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Unbiased truncated quadratic variation for volatility estimation in jump diffusion processes

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Abstract

The problem of integrated volatility estimation for an Ito semimartingale is considered under discrete high-frequency observations in short time horizon. We provide an asymptotic expansion for the integrated volatility that gives us, in detail, the contribution deriving from the jump part. The knowledge of such a contribution allows us to build an unbiased version of the truncated quadratic variation, in which the bias is visibly reduced. In earlier results to have the original truncated realized volatility well-performed the condition $\beta > \frac{1}{2(2-\alpha)}$ on β (that is such that $(\frac{1}{n})^{\beta}$ is the threshold of the truncated quadratic variation) and on the degree of jump activity α was needed (see Mancini, 2011; Jacod, 2008). In this paper we theoretically relax this condition and we show that our unbiased estimator achieves excellent numerical results for any couple (α, β).

Keywords: Lévy-driven SDE; Integrated variance; Threshold estimator; Convergence speed; High frequency data

1. Introduction

In this paper, we consider the problem of estimating the integrated volatility of a discretely-observed one-dimensional Itô semimartingale over a finite interval. The class of Itô semimartingales has many applications in various areas such as neuroscience, physics and finance. Indeed, it includes the stochastic Morris–Lecar neuron model [8] as well as important examples taken from finance such as the Barndorff–Nielsen–Shephard model [2], the Kou model [16] and the Merton model [19]; to name just a few.

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In this work we aim at estimating the integrated volatility based on discrete observations $X_{t_0}, \dots, X_{t_n}$ of the process X, with $t_i = i\frac{T}{n}$. Let X be a solution of

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t a_s dW_s + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \gamma(X_{s^-})\, z\, \tilde{\mu}(ds, dz), \quad t \in \mathbb{R}_+,$$

with $W = (W_t)_{t\geq 0}$ a one dimensional Brownian motion and $\tilde{\mu}$ a compensated Poisson random measure. We also require the volatility a_t to be an Itô semimartingale.

We consider here the setting of high frequency observations, i.e. $\Delta_n := \frac{T}{n} \to 0$ as $n \to \infty$. We want to estimate $IV := \frac{1}{T}\int_0^T a_s^2 f(X_s)ds$, where f is a polynomial growth function. Such a quantity has already been widely studied in the literature because of its great importance in finance. Indeed, taking $f \equiv 1$, IV turns out being the so called integrated volatility that has particular relevance in measuring and forecasting the asset risks; its estimation on the basis of discrete observations of X is one of the long-standing problems.

In the sequel we will present some known results denoting by IV the classical integrated volatility, that is we are assuming f equals 1.

When X is continuous, the canonical way for estimating the integrated volatility is to use the realized volatility or approximate quadratic variation at time T:

$$[X, X]_T^n := \sum_{i=0}^{n-1} (\Delta X_i)^2, \qquad \text{where } \Delta X_i = X_{t_{i+1}} - X_{t_i}.$$

Under very weak assumptions on b and a (namely when $\int_0^T b_s^2 ds$ and $\int_0^T a_s^4 ds$ are finite for all $t \in (0, T]$), we have a central limit theorem (CLT) with rate $\sqrt{n}$: the processes $\sqrt{n}([X, X]_T^n - IV)$ converge in the sense of stable convergence in law for processes, to a limit Z which is defined on an extension of the space and which conditionally is a centered Gaussian variable whose conditional law is characterized by its (conditional) variance $V_T := 2\int_0^T a_s^4 ds$.

When X has jumps, the variable $[X, X]_T^n$ no longer converges to IV. However, there are other known methods to estimate the integrated volatility.

The first type of jump-robust volatility estimators are the *Multipower variations* (cf [3,4,12]), which we do not explicitly recall here. These estimators satisfy a CLT with rate $\sqrt{n}$ but with a conditional variance bigger than V_T (so they are rate-efficient but not variance-efficient).

The second type of volatility estimators, introduced by Jacod and Todorov in [14], is based on estimating locally the volatility from the empirical characteristic function of the increments of the process over blocks of decreasing length but containing an increasing number of observations, and then summing the local volatility estimates.

Another method to estimate the integrated volatility in jump diffusion processes, introduced by Mancini in [17], is the use of the *truncated realized volatility* or *truncated quadratic variance* (see [12,18]):

$$\hat{IV}_T^n := \sum_{i=0}^{n-1} (\Delta X_i)^2 1_{\{|\Delta X_i| \leq v_n\}},$$

where v_n is a sequence of positive truncation levels, typically of the form $(\frac{1}{n})^\beta$ for some $\beta \in (0, \frac{1}{2})$.

Below we focus on the estimation of IV through the implementation of the truncated quadratic variation, that is based on the idea of summing only the squared increments of X whose absolute value is smaller than some threshold v_n.

It is shown in [11] that $I\hat{V}^n_T$ has exactly the same limiting properties as $[X, X]^n_T$ does for some $\alpha \in [0, 1)$ and $\beta \in [\frac{1}{2(2-\alpha)}, \frac{1}{2})$. The index α is the degree of jump activity or Blumenthal–Getoor index

$$\alpha := \inf\left\{r \in [0, 2] : \int_{|x|\leq 1} |x|^r F(dx) < \infty\right\},$$

where F is a Lévy measure which accounts for the jumps of the process and it is such that the compensator $\bar{\mu}$ has the form $\bar{\mu}(dt, dz) = F(z)dzdt$.

Mancini has proved in [18] that, when the jumps of X are those of a stable process with index $\alpha \geq 1$, the truncated quadratic variation is such that

$$(I\hat{V}^n_T - IV) \overset{\mathbb{P}}{\sim} (\frac{1}{n})^{\beta(2-\alpha)}. \tag{1}$$

This rate is less than $\sqrt{n}$ and no proper CLT is available in this case.

In this paper, in order to estimate $IV := \frac{1}{T}\int_0^T a_s^2 f(X_s)ds$, we consider in particular the truncated quadratic variation defined in the following way:

$$Q_n := \sum_{i=0}^{n-1} f(X_{t_i})(X_{t_{i+1}} - X_{t_i})^2 \varphi_{\Delta_n^\beta}(X_{t_{i+1}} - X_{t_i}),$$

where φ is a C^∞ function that vanishes when the increments of the data are too large compared to the typical increments of a continuous diffusion process, and thus can be used to filter the contribution of the jumps.

We aim to extend the results proved in short time in [18] characterizing precisely the noise introduced by the presence of jumps and finding consequently some corrections to reduce such a noise.

The main result of our paper is the asymptotic expansion for the integrated volatility. Compared to earlier results, our asymptotic expansion provides us precisely the limit to which $n^{\beta(2-\alpha)}(Q_n - IV)$ converges when $(\frac{1}{n})^{\beta(2-\alpha)} > \sqrt{n}$, which matches with the condition $\beta < \frac{1}{2(2-\alpha)}$.

Our work extends equation (1) (obtained in [18]). Indeed, we find

$$Q_n - IV = \frac{Z_n}{\sqrt{n}} + (\frac{1}{n})^{\beta(2-\alpha)} c_\alpha \int_{\mathbb{R}} \varphi(u)|u|^{1-\alpha}du \int_0^T |\gamma|^\alpha(X_s)f(X_s)ds + o_{\mathbb{P}}((\frac{1}{n})^{\beta(2-\alpha)}),$$

where $Z_n \xrightarrow{\mathcal{L}} N(0, 2\int_0^T a_s^4 f^2(X_s)ds)$ stably with respect to X. The asymptotic expansion here above allows us to deduce the behavior of the truncated quadratic variation for each couple (α, β), that is a plus compared to (1).

Furthermore, providing we know α (and if we do not it is enough to estimate it previously, see for example [23] or [20]), we can improve the performance of the truncated quadratic variation subtracting the bias due to the presence of jumps to the original estimator or taking particular functions φ that make the bias derived from the jump part equal to zero. Using the asymptotic expansion of the integrated volatility we also provide the rate of the error left after having applied the corrections. It derives from the Brownian increments mistakenly truncated away, when the truncation is tight.

Moreover, in the case where the volatility is constant, we show numerically that the corrections gained by the knowledge of the asymptotic expansion for the integrated volatility allows us to reduce visibly the noise for any $\beta \in (0, \frac{1}{2})$ and $\alpha \in (0, 2)$. It is a clear improvement because, if the original truncated quadratic variation was a well-performed estimator only if

$\beta > \frac{1}{2(2-\alpha)}$ (condition that never holds for $\alpha \geq 1$), the unbiased truncated quadratic variation achieves excellent results for any couple (α, β).

The outline of the paper is the following. In Section 2 we present the assumptions on the process X. In Section 3 we define the truncated quadratic variation and we state the main results of the paper. In Section 4 we show the numerical performance of the unbiased estimator. Section 5 is devoted to the statement of propositions useful for the proof of the main results, that is given in Section 6. In Section 7 we give some technical tools about Malliavin calculus, required for the proof of some propositions, while other proofs and some technical results are presented in the Appendix.

2. Model, assumptions

The underlying process X is a one dimensional Itô semimartingale on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where $(\mathcal{F}_t)_{t\geq 0}$ is a filtration, and observed at times $t_i = \frac{i}{n}$, for $i = 0, 1, \dots, n$.

Let X be a solution to

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t a_s dW_s + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \gamma(X_{s^-})\, z\, \tilde{\mu}(ds, dz), \quad t \in \mathbb{R}_+, \tag{2}$$

where $W = (W_t)_{t\geq 0}$ is a one dimensional Brownian motion and $\tilde{\mu}$ a compensated Poisson random measure on which conditions will be given later.

We will also require the volatility a_t to be an Itô semimartingale and it thus can be represented as

$$\begin{aligned} a_t = a_0 &+ \int_0^t \tilde{b}_s ds + \int_0^t \tilde{a}_s dW_s + \int_0^t \hat{a}_s d\hat{W}_s + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \tilde{\gamma}_s\, z\, \tilde{\mu}(ds, dz) \\ &+ \int_0^t \int_{\mathbb{R}\setminus\{0\}} \hat{\gamma}_s\, z\, \tilde{\mu}_2(ds, dz). \end{aligned} \tag{3}$$

The jumps of a_t are driven by the same Poisson compensated random measure $\tilde{\mu}$ as X plus another Poisson compensated measure $\tilde{\mu}_2$. We need also a second Brownian motion $\hat{W}$: in the case of "pure leverage" we would have $\hat{a} \equiv 0$ and $\hat{W}$ is not needed; in the case of "no leverage" we rather have $\tilde{a} \equiv 0$. In the mixed case both W and $\hat{W}$ are needed.

2.1. Assumptions

The first assumption is a structural assumption describing the driving terms W, $\hat{W}$, $\tilde{\mu}$ and $\tilde{\mu}_2$; the second one being a set of conditions on the coefficients implying in particular the existence of the various stochastic integrals involved above.

A1: The processes W and $\hat{W}$ are two independent Brownian motion, μ and μ_2 are Poisson random measures on $[0, \infty)\times\mathbb{R}$ associated to the Lévy processes $L = (L_t)_{t\geq 0}$ and $L_2 = (L_t^2)_{t\geq 0}$ respectively, with $L_t := \int_0^t \int_{\mathbb{R}} z\tilde{\mu}(ds, dz)$ and $L_t^2 := \int_0^t \int_{\mathbb{R}} z\tilde{\mu}_2(ds, dz)$. The compensated measures are $\tilde{\mu} = \mu - \bar{\mu}$ and $\tilde{\mu}_2 = \mu_2 - \bar{\mu}_2$; we suppose that the compensator has the following form: $\bar{\mu}(dt, dz) := F(dz)dt$, $\bar{\mu}_2(dt, dz) := F_2(dz)dt$. Conditions on the Levy measures F and F_2 will be given in A3 and A4. The initial condition X_0, a_0, W, $\hat{W}$, L and L_2 are independent. The Brownian motions and the Lévy processes are adapted with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. We suppose moreover that there exists X, solution of (2).

A2: The processes b, $\tilde{b}$, $\tilde{a}$, $\hat{a}$, $\tilde{\gamma}$, $\hat{\gamma}$ are bounded, γ is Lipschitz. The processes b, $\tilde{a}$ are cádlág adapted, γ, $\tilde{\gamma}$ and $\hat{\gamma}$ are predictable, $\tilde{b}$ and $\hat{a}$ are progressively measurable. Moreover it exists an $\mathcal{F}_t$ -measurable random variable K_t such that

$$\mathbb{E}[|b_{t+h} - b_t|^2 | \mathcal{F}_t] \le K_t\, |h|; \quad \forall p \ge 1,\ \mathbb{E}[|K_t|^p] < \infty.$$

We observe that the last condition on b holds true regardless if, for example, $b_t = b(X_t)$; $b : \mathbb{R} \to \mathbb{R}$ Lipschitz.

The next assumption ensures the existence of the moments:

A3: For all $q > 0$, $\int_{|z|>1} |z|^q F(dz) < \infty$ and $\int_{|z|>1} |z|^q F_2(dz) < \infty$. Moreover, $\mathbb{E}[|X_0|^q] < \infty$ and $\mathbb{E}[|a_0|^q] < \infty$.

A4 (Jumps):

1. The jump coefficient γ is bounded from below, that is $\inf_{x\in\mathbb{R}} |\gamma(x)| := \gamma_{min} > 0$.
2. The Lévy measures F and F_2 are absolutely continuous with respect to the Lebesgue measure and we denote $F(z) = \frac{F(dz)}{dz}$, $F_2(z) = \frac{F_2(dz)}{dz}$.
3. The Lévy measure F satisfies $F(dz) = \frac{\bar{g}(z)}{|z|^{1+\alpha}} dz$, where $\alpha \in (0, 2)$ and $\bar{g} : \mathbb{R} \to \mathbb{R}$ is a continuous symmetric nonnegative bounded function with $\bar{g}(0) = 1$.
4. The function $\bar{g}$ is differentiable on $\{0 < |z| \le \eta\}$ for some $\eta > 0$ with continuous derivative such that $\sup_{0<|z|\le\eta} |\frac{\bar{g}'}{\bar{g}}| < \infty$.
5. The jump coefficient γ is upper bounded, i.e. $\sup_{x\in\mathbb{R}} |\gamma(x)| := \gamma_{max} < \infty$.
6. The Levy measure F_2 satisfies $\int_{\mathbb{R}} |z|^2 F_2(z) dz < \infty$.

The first and fifth points of the assumptions here above are useful to compare size of jumps of X and L. The fourth point is required to use Malliavin calculus and it is satisfied by a large class of processes: α- stable process ($\bar{g} = 1$), truncated α-stable processes ($\bar{g} = \tau$, a truncation function), tempered stable process ($\bar{g}(z) = e^{-\lambda|z|}$, $\lambda > 0$).

In the following, we will use repeatedly some moment inequalities for jump diffusion, which are gathered in Lemma 1 and showed in the Appendix.

Lemma 1. *Suppose that A1–A4 hold. Then, for all $t > s$,*
(1) for all $p \ge 2$, $\mathbb{E}[|a_t - a_s|^p] \le c|t - s|$; for all $q > 0$ $\sup_{t\in[0,T]} \mathbb{E}[|a_t|^q] < \infty$.
(2) for all $p \ge 2$, $p \in \mathbb{N}$, $\mathbb{E}[|a_t - a_s|^p | \mathcal{F}_s] \le c|t - s|$.
(3) for all $p \ge 2$, $\mathbb{E}[|X_t - X_s|^p]^{\frac{1}{p}} \le c|t - s|^{\frac{1}{p}}$; for all $q > 0$ $\sup_{t\in[0,T]} \mathbb{E}[|X_t|^q] < \infty$,
(4) for all $p \ge 2$, $p \in \mathbb{N}$, $\mathbb{E}[|X_t - X_s|^p | \mathcal{F}_s] \le c|t - s|(1 + |X_s|^p)$.
(5) for all $p \ge 2$, $p \in \mathbb{N}$, $\sup_{h\in[0,1]} \mathbb{E}[|X_{s+h}|^p | \mathcal{F}_s] \le c(1 + |X_s|^p)$.
(6) for all $p > 1$, $\mathbb{E}[|X^c_t - X^c_s|^p]^{\frac{1}{p}} \le |t - s|^{\frac{1}{2}}$ and $\mathbb{E}[|X^c_t - X^c_s|^p | \mathcal{F}_s]^{\frac{1}{p}} \le c|t - s|^{\frac{1}{2}}(1 + |X_s|^p)$, where we have denoted by X^c the continuous part of the process X, which is such that

$$X^c_t - X^c_s := \int_s^t a_u dW_u + \int_s^t b_u du.$$

3. Setting and main results

The process X is observed at regularly spaced times $t_i = i\Delta_n = \frac{iT}{n}$ for $i = 0, 1, \ldots, n$, within a finite time interval $[0, T]$. We can assume, WLOG, that $T = 1$.

