be desirable to enhance the technique such that the results are robust for $\alpha\to 2-$. Under an assumption like (1.2), this has been achieved with analytic techniques in [15] and [11]. Note that Theorem 4.1 is uniform with respect to α.

Comparing our results to the local theory of second order partial differential equations, a natural question arises: What is a broad natural class of kernels n such that similar results hold true?

We call a kernel n of the above type nondegenerate if there is a function $N:(0,1)\to(0,\infty)$ with $\lim_{\rho\to 0+} N(\rho)=+\infty$ and $\lambda,\Lambda>0$ such that for every $\rho\in(0,1)$ and $x\in\mathbb{R}^d$ the symmetric matrix $[A^\rho_{ij}(x)]^d_{i,j=1}$ defined by

$$A^\rho_{ij}(x) = N(\rho)\int_{\{0<|h|\le\rho\}} h_i h_j n(x,h)\,dh$$

satisfies for every $\xi\in\mathbb{R}^d$

$$\lambda|\xi|^2 \le \sum_{i,j=1}^d A^\rho_{i,j}(x)\xi_i\xi_j \le \Lambda|\xi|^2. \tag{1.8}$$

If n depends only on h and $N(\rho)=\rho^{\alpha-2}$, then this condition implies that the corresponding Lévy process has a smooth density; see [14]. Note that condition (1.2) implies the nondegeneracy condition (1.8) with $N(\rho)=\rho^{\alpha-2}$ but is not necessary, just consider the example

$$n(x,h) = |h|^{-d-\alpha}\mathbb{1}_{\{|h_1|\ge 0.99|h|\}}. \tag{1.9}$$

Note that (1.8) holds under our assumptions.

Let us comment on other articles that generalize the results of [3]. Note that we do not include works on nonlocal Dirichlet forms. In [16] one can find conditions on Lévy processes and more general Markov jump processes such that the theory of [3] is applicable. In [1] the theory is extended to the variable order case and to situations where the lower and upper bounds in (1.2) behave differently for $|h|\to 0$. In these cases, regularity of harmonic functions does not hold. Regularity is established in [2] for variable order cases under additional assumptions. Fine potential theoretic results are obtained in [5,6] for stable processes. The case of Lévy processes with truncated stable Lévy densities is covered in [12] and generalized in [13]. As mentioned above there is an independent approach with analytic methods developed in [15,7] covering linear and fully nonlinear integro-differential operators.

Notation. For two functions f and g we write $f(t) \sim g(t)$ if $f(t)/g(t) \to 1$. For $A \subset \mathbb{R}^d$ open or closed τ_A denotes the first exit time of the Markov process under consideration. T_A denotes the first hitting time of the set A.

2. Some probabilistic estimates

In this section we prove useful auxiliary results. We follow closely the ideas of [3]. However, we need to provide several computations because of the appearance of a slowly varying function in (J1). The proofs of Propositions 2.6 and 2.8 are significantly different from their counterparts in [3].

The following proposition will be used often in obtaining probabilistic estimates.

Proposition 2.1. *Let $F: \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ be a measurable function that vanishes along the diagonal. Then for every bounded stopping time T*

$$\mathbb{E}^x \left[\sum_{s \le T} F(X_{s-}, X_s) \right] = \mathbb{E}^x \left[\int_0^T \int_{\mathbb{R}^d} F(X_s, u)\, n(X_s, u - X_s)\, du\, ds \right]$$

for every $x \in \mathbb{R}^d$.

For a proof see e.g. [8, Lemma 4.8].

The following result, taken from the theory of regular variation, will be repeatedly used throughout the paper.

Proposition 2.2. *Assume that $\ell: (0, 2) \to (0, \infty)$ varies slowly at 0 and let $\beta_1 > -1$ and $\beta_2 > 1$. Then the following is true:*

(i) $\int_0^r u^{\beta_1} \ell(u)\, du \sim \frac{r^{1+\beta_1}}{1+\beta_1} \ell(r)$ *as* $r \to 0+$,
(ii) $\int_r^1 u^{-\beta_2} \ell(u)\, du \sim \frac{r^{1-\beta_2}}{\beta_2 - 1} \ell(r)$ *as* $r \to 0+$.

Proof. By a change of variables and using [4, Proposition 1.5.10] we obtain

$$\int_0^r u^{\beta_1} \ell(u)\, du = \int_{r^{-1}}^{\infty} u^{-\beta_1 - 2} \ell(u^{-1})\, du \sim \frac{r^{1+\beta_1} \ell(r)}{1 + \beta_1},$$

since $u \mapsto \ell(u^{-1})$ varies slowly at infinity. This proves (i). Similarly, with the help of [4, Proposition 1.5.8] we obtain (ii). □

Remark. Using [4, Theorem 1.5.4] we conclude that for a function $\ell: (0, 2) \to (0, \infty)$ that varies slowly at 0 there exists a non-increasing function $\phi: (0, 2) \to (0, \infty)$ such that

$$\lim_{r \to 0+} \frac{r^{-d-\alpha} \ell(r)}{\phi(r)} = 1. \tag{2.1}$$

Before proving our main probabilistic estimates, note that (1.5) implies that there exists $\vartheta \in (0, \pi/2]$ such that for every $i \in \{1, \dots, N\}$

$$n(x,h) \geq j(|h|) \quad \text{for all } h \in \mathbb{R}^d, h \neq 0, \qquad \frac{|\langle h, \eta_i \rangle|}{|h|} \geq \cos\vartheta. \tag{2.2}$$

2.1. Exit time estimates

Proposition 2.3. *There exists a constant $C_1 > 0$ such that for every $x_0 \in \mathbb{R}^d$, $r \in (0,1)$ and $t > 0$*

$$\mathbb{P}^{x_0}(\tau_{B(x_0,r)} \leq t) \leq C_1 t \frac{\ell(r)}{r^\alpha}.$$

Proof. Again, we closely follow the ideas in [3]. Let $x_0 \in \mathbb{R}^d$, $r \in (0,1)$ and let $f \in C^2(\mathbb{R}^d)$ be a positive function such that

$$f(x) = \begin{cases} |x - x_0|^2, & |x - x_0| \leq \dfrac{r}{2} \\ r^2, & |x - x_0| \geq r \end{cases}$$

and

$$|f(x)| \leq c_1 r^2, \qquad \left| \frac{\partial f}{\partial x_i}(x) \right| \leq c_1 r \quad \text{and} \quad \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right| \leq c_1,$$

for some constant $c_1 > 0$.

Let $x \in \mathbb{R}^d$. We estimate $\mathcal{L}f(x)$ in a few steps.

First

$$\begin{aligned} &\int_{B(0,r)} \left(f(x+h) - f(x) - \langle \nabla f(x), h \rangle \mathbb{1}_{\{|h| \leq 1\}} \right) n(x,h)\, dh \\ &\leq c_2 \int_{B(0,r)} |h|^2 n(x,h)\, dh \leq c_2 \int_{B(0,r)} |h|^{2-d-\alpha} \ell(|h|)\, dh \\ &\leq c_3 r^{2-\alpha} \ell(r), \end{aligned}$$

where in the last line we have used Proposition 2.2(i). Similarly, by Proposition 2.2(ii) on $B(0,r)^c$ we get

$$\begin{aligned} &\int_{B(0,r)^c} (f(x+h) - f(x))\, n(x,h)\, dh \leq \|f\|_\infty \int_{B(0,r)^c} n(x,h)\, dh \\ &\leq \|f\|_\infty \left(c_4 \int_{B(0,1) \setminus B(0,r)} |h|^{-d-\alpha} \ell(|h|)\, dh + \int_{B(0,1)^c} n(x,h)\, dh \right) \\ &\leq c_1 r^2 \left(c_5 r^{-\alpha} \ell(r) + c_6 \right) \leq c_7 r^{2-\alpha} \ell(r). \end{aligned}$$

In the last inequality we have used the fact that $\lim_{r \to 0+} r^{-\alpha} \ell(r) = \infty$ (cf. [4, Proposition 1.3.6(v)]). Finally, by symmetry of the kernel, we have

$$\int_{B(0,1) \setminus B(0,r)} \langle h, \nabla f(x) \rangle n(x,h)\, dh = 0. \tag{2.3}$$

Therefore, by preceding estimates, we conclude that there is a constant $c_7 > 0$ such that for all $x \in \mathbb{R}^d$ and $r \in (0, 1)$

$$\mathcal{L} f(x) \leq c_8 r^{2-\alpha} \ell(r). \tag{2.4}$$

It follows from the optional stopping theorem that

$$\mathbb{E}^{x_0} f(X_{t\wedge\tau_{B(x_0,r)}}) - f(x_0) = \mathbb{E}^{x_0} \int_0^{t\wedge\tau_{B(x_0,r)}} \mathcal{L} f(X_s)\, ds \leq c_8 t r^{2-\alpha} \ell(r), \quad t > 0. \tag{2.5}$$

On $\{\tau_{B(x_0,r)} \leq t\}$ one has $X_{t\wedge\tau_{B(x_0,r)}} \notin B(x_0, r)$ and so $f(X_{t\wedge\tau_{B(x_0,r)}}) = r^2$. Then (2.5) gives

$$\mathbb{P}^{x_0}(\tau_{B(x_0,r)} \leq t) \leq c_8 t\, r^{-\alpha} \ell(r). \quad \square$$

Proposition 2.4. *There exists a constant $C_2 > 0$ such that for every $r \in (0, 1)$ and $x_0 \in \mathbb{R}^d$*

$$\inf_{y\in B(x_0,r/2)} \mathbb{E}^y \tau_{B(x_0,r)} \geq C_2 \frac{r^\alpha}{\ell(r)}.$$

Proof. Let $r \in (0, 1)$, $x_0 \in \mathbb{R}^d$ and $y \in B(x_0, r/2)$. Using Proposition 2.3 we obtain

$$\mathbb{P}^y(\tau_{B(x_0,r)} \leq t) \leq \mathbb{P}^y(\tau_{B(y,r/2)} \leq t) \leq C_1\, t\, r^{-\alpha} \ell(r) \quad \text{for } t > 0.$$

Let

$$t_0 = \frac{r^\alpha}{2C_1 \ell(r)}.$$

Then

$$\mathbb{E}^y \tau_{B(x_0,r)} \geq t_0 \mathbb{P}^y(\tau_{B(x_0,r)} > t_0) \geq \frac{r^\alpha}{4C_1 \ell(r)}. \quad \square$$

Proposition 2.5. *There exists a constant $C_3 > 0$ such that for every $r \in (0, \frac{1}{2})$ and $x_0 \in \mathbb{R}^d$*

$$\sup_{y\in B(x_0,r)} \mathbb{E}^y \tau_{B(x_0,r)} \leq C_3 \frac{r^\alpha}{\ell(r)}.$$

Proof. Let $r \in (0, \frac{1}{2})$, $x_0 \in \mathbb{R}^d$ and $y \in B(x_0, r)$. Denote by S the first time when process $(X_t)_{t\geq 0}$ has a jump larger than $2r$, i.e.

$$S = \inf\{t > 0 : |X_t - X_{t-}| > 2r\}.$$

Assume first that $\mathbb{P}^y(S \leq \frac{r^\alpha}{\ell(r)}) \leq \frac{1}{2}$. Then by Proposition 2.1

$$\begin{aligned}
\mathbb{P}^y\left(S \leq \frac{r^\alpha}{\ell(r)}\right) &= \mathbb{E}^y\left[\sum_{s\leq \frac{r^\alpha}{\ell(r)}\wedge S} \mathbb{1}_{\{|X_s - X_{s-}|>2r\}}\right] \\
&= \mathbb{E}^y\left[\int_0^{\frac{r^\alpha}{\ell(r)}\wedge S} \int_{B(0,2r)^c} n(X_s, h)\, dh\, ds\right].
\end{aligned} \tag{2.6}$$

Choose arbitrary $\xi_0 \in \{\eta_1, \dots, \eta_N\}$ and let ϑ be as in (2.2). Then

$$\begin{aligned}\int_{B(0,2r)^c} n(X_s, h)\, dh &\geq \int_{\left\{h\in\mathbb{R}^d : 2r\leq |h|<1, \frac{|\langle h,\xi_0\rangle|}{|h|}\geq \cos\vartheta\right\}} n(X_s, h)\, dh \\ &\geq \int_{\left\{h\in\mathbb{R}^d : 2r\leq |h|<1, \frac{|\langle h,\xi_0\rangle|}{|h|}\geq \cos\vartheta\right\}} \frac{\ell(|h|)}{|h|^{d+\alpha}}\, dh \\ &\geq c_1 \int_{2r}^1 \frac{\ell(t)}{t^{1+\alpha}}\, dt \geq c_2 \frac{\ell(r)}{r^\alpha},\end{aligned}$$

where in the last inequality we have used Proposition 2.2(ii). Using this estimate we get from (2.6) the following estimate

$$\begin{aligned}\mathbb{P}^y\left(S \leq \frac{r^\alpha}{\ell(r)}\right) &\geq c_2 \frac{\ell(r)}{r^\alpha} \mathbb{E}^y\left[\frac{r^\alpha}{\ell(r)} \wedge S\right] \\ &\geq c_2 \mathbb{P}^y\left(S > \frac{r^\alpha}{\ell(r)}\right) \geq \frac{c_2}{2}.\end{aligned}$$

Therefore, in any case the following inequality holds:

$$\mathbb{P}^y\left(S \leq \frac{r^\alpha}{\ell(r)}\right) \geq \frac{1}{2} \wedge \frac{c_2}{2}.$$

Since $S \geq \tau_{B(x_0,r)}$ we conclude

$$\mathbb{P}^y\left(\tau_{B(x_0,r)} \leq \frac{r^\alpha}{\ell(r)}\right) \geq \mathbb{P}^y\left(S \leq \frac{r^\alpha}{\ell(r)}\right) \geq c_3,$$

with $c_3 = \frac{1}{2} \wedge \frac{c_2}{2}$. By the Markov property, for $m \in \mathbb{N}$ we obtain

$$\begin{aligned}\mathbb{P}^y\left(\tau_{B(x_0,r)} > (m+1)\frac{r^\alpha}{\ell(r)}\right) &\leq \mathbb{P}^y\left(\tau_{B(x_0,r)} > m\frac{r^\alpha}{\ell(r)}, \tau_{B(x_0,r)} \circ \theta_{m\frac{r^\alpha}{\ell(r)}} > \frac{r^\alpha}{\ell(r)}\right) \\ &= \mathbb{E}^y\left[\mathbb{P}^{X_{m\frac{r^\alpha}{\ell(r)}}}\left(\tau_{B(x_0,r)} > \frac{r^\alpha}{\ell(r)}\right); \tau_{B(x_0,r)} > m\frac{r^\alpha}{\ell(r)}\right] \\ &\leq (1-c_3)\mathbb{P}^y\left(\tau_{B(x_0,r)} > m\frac{r^\alpha}{\ell(r)}\right),\end{aligned}$$

where θ_s denotes the usual shift operator. By iteration we obtain

$$\mathbb{P}^y\left(\tau_{B(x_0,r)} > m\frac{r^\alpha}{\ell(r)}\right) \leq (1-c_3)^m, \quad m \in \mathbb{N}.$$

Finally,

$$\begin{aligned}\mathbb{E}^y \tau_{B(x_0,r)} &\leq \frac{r^\alpha}{\ell(r)} \sum_{m=0}^\infty (m+1)\mathbb{P}^y\left(\tau_{B(x_0,r)} > m\frac{r^\alpha}{\ell(r)}\right) \\ &\leq \frac{r^\alpha}{\ell(r)} \sum_{m=0}^\infty (m+1)(1-c_3)^m \leq c_4 \frac{r^\alpha}{\ell(r)}. \quad \square\end{aligned}$$

2.2. Krylov–Safonov type estimate

Fix $\vartheta \in (0, \pi/2]$ such that (2.2) holds.