Our goal is to estimate the integrated volatility $IV := \frac{1}{T}\int_0^T a_s^2 f(X_s) ds$, where f is a polynomial growth function. To do it, we propose the estimator Q_n, based on the truncated quadratic variation introduced by Mancini in [17]. Given that the quadratic variation was a good estimator for the integrated volatility in the continuous framework, the idea is to filter the contribution of the jumps and to keep only the intervals in which we judge no jumps happened. We use the size of the increment of the process $X_{t_{i+1}} - X_{t_i}$ in order to judge if a jump occurred or not in the interval $[t_i, t_{i+1})$: as it is hard for the increment of X with continuous transition to overcome the threshold $\Delta_n^\beta = (\frac{1}{n})^\beta$ for $\beta \le \frac{1}{2}$, we can assert the presence of a jump in $[t_i, t_{i+1})$ if $|X_{t_{i+1}} - X_{t_i}| > \Delta_n^\beta$.

We set

$$Q_n := \sum_{i=0}^{n-1} f(X_{t_i})(X_{t_{i+1}} - X_{t_i})^2 \varphi_{\Delta_n^\beta}(X_{t_{i+1}} - X_{t_i}), \tag{4}$$

where

$$\varphi_{\Delta_n^\beta}(X_{t_{i+1}} - X_{t_i}) = \varphi(\frac{X_{t_{i+1}} - X_{t_i}}{\Delta_n^\beta}),$$

with φ a smooth version of the indicator function, such that $\varphi(\zeta) = 0$ for each ζ, with $|\zeta| \ge 2$ and $\varphi(\zeta) = 1$ for each ζ, with $|\zeta| \le 1$.

It is worth noting that, if we consider an additional constant k in φ (that becomes $\varphi_{k\Delta_n^\beta}(X_{t_{i+1}} - X_{t_i}) = \varphi(\frac{X_{t_{i+1}} - X_{t_i}}{k\Delta_n^\beta})$), the only difference is the interval on which the function is 1 or 0: it will be 1 for $|X_{t_{i+1}} - X_{t_i}| \le k\Delta_n^\beta$; 0 for $|X_{t_{i+1}} - X_{t_i}| \ge 2k\Delta_n^\beta$. Hence, for shortness in notations, we restrict the theoretical analysis to the situation where $k = 1$ while, for applications, we may take the threshold level as $k\Delta_n^\beta$ with $k \ne 1$.

3.1. Main results

The main result of this paper is the asymptotic expansion for the truncated integrated volatility.

We show first of all it is possible to decompose the truncated quadratic variation, separating the continuous part from the contribution of the jumps. We consider right after the difference between the truncated quadratic variation and the discretized volatility, showing it consists on the statistical error (which derives from the continuous part), on a noise term due to the jumps and on a third term which is negligible compared to the other two. From such an expansion it appears clearly the condition on (α, β) which specifies whether or not the truncated quadratic variation performs well for the estimation of the integrated volatility. It is also possible to build some unbiased estimators. Indeed, through Malliavin calculus, we identify the main bias term which arises from the presence of the jumps. We study then its asymptotic behavior and, by making it equal to zero or by removing it from the original truncated quadratic variation, we construct some corrected estimators.

We define as $\tilde{Q}_n^J$ the jumps contribution present in the original estimator Q_n:

$$\tilde{Q}_n^J := n^{\beta(2-\alpha)} \sum_{i=0}^{n-1} (\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}\setminus\{0\}} \gamma(X_{s^-})\, z\, \tilde{\mu}(ds, dz))^2 f(X_{t_i}) \varphi_{\Delta_n^\beta}(X_{t_{i+1}} - X_{t_i}). \tag{5}$$

Denoting as $o_{\mathbb{P}}((\frac{1}{n})^k)$ a quantity such that $\frac{o_{\mathbb{P}}((\frac{1}{n})^k)}{(\frac{1}{n})^k} \xrightarrow{\mathbb{P}} 0$, the following decomposition holds true:

Theorem 1. *Suppose that A1–A4 hold and that $\beta \in (0, \frac{1}{2})$ and $\alpha \in (0, 2)$ are given in definition (4) and in the third point of A4, respectively. Then, as $n \to \infty$,*

$$Q_n = \sum_{i=0}^{n-1} f(X_{t_i})(X^c_{t_{i+1}} - X^c_{t_i})^2 + (\frac{1}{n})^{\beta(2-\alpha)} \tilde{Q}^J_n + \mathcal{E}_n \tag{6}$$

$$= \sum_{i=0}^{n-1} f(X_{t_i})(\int_{t_i}^{t_{i+1}} a_s dW_s)^2 + (\frac{1}{n})^{\beta(2-\alpha)} \tilde{Q}^J_n + \mathcal{E}_n, \tag{7}$$

where $\mathcal{E}_n$ is both $o_{\mathbb{P}}((\frac{1}{n})^{\beta(2-\alpha)})$ and, for each $\tilde{\epsilon} > 0$, $o_{\mathbb{P}}((\frac{1}{n})^{(1-\alpha\beta-\tilde{\epsilon})\wedge(\frac{1}{2}-\tilde{\epsilon})})$.

To show Theorem 1 here above, the following lemma will be useful. It illustrates the error we commit when the truncation is tight and therefore the Brownian increments are mistakenly truncated away.

Lemma 2. *Suppose that A1–A4 hold. Then, $\forall \epsilon > 0$,*

$$\sum_{i=0}^{n-1} f(X_{t_i})(X^c_{t_{i+1}} - X^c_{t_i})^2 (\varphi_{\Delta_n^\beta}(X_{t_{i+1}} - X_{t_i}) - 1) = o_{\mathbb{P}}((\frac{1}{n})^{1-\alpha\beta-\epsilon}).$$

Theorem 1 anticipates that the size of the jumps part is $(\frac{1}{n})^{\beta(2-\alpha)}$ (see Theorem 3) while the size of the Brownian increments wrongly removed is upper bounded by $(\frac{1}{n})^{1-\alpha\beta-\epsilon}$ (see Lemma 2). As $\beta \in (0, \frac{1}{2})$, we can always find an $\epsilon > 0$ such that $1 - \alpha\beta - \epsilon > \beta(2 - \alpha)$ and therefore the bias derived from a tight truncation is always smaller compared to those derived from a loose truncation. However, as we will see, after having removed the contribution of the jumps such a small downward bias will represent the main error term if $\alpha\beta > \frac{1}{2}$.

In order to eliminate the bias arising from the jumps, we want to identify the term $\tilde{Q}^J_n$ in detail. For that purpose we introduce

$$\hat{Q}_n := (\frac{1}{n})^{\frac{2}{\alpha}-\beta(2-\alpha)} \sum_{i=0}^{n-1} f(X_{t_i}) \gamma^2(X_{t_i}) d(\gamma(X_{t_i}) n^{\beta-\frac{1}{\alpha}}), \tag{8}$$

where $d(\zeta) := \mathbb{E}[(S_1^\alpha)^2 \varphi(S_1^\alpha \zeta)]$; $(S_t^\alpha)_{t\geq 0}$ is an α-stable process.

We want to move from $\tilde{Q}^J_n$ to $\hat{Q}_n$. The idea is to move from our process, that in small time behaves like a conditional rescaled Lévy process, to an α stable distribution.

Proposition 1. *Suppose that A1–A4 hold. Let $(S_t^\alpha)_{t\geq 0}$ be an α-stable process. Let g be a measurable bounded function such that $\|g\|_{pol} := \sup_{x\in\mathbb{R}}(\frac{|g(x)|}{1+|x|^p}) < \infty$, for some $p \geq 1$, $p \geq \alpha$ hence*

$$|g(x)| \leq \|g\|_{pol} (|x|^p + 1). \tag{9}$$

Moreover we denote $\|g\|_\infty := \sup_{x\in\mathbb{R}} |g(x)|$. Then, for any $\epsilon > 0$, $0 < h < \frac{1}{2}$,

$$|\mathbb{E}[g(h^{-\frac{1}{\alpha}} L_h)] - \mathbb{E}[g(S_1^\alpha)]| \leq C_\epsilon h |\log(h)| \|g\|_\infty + C_\epsilon h^{\frac{1}{\alpha}} \|g\|_\infty^{1-\frac{\alpha}{p}-\epsilon} \|g\|_{pol}^{\frac{\alpha}{p}+\epsilon} |\log(h)| \tag{10}$$

$$+ C_\epsilon h^{\frac{1}{\alpha}} \|g\|_\infty^{1+\frac{1}{p}-\frac{\alpha}{p}+\epsilon} \|g\|_{pol}^{-\frac{1}{p}+\frac{\alpha}{p}-\epsilon} |\log(h)| 1_{\{\alpha>1\}},$$

where C_ϵ is a constant independent of h.

Proposition 1 requires some Malliavin calculus. The proof of Proposition 1 as well as some technical tools will be found in Section 7.

The previous proposition is an extension of Theorem 4.2 in [7] and it is useful when $\|g\|_\infty$ is large, compared to $\|g\|_{pol}$. For instance, it is the case if consider the function $g(x) := |x|^2 1_{|x|\le M}$ for M large.

We need Proposition 1 to prove the following theorem, in which we consider the difference between the truncated quadratic variation and the discretized volatility. We make explicit its decomposition into the statistical error and the noise term due to the jumps, identified as $\hat{Q}_n$.

Theorem 2. *Suppose that A1–A4 hold and that $\beta \in (0, \frac{1}{2})$ and $\alpha \in (0, 2)$ are given in Definition (4) and in the third point of A4, respectively. Then, as $n \to \infty$,*

$$Q_n - \frac{1}{n}\sum_{i=0}^{n-1} f(X_{t_i})a_{t_i}^2 = \frac{Z_n}{\sqrt{n}} + (\frac{1}{n})^{\beta(2-\alpha)}\hat{Q}_n + \mathcal{E}_n, \tag{11}$$

where $\mathcal{E}_n$ is always $o_{\mathbb{P}}((\frac{1}{n})^{\beta(2-\alpha)})$ and, adding the condition $\beta > \frac{1}{4-\alpha}$, it is also $o_{\mathbb{P}}((\frac{1}{n})^{(1-\alpha\beta-\tilde{\epsilon})\wedge(\frac{1}{2}-\tilde{\epsilon})})$. Moreover $Z_n \xrightarrow{\mathcal{L}} N(0, 2\int_0^T a_s^4 f^2(X_s)ds)$ stably with respect to X.

We recognize in the expansion (11) the statistical error of model without jumps given by Z_n, whose variance is equal to the so called quadricity. As said above, the term $\hat{Q}_n$ is a bias term arising from the presence of jumps and given by (8). From this explicit expression it is possible to remove the bias term (see Section 4).

The term $\mathcal{E}_n$ is an additional error term that is always negligible compared to the bias deriving from the jump part $(\frac{1}{n})^{\beta(2-\alpha)}\hat{Q}_n$ (that is of order $(\frac{1}{n})^{\beta(2-\alpha)}$ by Theorem 3).

The bias term admits a first order expansion that does not require the knowledge of the density of S^α.

Proposition 2. *Suppose that A1–A4 hold and that $\beta \in (0, \frac{1}{2})$ and $\alpha \in (0, 2)$ are given in Definition (4) and in the third point of Assumption 4, respectively. Then*

$$\hat{Q}_n = \frac{1}{n}c_\alpha \sum_{i=0}^{n-1} f(X_{t_i})|\gamma(X_{t_i})|^\alpha (\int_{\mathbb{R}} \varphi(u)|u|^{1-\alpha}du) + \tilde{\mathcal{E}}_n, \tag{12}$$

with

$$c_\alpha = \begin{cases} \frac{\alpha(1-\alpha)}{4\Gamma(2-\alpha)\cos(\frac{\alpha\pi}{2})} & \text{if } \alpha \neq 1,\ \alpha < 2 \\ \frac{1}{2\pi} & \text{if } \alpha = 1. \end{cases} \tag{13}$$

$\tilde{\mathcal{E}}_n = o_{\mathbb{P}}(1)$ and, if $\alpha < \frac{4}{3}$, it is also $n^{\beta(2-\alpha)}o_{\mathbb{P}}((\frac{1}{n})^{(1-\alpha\beta-\tilde{\epsilon})\wedge(\frac{1}{2}-\tilde{\epsilon})}) = o_{\mathbb{P}}((\frac{1}{n})^{(\frac{1}{2}-2\beta+\alpha\beta-\tilde{\epsilon})\wedge(1-2\beta-\tilde{\epsilon})})$.

We have not replaced directly the right hand side of (12) in (11), observing that $(\frac{1}{n})^{\beta(2-\alpha)}\tilde{\mathcal{E}}_n = \mathcal{E}_n$, because $(\frac{1}{n})^{\beta(2-\alpha)}\tilde{\mathcal{E}}_n$ is always $o_{\mathbb{P}}((\frac{1}{n})^{\beta(2-\alpha)})$ but to get it is also $o_{\mathbb{P}}((\frac{1}{n})^{(1-\alpha\beta-\tilde{\epsilon})\wedge(\frac{1}{2}-\tilde{\epsilon})})$ the additional condition $\alpha < \frac{4}{3}$ is required.

Proposition 2 provides the contribution of the jumps in detail, identifying a main term. Recalling we are dealing with some bias, it comes naturally to look for some conditions to make it equal to zero and to study its asymptotic behavior in order to remove its limit.

Corollary 1. *Suppose that A1–A4 hold and that* $\alpha \in (0, \frac{4}{3})$, $\beta \in (\frac{1}{4-\alpha}, (\frac{1}{2\alpha} \wedge \frac{1}{2}))$. *If* φ *is such that* $\int_{\mathbb{R}} |u|^{1-\alpha}\varphi(u)du = 0$ *then,* $\forall \tilde{\epsilon} > 0$,

$$Q_n - \frac{1}{n}\sum_{i=0}^{n-1} f(X_{t_i})a_{t_i}^2 = \frac{Z_n}{\sqrt{n}} + o_{\mathbb{P}}((\frac{1}{n})^{\frac{1}{2}-\tilde{\epsilon}}), \tag{14}$$

with Z_n *defined as in Theorem 2 here above.*

It is always possible to build a function φ for which the condition here above is respected (see Section 4).

We have supposed $\alpha < \frac{4}{3}$ in order to say that the error we commit identifying the contribution of the jumps as the first term in the right hand side of (12) is always negligible compared to the statistical error. Moreover, taking $\beta < \frac{1}{2\alpha}$ we get $1-\alpha\beta > \frac{1}{2}$ and therefore also the bias studied in Lemma 2 becomes upper bounded by a quantity which is roughly $o_{\mathbb{P}}(\frac{1}{\sqrt{n}})$.

Eq. (14) gives us the behavior of the unbiased estimator, that is the truncated quadratic variation after having removed the noise derived from the presence of jumps. Taking α and β as discussed above we have, in other words, reduced the error term $\mathcal{E}_n$ to be $o_{\mathbb{P}}((\frac{1}{n})^{\frac{1}{2}-\tilde{\epsilon}})$, which is roughly the same size as the statistical error.

We observe that, if $\alpha \geq \frac{4}{3}$ but $\gamma = k \in \mathbb{R}$, the result still holds if we choose φ such that

$$\int_{\mathbb{R}} u^2\varphi(u)\, f_\alpha(\frac{1}{k}u(\frac{1}{n})^{\beta-\frac{1}{\alpha}})du = 0,$$

where f_α is the density of the α-stable process. Indeed, following (8), the jump bias $\hat{Q}_n$ is now defined as

$$\begin{aligned}
&(\frac{1}{n})^{\frac{2}{\alpha}-\beta(2-\alpha)}\sum_{i=0}^{n-1} f(X_{t_i})k^2 d(k\, n^{\beta-\frac{1}{\alpha}}) \\
&\quad = (\frac{1}{n})^{\frac{2}{\alpha}-\beta(2-\alpha)}\sum_{i=0}^{n-1} f(X_{t_i})k^2 \int_{\mathbb{R}} z^2\varphi(zk(\frac{1}{n})^{\frac{1}{\alpha}-\beta}) f_\alpha(z)dz \\
&\quad = (\frac{1}{n})^{\frac{2}{\alpha}-\beta(2-\alpha)}\sum_{i=0}^{n-1} f(X_{t_i})k^2(\frac{1}{n})^{3(\beta-\frac{1}{\alpha})}\frac{1}{k^3}\int_{\mathbb{R}} u^2\varphi(u)\, f_\alpha(\frac{1}{k}u(\frac{1}{n})^{\beta-\frac{1}{\alpha}})du = 0,
\end{aligned}$$

where we have used a change of variable.

Another way to construct an unbiased estimator is to study how the main bias detailed in (12) asymptotically behaves and to remove it from the original estimator.

Theorem 3. *Suppose that A1–A4 hold. Then, as* $n \to \infty$,

$$\hat{Q}_n \xrightarrow{\mathbb{P}} c_\alpha \int_{\mathbb{R}} \varphi(u)|u|^{1-\alpha}du \int_0^T |\gamma(X_s)|^\alpha f(X_s)ds. \tag{15}$$

Moreover

$$Q_n - IV = \frac{Z_n}{\sqrt{n}} + (\frac{1}{n})^{\beta(2-\alpha)}c_\alpha \int_{\mathbb{R}} \varphi(u)|u|^{1-\alpha}du \int_0^T |\gamma(X_s)|^\alpha f(X_s)ds + o_{\mathbb{P}}((\frac{1}{n})^{\beta(2-\alpha)}), \tag{16}$$

where $Z_n \xrightarrow{\mathcal{L}} N(0, 2\int_0^T a_s^4 f^2(X_s)ds)$ *stably with respect to X.*

It is worth noting that, in both [13] and [18], the integrated volatility estimation in short time is dealt and they show that the truncated quadratic variation has rate $\sqrt{n}$ if $\beta > \frac{1}{2(2-\alpha)}$.