Proposition 2.6. *Let $\lambda \in \left(0, \frac{\sin\vartheta}{8}\right]$. There exists a constant $C_4 = C_4(\lambda) > 0$ such that for every $x_0 \in \mathbb{R}^d$, $r \in (0, \frac{1}{2})$, closed set $A \subset B(x_0, \lambda r)$ and $x \in B(x_0, \lambda r)$,*

$$\mathbb{P}^x(T_A < \tau_{B(x_0,r)}) \geq C_4 \frac{|A|}{|B(x_0, r)|}.$$

Proof. Choose arbitrary $\xi_0 \in \{\eta_1, \ldots, \eta_N\}$ and set $\tilde{x}_0 = x_0 - \frac{r}{2}\xi_0$. The idea is to choose $\lambda \in (0, \frac{1}{8}]$ such that (see Fig. 1)

$$\frac{|\langle u - v, \xi_0\rangle|}{|u - v|} \geq \cos\vartheta \tag{2.7}$$

for all $u \in B(x_0, 2\lambda r)$, $v \in B(\tilde{x}_0, 2\lambda r)$. Since for every $u \in B(x_0, 2\lambda r)$ and $v \in B(\tilde{x}_0, 2\lambda r)$

$$\frac{|\langle u - v, \xi_0\rangle|}{|u - v|} \geq \frac{\sqrt{\left(\frac{r}{4}\right)^2 - (2\lambda r)^2}}{\frac{r}{4}} = \sqrt{1 - (8\lambda)^2}$$

it is enough to choose $\lambda \in (0, \frac{1}{8}]$ such that

$$\sqrt{1 - (8\lambda)^2} \geq \cos\vartheta,$$

or, more explicitly,

$$\lambda \leq \frac{\sin\vartheta}{8}.$$

For $s > 0$ we denote $B(x_0, s)$ and $B(\tilde{x}_0, s)$ by B_s and $\tilde{B}_s$. Let $r \in (0, 1)$, $\lambda \in (0, \frac{\sin\vartheta}{8}]$, $x \in B_{\lambda r}$ and let $A \subset B_{\lambda r}$ be a closed subset. The strong Markov property now implies

$$\begin{aligned}\mathbb{P}^x(T_A < \tau_{B_r}) &\geq \mathbb{P}^x\left(X_{\tau_{B_{2\lambda r}}} \in \tilde{B}_{\lambda r}, X_{\tau_{\tilde{B}_{2\lambda r}}} \circ \theta_{\tau_{B_{2\lambda r}}} \in A\right) \\ &= \mathbb{E}^x\left[\mathbb{P}^{X_{\tau_{B_{2\lambda r}}}}(X_{\tau_{\tilde{B}_{2\lambda r}}} \in A); X_{\tau_{B_{2\lambda r}}} \in \tilde{B}_{\lambda r}\right].\end{aligned} \tag{2.8}$$

For every $y \in \tilde{B}_{\lambda r}$ and $t > 0$ Proposition 2.1 and (2.7) yield

$$\begin{aligned}\mathbb{P}^y(X_{\tau_{\tilde{B}_{2\lambda r}} \wedge t} \in A) &= \mathbb{E}^y\left[\sum_{s \leq \tau_{\tilde{B}_{2\lambda r}} \wedge t} \mathbb{1}_{\{X_{s-} \neq X_s, X_s \in A\}}\right] \\ &= \mathbb{E}^y\left[\int_0^{\tau_{\tilde{B}_{2\lambda r}} \wedge t} \int_A n(X_s, z - X_s)\, dz\, ds\right] \\ &\geq \mathbb{E}^y\left[\int_0^{\tau_{\tilde{B}_{2\lambda r}} \wedge t} \int_A \frac{\ell(|z - X_s|)}{|z - X_s|^{d+\alpha}}\, dz\, ds\right].\end{aligned}$$

Letting $t \to \infty$ and using the monotone convergence theorem we deduce

$$\mathbb{P}^y(X_{\tau_{\tilde{B}_{2\lambda r}}} \in A) \geq \mathbb{E}^y\left[\int_0^{\tau_{\tilde{B}_{2\lambda r}}} \int_A \frac{\ell(|z - X_s|)}{|z - X_s|^{d+\alpha}}\, dz\, ds\right].$$

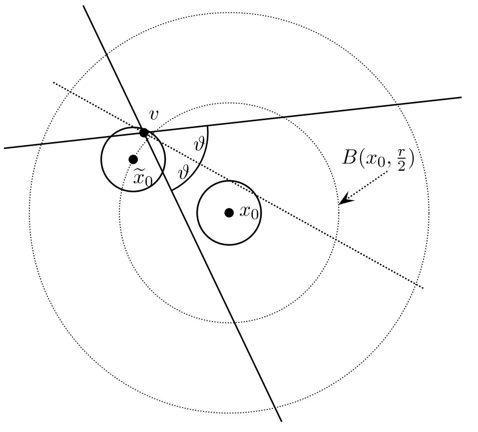

Fig. 1. The choice of $\widetilde{x}_0$ and λ.

Since $|z - X_s| \le r/2 + 4\lambda r \le r$, by (2.1) we conclude that

$$\begin{aligned}\mathbb{P}^y(X_{\tau_{\tilde{B}_{2\lambda r}}} \in A) &\ge c_1 \frac{\ell(r)}{r^{d+\alpha}} |A| \mathbb{E}^y \tau_{\tilde{B}_{2\lambda r}} \\ &\ge c_2 \ell(r) \frac{|A|}{|B_r|} r^{-\alpha} \mathbb{E}^y \tau_{\tilde{B}_{2\lambda r}}.\end{aligned}$$

Using Proposition 2.4 we deduce

$$\mathbb{P}^y(X_{\tau_{\tilde{B}_{2\lambda r}}} \in A) \ge c_3 \frac{\ell(r)}{\ell(2\lambda r)} \lambda^\alpha \frac{|A|}{|B_r|}. \tag{2.9}$$

Since ℓ varies slowly at 0 we finally obtain

$$\mathbb{P}^y(X_{\tau_{\tilde{B}_{2\lambda r}}} \in A) \ge c_4 \frac{|A|}{|B_r|} \quad \text{for all } y \in \tilde{B}_{\lambda r}, \tag{2.10}$$

for some constant $c_4 = c_4(\lambda) > 0$. By symmetry and (2.10) we deduce

$$\mathbb{P}^x(X_{\tau_{B_{2\lambda r}}} \in \tilde{B}_{\lambda r}) \ge c_4 \frac{|\tilde{B}_{\lambda r}|}{|\tilde{B}_r|} \quad \text{for all } x \in B_{\lambda r}. \tag{2.11}$$

Finally, by (2.8), (2.10) and (2.11) we get

$$\mathbb{P}^x(T_A < \tau_{B_r}) \ge c_4^2 \lambda^d \frac{|A|}{|B_r|}. \quad \square$$

2.3. Restricted Harnack inequality

The aim of this subsection is to establish a Harnack inequality for a restricted class of harmonic functions.

The following lemma can be proved similarly as [13, Lemma 2.7].

Lemma 2.7. *Let* $g\colon (0, \infty) \to [0, \infty)$ *be a function satisfying*

$$g(s) \le c g(t) \quad \textit{for all } 0 < t \le s,$$

for some constant $c > 0$. There is a constant $c' > 0$ such that for any $x_0 \in \mathbb{R}^d$ and $r > 0$ we have

$$g(|z-x|) \le c' r^{-d} \int_{B(x_0,r)} g(|z-u|)\,du,$$

for all $x \in B(x_0, r/2)$ and $z \in B(x_0, 2r)^c$.

Proposition 2.8. *There is a constant $\lambda_0 \in (0, \frac{1}{16})$ so that for every $\lambda \in (0, \lambda_0]$ there exists a constant $C_5 = C_5(\lambda) \ge 1$ such that for all $x_0 \in \mathbb{R}^d$, $r \in (0, \frac{1}{2})$ and $x, y \in B(x_0, \lambda r)$*

$$\mathbb{E}^x[H(X_{\tau_{B(x_0,\lambda r)}})] \le C_5 \mathbb{E}^y[H(X_{\tau_{B(x_0,r)}})],$$

for every non-negative function $H: \mathbb{R}^d \to [0, \infty)$ supported in $B(x_0, 3r/2)^c$.

Proof. Let $x_0 \in \mathbb{R}^d$, $r \in (0, \frac{1}{2})$ and let $x, y \in B(x_0, \lambda r)$, where $\lambda \in (0, \lambda_0)$ and $\lambda_0 \in (0, \frac{1}{16})$ is chosen later. λ_0 will depend only on constants in our main assumptions. Take $z \in B(x_0, 3r/2)^c$. There are only two cases.

Case 1: There exists $u_0 \in B(x_0, \lambda r)$ so that $n(u_0, z - u_0) > 0$.

Case 2: $n(u, z - u) = 0$ for all $u \in B(x_0, \lambda r)$.

We consider Case 1. By (1.4) and (1.5) there exist $\xi' \in \{\pm\eta_1, \dots, \pm\eta_N\}$ and $\vartheta' \in (0, \frac{\pi}{2}]$ with

$$\frac{\langle z - u_0, \xi' \rangle}{|z - u_0|} \ge \cos\vartheta'.$$

Note that ξ', ϑ' depend on u_0, z, x_0 and r but $\vartheta' \ge \vartheta$ uniformly with ϑ as in (2.2).

Set $\tilde{x_0} = x_0 - \frac{r}{2}\xi'$ and take $\lambda_0 \le \frac{\sin\vartheta}{16}$. Let $B_s := B(x_0, s)$ and $\tilde{B}_s := B(\tilde{x_0}, s)$. As in (2.7), for $\lambda \le \lambda_0$ we have

$$\frac{|\langle u - v, \xi' \rangle|}{|u - v|} \ge \cos\vartheta' \quad \text{for all } u \in B_{2\lambda r}, v \in \tilde{B}_{2\lambda r}.$$

Choose $\tilde{z_0} \in \partial B_{r/2}$ so that the following conditions hold:

$$\begin{aligned}
&|z - w| \le |z - u| \quad \text{for all } u \in B_{2\lambda r}, w \in B\left(\tilde{z_0}, \frac{\lambda r}{4}\right), \\
&\frac{\langle w - v, \xi' \rangle}{|w - v|} \ge \cos\vartheta' \quad \text{for all } v \in \tilde{B}_{2\lambda r}, w \in B\left(\tilde{z_0}, \frac{\lambda r}{4}\right), \\
&\frac{\langle z - w, \xi' \rangle}{|z - w|} \ge \cos\vartheta' \quad \text{for all } w \in B\left(\tilde{z_0}, \frac{\lambda r}{4}\right).
\end{aligned} \tag{2.12}$$

In the Appendix we briefly explain the geometric argument behind the choice of $\tilde{z_0} \in \partial B_{r/2}$.

Let $B'_s = B(\tilde{z_0}, s)$. By the strong Markov property,

$$\begin{aligned}
\mathbb{E}^y\left[\int_0^{\tau_{B_r}} n(X_s, z - X_s)\,ds\right] &\ge \mathbb{E}^y\left[\int_{\tau_{B_{2\lambda r}}}^{\tau_{B_r}} n(X_s, z - X_s)\,ds;\, X_{\tau_{B_{2\lambda r}}} \in \tilde{B}_{\lambda r}\right] \\
&= \mathbb{E}^y\left[\left\{\int_0^{\tau_{B_r}} n(X_s, z - X_s)\,ds\right\} \circ \theta_{\tau_{B_{2\lambda r}}};\, X_{\tau_{B_{2\lambda r}}} \in \tilde{B}_{\lambda r}\right] \\
&= \mathbb{E}^y\left[\mathbb{E}^{X_{\tau_{B_{2\lambda r}}}}\left[\int_0^{\tau_{B_r}} n(X_s, z - X_s)\,ds\right];\, X_{\tau_{B_{2\lambda r}}} \in \tilde{B}_{\lambda r}\right].
\end{aligned} \tag{2.13}$$

Similarly, for $v \in \tilde{B}_{\lambda r}$ we have

$$\mathbb{E}^v\left[\int_0^{\tau_{B_r}} n(X_s, z - X_s)\,ds\right] \geq \mathbb{E}^v\left[\mathbb{E}^{X_{\tau_{\tilde{B}_{2\lambda r}}}}\left[\int_0^{\tau_{B_r}} n(X_s, z - X_s)\,ds\right]; X_{\tau_{\tilde{B}_{2\lambda r}}} \in B'_{\frac{\lambda r}{8}}\right]. \tag{2.14}$$

Let $w \in B'_{\frac{\lambda r}{8}}$. Then (J1), (J2), Proposition 2.4 and (2.12) yield

$$\begin{aligned}\mathbb{E}^w\left[\int_0^{\tau_{B_r}} n(X_s, z - X_s)\,ds\right] &\geq \mathbb{E}^w\left[\int_0^{\tau_{B'_{\frac{\lambda r}{4}}}} n(X_s, z - X_s)\,ds\right]\\ &\geq c_1\mathbb{E}^w\left[\int_0^{\tau_{B'_{\frac{\lambda r}{4}}}} j(|z - X_s|)\,ds\right]\\ &\geq c_2\mathbb{E}^w\tau_{B'_{\frac{\lambda r}{4}}}(2\lambda r)^{-d}\int_{B_{2\lambda r}} j(|z-u|)\,du\\ &\geq c_3\lambda^{\alpha-d}\frac{r^{\alpha-d}}{\ell\left(\frac{\lambda r}{4}\right)}\int_{B_{2\lambda r}} j(|z-u|)\,du. \end{aligned}\tag{2.15}$$

Combining (2.13)–(2.15) we obtain

$$\begin{aligned}&\mathbb{E}^y\left[\int_0^{\tau_{B_r}} n(X_s, z - X_s)\,ds\right]\\ &\quad\geq c_3\lambda^{\alpha-d}\frac{r^{\alpha-d}}{\ell\left(\frac{\lambda r}{4}\right)}\int_{B_{2\lambda r}} j(|z-u|)\,du\,\mathbb{E}^y\left[\mathbb{P}^{X_{\tau_{B_{2\lambda r}}}}(X_{\tau_{\tilde{B}_{2\lambda r}}} \in B'_{\frac{\lambda r}{8}}); X_{\tau_{B_{2\lambda r}}} \in \tilde{B}_{\lambda r}\right].\end{aligned}$$

Similarly as in the proof of Proposition 2.6 we obtain, for some $c_4 = c_4(\lambda) > 0$

$$\mathbb{P}^v(X_{\tau_{\tilde{B}_{2\lambda r}}} \in B'_{\frac{\lambda r}{8}}) \geq c_4 \quad \text{for all } v \in \tilde{B}_{\lambda r}$$

and

$$\mathbb{P}^u(X_{\tau_{B_{2\lambda r}}} \in \tilde{B}_{\lambda r}) \geq c_4 \quad \text{for all } u \in B_{\lambda r}.$$

Therefore,

$$\mathbb{E}^y\left[\int_0^{\tau_{B_r}} n(X_s, z - X_s)\,ds\right] \geq c_5\frac{r^{\alpha-d}}{\ell\left(\frac{\lambda r}{4}\right)}\int_{B_{2\lambda r}} j(|z-u|)\,du. \tag{2.16}$$

On the other hand, by Proposition 2.5 and Lemma 2.7,

$$\begin{aligned}\mathbb{E}^x\left[\int_0^{\tau_{B_{\lambda r}}} n(X_s, z - X_s)\,ds\right] &\leq c_6\mathbb{E}^x\left[\int_0^{\tau_{B_{\lambda r}}} j(|z - X_s|)\,ds\right]\\ &\leq c_7\mathbb{E}^x\tau_{B_{\lambda r}}(2r)^{-d}\int_{B_{2\lambda r}} j(|z-u|)\,du\\ &\leq c_8\frac{r^{\alpha-d}}{\ell(2\lambda r)}\int_{B_{4\lambda r}} j(|z-u|)\,du.\end{aligned}\tag{2.17}$$

It follows from (2.16) and (2.17) that

$$\mathbb{E}^x\left[\int_0^{\tau_{B_{\lambda r}}} n(X_s, z - X_s)\,ds\right] \leq c_9\mathbb{E}^y\left[\int_0^{\tau_{B_r}} n(X_s, z - X_s)\,ds\right]. \tag{2.18}$$

Next, we consider Case 2, i.e. $n(u, z-u) = 0$ for all $u \in B(x_0, \lambda r)$. Also in this case, assertion (2.18) holds true, because

$$\begin{aligned} &\mathbb{E}^y\left[\int_0^{\tau_{B_r}} n(X_s, z - X_s)\,ds\right] \geq 0, \\ &\mathbb{E}^x\left[\int_0^{\tau_{B_{\lambda r}}} n(X_s, z - X_s)\,ds\right] = 0. \end{aligned} \tag{2.19}$$

We have shown that (2.18) always holds. It is enough to prove the proposition for $H = \mathbb{1}_A$, where $A \subset B(x_0, 3r/2)^c$. We conclude from Proposition 2.1 and (2.18) that

$$\begin{aligned} \mathbb{P}^y(X_{\tau_{B_r}} \in A) &= \int_A \mathbb{E}^y\left[\int_0^{\tau_{B_r}} n(X_s, z - X_s)\,ds\right] dz \\ &\geq c_9^{-1} \int_A \mathbb{E}^x\left[\int_0^{\tau_{B_{\lambda r}}} n(X_s, z - X_s)\,ds\right] dz \\ &= c_9^{-1}\mathbb{P}^x(X_{\tau_{B_{\lambda r}}} \in A). \quad \Box \end{aligned}$$

3. Harnack inequality

In this section we prove Theorem 1.1.