We remark that the jump part is negligible compared to the statistic error if $n^{-1} < n^{-\frac{1}{2\beta(2-\alpha)}}$ and so $\beta > \frac{1}{2(2-\alpha)}$, that is the same condition given in the literature.

However, if we take (α, β) for which such a condition does not hold, we can still use that we know in detail the noise deriving from jumps to implement corrections that still make the unbiased estimator well-performed (see Section 4).

We require the activity α to be known, for conducting bias correction. If it is unknown, we need to estimate it previously (see for example the methods proposed by Todorov in [23] and by Mies in [20]). Then, a question could be how the estimation error in α would affect the rate of the bias-corrected estimator. We therefore assume that $\hat{\alpha}_n = \alpha + O_{\mathbb{P}}(a_n)$, for some rate sequence a_n. Replacing $\hat{\alpha}_n$ in (16) it turns out that the error derived from the estimation of α does not affect the correction if $a_n(\frac{1}{n})^{\beta(2-\alpha)} < (\frac{1}{n})^{\frac{1}{2}}$, which means that a_n has to be smaller than $(\frac{1}{n})^{\frac{1}{2}-\beta(2-\alpha)}$. We recall that $\beta \in (0, \frac{1}{2})$ and $\alpha \in (0, 2)$. Hence, such a condition is not a strong requirement and it becomes less and less restrictive when α gets smaller or β gets bigger.

4. Unbiased estimation in the case of constant volatility

In this section we consider a concrete application of the unbiased volatility estimator in a jump diffusion model and we investigate its numerical performance.

We consider our model (2) in which we assume, in addition, that the functions a and γ are both constants.

Suppose that we are given a discrete sample $X_{t_0}, \dots, X_{t_n}$ with $t_i = i\Delta_n = \frac{i}{n}$ for $i = 0, \dots, n$.

We now want to analyze the estimation improvement; to do it we compare the classical error committed using the truncated quadratic variation with the unbiased estimation derived by our main results.

We define the estimator we are going to use, in which we have clearly taken $f \equiv 1$ and we have introduced a threshold k in the function φ, so it is

$$Q_n = \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})^2 \varphi_{k\Delta_n^\beta}(X_{t_{i+1}} - X_{t_i}). \tag{17}$$

If normalized, the error committed estimating the volatility is $E_1 := (Q_n - \sigma^2)\sqrt{n}$.

We start from (12) that in our case, taking into account the presence of k, is

$$\hat{Q}_n = c_\alpha \gamma^\alpha k^{2-\alpha} (\int_{\mathbb{R}} \varphi(u)|u|^{1-\alpha} du) + \tilde{\mathcal{E}}_n. \tag{18}$$

We now get different methods to make the error smaller.

First of all we can replace (18) in (11) and so we can reduce the error by subtracting a correction term, building the new estimator $Q_n^c := Q_n - (\frac{1}{n})^{\beta(2-\alpha)} c_\alpha \gamma^\alpha k^{2-\alpha} (\int_{\mathbb{R}} \varphi(u)|u|^{1-\alpha} du)$. The error committed estimating the volatility with such a corrected estimator is $E_2 := (Q_n^c - \sigma^2)\sqrt{n}$.

Another approach consists of taking a particular function $\tilde{\varphi}$ that makes the main contribution of $\hat{Q}_n$ equal to 0. We define $\tilde{\varphi}(\zeta) = \varphi(\zeta) + c\psi(\zeta)$, with ψ a $\mathcal{C}^\infty$ function such that $\psi(\zeta) = 0$ for each ζ, $|\zeta| \geq 2$ or $|\zeta| \leq 1$. In this way, for any $c \in \mathbb{R} \setminus \{0\}$, $\tilde{\varphi}$ is still a smooth version of the indicator function such that $\tilde{\varphi}(\zeta) = 0$ for each ζ, $|\zeta| \geq 2$ and $\tilde{\varphi}(\zeta) = 1$ for each ζ, $|\zeta| \leq 1$.

We can therefore leverage the arbitrariness in c to make the main contribution of $\hat{Q}_n$ equal to zero, choosing $\tilde{c} := -\frac{\int_{\mathbb{R}} \varphi(u)|u|^{1-\alpha} du}{\int_{\mathbb{R}} \psi(u)|u|^{1-\alpha} du}$, which is such that $\int_{\mathbb{R}} (\varphi + \tilde{c}\psi(u))|u|^{1-\alpha} du = 0$.

Hence, it is possible to achieve an improved estimation of the volatility by using the truncated quadratic variation $Q_{n,c} := \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})^2 (\varphi + \tilde{c}\psi)(\frac{X_{t_{i+1}} - X_{t_i}}{k\Delta_n^\beta})$. To make it clear we will analyze the quantity $E_3 := (Q_{n,c} - \sigma^2)\sqrt{n}$.

Another method widely used in numerical analysis to improve the rate of convergence of a sequence is the so-called Richardson extrapolation. We observe that the first term on the right hand side of (18) does not depend on n and so we can just write $\hat{Q}_n = \hat{Q} + \tilde{\mathcal{E}}_n$. Replacing it in (11) we get

$$Q_n = \sigma^2 + \frac{Z_n}{\sqrt{n}} + \frac{1}{n^{\beta(2-\alpha)}} \hat{Q} + \mathcal{E}_n \qquad \text{and}$$

$$Q_{2n} = \sigma^2 + \frac{Z_{2n}}{\sqrt{2n}} + \frac{1}{2^{\beta(2-\alpha)}} \frac{1}{n^{\beta(2-\alpha)}} \hat{Q} + \mathcal{E}_{2n},$$

where we have also used that $(\frac{1}{n})^{\beta(2-\alpha)} \tilde{\mathcal{E}}_n = \mathcal{E}_n$. We can therefore use $\frac{Q_n - 2^{\beta(2-\alpha)} Q_{2n}}{1 - 2^{\beta(2-\alpha)}}$ as improved estimator of σ^2.

We give simulation results for E_1, E_2 and E_3 in the situation where $\sigma = 1$. The given mean and the deviation standard are each based on 500 Monte Carlo samples. We choose to simulate a tempered stable process (that is F satisfies $F(dz) = \frac{e^{-|z|}}{|z|^{1+\alpha}}$) in the case $\alpha < 1$ while, in the interest of computational efficiency, we will exhibit results gained from the simulation of a stable Lévy process in the case $\alpha \geq 1$ ($F(dz) = \frac{1}{|z|^{1+\alpha}}$).

We have taken the smooth functions φ and ψ as below:

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ e^{\frac{1}{3} + \frac{1}{|x|^2 - 4}} & \text{if } 1 \leq |x| < 2 \\ 0 & \text{if } |x| \geq 2 \end{cases} \tag{19}$$

$$\psi_M(x) = \begin{cases} 0 & \text{if } |x| \leq 1 \text{ or } |x| \geq M \\ e^{\frac{1}{3} + \frac{1}{|3-x|^2 - 4}} & \text{if } 1 < |x| \leq \frac{3}{2} \\ e^{\frac{1}{|x|^2 - M} - \frac{5}{21} + \frac{4}{4M^2 - 9}} & \text{if } \frac{3}{2} < |x| < M; \end{cases} \tag{20}$$

choosing opportunely the constant M in the definition of ψ_M we can make its decay slower or faster. We observe that the theoretical results still hold even if the support of $\tilde{\varphi}$ changes as M changes and so it is $[-M, M]$ instead of $[-2, 2]$.

Concerning the constant k in the definition of φ, we fix it equal to 3 in the simulation of the tempered stable process, while its value is 2 in the case $\alpha > 1$, $\beta = 0.2$ and, in the case $\alpha > 1$ and $\beta = 0.49$, it increases as α and γ increase.

The results of the simulations are given in columns 3–6 of Table 1a for $\beta = 0.2$ and in columns 3–6 of Table 1b for $\beta = 0.49$.

It appears that the estimation we get using the truncated quadratic variation performs worse as soon as α and γ become bigger (see column 3 in both Table 1a and b). However, after having applied the corrections, the error seems visibly reduced. A proof of which lies, for example, in the comparison between the error and the root mean square: before the adjustment in both Table 1a and b the third column dominates the fourth one, showing that the bias of the original estimator dominates the standard deviation while, after the implementation of our main results, we get E_2 and E_3 for which the bias is much smaller.

Table 1
Monte Carlo estimates of E_1, E_2 and E_3 from 500 samples. We have here fixed $n = 700$; $\beta = 0.2$ in the first table and $\beta = 0.49$ in the second one.

(a) $\beta = 0.2$

α	γ	Mean E_1	Rms E_1	Mean E_2	Mean E_3
0.1	1	3.820	3.177	0.831	0.189
	3	5.289	3.388	1.953	−0.013
0.5	1	15.168	9.411	0.955	1.706
	3	14.445	5.726	2.971	0.080
0.9	1	13.717	4.573	4.597	0.311
	3	42.419	6.980	13.664	−0.711
1.2	1	32.507	11.573	0.069	2.137
	3	112.648	21.279	−0.915	0.800
1.5	1	50.305	12.680	0.195	0.923
	3	250.832	27.170	−5.749	3.557
1.9	1	261.066	20.729	−0.530	9.139
	3	2311.521	155.950	−0.304	−35.177

(b) $\beta = 0.49$

α	γ	Mean E_1	Rms E_1	Mean E_2	Mean E_3
0.1	1	1.092	1.535	0.307	−0.402
	3	1.254	1.627	0.378	−0.372
0.5	1	2.503	1.690	0.754	−0.753
	3	4.680	2.146	1.651	−0.824
0.9	1	2.909	1.548	0.217	0.416
	3	8.042	1.767	0.620	−0.404
1.2	1	7.649	1.992	−0.944	−0.185
	3	64.937	9.918	−1.692	−2.275
1.5	1	25.713	3.653	−1.697	3.653
	3	218.591	21.871	−4.566	−13.027
1.9	1	238.379	14.860	−6.826	16.330
	3	2357.553	189.231	3.827	−87.353

We observe that for $\alpha < 1$, in both cases $\beta = 0.2$ and $\beta = 0.49$, it is possible to choose opportunely M (on which ψ's decay depends) to make the error E_3 smaller than E_2. On the other hand, for $\alpha > 1$, the approach which consists of subtracting the jump part to the error results better than the other, since E_3 is in this case generally bigger than E_2, but to use this method the knowledge of γ is required. It is worth noting that both the approaches used, that lead us respectively to E_2 and E_3, work well for any $\beta \in (0, \frac{1}{2})$.

We recall that, in [13], the condition found on β to get a well-performed estimator was

$$\beta > \frac{1}{2(2-\alpha)}, \tag{21}$$

that is not respected in the case $\beta = 0.2$. Our results match the ones in [13], since the third column in Table 1b (where $\beta = 0.49$) is generally smaller than the third one in Table 1a (where $\beta = 0.2$). We emphasize nevertheless that, comparing columns 5 and 6 in the two tables, there is no evidence of a dependence on β of E_2 and E_3.

The price you pay is that, to implement our corrections, the knowledge of α is request. Such corrections turn out to be a clear improvement also because for α that is less than 1 the original estimator (17) is well-performed only for those values of the couple (α, β) which respect the condition (21) while, for $\alpha \geq 1$, there is no $\beta \in (0, \frac{1}{2})$ for which such a condition can hold. That is the reason why, in the lower part of both Table 1a and b, E_1 is so big.

Using our main results, instead, we get E_2 and E_3 that are always small and so we obtain two corrections which make the unbiased estimator always well-performed without adding any requirement on α or β.

5. Preliminary results

In the sequel, for $\delta \geq 0$, we will denote as $R_i(\Delta_n^\delta)$ any random variable which is $\mathcal{F}_{t_i}$ measurable and such that, for any $q \geq 1$,

$$\exists c > 0 : \quad \left\| \frac{R_i(\Delta_n^\delta)}{\Delta_n^\delta} \right\|_{L^q} \leq c < \infty, \tag{22}$$

with c independent of i, n.

R_i represent the term of rest and have the following useful property, consequence of the just given definition:

$$R_i(\Delta_n^\delta) = \Delta_n^\delta R_i(\Delta_n^0). \tag{23}$$

We point out that it does not involve the linearity of R_i, since the random variables R_i on the left and on the right side are not necessarily the same but only two on which the control (22) holds with Δ_n^δ and Δ_n^0, respectively.

In order to prove the main result, the following proposition will be useful.

We define, for $i \in \{0, \dots, n-1\}$,

$$\begin{aligned} \Delta X_i^J &:= \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}\setminus\{0\}} \gamma(X_{s^-})\, z\, \tilde{\mu}(ds, dz) \qquad \text{and} \\ \Delta \tilde{X}_i^J &:= \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}\setminus\{0\}} \gamma(X_{t_i})\, z\, \tilde{\mu}(ds, dz). \end{aligned} \tag{24}$$

We want to bound the error we commit moving from ΔX_i^J to $\Delta \tilde{X}_i^J$, denoting as $o_{L^1}(\Delta_n^k)$ a quantity such that $\mathbb{E}_i[|o_{L^1}(\Delta_n^k)|] = R_i(\Delta_n^k)$, with the notation $\mathbb{E}_i[.] = \mathbb{E}[.|\mathcal{F}_{t_i}]$.

Proposition 3. *Suppose that A1–A4 hold. Then*

$$(\Delta X_i^J)^2 \varphi_{\Delta_n^\beta}(\Delta X_i) = (\Delta \tilde{X}_i^J)^2 \varphi_{\Delta_n^\beta}(\Delta \tilde{X}_i^J) + o_{L^1}(\Delta_n^{\beta(2-\alpha)+1}), \tag{25}$$

$$\Big(\int_{t_i}^{t_{i+1}} a_s dW_s\Big) \Delta X_i^J \varphi_{\Delta_n^\beta}(\Delta X_i) = \Big(\int_{t_i}^{t_{i+1}} a_s dW_s\Big) \Delta \tilde{X}_i^J \varphi_{\Delta_n^\beta}(\Delta \tilde{X}_i^J) + o_{L^1}(\Delta_n^{\beta(2-\alpha)+1}). \tag{26}$$

Moreover, for each $\tilde{\epsilon} > 0$ and f the function introduced in the definition of Q_n,

$$\sum_{i=0}^{n-1} f(X_{t_i})(\Delta X_i^J)^2 \varphi_{\Delta_n^\beta}(\Delta X_i) = \sum_{i=0}^{n-1} f(X_{t_i})(\Delta \tilde{X}_i^J)^2 \varphi_{\Delta_n^\beta}(\Delta \tilde{X}_i^J) + o_{\mathbb{P}}(\Delta_n^{(1-\alpha\beta-\tilde{\epsilon})\wedge(\frac{1}{2}-\tilde{\epsilon})}), \tag{27}$$

$$\begin{aligned} \sum_{i=0}^{n-1} f(X_{t_i})\Big(\int_{t_i}^{t_{i+1}} a_s dW_s\Big) \Delta X_i^J \varphi_{\Delta_n^\beta}(\Delta X_i) &= \sum_{i=0}^{n-1} f(X_{t_i})\Big(\int_{t_i}^{t_{i+1}} a_s dW_s\Big) \Delta \tilde{X}_i^J \varphi_{\Delta_n^\beta}(\Delta \tilde{X}_i^J) \\ &\quad + o_{\mathbb{P}}(\Delta_n^{(1-\alpha\beta-\tilde{\epsilon})\wedge(\frac{1}{2}-\tilde{\epsilon})}). \end{aligned} \tag{28}$$

Proposition 3 will be showed in the Appendix.

In the proof of our main results, also the following lemma will be repeatedly used.

Lemma 3. *Let us consider ΔX_i^J and $\Delta \tilde{X}_i^J$ as defined in (24). Then*

1. *For each $q \geq 2$ $\exists \epsilon > 0$ such that*

$$\mathbb{E}[|\Delta X_i^J 1_{\{|\Delta X_i^J| \leq 4\Delta_n^\beta\}}|^q | \mathcal{F}_{t_i}] = R_i(\Delta_n^{1+\beta(q-\alpha)}) = R_i(\Delta_n^{1+\epsilon}). \tag{29}$$

$$\mathbb{E}[|\Delta \tilde{X}_i^J 1_{\{|\Delta \tilde{X}_i^J| \leq 4\Delta_n^\beta\}}|^q | \mathcal{F}_{t_i}] = R_i(\Delta_n^{1+\beta(q-\alpha)}) = R_i(\Delta_n^{1+\epsilon}). \tag{30}$$

2. *For each $q \geq 1$ we have*

$$\mathbb{E}[|\Delta X_i^J 1_{\left\{\frac{\Delta_n^\beta}{4} \leq |\Delta X_i^J| \leq 4\Delta_n^\beta\right\}}|^q | \mathcal{F}_{t_i}] = R_i(\Delta_n^{1+\beta(q-\alpha)}). \tag{31}$$

Proof. Reasoning as in Lemma 10 in [1] we easily get (29). Observing that $\Delta\tilde{X}_i^J$ is a particular case of ΔX_i^J where γ is fixed, evaluated in X_{t_i}, it follows that (30) can be obtained in the same way of (29). Using the bound on ΔX_i^J obtained from the indicator function we get that the left hand side of (31) is upper bounded by

$$c\Delta_n^{\beta q}\mathbb{E}[1_{\left\{\frac{\Delta_n^\beta}{4}\le|\Delta X_i^J|\le 4\Delta_n^\beta\right\}}|\mathcal{F}_{t_i}]\le \Delta_n^{\beta q}R_i(\Delta_n^{1-\alpha\beta}),$$

where in the last inequality we have used Lemma 11 in [1] on the interval $[t_i, t_{i+1}]$ instead of on $[0, h]$. From property (23) of R_i we get (31). □

6. Proof of main results

We show Lemma 2, required for the proof of Theorem 1.