Proof of Theorem 1.1. Since f is non-negative in $B(x_0, 4r)$, we may assume that $\inf_{x\in B(x_0,r)} f(x)$ is positive. If not, we would prove the claim for $f_\varepsilon = f + \varepsilon$ and then consider $\varepsilon \to 0+$. By taking a constant multiple of f we may further assume $\inf_{x\in B(x_0,r)} f(x) = \frac{1}{2}$.

Choose $u \in B(x_0, r)$ such that $f(u) \leq 1$. By Proposition 2.5 and using properties of slowly varying functions we can find a constant $c_1 > 0$ such that for all $u, v \in \mathbb{R}^d$ and $s \in (0, r]$

$$\mathbb{E}^u \tau_{B(v,2s)} \leq c_1 \frac{s^\alpha}{\ell(s)} \quad \text{and} \quad \mathbb{E}^u \tau_{B(v,s)} \leq c_1 \frac{r^\alpha}{\ell(r)}. \tag{3.1}$$

From Proposition 2.6 we deduce that there is a constant $c_2 > 0$ and $\lambda \in (0, \frac{\sin\vartheta}{16}]$ such that for all $A \subset B(x_0, 2\lambda r)$ and $y \in B(x_0, 2\lambda r)$

$$\mathbb{P}^y(T_A < \tau_{B(x_0,2r)}) \geq c_2 \frac{|A|}{|B(x_0, 2r)|}. \tag{3.2}$$

Similarly, by Proposition 2.6 we see that there exists a constant $c_3 \in (0, 1)$ such that for every $x \in \mathbb{R}^d$, $s < r$ and $C \subset B(x, \lambda s)$ with $|C|/|B(x, \lambda s)| \geq \frac{1}{3}$

$$\mathbb{P}^x(T_C < \tau_{B(x,s)}) \geq c_3.$$

The idea of the proof is to show that f is bounded from the above in $B(x_0, r)$ by

$$c_4\left(1 + \frac{r^\alpha}{\ell(r)} \sup_{v\in B(x_0,2r)} \int_{B(x_0,4r)^c} f^-(z) n(v, z - v)\,dz\right),$$

for some constant $c_4 > 0$ that does not depend on f. This will be proved by contradiction.

Define

$$\eta = \frac{c_3}{3} \quad \text{and} \quad \zeta = \frac{\eta}{2C_5}, \tag{3.3}$$

where C_5 is taken from Proposition 2.8.

Assume that there exists $x \in B(x_0, \frac{3r}{2})$ such that $f(x) = K$ for some

$$K > \max\left\{\frac{K_0}{\zeta}, \frac{2 \cdot 8^d \lambda^{-d} K_0}{c_2 \zeta}\right\},$$

where

$$K_0 = 1 + c_1 \frac{r^\alpha}{\ell(r)} \sup_{v \in B(x_0,2r)} \int_{B(x_0,4r)^c} f^-(z) n(v, z - v)\, dz. \tag{3.4}$$

Let $s = \left(\frac{2K_0}{c_2 \zeta K}\right)^{1/d} 2\lambda^{-1} r$. Then $s < \frac{r}{4}$ and

$$|B(x, \lambda s)| = \frac{2K_0}{c_2 \zeta\ K} |B(x_0, 2r)|.$$

Set $B_s := B(x, s)$ and $\tau_s := \tau_{B(x,s)}$. Let A be a compact subset of

$$A' = \{w \in B(x, \lambda s) \colon f(w) \geq \zeta K\}.$$

By the optional stopping theorem, (3.1) and (3.2) and Proposition 2.1

$$\begin{aligned}
1 &\geq f(u) = \mathbb{E}^u[f(X_{T_A \wedge \tau_{B(x_0,2r)}})] \\
&\geq \mathbb{E}^u[f(X_{T_A \wedge \tau_{B(x_0,2r)}}); T_A < \tau_{B(x_0,2r)}] - \mathbb{E}^u[f^-(X_{T_A \wedge \tau_{B(x_0,2r)}}); T_A > \tau_{B(x_0,2r)}] \\
&\geq \zeta K\, \mathbb{P}^u(T_A < \tau_{B(x_0,2r)}) - \mathbb{E}^u[f^-(X_{\tau_{B(x_0,2r)}})] \\
&= \zeta K\, \mathbb{P}^u(T_A < \tau_{B(x_0,2r)}) - \mathbb{E}^u\left[\int_0^{\tau_{B(x_0,2r)}} \int_{B(x_0,4r)^c} f^-(z) n(X_t, z - X_t)\, dz\, dt\right] \\
&\geq c_2\, \zeta K \frac{|A|}{|B(x_0, 2r)|} - c_1 \frac{r^\alpha}{\ell(r)} \sup_{v \in B(x_0,2r)} \int_{B(x_0,4r)^c} f^-(z) n(v, z - v)\, dz.
\end{aligned}$$

Using (3.4) we obtain

$$\begin{aligned}
\frac{|A|}{|B(x, \lambda s)|} &\leq \left(1 + c_1 \frac{r^\alpha}{\ell(r)} \sup_{v \in B(x_0,2r)} \int_{B(x_0,4r)^c} f^-(z) n(v, z - v)\, dz\right) \frac{|B(x_0, 2r)|}{c_2 \zeta K |B(x, \lambda s)|} \\
&= \frac{K_0}{c_2 \zeta K} \frac{|B(x_0, 2r)|}{|B(x, \lambda s)|} = \frac{1}{2},
\end{aligned}$$

which implies

$$\frac{|A'|}{|B(x, \lambda s)|} \leq \frac{1}{2}.$$

Let $C \subset B(x, \lambda s) \setminus A'$ be a compact subset such that

$$\frac{|C|}{|B(x, \lambda s)|} \geq \frac{1}{3}. \tag{3.5}$$

Let $H = f^+ \mathbb{1}_{B^c_{3s/2}}$. Assume that

$$\mathbb{E}^x[H(X_{\tau_{\lambda s}})] > \eta K. \tag{3.6}$$

Then for any $y \in B(x, \lambda s)$ we have

$$\begin{aligned} f(y) &= \mathbb{E}^y f(X_{\tau_s}) = \mathbb{E}^y f^+(X_{\tau_s}) - \mathbb{E}^y f^-(X_{\tau_s}) \\ &= \mathbb{E}^y f^+(X_{\tau_s}) - \mathbb{E}^y[f^-(X_{\tau_s}); X_{\tau_s} \notin B(x_0, 4r)] \\ &\geq \mathbb{E}^y[f^+(X_{\tau_s}); X_{\tau_s} \notin B_{3s/2}] - \mathbb{E}^y[f^-(X_{\tau_s}); X_{\tau_s} \notin B(x_0, 4r)]. \end{aligned}$$

Applying Proposition 2.8 to H it follows that

$$\begin{aligned} f(y) \geq\ & C_5^{-1} \mathbb{E}^x[f^+(X_{\tau_{\lambda s}}); X_{\tau_{\lambda s}} \notin B_{3s/2}] \\ & - c_1 \frac{r^\alpha}{\ell(r)} \sup_{v \in B(x_0, 2r)} \int_{B(x_0, 4r)^c} f^-(z) n(v, z - v)\, dz. \end{aligned}$$

Combining the last display with the assumption (3.6) and the definition of ζ in (3.3) gives

$$f(y) \geq C_5^{-1} \eta K - K_0 = \zeta K \left(2 - \frac{K_0}{\zeta K}\right) \geq \zeta K \quad \text{for all } y \in B(x, \lambda s),$$

which is a contradiction to (3.5). Therefore $\mathbb{E}^x[H(X_{\tau_{\lambda s}})] \leq \eta K$.