6.1. *Proof of Lemma 2*

Proof. By the definition of X^c we have

$$|\sum_{i=0}^{n-1} f(X_{t_i})(X^c_{t_{i+1}}-X^c_{t_i})^2(\varphi_{\Delta_n^\beta}(\Delta X_i)-1)|$$
$$\le c\sum_{i=0}^{n-1}|f(X_{t_i})|\big(|\int_{t_i}^{t_{i+1}}a_s dW_s|^2+|\int_{t_i}^{t_{i+1}}b_s ds|^2\big)|\varphi_{\Delta_n^\beta}(\Delta X_i)-1| =: |I_{2,1}^n|+|I_{2,2}^n|.$$

In the sequel the constant c may change value from line to line.

Concerning $I_{2,1}^n$, using Holder inequality we have

$$\mathbb{E}[|I_{2,1}^n|]\le c\sum_{i=0}^{n-1}\mathbb{E}[|f(X_{t_i})|\mathbb{E}_i[|\int_{t_i}^{t_{i+1}}a_s dW_s|^{2p}]^{\frac{1}{p}}\mathbb{E}_i[|\varphi_{\Delta_n^\beta}(\Delta X_i)-1|^q]^{\frac{1}{q}}],\tag{32}$$

where $\mathbb{E}_i$ is the conditional expectation wit respect to $\mathcal{F}_{t_i}$.

We now use Burkholder–Davis–Gundy inequality to get, for $p_1\ge 2$,

$$\mathbb{E}_i[|\int_{t_i}^{t_{i+1}}a_s dW_s|^{p_1}]^{\frac{1}{p_1}}\le \mathbb{E}_i[|\int_{t_i}^{t_{i+1}}a_s^2 ds|^{\frac{p_1}{2}}]^{\frac{1}{p_1}}\le R_i(\Delta_n^{\frac{p_1}{2}})^{\frac{1}{p_1}}=R_i(\Delta_n^{\frac{1}{2}}),\tag{33}$$

where in the last inequality we have used that a_s^2 has bounded moments as a consequence of Lemma 1. We now observe that, from the definition of φ we know that $\varphi_{\Delta_n^\beta}(\Delta X_i)-1$ is different from 0 only if $|\Delta X_i|>\Delta_n^\beta$. We consider two different sets: $|\Delta X_i^J|<\frac{1}{2}\Delta_n^\beta$ and $|\Delta X_i^J|\ge\frac{1}{2}\Delta_n^\beta$. We recall that $\Delta X_i=\Delta X_i^c+\Delta X_i^J$ and so, if $|\Delta X_i|>\Delta_n^\beta$ and $|\Delta X_i^J|<\frac{1}{2}\Delta_n^\beta$, then it means that $|\Delta X_i^c|$ must be more than $\frac{1}{2}\Delta_n^\beta$. Using a conditional version of Tchebychev inequality we have that, $\forall r>1$,

$$\mathbb{P}_i(|\Delta X_i^c|\ge\frac{1}{2}\Delta_n^\beta)\le c\frac{\mathbb{E}_i[|\Delta X_i^c|^r]}{\Delta_n^{\beta r}}\le R_i(\Delta_n^{(\frac{1}{2}-\beta)r}),\tag{34}$$

where $\mathbb{P}_i$ is the conditional probability with respect to $\mathcal{F}_{t_i}$; the last inequality follows from the sixth point of Lemma 1. If otherwise $|\Delta X_i^J|\ge\frac{1}{2}\Delta_n^\beta$, then we introduce the set $N_{i,n}:=\left\{|\Delta L_s|\le\frac{2\Delta_n^\beta}{\gamma_{min}};\forall s\in(t_i,t_{i+1}]\right\}$. We have $\mathbb{P}_i(\left\{|\Delta X_i^J|\ge\frac{1}{2}\Delta_n^\beta\right\}\cap(N_{i,n})^c)\le\mathbb{P}_i((N_{i,n})^c)$, with

$$\mathbb{P}_i((N_{i,n})^c) = \mathbb{P}_i(\exists s \in (t_i, t_{i+1}] : |\Delta L_s| > \frac{\Delta_n^\beta}{2\gamma_{min}}) \le c \int_{t_i}^{t_{i+1}} \int_{\frac{\Delta_n^\beta}{2\gamma_{min}}}^{\infty} F(z) dz ds \le c\Delta_n^{1-\alpha\beta}, \tag{35}$$

where we have used the third point of A4. Furthermore, using Markov inequality,

$$\begin{aligned}\mathbb{P}_i(\left\{|\Delta X_i^J| \ge \frac{1}{2}\Delta_n^\beta\right\} \cap N_{i,n}) &\le c\mathbb{E}_i[|\Delta X_i^J|^r 1_{N_{i,n}}]\Delta_n^{-\beta r} \le R_i(\Delta_n^{-\beta r+1+\beta(r-\alpha)}) \\ &= R_i(\Delta_n^{1-\beta\alpha}),\end{aligned} \tag{36}$$

where we have used the first point of Lemma 3, observing that $1_{N_{i,n}}$ acts like the indicator function in (29) (see also (219) in [1]). Now using (34), (35), (36) and the arbitrariness of r we have

$$\begin{aligned}\mathbb{P}_i(|\Delta X_i| > \Delta_n^\beta) &= \mathbb{P}_i(|\Delta X_i| > \Delta_n^\beta, |\Delta X_i^J| < \frac{1}{2}\Delta_n^\beta) + \mathbb{P}_i(|\Delta X_i| > \Delta_n^\beta, |\Delta X_i^J| \ge \frac{1}{2}\Delta_n^\beta) \\ &\le R_i(\Delta_n^{1-\alpha\beta}).\end{aligned} \tag{37}$$

Taking p big and q next to 1 in (32) and replacing there (33) with $p_1 = 2p$ and (37) we get, $\forall \epsilon > 0$,

$$\begin{aligned}n^{1-\alpha\beta-\tilde{\epsilon}}\mathbb{E}[|I_{2,1}^n|] &\le n^{1-\alpha\beta-\tilde{\epsilon}} c \sum_{i=1}^{n-1} \mathbb{E}[|f(X_{t_i})| R_i(\Delta_n) R_i(\Delta_n^{1-\alpha\beta-\epsilon})] \\ &\le (\frac{1}{n})^{\tilde{\epsilon}-\epsilon} \frac{c}{n} \sum_{i=1}^{n-1} \mathbb{E}[|f(X_{t_i})| R_i(1)].\end{aligned}$$

Now, for each $\tilde{\epsilon} > 0$, we can always find an ϵ smaller than it, that is enough to get that $\frac{I_{2,1}^n}{(\frac{1}{n})^{1-\alpha\beta-\tilde{\epsilon}}}$ goes to zero in L^1 and so in probability. Let us now consider $I_{2,2}^n$. We recall that b is uniformly bounded by a constant, therefore

$$(\int_{t_i}^{t_{i+1}} b_s ds)^2 \le c\Delta_n^2. \tag{38}$$

Acting moreover on $|\varphi_{\Delta_{n,i}^\beta}(\Delta X_i) - 1|$ as we did here above it follows

$$\begin{aligned}n^{1-\alpha\beta-\tilde{\epsilon}}\mathbb{E}[|I_{2,2}^n|] &\le n^{1-\alpha\beta-\tilde{\epsilon}} c \sum_{i=1}^{n-1} \mathbb{E}[|f(X_{t_i})| R_i(\Delta_n^2) R_i(\Delta_n^{1-\alpha\beta-\epsilon})] \\ &\le (\frac{1}{n})^{1+\tilde{\epsilon}-\epsilon} \frac{c}{n} \sum_{i=1}^{n-1} \mathbb{E}[|f(X_{t_i})| R_i(1)]\end{aligned}$$

and so $I_{2,2}^n = o_{\mathbb{P}}((\frac{1}{n})^{1-\alpha\beta-\tilde{\epsilon}})$. $\square$

6.2. Proof of Theorem 1

We observe that, using the dynamic (2) of X and the definition of the continuous part X^c, we have that

$$X_{t_{i+1}} - X_{t_i} = (X_{t_{i+1}}^c - X_{t_i}^c) + \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}\setminus\{0\}} \gamma(X_{s^-})\, z\, \tilde{\mu}(ds, dz). \tag{39}$$

Replacing (39) in definition (4) of Q_n we have

$$Q_n = \sum_{i=0}^{n-1} f(X_{t_i})(X^c_{t_{i+1}} - X^c_{t_i})^2 + \sum_{i=0}^{n-1} f(X_{t_i})(X^c_{t_{i+1}} - X^c_{t_i})^2(\varphi_{\Delta_n^\beta}(\Delta X_i) - 1)$$
$$+ 2\sum_{i=0}^{n-1} f(X_{t_i})(X^c_{t_{i+1}} - X^c_{t_i})(\Delta X_i^J)\varphi_{\Delta_n^\beta}(\Delta X_i)$$
$$+ \sum_{i=0}^{n-1} f(X_{t_i})(\Delta X_i^J)^2\varphi_{\Delta_n^\beta}(\Delta X_i) =: \sum_{j=1}^{4} I_j^n. \tag{40}$$

Comparing (40) with (6), using also definition (5) of $\tilde{Q}_n$, it follows that our goal is to show that $I_2^n + I_3^n = \mathcal{E}_n$, that is both $o_{\mathbb{P}}(\Delta_n^{\beta(2-\alpha)})$ and $o_{\mathbb{P}}(\Delta_n^{(1-\alpha\beta-\tilde{\epsilon})\wedge(\frac{1}{2}-\tilde{\epsilon})})$. We have already shown in Lemma 2 that $I_2^n = o_{\mathbb{P}}(\Delta_n^{1-\alpha\beta-\tilde{\epsilon}})$. As $(1-\alpha\beta-\tilde{\epsilon})\wedge(\frac{1}{2}-\tilde{\epsilon}) < 1-\alpha\beta-\tilde{\epsilon}$ and $\beta(2-\alpha) < 1-\alpha\beta-\tilde{\epsilon}$, we immediately get $I_2^n = \mathcal{E}_n$.

Let us now consider I_3^n. From the definition of the process (X_t^c) it is

$$2\sum_{i=0}^{n-1} f(X_{t_i})[\int_{t_i}^{t_{i+1}} b_s ds + \int_{t_i}^{t_{i+1}} a_s dW_s]\Delta X_i^J \varphi_{\Delta_n^\beta}(\Delta X_i) =: I_{3,1}^n + I_{3,2}^n.$$

We use on $I_{3,1}^n$ Cauchy–Schwarz inequality, (38) and Lemma 10 in [1], getting

$$\mathbb{E}[|I_{3,1}^n|] \le 2\sum_{i=0}^{n-1} \mathbb{E}[|f(X_{t_i})|R_i(\Delta_n^{1+\beta(2-\alpha)})^{\frac{1}{2}} R_i(\Delta_n^2)^{\frac{1}{2}}]$$
$$\le \Delta_n^{\frac{1}{2}+\frac{\beta}{2}(2-\alpha)} \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}[|f(X_{t_i})|R_i(1)],$$

where we have also used property (23) on R. We observe it is $\frac{1}{2} + \beta - \frac{\alpha\beta}{2} > \frac{1}{2}$ if and only if $\beta(1 - \frac{\alpha}{2}) > 0$, that is always true. We can therefore say that $I_{3,1}^n = o_{\mathbb{P}}(\Delta_n^{\frac{1}{2}})$ and so

$$I_{3,1}^n = o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}). \tag{41}$$

Moreover,

$$\frac{\mathbb{E}[|I_{3,1}^n|]}{\Delta_n^{\beta(2-\alpha)}} \le \Delta_n^{\frac{1}{2}-\beta+\frac{\alpha\beta}{2}} \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}[|f(X_{t_i})|R_i(1)], \tag{42}$$

that goes to zero using the polynomial growth of f, the definition of R, the fifth point of Lemma 1. Moreover, we have observed that the exponent on Δ_n is positive for $\beta < \frac{1}{2}\frac{1}{(1-\frac{\alpha}{2})}$, that is always true.

Concerning $I_{3,2}^n$, we start proving that $I_{3,2}^n = o_{\mathbb{P}}(\Delta_n^{\beta(2-\alpha)})$. From (26) in Proposition 3 we have

$$\frac{I_{3,2}^n}{\Delta_n^{\beta(2-\alpha)}} = \frac{2}{\Delta_n^{\beta(2-\alpha)}} \sum_{i=0}^{n-1} f(X_{t_i})\Delta \tilde{X}_i^J \varphi_{\Delta_n^\beta}(\Delta \tilde{X}_i^J) \int_{t_i}^{t_{i+1}} a_s dW_s$$
$$+ \frac{2}{\Delta_n^{\beta(2-\alpha)}} \sum_{i=0}^{n-1} f(X_{t_i}) o_{L^1}(\Delta_n^{\beta(2-\alpha)+1}). \tag{43}$$

By the definition of o_{L^1} the last term here above goes to zero in norm 1 and so in probability. The first term of (43) can be seen as

$$\frac{2}{\Delta_n^{\beta(2-\alpha)}}\sum_{i=0}^{n-1} f(X_{t_i})\Delta\tilde{X}_i^J\varphi_{\Delta_n^\beta}(\Delta\tilde{X}_i^J)[\int_{t_i}^{t_{i+1}} a_{t_i}dW_s+\int_{t_i}^{t_{i+1}}(a_s-a_{t_i})dW_s]. \tag{44}$$

On the first term of (44) here above we want to use Lemma 9 of [9] in order to get that it converges to zero in probability, so we have to show the following:

$$\frac{2}{\Delta_n^{\beta(2-\alpha)}}\sum_{i=0}^{n-1}\mathbb{E}_i[f(X_{t_i})\Delta\tilde{X}_i^J\varphi_{\Delta_n^\beta}(\Delta\tilde{X}_i^J)\int_{t_i}^{t_{i+1}} a_{t_i}dW_s]\xrightarrow{\mathbb{P}}0, \tag{45}$$

$$\frac{4}{\Delta_n^{2\beta(2-\alpha)}}\sum_{i=0}^{n-1}\mathbb{E}_i[f^2(X_{t_i})(\Delta\tilde{X}_i^J)^2\varphi^2_{\Delta_n^\beta}(\Delta\tilde{X}_i^J)(\int_{t_i}^{t_{i+1}} a_{t_i}dW_s)^2]\xrightarrow{\mathbb{P}}0, \tag{46}$$

where $\mathbb{E}_i[.]=\mathbb{E}[.|\mathcal{F}_{t_i}]$.

Using the independence between W and L we have that the left hand side of (45) is

$$\frac{2}{\Delta_n^{\beta(2-\alpha)}}\sum_{i=0}^{n-1} f(X_{t_i})\mathbb{E}_i[\Delta\tilde{X}_i^J\varphi_{\Delta_n^\beta}(\Delta\tilde{X}_i^J)]\mathbb{E}_i[\int_{t_i}^{t_{i+1}} a_{t_i}dW_s]=0. \tag{47}$$

Now, in order to prove (46), we use Holder inequality with p big and q next to 1 on its left hand side, getting it is upper bounded by

$$\begin{aligned}&\Delta_n^{-2\beta(2-\alpha)}\sum_{i=0}^{n-1} f^2(X_{t_i})\mathbb{E}_i[(\int_{t_i}^{t_{i+1}} a_{t_i}dW_s)^{2p}]^{\frac{1}{p}}\mathbb{E}_i[|\Delta\tilde{X}_i^J\varphi_{\Delta_n^\beta}(\Delta\tilde{X}_i^J)|^{2q}]^{\frac{1}{q}}\\ &\quad\le\Delta_n^{-2\beta(2-\alpha)}\sum_{i=0}^{n-1} f^2(X_{t_i})R_i(\Delta_n)R_i(\Delta_n^{\frac{1}{q}+\frac{\beta}{q}(2q-\alpha)})\\ &\quad\le\Delta_n^{1-2\beta(2-\alpha)+2\beta-\alpha\beta-\epsilon}\frac{1}{n}\sum_{i=0}^{n-1} f^2(X_{t_i})R_i(1),\end{aligned} \tag{48}$$

where we have used (33), (30) and property (23) of R. We observe that the exponent on Δ_n is positive if $\beta<\frac{1}{2-\alpha}-\epsilon$ and we can always find an $\epsilon>0$ such that it is true. Hence (48) goes to zero in norm 1 and so in probability.

Concerning the second term of (44), using Cauchy–Schwarz inequality and (30) we have

$$\begin{aligned}&\mathbb{E}_i[|\Delta\tilde{X}_i^J\varphi_{\Delta_n^\beta}(\Delta\tilde{X}_i^J)||\int_{t_i}^{t_{i+1}}[a_s-a_{t_i}]dW_s|]\\ &\quad\le\mathbb{E}_i[|\Delta\tilde{X}_i^J\varphi_{\Delta_n^\beta}(\Delta\tilde{X}_i^J)|^2]^{\frac{1}{2}}\mathbb{E}_i[|\int_{t_i}^{t_{i+1}}[a_s-a_{t_i}]dW_s|^2]^{\frac{1}{2}}\\ &\quad\le R_i(\Delta_n^{\frac{1}{2}+\frac{\beta}{2}(2-\alpha)})\mathbb{E}_i[\int_{t_i}^{t_{i+1}}|a_s-a_{t_i}|^2ds]^{\frac{1}{2}}\le\Delta_n^{\frac{1}{2}+\frac{\beta}{2}(2-\alpha)}R_i(1)\Delta_n\le\Delta_{n,i}^{\frac{3}{2}+\frac{\beta}{2}(2-\alpha)}R_i(1),\end{aligned} \tag{49}$$

where we have also used the second point of Lemma 1 and the property (23) of R. Replacing (49) in the second term of (44) we get it is upper bounded in norm 1 by

$$\Delta_n^{\frac{1}{2}-\beta+\frac{\alpha\beta}{2}}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[|f(X_{t_i})|R_i(1)], \tag{50}$$

that goes to zero since the exponent on Δ_n is more than 0 for $\beta < \frac{1}{2}\frac{1}{(1-\frac{\alpha}{2})}$, that is always true. Using (43)–(46) and (50) we get

$$\frac{I_{3,2}^n}{\Delta_n^{\beta(2-\alpha)}} \xrightarrow{\mathbb{P}} 0. \tag{51}$$

We now want to show that $I_{3,2}^n$ is also $o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})})$.

Using (28) in Proposition 3 we get it is enough to prove that

$$\frac{1}{\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}}\sum_{i=0}^{n-1} f(X_{t_i})[\Delta\tilde{X}_i^J \varphi_{\Delta_n^\beta}(\Delta\tilde{X}_i^J)\int_{t_i}^{t_{i+1}} a_s dW_s] \xrightarrow{\mathbb{P}} 0, \tag{52}$$

where the left hand side here above can be seen as (44), with the only difference that now we have $\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}$ instead of $\Delta_n^{\beta(2-\alpha)}$. We have again, acting like we did in (47) and (48),

$$\frac{2}{\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}}\sum_{i=0}^{n-1} f(X_{t_i})\mathbb{E}_i[\Delta\tilde{X}_i^J \varphi_{\Delta_n^\beta}(\Delta\tilde{X}_i^J)\int_{t_i}^{t_{i+1}} a_{t_i} dW_s] \xrightarrow{\mathbb{P}} 0 \tag{53}$$

and

$$\frac{4}{\Delta_n^{2(\frac{1}{2}-\tilde{\epsilon})}}\sum_{i=0}^{n-1} \mathbb{E}_i[f^2(X_{t_i})(\Delta\tilde{X}_i^J)^2 \varphi^2_{\Delta_n^\beta}(\Delta\tilde{X}_i^J)(\int_{t_i}^{t_{i+1}} a_{t_i} dW_s)^2]$$
$$\le \Delta_n^{2\tilde{\epsilon}+2\beta-\alpha\beta-\epsilon}\frac{1}{n}\sum_{i=0}^{n-1} f^2(X_{t_i})R_i(1), \tag{54}$$

that goes to zero in norm 1 and so in probability. Using also (49) we have that

$$\frac{2}{\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}}\sum_{i=0}^{n-1} \mathbb{E}_i[|f(X_{t_i})\Delta\tilde{X}_i^J \varphi_{\Delta_n^\beta}(\Delta\tilde{X}_i^J)\int_{t_i}^{t_{i+1}} [a_s - a_{t_i}]dW_s|]$$
$$\le \Delta_n^{\frac{\beta}{2}(2-\alpha)+\tilde{\epsilon}}\frac{1}{n}\sum_{i=0}^{n-1} |f(X_{t_i})|R_i(1), \tag{55}$$

that, again, goes to zero in norm 1 and so in probability since the exponent on Δ_n is always positive. Using (52)–(55) we get $I_{3,2}^n = o_{\mathbb{P}}(\Delta_n^{\frac{1}{2}-\tilde{\epsilon}})$ and so

$$I_{3,2}^n = o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}). \tag{56}$$

From Lemma 2, (41), (42), (51) and (56) it follows (6).

Now, in order to prove (7), we recall the definition of X_t^c:

$$X_{t_{i+1}}^c - X_{t_i}^c = \int_{t_i}^{t_{i+1}} b_s ds + \int_{t_i}^{t_{i+1}} a_s dW_s. \tag{57}$$

Replacing (57) in (6) and comparing it with (7) it follows that our goal is to show that

$$A_1^n + A_2^n := \sum_{i=0}^{n-1} f(X_{t_i})(\int_{t_i}^{t_{i+1}} b_s ds)^2 + 2\sum_{i=0}^{n-1} f(X_{t_i})(\int_{t_i}^{t_{i+1}} b_s ds)(\int_{t_i}^{t_{i+1}} a_s dW_s) = \mathcal{E}_n.$$

Using (38) and property (23) of R we know that

$$\frac{\mathbb{E}[|A_1^n|]}{\Delta_n^{\beta(2-\alpha)}} \le \frac{1}{\Delta_n^{\beta(2-\alpha)}} \sum_{i=0}^{n-1} \mathbb{E}[|f(X_{t_i})| R_i(\Delta_n^2)] \le \Delta_n^{1-\beta(2-\alpha)} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}[|f(X_{t_i})| R_i(1)] \tag{58}$$

and

$$\frac{\mathbb{E}[|A_1^n|]}{\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}} \le \Delta_n^{\frac{1}{2}+\tilde{\epsilon}} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}[|f(X_{t_i})| R_i(1)], \tag{59}$$

that go to zero since the exponent on Δ_n is always more than 0, f has both polynomial growth and the moments are bounded.

Let us now consider A_2^n. By adding and subtracting b_{t_i} in the first integral, as we have already done, we get that

$$A_2^n = \sum_{i=0}^{n-1} \zeta_{n,i} + A_{2,2}^n := 2\sum_{i=0}^{n-1} f(X_{t_i})(\int_{t_i}^{t_{i+1}} b_{t_i} ds)(\int_{t_i}^{t_{i+1}} a_s dW_s)$$
$$+ 2\sum_{i=0}^{n-1} f(X_{t_i})(\int_{t_i}^{t_{i+1}} [b_s - b_{t_i}] ds)(\int_{t_i}^{t_{i+1}} a_s dW_s).$$

Using Lemma 9 in [9], we want to show that

$$\sum_{i=0}^{n-1} \zeta_{n,i} = \mathcal{E}_n \tag{60}$$

and so that the following convergences hold:

$$\frac{1}{\Delta_n^{\beta(2-\alpha)}} \sum_{i=0}^{n-1} \mathbb{E}_i[\zeta_{n,i}] \xrightarrow{\mathbb{P}} 0 \qquad \frac{1}{\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}} \sum_{i=0}^{n-1} \mathbb{E}_i[\zeta_{n,i}] \xrightarrow{\mathbb{P}} 0; \tag{61}$$

$$\frac{1}{\Delta_n^{2\beta(2-\alpha)}} \sum_{i=0}^{n-1} \mathbb{E}_i[\zeta_{n,i}^2] \xrightarrow{\mathbb{P}} 0 \qquad \frac{1}{\Delta_n^{2(\frac{1}{2}-\tilde{\epsilon})}} \sum_{i=0}^{n-1} \mathbb{E}_i[\zeta_{n,i}^2] \xrightarrow{\mathbb{P}} 0. \tag{62}$$

We have

$$\sum_{i=0}^{n-1} \mathbb{E}_i[\zeta_{n,i}] = \frac{2}{\Delta_n^{\beta(2-\alpha)}} \sum_{i=0}^{n-1} f(X_{t_i}) \Delta_n b_{t_i} \mathbb{E}_i[\int_{t_i}^{t_{i+1}} a_s dW_s] = 0$$

and so the two convergences in (61) both hold. Concerning (62), using (33) we have

$$\Delta_n^{1-2\beta(2-\alpha)} \frac{c}{n} \sum_{i=0}^{n-1} f^2(X_{t_i}) b_{t_i}^2 \mathbb{E}_i[(\int_{t_i}^{t_{i+1}} a_s dW_s)^2] \le \Delta_n^{2-2\beta(2-\alpha)} \frac{c}{n} \sum_{i=0}^{n-1} f^2(X_{t_i}) b_{t_i}^2 R_i(1)$$

and

$$\Delta_n^{1-2(\frac{1}{2}-\tilde{\epsilon})} \frac{c}{n} \sum_{i=0}^{n-1} f^2(X_{t_i}) b_{t_i}^2 \mathbb{E}_i[(\int_{t_i}^{t_{i+1}} a_s dW_s)^2] \le \Delta_n^{1+2\tilde{\epsilon}} \frac{c}{n} \sum_{i=0}^{n-1} f^2(X_{t_i}) b_{t_i}^2 R_i(1),$$

that go to zero in norm 1 and so in probability since Δ_n is always positive. It follows (62) and so (60). Concerning $A_{2,2}^n$, using Holder inequality, (33), the assumption on b gathered in A2 and Jensen inequality it is

$$\begin{aligned}\mathbb{E}[|A^n_{2,2}|] &\le c\sum_{i=0}^{n-1}\mathbb{E}[|f(X_{t_i})|\mathbb{E}_i[(\int_{t_i}^{t_{i+1}}|b_s-b_{t_i}|ds)^q]^{\frac{1}{q}}R_i(\Delta_n^{\frac{1}{2}})]\\ &\le c\sum_{i=0}^{n-1}\mathbb{E}[|f(X_{t_i})|(\Delta_n^{q-1}\int_{t_i}^{t_{i+1}}\mathbb{E}_i[|b_s-b_{t_i}|^q]ds)^{\frac{1}{q}}R_i(\Delta_n^{\frac{1}{2}})]\\ &\le c\sum_{i=0}^{n-1}\mathbb{E}[|f(X_{t_i})|(\Delta_n^{q-1}\int_{t_i}^{t_{i+1}}\Delta_n ds)^{\frac{1}{q}}R_i(\Delta_n^{\frac{1}{2}})].\end{aligned}$$

So we get

$$\frac{\mathbb{E}[|A^n_{2,2}|]}{\Delta_n^{\beta(2-\alpha)}}\le\Delta_n^{\frac{1}{q}+\frac{1}{2}-\beta(2-\alpha)}\frac{c}{n}\sum_{i=0}^{n-1}\mathbb{E}[|f(X_{t_i})|R_i(1)]\qquad\text{and}\tag{63}$$

$$\frac{\mathbb{E}[|A^n_{2,2}|]}{\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}}\le\Delta_n^{\frac{1}{q}+\tilde{\epsilon}}\frac{c}{n}\sum_{i=0}^{n-1}\mathbb{E}[|f(X_{t_i})|R_i(1)].\tag{64}$$

Since it holds for $q\ge 2$, the best choice is to take $q=2$, in this way we get that (63) and (64) go to 0 in norm 1, using the polynomial growth of f, the boundedness of the moments, the definition of R_i and the fact that the exponent on Δ_n is in both cases more than zero, because of $\beta<\frac{1}{2-\alpha}$.

From (58), (59), (61), (63) and (64) it follows (7).

6.3. *Proof of Theorem 2*

Proof. From Theorem 1 it is enough to prove that

$$\sum_{i=0}^{n-1}f(X_{t_i})(\int_{t_i}^{t_{i+1}}a_s dW_s)^2-\frac{1}{n}\sum_{i=0}^{n-1}f(X_{t_i})a^2_{t_i}=\frac{Z_n}{\sqrt{n}}+\mathcal{E}_n,\tag{65}$$

and

$$\tilde{Q}^J_n=\hat{Q}_n+\frac{1}{\Delta_n^{\beta(2-\alpha)}}\mathcal{E}_n,$$

where $\mathcal{E}_n$ is always $o_{\mathbb{P}}(\Delta_n^{\beta(2-\alpha)})$ and, if $\beta>\frac{1}{4-\alpha}$, then it is also $o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})})$. We can rewrite the last equation here above as

$$\tilde{Q}^J_n=\hat{Q}_n+o_{\mathbb{P}}(1)\tag{66}$$

and, for $\beta>\frac{1}{4-\alpha}$,

$$\tilde{Q}^J_n=\hat{Q}_n+\frac{1}{\Delta_n^{\beta(2-\alpha)}}o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}).\tag{67}$$

Indeed, using them and (7) it follows (11). Hence we are now left to prove (65)–(67).

Proof of (65). We can see the left hand side of (65) as

$$\sum_{i=0}^{n-1}f(X_{t_i})[(\int_{t_i}^{t_{i+1}}a_s dW_s)^2-\int_{t_i}^{t_{i+1}}a^2_s ds]+\sum_{i=0}^{n-1}f(X_{t_i})\int_{t_i}^{t_{i+1}}[a^2_s-a^2_{t_i}]ds=:M^Q_n+B_n.\tag{68}$$

We want to show that $B_n = \mathcal{E}_n$, it means that it is both $o_{\mathbb{P}}(\Delta_n^{\beta(2-\alpha)})$ and $o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})})$. We write

$$a_s^2 - a_{t_i}^2 = 2a_{t_i}(a_s - a_{t_i}) + (a_s - a_{t_i})^2, \tag{69}$$

replacing (69) in the definition of B_n it is $B_n = B_1^n + B_2^n$. We start by proving that $B_2^n = o_{\mathbb{P}}(\Delta_n^{\beta(2-\alpha)})$. Indeed, from the second point of Lemma 1, it is

$$\mathbb{E}[|B_2^n|] \le c\sum_{i=0}^{n-1}\mathbb{E}[|f(X_{t_i})|\int_{t_i}^{t_{i+1}}\mathbb{E}_i[|a_s - a_{t_i}|^2]ds] \le c\Delta_n^2\sum_{i=0}^{n-1}\mathbb{E}[|f(X_{t_i})|].$$

It follows

$$\frac{\mathbb{E}[|B_2^n|]}{\Delta_n^{\beta(2-\alpha)}} \le \Delta_n^{1-\beta(2-\alpha)}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[|f|(X_{t_i})] \qquad \text{and} \qquad \frac{\mathbb{E}[|B_2^n|]}{\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}} \le \Delta_n^{\frac{1}{2}+\tilde{\epsilon}}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[|f|(X_{t_i})], \tag{70}$$

that go to zero using the polynomial growth of f and the fact that the moments are bounded. We have also observed that the exponent on Δ_n is always more than 0.

Concerning B_1^n, we recall that from (3) it follows

$$a_s - a_{t_i} = \int_{t_i}^s \tilde{b}_u du + \int_{t_i}^s \tilde{a}_u dW_u + \int_{t_i}^s \hat{a}_u d\hat{W}_u + \int_{t_i}^s\int_{\mathbb{R}\setminus\{0\}}\tilde{\gamma}_u\, z\, \tilde{\mu}(du,dz)$$
$$+ \int_{t_i}^s\int_{\mathbb{R}\setminus\{0\}}\hat{\gamma}_u\, z\, \tilde{\mu}_2(du,dz)$$

and so, replacing it in the definition of B_1^n, we get $B_1^n := I_1^n + I_2^n + I_3^n + I_4^n + I_5^n$.

We start considering I_1^n on which we use that $\tilde{b}$ is bounded

$$\mathbb{E}[|I_1^n|] \le 2\sum_{i=0}^{n-1}\mathbb{E}[|f(X_{t_i})||a_{t_i}|\int_{t_i}^{t_{i+1}}(\int_{t_i}^s c du)ds] \le \Delta_n\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[|f(X_{t_i})||a_{t_i}|].$$

It follows

$$\frac{\mathbb{E}[|I_1^n|]}{\Delta_n^{\beta(2-\alpha)}} \le \Delta_n^{1-\beta(2-\alpha)}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[|f(X_{t_i})||a_{t_i}|] \qquad \text{and} \tag{71}$$

$$\frac{\mathbb{E}[|I_1^n|]}{\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}} \le \Delta_n^{\frac{1}{2}+\tilde{\epsilon}}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[|f(X_{t_i})||a_{t_i}|], \tag{72}$$

that go to zero because of the polynomial growth of f, the boundedness of the moments and the fact that $1-\beta(2-\alpha) > 0$.