Let $M = \sup_{v \in B_{3s/2}} f(v)$. Then

$$\begin{aligned} K &= f(x) = \mathbb{E}^x[f(X_{T_C}); T_C < \tau_s] + \mathbb{E}^x[f(X_{\tau_s}); \tau_s < T_C, X_{\tau_s} \in B_{3s/2}] \\ &\quad + \mathbb{E}^x[f(X_{\tau_s}); \tau_s < T_C, X_{\tau_s} \notin B_{3s/2}] \\ &\leq \zeta K\, \mathbb{P}^x(T_C < \tau_s) + M(1 - \mathbb{P}^x(T_C < \tau_s)) + \eta K \end{aligned}$$

and thus

$$\frac{M}{K} \geq \frac{1 - \eta - \zeta\, \mathbb{P}^x(T_C < \tau_s)}{1 - \mathbb{P}^x(T_C < \tau_s)}.$$

From the last display we conclude that $M \geq K(1 + 2\beta)$ with $\beta = \frac{c_3}{6(1-c_3)} + \frac{\zeta}{2} > 0$. Thus there exists $x' \in B(x, \frac{3s}{2})$ so that $f(x') \geq K(1 + \beta)$.

Using this procedure we obtain sequences (x_n) and (s_n) such that $x_{n+1} \in B(x_n, \frac{3s_n}{2})$ and $K_n := f(x_n) \geq (1 + \beta)^{n-1} K$. Thus

$$\sum_{n=1}^{\infty} |x_{n+1} - x_i| \leq \frac{3}{2} \sum_{n=1}^{\infty} s_i \leq c_5 \left(\frac{K_0}{K}\right)^{1/d} r,$$

for some constant $c_5 > 0$.

If $K > K_0\, c_5^d$, then (x_n) is a sequence in $B(x_0, \frac{3r}{2})$ such that

$$\lim_{n \to +\infty} f(x_n) \geq \lim_{n \to +\infty} (1 + \beta)^{n-1} K_1 = \infty.$$

This contradicts the boundedness of f and so $K \leq c_5^d K_0$. Thus

$$\begin{aligned} \sup_{v \in B(x_0, r)} f(v) &\leq c_5^d\, K_0 \\ &= c_5^d \left(1 + \frac{r^\alpha}{\ell(r)} \sup_{v \in B(x_0, 2r)} \int_{B(x_0, 4r)^c} f^-(z) n(v, z - v)\, dz\right). \end{aligned}$$

Now, let $x, y \in B(x_0, r)$. Then

$$f(x) \le c_5^d \left(1 + \frac{r^\alpha}{\ell(r)} \sup_{v \in B(x_0,2r)} \int_{B(x_0,4r)^c} f^-(z) n(v, z-v)\, dz\right)$$
$$\le 2c_5^d f(y) + c_5^d \frac{r^\alpha}{\ell(r)} \sup_{v \in B(x_0,2r)} \int_{B(x_0,4r)^c} f^-(z) n(v, z-v)\, dz.$$

The proof is complete. □

4. Harnack and Hölder

In this section we prove a general tool that allows to deduce regularity estimates from our version of the Harnack equality given in Theorem 1.1. This approach is developed in [10]; see also [9]. We show that the implication

Harnack inequality ⇒ Hölder regularity estimates

holds true for nonlocal operators. Since this implication is of general interest, we formulate the set-up independently of Theorem 1.2.

Let $m\colon \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \to [0, \infty)$ be a measurable function. Assume there is a function $\gamma : (0, \infty) \to (0, \infty)$ such that $\int_{\mathbb{R}^d} (|u|^2 \wedge 1)\gamma(|u|)\, du < \infty$ and for all $x, h \in \mathbb{R}^d$, $h \neq 0$,

$$k\left(\frac{h}{|h|}\right) \gamma(|h|) \le m(x, h) \le \gamma(|h|), \tag{4.1}$$

where $k : S^{d-1} \to [0, \infty)$ is a measurable bounded symmetric function such that $k \ge 1$ on a non-empty open set $I \subset S^{d-1}$. Note that this is a very weak assumption.

We assume that there exist $\varepsilon > 0$ and $L \ge 1$ such that for $0 < r < 1$ and $r < R$ the following estimate holds:

$$\frac{\int_R^\infty s^{d-1}\gamma(s)\, ds}{\int_r^\infty s^{d-1}\gamma(s)\, ds} \le L \left(\frac{R}{r}\right)^{-\varepsilon}. \tag{4.2}$$

Finally, let $\mathcal{L}$ be a non-local operator defined by

$$\mathcal{L} f(x) = \int_{\mathbb{R}^d \setminus \{0\}} (f(x+h) - f(x) - \langle \nabla f(x), h\rangle\, \mathbb{1}_{\{|h| \le 1\}}) m(x, h)\, dh \tag{4.3}$$

for $f \in C_b^2(\mathbb{R}^d)$.

Theorem 4.1. *Assume (4.1)–(4.3). Assume that harmonic functions with respect to $\mathcal{L}$ satisfy a Harnack inequality in the following sense: there exists a constant $c \ge 1$ such that for every $x_0 \in \mathbb{R}^d$, $r \in (0, \frac{1}{4})$ and for every bounded function $f\colon \mathbb{R}^d \to \mathbb{R}$ which is non-negative in $B(x_0, 4r)$ and harmonic in $B(x_0, 4r)$ the following inequality holds for all $x, y \in B(x_0, r)$*

$$f(x) \le cf(y) + cM(x_0, r) \sup_{v \in B(x_0,2r)} \int_{B(x_0,4r)^c} f^-(z) m(v, z-v)\, dz, \tag{4.4}$$

where $M(x_0, r) = (\int_{B(x_0,4r)^c} m(x_0, z - x_0)\, dz)^{-1}$.

Then there exist $\beta \in (0,1)$, $c_0 \geq 1$ *such that for every* $x_0 \in \mathbb{R}^d$, *every* $R \in (0,1)$, *every function* $f : \mathbb{R}^d \to \mathbb{R}$ *which is harmonic in* $B(x_0, R)$ *and every* $\rho \in (0, R/2)$

$$\sup_{x,y \in B(x_0,\rho)} |f(x) - f(y)| \leq c_0 \|f\|_\infty (\rho/R)^\beta. \tag{4.5}$$

Note that conditions (4.1)–(4.3) do not imply in general that $\mathcal{L}$ satisfies a Harnack inequality; see the discussion of Example 2.

Let us illustrate the above result.

Example 3. Assume $m(x,h) = |h|^{-d-\alpha}$, i.e. $k \equiv 1$, $\gamma(t) = t^{-d-\alpha}$, $\varepsilon = \alpha$. Then $\mathcal{L} = -c(-\Delta)^{\alpha/2}$. The Harnack inequality (4.4) then becomes

$$f(x) \leq c_1 f(y) + c_2 r^\alpha \int_{B(x_0,4r)^c} f^-(z)|z - x_0|^{-d-\alpha}\, dz. \tag{4.6}$$

Theorem 4.1 says that (4.6) implies a Hölder regularity estimate. Note that (4.4) follows from the more classical Harnack inequality for positive functions; see [9].

Proof of Theorem 4.1. Let $x_0 \in \mathbb{R}^d$. For $s \in (0,1)$ and $x \in B(x_0, s/2)$ we define a measure

$$\nu_s^x(A) = \frac{\int_A \gamma(|z-x|)\, dz}{\int_{B(x_0,s)^c} \gamma(|z-x_0|)\, dz} \quad \text{for measurable } A \subset B(x_0,s)^c.$$

Note that, by the assumption $k \geq 1$ on $I \subset S^{d-1}$ we can deduce

$$\int_{B(x_0,s)^c} m(x_0, z - x_0)\, dz \geq c_0 \int_{B(x_0,s)^c} \gamma(|z-x_0|)\, dz,$$

with a constant $c_0 > 0$ depending on I.