We now act on I_2^n and I_3^n in the same way. Considering I_2^n, we define $\zeta_{n,i} := 2f(X_{t_i})a_{t_i}\int_{t_i}^{t_{i+1}}(\int_{t_i}^s \tilde{a}_u dW_u)ds$. We want to use Lemma 9 in [9] to get that

$$\frac{I_2^n}{\Delta_n^{\beta(2-\alpha)}} \xrightarrow{\mathbb{P}} 0 \qquad \text{and} \qquad \frac{I_2^n}{\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}} \xrightarrow{\mathbb{P}} 0 \tag{73}$$

and so we have to show the following:

$$\frac{1}{\Delta_n^{\beta(2-\alpha)}}\sum_{i=0}^{n-1}\mathbb{E}_i[\zeta_{n,i}] \xrightarrow{\mathbb{P}} 0, \qquad \frac{1}{\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}}\sum_{i=0}^{n-1}\mathbb{E}_i[\zeta_{n,i}] \xrightarrow{\mathbb{P}} 0; \tag{74}$$

$$\frac{1}{\Delta_n^{2\beta(2-\alpha)}}\sum_{i=0}^{n-1}\mathbb{E}_i[\zeta_{n,i}^2]\xrightarrow{\mathbb{P}}0, \tag{75}$$

$$\frac{1}{\Delta_n^{2(\frac{1}{2}-\tilde{\epsilon})}}\sum_{i=0}^{n-1}\mathbb{E}_i[\zeta_{n,i}^2]\xrightarrow{\mathbb{P}}0. \tag{76}$$

By the definition of $\zeta_{n,i}$ it is $\mathbb{E}_i[\zeta_{n,i}]=0$ and so (74) is clearly true. The left hand side of (75) is

$$\Delta_n^{-2\beta(2-\alpha)}4\sum_{i=0}^{n-1}f^2(X_{t_i})a_{t_i}^2\mathbb{E}_i[(\int_{t_i}^{t_{i+1}}(\int_{t_i}^{s}\tilde{a}_u dW_u)ds)^2]. \tag{77}$$

Using Fubini theorem and Ito isometry we have

$$\mathbb{E}_i[(\int_{t_i}^{t_{i+1}}(\int_{t_i}^{s}\tilde{a}_u dW_u)ds)^2]=\mathbb{E}_i[(\int_{t_i}^{t_{i+1}}(t_{i+1}-s)\tilde{a}_s dW_s)^2]=\mathbb{E}_i[\int_{t_i}^{t_{i+1}}(t_{i+1}-s^2)\tilde{a}_s^2 ds]$$
$$\leq R_i(\Delta_n^3). \tag{78}$$

Because of (78), we get that (77) is upper bounded by

$$\Delta_n^{2-2\beta(2-\alpha)}\frac{1}{n}\sum_{i=0}^{n-1}f^2(X_{t_i})a_{t_i}^2R_i(1),$$

that converges to zero in norm 1 and so (75) follows, since $2-2\beta(2-\alpha)>0$ for $\beta<\frac{1}{2-\alpha}$, that is always true. Acting in the same way we get that the left hand side of (76) is upper bounded by

$$\Delta_n^{1+2\tilde{\epsilon}}\frac{1}{n}\sum_{i=0}^{n-1}f^2(X_{t_i})a_{t_i}^2R_i(1),$$

that goes to zero in norm 1. The same holds clearly for I_3^n instead of I_2^n. In order to show also

$$\frac{I_4^n}{\Delta_n^{\beta(2-\alpha)}}\xrightarrow{\mathbb{P}}0 \quad\text{ and }\quad \frac{I_4^n}{\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}}\xrightarrow{\mathbb{P}}0, \tag{79}$$

we define $\tilde{\zeta}_{n,i}:=2f(X_{t_i})a_{t_i}\int_{t_i}^{t_{i+1}}(\int_{t_i}^{s}\int_{\mathbb{R}}\tilde{\gamma}_u z\tilde{\mu}(du,dz))ds$. We have again $\mathbb{E}_i[\tilde{\zeta}_{n,i}]=0$ and so (74) holds with $\tilde{\zeta}_{n,i}$ in place of $\zeta_{n,i}$. We now act like we did in (78), using Fubini theorem and Ito isometry. It follows

$$\mathbb{E}_i[(\int_{t_i}^{t_{i+1}}(\int_{t_i}^{s}\int_{\mathbb{R}}\tilde{\gamma}_u z\tilde{\mu}(du,dz)ds)^2)]=\mathbb{E}_i[(\int_{t_i}^{t_{i+1}}\int_{\mathbb{R}}(t_{i+1}-s)\tilde{\gamma}_s z\tilde{\mu}(ds,dz))^2]$$
$$=\mathbb{E}_i[\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^2\tilde{\gamma}_s^2 ds(\int_{\mathbb{R}}z^2F(z)dz)]\leq R_i(\Delta_n^3), \tag{80}$$

having used in the last inequality the definition of $\bar{\mu}(ds,dz)$, the fact that $\int_{\mathbb{R}}z^2F(z)dz<\infty$ and the boundedness of $\tilde{\gamma}$. Replacing (80) in the left hand side of (75) and (76), with $\tilde{\zeta}_{n,i}$ in place of $\zeta_{n,i}$, we have

$$\frac{1}{\Delta_n^{2\beta(2-\alpha)}}\sum_{i=0}^{n-1}\mathbb{E}_i[\tilde{\zeta}_{n,i}^2]\leq c\Delta_n^{-2\beta(2-\alpha)}\sum_{i=0}^{n-1}f^2(X_{t_i})a_{t_i}^2R_i(\Delta_n^3)$$
$$\leq\Delta_n^{2-2\beta(2-\alpha)}\frac{1}{n}\sum_{i=0}^{n-1}f^2(X_{t_i})a_{t_i}^2R_i(1)$$

$$\text{and } \frac{1}{\Delta_n^{1-2\tilde{\epsilon}}}\sum_{i=0}^{n-1}\mathbb{E}_i[\tilde{\zeta}_{n,i}^2] \le \Delta_n^{1+2\tilde{\epsilon}}\frac{1}{n}\sum_{i=0}^{n-1} f^2(X_{t_i})a_{t_i}^2 R_i(1).$$

Again, they converge to zero in norm 1 and thus in probability since $2-2\beta(2-\alpha)>0$ always holds. Therefore, we get (79). Clearly, (79) holds also with I_5^n replacing I_4^n; the reasoning here above joint with the sixth point of A4 on F_2 is proof of that.

From (70), (71), (72), (73) and (79) it follows that

$$B_n = \mathcal{E}_n. \tag{81}$$

Concerning $M_n^Q := \sum_{i=0}^{n-1}\hat{\zeta}_{n,i}$, Genon-Catalot and Jacod have proved in [9] that, in the continuous framework, the following conditions are enough to get $\sqrt{n}M_n^Q \to N(0, 2\int_0^T f^2(X_s) a_s^4 ds)$ stably with respect to X:

- $\mathbb{E}_i[\hat{\zeta}_{n,i}] = 0$;
- $\sum_{i=0}^{n-1}\mathbb{E}_i[\hat{\zeta}_{n,i}^2] \xrightarrow{\mathbb{P}} 2\int_0^T f^2(X_s)a_s^4 ds$;
- $\sum_{i=0}^{n-1}\mathbb{E}_i[\hat{\zeta}_{n,i}^4] \xrightarrow{\mathbb{P}} 0$;
- $\sum_{i=0}^{n-1}\mathbb{E}_i[\hat{\zeta}_{n,i}(W_{t_{i+1}} - W_{t_i})] \xrightarrow{\mathbb{P}} 0$;
- $\sum_{i=0}^{n-1}\mathbb{E}_i[\hat{\zeta}_{n,i}(\hat{W}_{t_{i+1}} - \hat{W}_{t_i})] \xrightarrow{\mathbb{P}} 0$.

Theorem 2.2.15 in [12] adapts the previous theorem to our framework, in which there is the presence of jumps.

We observe that the conditions here above are respected, hence

$$M_n^Q = \frac{Z_n}{\sqrt{n}}, \text{ where } Z_n \xrightarrow{n} N(0, 2\int_0^T f^2(X_s)a_s^4 ds), \tag{82}$$

stably with respect to X. From (81) and (82), it follows (65).

Proof of (66). We use Proposition 3 replacing (25) in the definition (5) of $\tilde{Q}_n^J$. Recalling that the convergence in norm 1 implies the convergence in probability it is clear that we have to prove the result on

$$\begin{aligned} & n^{\beta(2-\alpha)}\sum_{i=0}^{n-1} f(X_{t_i})(\Delta\tilde{X}_i^J)^2\varphi_{\Delta_n^\beta}(\Delta\tilde{X}_i^J) \\ &= n^{\beta(2-\alpha)}\sum_{i=0}^{n-1} f(X_{t_i})\gamma^2(X_{t_i})\Delta_n^{\frac{2}{\alpha}}(\frac{\Delta\tilde{X}_i^J}{\gamma(X_{t_i})\Delta_n^{\frac{1}{\alpha}}})^2\varphi_{\Delta_n^\beta}(\frac{\Delta\tilde{X}_i^J}{\gamma(X_{t_i})\Delta_n^{\frac{1}{\alpha}}}\gamma(X_{t_i})\Delta_n^{\frac{1}{\alpha}}), \end{aligned} \tag{83}$$

where we have also rescaled the process in order to apply Proposition 1. We now define

$$g_{i,n}(y) := y^2\varphi_{\Delta_n^\beta}(y\gamma(X_{t_i})\Delta_n^{\frac{1}{\alpha}}), \tag{84}$$

hence we can rewrite (83) as

$$\begin{aligned} & (\frac{1}{n})^{\frac{2}{\alpha}-\beta(2-\alpha)}\sum_{i=0}^{n-1} f(X_{t_i})\gamma^2(X_{t_i})[g_{i,n}(\frac{\Delta\tilde{X}_i^J}{\gamma(X_{t_i})\Delta_n^{\frac{1}{\alpha}}}) - \mathbb{E}[g_{i,n}(S_1^\alpha)]] \\ & + (\frac{1}{n})^{\frac{2}{\alpha}-\beta(2-\alpha)}\sum_{i=0}^{n-1} f(X_{t_i})\gamma^2(X_{t_i})\mathbb{E}[g_{i,n}(S_1^\alpha)] =: \sum_{i=0}^{n-1} A_{1,i}^n + \hat{Q}_n, \end{aligned} \tag{85}$$

where S_1^α is the α-stable process at time $t = 1$. We want to show that $\sum_{i=0}^{n-1} A_{1,i}^n$ converges to zero in probability. With this purpose in mind, we take the conditional expectation of $A_{1,i}^n$ and we apply Proposition 1 on the interval $[t_i, t_{i+1}]$ instead of on $[0, h]$, observing that property (9) holds on $g_{i,n}$ for $p = 2$. By the definition (84) of $g_{i,n}$, we have $\left\| g_{i,n} \right\|_\infty = R_i(\Delta_n^{2(\beta-\frac{1}{\alpha})})$ and $\left\| g_{i,n} \right\|_{pol} = R_i(1)$. Replacing them in (10) we have that

$$|\mathbb{E}_i[g_{i,n}(\frac{\Delta\tilde{X}_i^J}{\gamma(X_{t_i})\Delta_n^{\frac{1}{\alpha}}})] - \mathbb{E}[g_{i,n}(S_1^\alpha)]| \le c_{\epsilon,\alpha}\Delta_n|\log(\Delta_n)|R_i(\Delta_n^{2(\beta-\frac{1}{\alpha})})$$
$$+ c_{\epsilon,\alpha}\Delta_n^{\frac{1}{\alpha}}|\log(\Delta_n)|R_i(\Delta_n^{2(\beta-\frac{1}{\alpha})(1-\frac{\alpha}{2}-\epsilon)}) + c_{\epsilon,\alpha}\Delta_n^{\frac{1}{\alpha}}|\log(\Delta_n)|R_i(\Delta_n^{2(\beta-\frac{1}{\alpha})(\frac{3}{2}-\frac{\alpha}{2}-\epsilon)})1_{\alpha>1}.$$

To get $\sum_{i=0}^{n-1} A_{1,i}^n := o_{\mathbb{P}}(1)$, we want to use Lemma 9 of [9]. We have

$$\sum_{i=0}^{n-1}|\mathbb{E}_i[A_{1,i}^n]| \le (\frac{1}{n})^{\frac{2}{\alpha}-\beta(2-\alpha)}\sum_{i=0}^{n-1}|f(X_{t_i})||\gamma^2(X_{t_i})||\log(\Delta_n)|[\Delta_n^{1+2(\beta-\frac{1}{\alpha})}$$
$$+ \Delta_n^{\frac{1}{\alpha}+(2-\alpha-\epsilon)(\beta-\frac{1}{\alpha})}$$
$$+ \Delta_n^{\frac{1}{\alpha}+(3-\alpha-\epsilon)(\beta-\frac{1}{\alpha})}1_{\alpha>1}]R_i(1) \le (\Delta_n^{\alpha\beta} + \Delta_n^{\frac{1}{\alpha}-\epsilon} + \Delta_n^{\beta-\epsilon}1_{\alpha>1})\frac{|\log(\Delta_n)|}{n}$$
$$\times \sum_{i=0}^{n-1}|f(X_{t_i})||\gamma^2(X_{t_i})|R_i(1), \tag{86}$$

where we have used property (23). Using the polynomial growth of f, the boundedness of the moments and the fifth point of Assumption 4 in order to bound γ, (86) converges to 0 in norm 1 and so in probability since $\Delta_n^{\alpha\beta}\log(\Delta_n) \to 0$ for $n \to \infty$ and we can always find an $\epsilon > 0$ such that $\Delta_n^{\frac{1}{\alpha}-\epsilon}$ does the same.

To use Lemma 9 of [9] we have also to show that

$$(\frac{1}{n})^{\frac{4}{\alpha}-2\beta(2-\alpha)}\sum_{i=0}^{n-1} f^2(X_{t_i})\gamma^4(X_{t_i})\mathbb{E}_i[(g_{i,n}(\frac{\Delta\tilde{X}_i^J}{\gamma(X_{t_i})\Delta_n^{\frac{1}{\alpha}}}) - \mathbb{E}[g_{i,n}(S_1^\alpha)])^2] \xrightarrow{P} 0. \tag{87}$$

We observe that $\mathbb{E}_i[(g_{i,n}(\frac{\Delta\tilde{X}_i^J}{\gamma(X_{t_i})\Delta_n^{\frac{1}{\alpha}}}) - \mathbb{E}[g_{i,n}(S_1^\alpha)])^2] \le c\mathbb{E}_i[g_{i,n}^2(\frac{\Delta\tilde{X}_i^J}{\gamma(X_{t_i})\Delta_n^{\frac{1}{\alpha}}})] + c\mathbb{E}_i[\mathbb{E}[g_{i,n}(S_1^\alpha)]^2]$. Now, using Eq. (30) of Lemma 3, we observe it is

$$\mathbb{E}_i[g_{i,n}^2(\frac{\Delta\tilde{X}_i^J}{\gamma(X_{t_i})\Delta_n^{\frac{1}{\alpha}}})] = \frac{\Delta_n^{-\frac{4}{\alpha}}}{\gamma^4(X_{t_i})}\mathbb{E}_i[(\Delta\tilde{X}_i^J)^4\varphi^2_{\Delta_n^\beta}(\Delta\tilde{X}_i^J)] = \frac{\Delta_n^{-\frac{4}{\alpha}}}{\gamma^4(X_{t_i})}R_i(\Delta_n^{1+\beta(4-\alpha)}), \tag{88}$$

where φ acts as the indicator function. Moreover we observe that

$$\mathbb{E}[g_{i,n}(S_1^\alpha)] = \int_{\mathbb{R}} z^2\varphi(\Delta_n^{\frac{1}{\alpha}-\beta}\gamma(X_{t_i})z)f_\alpha(z)dz = d(\gamma(X_{t_i})\Delta_n^{\frac{1}{\alpha}-\beta}), \tag{89}$$

with $f_\alpha(z)$ the density of the stable process. We now introduce the following lemma, that will be shown in the Appendix:

Lemma 4. *Suppose that Assumptions 1–4 hold. Then, for each ζ_n such that $\zeta_n \to 0$ and for each $\hat{\epsilon} > 0$,*

$$d(\zeta_n) = |\zeta_n|^{\alpha-2} c_\alpha \int_{\mathbb{R}} |u|^{1-\alpha} \varphi(u) du + o(|\zeta_n|^{-\hat{\epsilon}} + |\zeta_n|^{2\alpha-2-\hat{\epsilon}}), \tag{90}$$

where c_α has been defined in (13).

Since $\frac{1}{\alpha} - \beta > 0$, $\gamma(X_{t_i})\Delta_n^{\frac{1}{\alpha}-\beta}$ goes to zero for $n \to \infty$ and so we can take ζ_n as $\gamma(X_{t_i})$ $\Delta_n^{\frac{1}{\alpha}-\beta}$, getting that

$$\mathbb{E}[g_{i,n}(S_1^\alpha)] = d(\gamma(X_{t_i})\Delta_n^{\frac{1}{\alpha}-\beta}) = R_i(\Delta_n^{(\frac{1}{\alpha}-\beta)(\alpha-2)}). \tag{91}$$

Replacing (88) and (91) in the left hand side of (87) we get it is upper bounded by

$$\begin{aligned}\sum_{i=0}^{n-1} \mathbb{E}_i[(A_{1,i}^n)^2] &= (\frac{1}{n})^{\frac{4}{\alpha}-2\beta(2-\alpha)} \sum_{i=0}^{n-1} f^2(X_{t_i})\gamma^4(X_{t_i})(R_i(\Delta_n^{1+\beta(4-\alpha)}) + R_i(\Delta_n^{4\beta-\frac{4}{\alpha}+2-2\alpha\beta})) \\ &\leq \Delta_n^{\alpha\beta\wedge 1} \frac{1}{n} \sum_{i=0}^{n-1} f^2(X_{t_i})\gamma^4(X_{t_i}) R_i(1),\end{aligned} \tag{92}$$

that converges to zero in norm 1 and so in probability, as a consequence of the polynomial growth of f and the fact that the exponent on Δ_n is always positive. From (86) and (92) it follows

$$\sum_{i=0}^{n-1} A_{1,i}^n = o_{\mathbb{P}}(1). \tag{93}$$

and so (66).