Let $r \in (0,1)$ and let $f : \mathbb{R}^d \to \mathbb{R}$ be a bounded function that is harmonic in $B(x_0, r)$. Then

$$M(x_0, r/4) \sup_{x \in B(x_0,r/2)} \int_{B(x_0,r)^c} f^-(z) m(x, z-x)\, dz \leq c' \frac{\sup_{x \in B(x_0,r/2)} \int_{B(x_0,r)^c} f^-(z)\gamma(|z-x|)\, dz}{\int_{B(x_0,r)^c} \gamma(|z-x_0|)\, , dz}.$$

By the Harnack inequality (4.4) with r replaced by $r/4$ we get

$$\sup_{B(x_0,r/4)} f \leq c \inf_{B(x_0,r/4)} f + c \sup_{B(x_0,r/2)} \int_{B(x_0,r)^c} f^-(z) \nu_r^x(dz). \tag{4.7}$$

Set $\kappa = \frac{1}{2c}$. We will choose $\beta \in (0,1)$ in the course of the proof such that

$$(1 - \frac{\kappa}{2})4^\beta \leq 1.$$

The main idea is to construct an increasing sequence $(m_n)_{n \in \mathbb{N}_0}$ and a decreasing sequence $(M_n)_{n \in \mathbb{N}_0}$ so that for all $n \in \mathbb{N}_0$

$$\begin{aligned} &m_n \leq f(x) \leq M_n \quad \text{for all } x \in B_{4^{-n}r}, \\ &M_n - m_n = 4^{-n\beta} K, \end{aligned} \tag{4.8}$$

where $M_0 = \sup_{\mathbb{R}^d} f(x)$, $m_0 = \inf_{\mathbb{R}^d} f(x)$, $K = M_0 - m_0 \in [0, 2\|f\|_\infty]$ and $B_s = B(x_0, s)$. Set $m_{-n} = m_0$ and $M_{-n} = M_0$ for $n \in \mathbb{N}$.

Assume that there are $k \in \mathbb{N}$, $m_0 \le m_1 \le \cdots \le m_{k-1}$ and $M_0 \ge M_1 \ge \cdots \ge M_{k-1}$ such that (4.8) holds for $n \le k-1$.

We need to choose $m_k \ge m_{k-1}$ and $M_k \le M_{k-1}$ such that (4.8) holds for $n = k$.

Set

$$g(x) = \left(f(x) - \frac{m_{k-1} + M_{k-1}}{2} \right) \frac{2 \cdot 4^{(k-1)\beta}}{K}.$$

Then for $x \in B_{4^{-(k-1)}r}$

$$g(x) \le \frac{M_{k-1} - m_{k-1}}{2} \frac{2 \cdot 4^{(k-1)\beta}}{K} = 1$$

$$g(x) \ge \frac{m_{k-1} - M_{k-1}}{2} \frac{2 \cdot 4^{(k-1)\beta}}{K} = -1, \quad \text{i.e. } |g(x)| \le 1.$$

Let $y \in \mathbb{R}^d$ be such that $|y - x_0| \ge 4^{-(k-1)}r$. Then there exists $j \in \mathbb{N}$ such that

$$4^{-k+j} r \le |y - x_0| \le 4^{-k+j+1} r.$$

Therefore, since $f(y) \le M_{k-j-1}$ and $m_{k-j-1} \le m_{k-1}$,

$$\begin{aligned} \frac{K}{2 \cdot 4^{(k-1)\beta}} g(y) &= f(y) - \frac{m_{k-1} + M_{k-1}}{2} \\ &\le M_{k-j-1} - m_{k-j-1} - \frac{M_{k-1} - m_{k-1}}{2} \\ &= 4^{-(k-j-1)\beta} K - 4^{-(k-1)\beta} \frac{K}{2} \end{aligned}$$

and so

$$g(y) \le 2 \cdot 4^{j\beta} - 1.$$

Similarly,

$$g(y) \ge 1 - 2 \cdot 4^{j\beta}.$$

Now there are two cases:

Case 1: $|\{x \in B_{4^{-k}r} : g(x) \le 0\}| \ge \frac{1}{2} |B_{4^{-k}r}|$.

Case 2: $|\{x \in B_{4^{-k}r} : g(x) > 0\}| \ge \frac{1}{2} |B_{4^{-k}r}|$.

We work out details for Case 1 and comment afterwards on Case 2. In Case 1 our aim is to show $g(x) \le 1 - \kappa$ for every $x \in B_{4^{-k}r}$ and $\kappa = \frac{1}{2c}$. Because then for every $x \in B_{4^{-k}r}$ we obtain

$$\begin{aligned} f(x) &\le \frac{M_{k-1} + m_{k-1}}{2} + \frac{(1-\kappa)K}{2} 4^{-(k-1)\beta} \\ &= m_{k-1} + \frac{M_{k-1} - m_{k-1}}{2} + \frac{(1-\kappa)K}{2} 4^{-(k-1)\beta} \\ &= m_{k-1} + \frac{K}{2} 4^{-(k-1)\beta} + \frac{(1-\kappa)K}{2} 4^{-(k-1)\beta} \\ &= m_{k-1} + \left(1 - \frac{\kappa}{2}\right) 4^{-(k-1)\beta} K \\ &\le m_{k-1} + 4^{-k\beta} K. \end{aligned} \tag{4.9}$$

In this case we set $m_k = m_{k-1}$ and $M_k = m_k + 4^{-k\beta}K$ and obtain, using (4.9), $m_k \le f(x) \le M_k$ for every $x \in B_{4^{-k}r}$ as desired.

Thus we need to prove

$$g(x) \le 1-\kappa \quad \text{for every } x \in B_{4^{-k}r}.$$

Define $w = 1-g$. Then w is non-negative and harmonic in $B_{4^{-k+1}r}$ and thus, by (4.7), we deduce

$$\sup_{B_{4^{-k}r}} w \le c \inf_{B_{4^{-k}r}} w + c \sup_{x\in B_{2\cdot 4^{-k}r}} \int_{B^c_{4^{-k+1}r}} w^-(y)\nu^x_{4^{-k+1}r}(dy).$$

Since $\sup_{B_{4^{-k}r}} w \ge 1$, we get

$$\begin{aligned}\inf_{B_{4^{-k}r}} w &\ge \frac{1}{c} - \sum_{j=1}^{\infty} \sup_{x\in B_{2\cdot 4^{-k}r}} \int_{A(x_0,4^{-k+j}r,4^{-k+j+1}r)} (1-g(y))^-\nu^x_{4^{-(k-1)}r}(dy)\\ &\ge \frac{1}{c} - 2\sum_{j=1}^{\infty}(4^{j\beta}-1) \sup_{x\in B_{2\cdot 4^{-k}r}} \nu^x_{4^{-(k-1)}r}(A(x_0,4^{-k+j}r,4^{-k+j+1}r)),\end{aligned}$$

where $A(x_0,s_1,s_2) = \{y\in\mathbb{R}^d : s_1 \le |y-x_0| < s_2\}$.

By assumption (4.2) and the definition of $\nu^x_{4^{-(k-1)}r}$ we obtain

$$\begin{aligned}&\sup_{x\in B_{2\cdot 4^{-k}r}} \nu^x_{4^{-(k-1)}r}(A(x_0,4^{-k+j}r,4^{-k+j+1}r))\\ &\quad\le c_1 \frac{\int_{B(0,4^{-k+j-1}r)^c}\gamma(|u|)\,du}{\int_{B(0,4^{-k+1}r)^c}\gamma(|u|)\,du} \le c_1 L 4^{-\varepsilon(j-2)}.\end{aligned}$$

Choose $\beta_0 \in (0,\varepsilon)$. Then for $\beta\in(0,\beta_0)$

$$\begin{aligned}&\sum_{j=1}^{\infty}(4^{j\beta}-1) \sup_{x\in B_{2\cdot 4^{-k}r}} \nu^x_{4^{-(k-1)}r}(A(x_0,4^{-k+j}r,4^{-k+j+1}r))\\ &\quad\le c_1 4^{2\varepsilon} L \sum_{j=1}^{\infty} 4^{-j(\varepsilon-\beta_0)} < \infty.\end{aligned}$$

Choose $l = l(c,c_1,L,\beta_0,\varepsilon)\in\mathbb{N}$ so that

$$c_1 4^{2\varepsilon} L \sum_{j=l+1}^{\infty} 4^{-j(\varepsilon-\beta_0)} \le \frac{1}{8c}$$

and then $\beta\in(0,\beta_0)$ small enough so that

$$\begin{aligned}&\sum_{j=1}^{l}(4^{j\beta}-1) \sup_{x\in B_{2\cdot 4^{-k}r}} \nu^x_{4^{-(k-1)}r}(A(x_0,4^{-k+j}r,4^{-k+j+1}r))\\ &\quad\le c_1 4^{2\varepsilon} L \sum_{j=1}^{l}(4^{j\beta}-1)4^{-\varepsilon j} \le \frac{1}{8c}.\end{aligned}$$

Therefore,

$$\inf_{B_{4^{-k}r}} w \geq \frac{1}{c} - 2\left(\frac{1}{8c} + \frac{1}{8c}\right) = \kappa,$$

i.e.

$$g \leq 1 - \kappa \text{ for all } x \in B_{4^{-k}r}.$$

In Case 2 our aim is to show $g(x) \geq -1 + \kappa$. This time, set $w = 1 + v$. Following the strategy above one sets $M_k = M_{k-1}$ and $m_k = M_k - 4^{-k\beta}K$ leading to the desired result.