Proof of (67). We use Proposition 3 replacing (27) in definition (5) of $\tilde{Q}_n^J$. Our goal is to prove that

$$n^{\beta(2-\alpha)} \sum_{i=0}^{n-1} f(X_{t_i})(\Delta\tilde{X}_i^J)^2 \varphi_{\Delta_n^\beta}(\Delta\tilde{X}_i^J) = \hat{Q}_n + o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-2\beta+\alpha\beta-\tilde{\epsilon})\wedge(1-2\beta-\tilde{\epsilon})}).$$

On the left hand side of the equation here above we can act like we did in (83)–(85). To get (67) we are therefore left to show that, if $\beta > \frac{1}{4-\alpha}$, then $\sum_{i=0}^{n-1} A_{1,i}^n$ is also $o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-2\beta+\alpha\beta-\tilde{\epsilon})\wedge(1-2\beta-\tilde{\epsilon})})$. To prove it, we want to use Lemma 9 of [9], hence we want to prove the following:

$$\frac{1}{\Delta_n^{\frac{1}{2}-2\beta+\alpha\beta-\tilde{\epsilon}}} \sum_{i=0}^{n-1} \mathbb{E}_i[A_{1,i}^n] \xrightarrow{\mathbb{P}} 0 \quad \text{and} \tag{94}$$

$$\frac{1}{\Delta_n^{2(\frac{1}{2}-2\beta+\alpha\beta-\tilde{\epsilon})}} \sum_{i=0}^{n-1} \mathbb{E}_i[(A_{1,i}^n)^2] \xrightarrow{\mathbb{P}} 0. \tag{95}$$

Using (86) we have that, if $\alpha > 1$, then the left hand side of (94) is in module upper bounded by

$$\begin{aligned}&\frac{\Delta_n^{\beta-\epsilon}|\log(\Delta_n)|}{\Delta_n^{\frac{1}{2}-2\beta+\alpha\beta-\tilde{\epsilon}}} \frac{1}{n} \sum_{i=0}^{n-1} |f(X_{t_i})||\gamma^2(X_{t_i})| R_i(1) \\ &= \Delta_n^{3\beta-\alpha\beta-\frac{1}{2}+\tilde{\epsilon}-\epsilon} |\log(\Delta_n)| \frac{1}{n} \sum_{i=0}^{n-1} |f(X_{t_i})||\gamma^2(X_{t_i})| R_i(1),\end{aligned}$$

that goes to zero since we have chosen $\beta > \frac{1}{4-\alpha} > \frac{1}{2(3-\alpha)}$. Otherwise, if $\alpha \le 1$, then (86) gives us that the left hand side of (94) is in module upper bounded by

$$\frac{\Delta_n^{\alpha\beta}|\log(\Delta_n)|}{\Delta_n^{\frac{1}{2}-2\beta+\alpha\beta-\tilde{\epsilon}}}\frac{1}{n}\sum_{i=0}^{n-1}|f(X_{t_i})||\gamma^2(X_{t_i})|R_i(1)$$

$$= \Delta_n^{2\beta-\frac{1}{2}+\tilde{\epsilon}}|\log(\Delta_n)|\frac{1}{n}\sum_{i=0}^{n-1}|f(X_{t_i})||\gamma^2(X_{t_i})|R_i(1),$$

that goes to zero because $\beta > \frac{1}{4-\alpha} > \frac{1}{4}$.

Using also (92), the left hand side of (95) turns out to be upper bounded by $\Delta_n^{-1+4\beta-2\alpha\beta+2\tilde{\epsilon}}\Delta_n^{\alpha\beta\wedge 1}\frac{1}{n}\sum_{i=0}^{n-1} f^2(X_{t_i})\gamma^4(X_{t_i})R_i(1)$, that goes to zero in norm 1 and so in probability since we have chosen $\beta > \frac{1}{4-\alpha}$. It follows (95) and so (11). □

6.4. Proof of Proposition 2

Proof. To prove the proposition we replace (90) in the definition of $\hat{Q}_n$. It follows that our goal is to show that

$$I_1^n + I_2^n := (\frac{1}{n})^{\frac{2}{\alpha}-\beta(2-\alpha)}\sum_{i=0}^{n-1} f(X_{t_i})\gamma^2(X_{t_i})(o(|\Delta_n^{\frac{1}{\alpha}-\beta}\gamma(X_{t_i})|^{-\hat{\epsilon}}$$
$$+|\Delta_n^{\frac{1}{\alpha}-\beta}\gamma(X_{t_i})|^{2\alpha-2-\hat{\epsilon}})) = \tilde{\mathcal{E}}_n,$$

where $\tilde{\mathcal{E}}_n$ is always $o_{\mathbb{P}}(1)$ and, if $\alpha < \frac{4}{3}$, it is also $\frac{1}{\Delta_n^{\beta(2-\alpha)}}o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})})$.

We have that $I_1^n = o_{\mathbb{P}}(1)$ since it is upper bounded by

$$\Delta_n^{\frac{2}{\alpha}-1-2\beta+\alpha\beta-\hat{\epsilon}(\frac{1}{\alpha}-\beta)}\frac{1}{n}\sum_{i=0}^{n-1} R_i(1)\,o(1),$$

that goes to zero in norm 1 and so in probability since we can always find an $\hat{\epsilon} > 0$ such that the exponent on Δ_n is positive.

Also I_2^n is $o_{\mathbb{P}}(1)$. Indeed it is upper bounded by

$$\Delta_n^{\frac{2}{\alpha}-1-2\beta+\alpha\beta-2(\frac{1}{\alpha}-\beta)+2(1-\alpha\beta)-\hat{\epsilon}(\frac{1}{\alpha}-\beta)}\frac{1}{n}\sum_{i=0}^{n-1} R_i(1)\,o(1). \tag{96}$$

We observe that the exponent on Δ_n is $1-\alpha\beta-\hat{\epsilon}(\frac{1}{\alpha}-\beta)$ and we can always find $\hat{\epsilon}$ such that it is more than zero, hence (96) converges in norm 1 and so in probability.

In order to show that $I_1^n = \frac{1}{\Delta_n^{\beta(2-\alpha)}}o_{\mathbb{P}}(\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}) = o_{\mathbb{P}}(\Delta_n^{\frac{1}{2}-\tilde{\epsilon}-\beta(2-\alpha)})$ we observe that

$$\frac{I_1^n}{\Delta_n^{\frac{1}{2}-\tilde{\epsilon}-\beta(2-\alpha)}} \le \Delta_n^{\frac{2}{\alpha}-1-\frac{1}{2}+\tilde{\epsilon}-\hat{\epsilon}(\frac{1}{\alpha}-\beta)}\frac{1}{n}\sum_{i=0}^{n-1} R_i(1)\,o(1).$$

If $\alpha < \frac{4}{3}$ we can always find $\tilde{\epsilon}$ and $\hat{\epsilon}$ such that the exponent on Δ_n is more than zero, getting the convergence wanted. It follows $I_1^n = \frac{1}{\Delta_n^{\beta(2-\alpha)}}o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})})$.

To conclude, $I_2^n = \frac{1}{\Delta_n^{\beta(2-\alpha)}} o_{\mathbb{P}}(\Delta_n^{1-\alpha\beta-\tilde{\epsilon}}) = o_{\mathbb{P}}(\Delta_n^{1-2\beta-\tilde{\epsilon}})$. Indeed,

$$\frac{I_2^n}{\Delta_n^{1-2\beta-\tilde{\epsilon}}} \leq \Delta_n^{\frac{2}{\alpha}-1-1+\alpha\beta+\tilde{\epsilon}-2(\frac{1}{\alpha}-\beta)+2(1-\alpha\beta)-\hat{\epsilon}(\frac{1}{\alpha}-\beta)} \frac{1}{n}\sum_{i=0}^{n-1} R_i(1)\, o(1). \tag{97}$$

The exponent on Δ_n is $2\beta-\alpha\beta+\tilde{\epsilon}-\hat{\epsilon}(\frac{1}{\alpha}-\beta)$ and so we can always find $\tilde{\epsilon}$ and $\hat{\epsilon}$ such that it is positive. It follows the convergence in norm 1 and so in probability of (97). The proposition is therefore proved. □

6.5. Proof of Corollary 1

Proof. We observe that (14) is a consequence of (12) in the case where $\hat{Q}_n = 0$. Moreover, $\beta < \frac{1}{2\alpha}$ implies that $\Delta_n^{1-\alpha\beta-\tilde{\epsilon}}$ is negligible compared to $\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}$. It follows (14). □

6.6. Proof of Theorem 3

Proof. The convergence (15) clearly follows from (12).

Concerning the proof of (16), we can see its left hand side as

$$Q_n - \frac{1}{n}\sum_{i=0}^{n-1} f(X_{t_i})a_{t_i}^2 + \frac{1}{n}\sum_{i=0}^{n-1} f(X_{t_i})a_{t_i}^2 - IV_1$$

and so, using (11) and the definition of IV_1, it turns out that our goal is to show that

$$\frac{1}{n}\sum_{i=0}^{n-1} f(X_{t_i})a_{t_i}^2 - \int_0^1 f(X_s)\, a_s^2 ds = o_{\mathbb{P}}(\Delta_n^{\beta(2-\alpha)}). \tag{98}$$

The left hand side of (98) is

$$\sum_{i=0}^{n-1} f(X_{t_i})\int_{t_i}^{t_{i+1}} (a_{t_i}^2 - a_s^2)ds + \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} a_s^2(f(X_{t_i}) - f(X_s))ds =: B_n + R_n.$$

B_n in the equation here above is exactly the same term we have already dealt with in the proof of Theorem 2 (see (68)). As showed in (81) it is $\mathcal{E}_n$ and so, in particular, it is also $o_{\mathbb{P}}(\Delta_n^{\beta(2-\alpha)})$.

On R_n we act like we did on B_n, considering this time the development up to second order of the function f, getting

$$f(X_s) = f(X_{t_i}) + f'(X_{t_i})(X_s - X_{t_i}) + \frac{f''(\tilde{X}_{t_i})}{2}(X_s - X_{t_i})^2, \tag{99}$$

where $\tilde{X}_{t_i} \in [X_{t_i}, X_s]$. Replacing (99) in R_n we get two terms that we denote R_n^1 and R_n^2. On them we can act like we did on (69). The estimations gathered in Lemma 1 about the increments of X and of a have the same size (see points 2 and 4) and provide on B_2^n and R_n^2 the same upper bound:

$$\mathbb{E}[|R_n^2|] \leq c\sum_{i=0}^{n-1}\mathbb{E}[|f''(X_{t_i})|\int_{t_i}^{t_{i+1}}\mathbb{E}_i[|a_s||X_s - X_{t_i}|^2]ds] \leq c\Delta_n^2\sum_{i=0}^{n-1}\mathbb{E}[|f''(X_{t_i})|R_i(1)],$$

where we have used Cauchy–Schwarz inequality and the fourth point of Lemma 1. It yields $R_n^2 = o_{\mathbb{P}}(\Delta_n^{\beta(2-\alpha)})$, which is the same result found in the first inequality of (70) for the increments of a.

To deal with R^1_n we replace the dynamic of X (as done with the dynamic of a for B^n_1). Even if the volatility coefficient in the dynamic of X is no longer bounded, the condition $\sup_{s\in[t_i,t_{i+1}]}\mathbb{E}_i[|a_s|] < \infty$ (which is true according with Lemma 1) is enough to say that (78) keep holding.

Following the method provided in the proof of Theorem 2 to show that $B^n_1 = \mathcal{E}_n$ we obtain $R^1_n = \mathcal{E}_n$ and therefore $R^1_n = o_{\mathbb{P}}(\Delta_n^{\beta(2-\alpha)})$. It yields (98) and so the theorem is proved. $\square$

7. Proof of Proposition 1

This section is dedicated to the proof of Proposition 1. To do it, it is convenient to introduce an adequate truncation function and to consider a rescaled process, as explained in the next subsections. Moreover, the proof of Proposition 1 requires some Malliavin calculus; we recall in what follows all the technical tools to make easier the understanding of the paper.

7.1. Localization and rescaling

We introduce a truncation function in order to suppress the big jumps of (L_t). Let $\tau : \mathbb{R} \to [0, 1]$ be a symmetric function, continuous with continuous derivative, such that $\tau = 1$ on $\left\{|z| \le \frac{1}{4}\eta\right\}$ and $\tau = 0$ on $\left\{|z| \ge \frac{1}{2}\eta\right\}$, with η defined in the fourth point of Assumption 4.

On the same probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ we consider the Lévy process (L_t) defined below (2) whose measure is $F(dz) = \frac{\bar{g}(z)}{|z|^{1+\alpha}}1_{\mathbb{R}\setminus\{0\}}(z)dz$, according with the third point of A4, and the truncated Lévy process (L^τ_t) with measure $F^\tau(dz)$ given by $F^\tau(dz) = \frac{\bar{g}(z)\tau(z)}{|z|^{1+\alpha}}1_{\mathbb{R}\setminus\{0\}}(z)dz$. This can be done by setting $L_t := \int_0^t\int_{\mathbb{R}} z\tilde{\mu}(ds, dz)$, as we have already done, and $L^\tau_t := \int_0^t\int_{\mathbb{R}} z\tilde{\mu}^\tau(ds, dz)$, where $\tilde{\mu}$ and $\tilde{\mu}^\tau$ are the compensated Poisson random measures associated respectively to

$$\mu(A) := \int_{[0,1]}\int_{\mathbb{R}}\int_{[0,T]} 1_A(t, z)\mu^{\bar{g}}(dt, dz, du), \qquad A \subset [0, T] \times \mathbb{R},$$

$$\mu^\tau(A) := \int_{[0,1]}\int_{\mathbb{R}}\int_{[0,T]} 1_A(t, z)1_{u\le\tau(z)}\mu^{\bar{g}}(dt, dz, du), \qquad A \subset [0, T] \times \mathbb{R},$$

for $\mu^{\bar{g}}$ a Poisson random measure on $[0, T] \times \mathbb{R} \times [0, 1]$ with compensator $\bar{\mu}^{\bar{g}}(dt, dz, du) = dt\frac{\bar{g}(z)}{|z|^{1+\alpha}}1_{\mathbb{R}\setminus\{0\}}(z)dzdu$.

By construction, the restrictions of the measures μ and μ^τ to $[0, h] \times \mathbb{R}$ coincide on the set $\{(u, z)$ such that $u \le \tau(z)\}$, and thus coincide on the event

$$\Omega_h := \left\{\omega \in \Omega;\, \mu^{\bar{g}}([0, h] \times \left\{z \in \mathbb{R} : |z| \ge \frac{\eta}{4}\right\} \times [0, 1]) = 0\right\}.$$

Since $\mu^{\bar{g}}([0, h] \times \left\{z \in \mathbb{R} : |z| \ge \frac{\eta}{4}\right\} \times [0, 1])$ has a Poisson distribution with parameter

$$\lambda_h := \int_0^h\int_{|z|\ge\frac{\eta}{4}}\int_0^1 \frac{\bar{g}(z)}{|z|^{1+\alpha}}du\,dz\,dt \le ch;$$

we deduce that

$$\mathbb{P}(\Omega_h^c) \le c\,h. \tag{100}$$

Then we have

$$\mathbb{P}((L_t)_{t\le h} \ne (L^\tau_t)_{t\le h}) \le \mathbb{P}(\Omega_h^c) \le c\,h. \tag{101}$$

To prove Proposition 1 we have to rescale the process $(L_t)_{t\in[0,1]}$, we therefore introduce an auxiliary Lévy process $(L^h_t)_{t\in[0,1]}$ defined possibly on another filtered space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}})$ and admitting the decomposition $L^h_t := \int_0^t \int_{\mathbb{R}} z\tilde{\mu}^h(dt, dz)$, with $t \in [0, 1]$; where $\tilde{\mu}^h$ is a compensated Poisson random measure $\tilde{\mu}^h = \mu^h - \bar{\mu}^h$, with compensator

$$\bar{\mu}^h(dt, dz) = dt\frac{\bar{g}(zh^{\frac{1}{\alpha}})}{|z|^{1+\alpha}}\tau(zh^{\frac{1}{\alpha}})1_{\mathbb{R}\setminus\{0\}}(z)dz. \tag{102}$$

By construction, the process $(L^h_t)_{t\in[0,1]}$ is equal in law to the rescaled truncated process $(h^{-\frac{1}{\alpha}}L^\tau_{ht})_{t\in[0,1]}$ that coincides with $(h^{-\frac{1}{\alpha}}L_{ht})_{t\in[0,1]}$ on Ω_n.