Let us show how (4.8) proves the assertion of the theorem. Let $\rho \in (0, r/2)$. Choose $m \in \mathbb{N}_0$ with $4^{-(m+1)}r \leq \rho < 4^{-m}r$. Then condition (4.8) implies

$$\sup_{x,y \in B_\rho(x_0)} |f(x) - f(y)| \leq 4^{-m\beta}K = (4^{-m-1}r)^\beta r^{-\beta} 4^\beta K \leq 4^\beta K \left(\frac{\rho}{r}\right)^\beta.$$

The assertion of the lemma follows and the proof is complete. □

Now we are finally able to prove Theorem 1.2.

Proof of Theorem 1.2. We apply Theorem 4.1. Let $k = k_1$ as in (1.4) and $I = S_1$ as in (1.5). Set $m(x, h) = n(x, h)$, $\gamma(t) = j(t)$. We need to check condition (4.2). We will show that there is $\varepsilon > 0$ with the desired property.

Let $r \in (0, 1)$. Using condition (J1) and Proposition 2.2(ii) we obtain

$$\int_r^\infty s^{d-1} j(s)\, ds \geq \int_r^1 s^{-\alpha-1} \ell(s)\, ds \geq c_1 r^{-\alpha} \ell(r). \tag{4.10}$$

Assume first that $R \in (r, 1)$. Then

$$\int_R^\infty s^{d-1} j(s)\, ds \leq c_2 \int_R^1 s^{-\alpha-1} \ell(s) + c_3 \leq c_4 R^{-\alpha} \ell(R).$$

Choose $\delta_1 \in (0, \alpha)$. By the theorem of Potter (see [4, Theorem 1.5.6(ii)]) there is a constant $c_5 > 0$ such that $\frac{\ell(R)}{\ell(r)} \leq c_5 \left(\frac{R}{r}\right)^{\delta_1}$ for all $0 < r < R < 1$. Therefore,

$$\frac{\int_R^\infty s^{d-1} j(s)\, ds}{\int_r^\infty s^{d-1} j(s)\, ds} \leq \frac{c_4 R^{-\alpha}}{c_1 r^{-\alpha}} \frac{\ell(R)}{\ell(r)} \leq c_6 \left(\frac{R}{r}\right)^{-(\alpha-\delta_1)}.$$

Next we treat the case $R > 1$. (J3) implies $\int_R^\infty s^{d-1} j(s)\, ds \leq c_7 R^{-\sigma}$. Choose $\delta_2 \in (0, \alpha - \sigma)$ with $\sigma \in (0, \alpha)$ as in (J3). The theorem of Potter and (4.10) imply that there is a constant $c_8 > 0$ with

$$\int_r^\infty s^{d-1} j(s)\, ds \geq c_1 \ell(1) r^{-\alpha} \frac{\ell(r)}{\ell(1)} \geq c_1 \ell(1) c_8 r^{-(\alpha-\delta_2)}.$$

In this case,

$$\frac{\int_R^\infty s^{d-1} j(s)\, ds}{\int_r^\infty s^{d-1} j(s)\, ds} \leq c_9 \left(\frac{R}{r}\right)^{-\sigma} r^{\alpha-\delta_2-\sigma} \leq c_9 \left(\frac{R}{r}\right)^{-\sigma},$$

since $r < 1$ and $\alpha - \delta_2 - \sigma > 0$.

Therefore, condition (4.2) is satisfied for $\varepsilon = \min\{\alpha - \delta_1, \sigma\}$. □

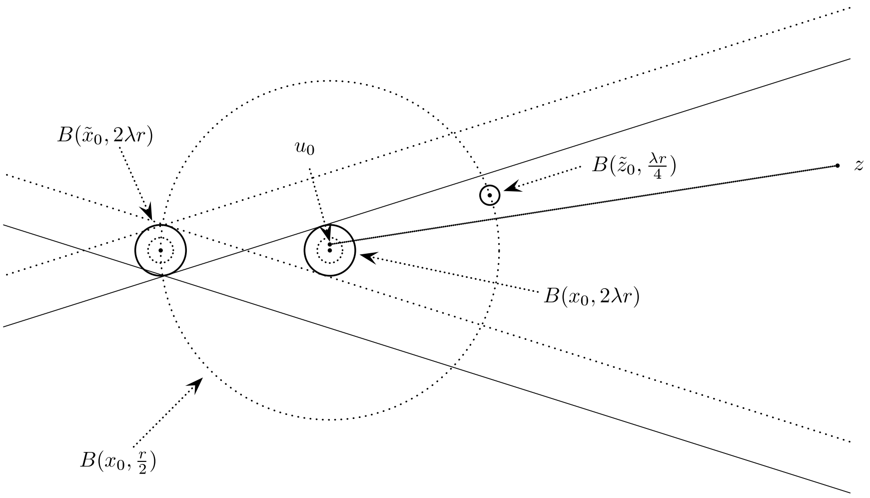

Fig. 2. The choice of $\tilde{x}_0$ and $\tilde{z}_0$.

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Appendix

We explain the geometric arguments behind the proof of Proposition 2.8.

Given $\eta \in S^{d-1}$ and $\rho > 0$ we define a cone $V(\eta, \rho) \subset \mathbb{R}^d$ as follows. Set

$$S(\eta, \rho) = (B(\eta, \rho) \cup B(-\eta, \rho)) \cap S^{d-1} \quad \text{and}$$
$$V(\eta, \rho) = \left\{ x \in \mathbb{R}^d | x \neq 0, \frac{x}{|x|} \in S(\eta, \rho) \right\}.$$

From now on, we keep $\eta \in S^{d-1}$ and $\rho > 0$ fixed and write V instead of $V(\eta, \rho)$. Choose $\vartheta \in (0, \frac{\pi}{2}]$ so that $\rho^2 = 2(1 - \cos\vartheta)$.

Using a simple geometric argument one can establish the following fact.

Let $\lambda \in (0, \frac{\sin\vartheta}{8})$, $x_0 \in \mathbb{R}^d$, $r \in (0, 2)$, $u_0 \in B_{\lambda r}(x_0)$ and $z \in B(x_0, \frac{3r}{2})^c$. Assume $z \in u_0 + V$. Set $\widetilde{x_0} = x_0 - \frac{r}{2}\xi \in \partial B(x_0, \frac{r}{2})$ where $\xi \in \{+\eta, -\eta\}$ is chosen so that $\langle z - u_0, \xi \rangle > 0$; see Fig. 2. Then the choice of λ implies

(1) $B(\tilde{x}_0, 2\lambda r) \subset \bigcap_{u \in B(x_0, 2\lambda r)} (u + V)$.

Moreover, there is $\tilde{z}_0 \in \partial B(x_0, \frac{r}{2})$ such that

(2) $B(\tilde{z}_0, \frac{\lambda r}{4}) \subset \bigcap_{v \in B(\tilde{x}_0, 2\lambda r)} (v + V)$,
(3) $z \in \bigcap_{w \in B(\tilde{z}_0, \frac{\lambda r}{4})} (w + V)$,
(4) $|z - \tilde{z}_0| < |z - x_0|$ and thus $|z - w| < |z - u|$ for all $u \in B(x_0, 4\lambda r)$, $w \in B(\tilde{z}_0, \frac{\lambda r}{4})$.

These conditions ensure that the Markov jump process under consideration has a strictly positive probability to jump from a neighborhood of x_0 via neighborhoods of $\tilde{x}_0$ and $\tilde{z}_0$ to z. One could avoid the introduction of $\tilde{z}_0$ and let the process jump directly from the neighborhood of $\tilde{x}_0$ to z but this would result in a slightly stronger assumption than (J2).

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