7.2. Malliavin calculus

In this section, we recall some results on Malliavin calculus for jump processes. We refer to [6] for a complete presentation and to [7] for the adaptation to our framework. We will work on the Poisson space associated to the measure μ^h defining the process $(L^h_t)_{t\in[0,1]}$ of the previous section, assuming that h is fixed. By construction, the support of μ^h is contained in $[0, 1] \times E_h$, where $E_h := \left\{z \in \mathbb{R} : |z| < \frac{\eta}{2}\frac{1}{h^{\frac{1}{\alpha}}}\right\}$, with η defined in the fourth point of A4. We recall that the measure μ^h has compensator

$$\bar{\mu}^h(dt, dz) = dt\frac{\bar{g}(zh^{\frac{1}{\alpha}})}{|z|^{1+\alpha}}\tau(zh^{\frac{1}{\alpha}})1_{\mathbb{R}\setminus\{0\}}(z)dz := dtF_h(z)dz. \tag{103}$$

In this section we assume that the truncation function τ satisfies the additional assumption

$$\int_{\mathbb{R}} |\frac{\tau'(z)}{\tau(z)}|^p\tau(z)dz < \infty, \qquad \forall p \geq 1.$$

We now define the Malliavin operators L and Γ (omitting their dependence in h) and their basic properties (see [6] Chapter IV, sections 8–9–10). For a test function $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ measurable, $\mathcal{C}^2$ with respect the second variable, with bounded derivative and such that $f \in \cap_{p\geq1}L^p(\bar{\mu}^h(dt, dz))$, we set $\mu^h(f) = \int_0^1 \int_{\mathbb{R}} f(t, z)\mu^h(dt, dz)$. As auxiliary function, we consider $\rho : \mathbb{R} \to [0, \infty)$ such that ρ is symmetric, two times differentiable and such that $\rho(z) = z^4$ if $z \in [0, \frac{1}{2}]$ and $\rho(z) = z^2$ if $z \geq 1$. Thanks to the truncation τ, we do not need that ρ vanishes at infinity. Assuming the fourth point of Assumption 4, we check that ρ, ρ' and $\rho\frac{F'_h}{F_h}$ belong to $\cap_{p\geq1}L^p(F_h(z)dz)$. With these notations, we define the Malliavin operator L on the functional $\mu^h(f)$ as follows:

$$L(\mu^h(f)) := \frac{1}{2}\mu^h(\rho'f' + \rho\frac{F'_h}{F_h}f' + \rho f''),$$

where f' and f'' are derivative with respect to the second variable. This definition permits to construct a linear operator on the space $D \subset \cap_{p\geq1}L^p(F_h(z)dz)$ which is self-adjoint: $\forall\Phi, \Psi \in D$, $\mathbb{E}\Phi L\Psi = \mathbb{E}L\Phi\Psi$ (see Section 8 in [6] for the details on the construction of D).

We associate to L the symmetric bilinear operator Γ:

$$\Gamma(\Phi, \Psi) = L(\Phi, \Psi) - \Phi L(\Psi) - \Psi L(\Phi).$$

If f and g are two test functions, we have

$$\Gamma(\mu^h(f), \mu^h(g)) = \mu^h(\rho f'g'). \tag{104}$$

The operators L and Γ satisfy the chain rule property:

$$LF(\Phi) = F'(\Phi)L\Phi + \frac{1}{2}F''(\Phi)\Gamma(\Phi, \Phi), \qquad \Gamma(F(\Phi), \Psi) = F'(\Phi)\Gamma(\Phi, \Psi).$$

These operators permit to establish the following integration by parts formula (see [6] Theorem 8–10 p.103).

Theorem 4. *Let Φ and Ψ be random variable in D and f be a bounded function with bounded derivatives up to order two. If $\Gamma(\Phi, \Phi)$ is invertible and $\Gamma^{-1}(\Phi, \Phi) \in \cap_{p\geq 1} L^p$, then we have*

$$\mathbb{E}f'(\Phi)\Psi = \mathbb{E}f(\Phi)\mathcal{H}_\Phi(\Psi), \tag{105}$$

with

$$\mathcal{H}_\Phi(\Psi) = -2\Psi\Gamma^{-1}(\Phi, \Phi)L\Phi - \Gamma(\Phi, \Psi\Gamma^{-1}(\Phi, \Phi)). \tag{106}$$

The random variable L_1^h belongs to the domain of the operators L and Γ. Computing $L(L_1^h)$, $\Gamma(L_1^h, L_1^h)$ and $\mathcal{H}_{L_1^h}(1)$ it is possible to deduce the following useful inequalities, proved in Lemma 4.3 in [7].

Lemma 5. *We have*

$$\sup_n \mathbb{E}|\mathcal{H}_{L_1^h}(1)|^p \leq C_p \qquad \forall p \geq 1,$$

$$\sup_n \mathbb{E}|\int_0^1 \int_{|z|>1} |z|\mu^h(ds, dz)\mathcal{H}_{L_1^h}(1)|^p \leq C_p \qquad \forall p \geq 1.$$

With this background we can proceed to the proof of Proposition 1.

7.3. Proof of Proposition 1

Proof. The first step is to construct on the same probability space two random variables whose laws are close to the laws of $h^{-\frac{1}{\alpha}}L_h$ and S_1^α. We recall briefly the notation of Section 7.1: μ^h is a Poisson random measure with compensator $\bar{\mu}^h(dt, dz)$ defined in (102) and the process L_t^h is defined by

$$L_t^h = \int_0^t \int_{\mathbb{R}} z\tilde{\mu}^h(ds, dz) = \int_0^t \int_{|z|\leq h^{-\frac{1}{\alpha}}\frac{\eta}{2}} z\tilde{\mu}^h(ds, dz) \tag{107}$$

with $\tilde{\mu}^h = \mu^h - \bar{\mu}^h$. Using triangle inequality we have

$$|\mathbb{E}[g(h^{-\frac{1}{\alpha}}L_h)] - \mathbb{E}[g(S_1^\alpha)]| \leq |\mathbb{E}[g(h^{-\frac{1}{\alpha}}L_h)] - \mathbb{E}[g(L_1^h)]| + |\mathbb{E}[g(L_1^h) - g(S_1^\alpha)]|. \tag{108}$$

By the definition of L_1^h it is

$$|\mathbb{E}[g(h^{-\frac{1}{\alpha}}L_h)] - \mathbb{E}[g(L_1^h)]| = |\mathbb{E}[g(h^{-\frac{1}{\alpha}}L_h) - g(h^{-\frac{1}{\alpha}}L_h^\tau)]| \leq 2\left\|g\right\|_\infty \mathbb{P}(\Omega_n^c) \leq c\left\|g\right\|_\infty h, \tag{109}$$

where in the last inequality we have used (101). In order to get an estimation to the second term of (108) we now construct a variable approximating the law of S_1^α and based on the Poisson measure μ^h:

$$L_t^{\alpha,h} := \int_0^t \int_{|z|\leq h^{-\frac{1}{\alpha}}\frac{\eta}{2}} g_h(z)\tilde{\mu}^h(ds, dz), \tag{110}$$

where g_h is an odd function built in the proof of Theorem 4.1 in [7] for which the following lemma holds:

Lemma 6.

1. *For each test function f, defined as in Section 7.2, we have*
$$\int_0^1 \int_{|z|\leq \frac{\eta}{2}h^{-\frac{1}{\alpha}}} f(t, g_h(z))\bar{\mu}^h(dt, dz) = \int_0^1 \int_{|\omega|\leq \frac{\eta}{2}h^{-\frac{1}{\alpha}}} f(t,\omega)\bar{\mu}^{\alpha,h}(dt, d\omega), \qquad (111)$$
where $\bar{\mu}^h(dt, dz)$ is the compensator defined in (102) and
$$\bar{\mu}^{\alpha,h}(dt, d\omega) = dt\frac{\tau(\omega h^{\frac{1}{\alpha}})}{|\omega|^{1+\alpha}}d\omega$$
is the compensator of a measure associated to an α-stable process whose jumps are truncated with the function τ.
2. *There exists $\epsilon_0 > 0$ such that, for $|z| \leq \epsilon_0 h^{-\frac{1}{\alpha}}$,*
$$|g_h(z) - z| \leq cz^2 h^{\frac{1}{\alpha}} + c|z|^{1+\alpha}h \qquad \text{if } \alpha \neq 1,$$
$$|g_h(z) - z| \leq cz^2 h|\log(|z|h)| \qquad \text{if } \alpha = 1.$$
3. *The function g_h is $\mathcal{C}^1$ on $(-\epsilon_0 h^{-\frac{1}{\alpha}}, \epsilon_0 h^{-\frac{1}{\alpha}})$ and for $|z| < \epsilon_0 h^{-\frac{1}{\alpha}}$,*
$$|g_h'(z) - 1| \leq c|z|h^{\frac{1}{\alpha}} + c|z|^{\alpha}h \qquad \text{if } \alpha \neq 1,$$
$$|g_h'(z) - 1| \leq c|z|h|\log(|z|h)| \qquad \text{if } \alpha = 1.$$

The second and the third point of the lemma here above are proved in Lemma 4.5 of [7], while the first point is proved in Theorem 4.1 [7] and it shows us, using the exponential formula for Poisson measure, that g_h is the function that turns our measure μ^h into the measure associated to an α-stable process truncated with the function τ. Thus $(L_t^{\alpha,h})_{t\in[0,1]}$ is a Lévy process with jump intensity $\omega \mapsto \frac{\tau(\omega h^{\frac{1}{\alpha}})}{|\omega|^{1+\alpha}}$ and we recognize the law of an α-stable truncated process. We deduce, similarly to (109),
$$|\mathbb{E}[g(L_1^{\alpha,h})] - \mathbb{E}[g(S_1^{\alpha})]| \leq c\,\|g\|_\infty\, h. \qquad (112)$$
Proposition 1 is a consequence of (108), (109), (112) and the following lemma:

Lemma 7. *Suppose that Assumptions 1 to 4 hold. Let g be as in Proposition 1. Then, for any $\epsilon > 0$ and for $p \geq \alpha$,*
$$|\mathbb{E}[g(L_1^h) - g(L_1^{\alpha,h})]| \leq C_\epsilon h|\log(h)|\,\|g\|_\infty + C_\epsilon h^{\frac{1}{\alpha}}\,\|g\|_\infty^{1-\frac{\alpha}{p}+\epsilon}\,\|g\|_{pol}^{\frac{\alpha}{p}-\epsilon}\,|\log(h)|$$
$$+ C_\epsilon h^{\frac{1}{\alpha}}\,\|g\|_\infty^{1+\frac{1}{p}-\frac{\alpha}{p}+\epsilon}\,\|g\|_{pol}^{-\frac{1}{p}+\frac{\alpha}{p}-\epsilon}\,|\log(h)|1_{\alpha>1}.$$

Proof. The proof is based of the comparison of the representation of (107) and (110). Since in Lemma 6 the difference $g_h(z) - z$ is controlled for $|z| \leq \epsilon_0 h^{-\frac{1}{\alpha}}$, we need to introduce a localization procedure consisting in regularizing $1_{\left\{\mu^h([0,1]\times\left\{z\in\mathbb{R}:|z|>\epsilon_0 h^{-\frac{1}{\alpha}}\right\})=0\right\}}$. Let $\mathcal{I}$ be a smooth function defined on $\mathbb{R}$ and with values in $[0, 1]$, such that $\mathcal{I}(x) = 1$ for $x \leq \frac{1}{2}$ and

$\mathcal{I}(x) = 0$ for $x \geq 1$. Moreover, we denote by ζ a smooth function on $\mathbb{R}$, with values in [0, 1] such that $\zeta(z) = 0$ for $|z| \leq \frac{1}{2}$ and $\zeta(z) = 1$ for $|z| \geq 1$ and we set

$$V^h := \int_0^1 \int_{\mathbb{R}} \zeta(\frac{zh^{\frac{1}{\alpha}}}{\epsilon_0})\mu^h(ds, dz) = \int_0^1 \int_{\left\{\frac{1}{2}\epsilon_0 h^{-\frac{1}{\alpha}} \leq |z| \leq \epsilon_0 h^{-\frac{1}{\alpha}}\right\}} \zeta(\frac{zh^{\frac{1}{\alpha}}}{\epsilon_0})\mu^h(ds, dz)$$
$$+ \int_0^1 \int_{\left\{|z| \geq \epsilon_0 h^{-\frac{1}{\alpha}}\right\}} \mu^h(ds, dz),$$

$$W^h := \mathcal{I}(V^h).$$

From the construction, W^h is a Malliavin differentiable random variable such that $W^h \neq 0$ implies $\mu^h([0, 1] \times \left\{z \in \mathbb{R} : |z| > \epsilon_0 h^{-\frac{1}{\alpha}}\right\}) = 0$. It is possible to show, acting as we did in (100), that $\mathbb{P}(W^h \neq 1) \leq \mathbb{P}(\mu^h$ has a jump of size $> \frac{1}{2}\epsilon_0 h^{-\frac{1}{\alpha}}) = O(h)$. From the latter, it is clear that the proof of the lemma reduces in proving the result on $|\mathbb{E}[g(L_1^h)W^h] - \mathbb{E}[g(L_1^{\alpha,h})W^h]|$. Considering a regularizing sequence (g_p) converging to g in L^1 norm, such that $\forall p$ g_p is $\mathcal{C}^1$ with bounded derivative and $\left\|g_p\right\|_\infty \leq \|g\|_\infty$, we may assume that g is $\mathcal{C}^1$ with bounded derivative too. Using the integration by parts formula (105) and denoting by G any primitive function of g we can write $\mathbb{E}[g(L_1^h)W^h] = \mathbb{E}[G(L_1^h)\mathcal{H}_{L_1^h}(W^h)]$ where the Malliavin weight can be written, using (106) and the chain rule property of the operator Γ, as

$$\mathcal{H}_{L_1^h}(W^h) = W^h \mathcal{H}_{L_1^h}(1) - \frac{\Gamma(W^h, L_1^h)}{\Gamma(L_1^h, L_1^h)}. \tag{113}$$

Using the triangle inequality, we are now left to find upper bounds for the following two terms:

$$\tilde{T}_1 := |\mathbb{E}[g(L_1^{\alpha,h})W^h] - \mathbb{E}[G(L_1^{\alpha,h})\mathcal{H}_{L_1^h}(W^h)]|,$$

$$\tilde{T}_2 := |\mathbb{E}[G(L_1^{\alpha,h})\mathcal{H}_{L_1^h}(W^h)] - \mathbb{E}[G(L_1^h)\mathcal{H}_{L_1^h}(W^h)]|.$$

Let us start considering $\tilde{T}_2$. Using the Lipschitz property of the function G and (113) we have it is upper bounded by

$$\begin{aligned}\mathbb{E}[|g(\hat{L}_1)\| L_1^{\alpha,h} - L_1^h \| \mathcal{H}_{L_1^h}(W^h)|] &\leq \mathbb{E}[|g(\hat{L}_1)\| L_1^{\alpha,h} - L_1^h \| W^h \mathcal{H}_{L_1^h}(1)|] \\ &\quad + \mathbb{E}[|g(\hat{L}_1)\| L_1^{\alpha,h} - L_1^h \| \frac{\Gamma(W^h, L_1^h)}{\Gamma(L_1^h, L_1^h)}|] \\ &=: \tilde{T}_{2,1} + \tilde{T}_{2,2},\end{aligned}$$

where $\hat{L}_1$ is between $L_1^{\alpha,h}$ and L_1^h. We focus on $\tilde{T}_{2,1}$. Using the definitions (107) and (110) of L_1^h and $L_1^{\alpha,h}$ it is

$$\begin{aligned}\tilde{T}_{2,1} &\leq \mathbb{E}[|g(\hat{L}_1)\| \int_0^1 \int_{\mathbb{R}} (g_h(z) - z)\tilde{\mu}^h(ds, dz) \| \mathcal{H}_{L_1^h}(1) W^h|] \\ &\leq \mathbb{E}[|g(\hat{L}_1)\| \int_0^1 \int_{|z| \leq 1} (g_h(z) - z)\tilde{\mu}^h(ds, dz) \| \mathcal{H}_{L_1^h}(1) W^h|] \\ &\quad + \mathbb{E}[|g(\hat{L}_1)\| \int_0^1 \int_{1 \leq |z| \leq \epsilon_0 h^{-\frac{1}{\alpha}}} (g_h(z) - z)\mu^h(ds, dz) \| \mathcal{H}_{L_1^h}(1) W^h|],\end{aligned} \tag{114}$$

where we have used that g_h is an odd function with the symmetry of the compensator $\bar{\mu}^h$ and the fact that on $W_h \neq 0$ we have $\mu^h([0,1] \times \left\{z \in \mathbb{R} : |z| > \epsilon_0 h^{-\frac{1}{\alpha}}\right\}) = 0$. For the sake of shortness, we only give the details of the proof in the case $\alpha \neq 1$. In the case $\alpha = 1$, one needs to modify this control with an additional logarithmic term. For the small jumps term, from inequality 2.1.37 in [12] and the second point of Lemma 6 we deduce $\mathbb{E}[|\int_0^1 \int_{|z|\le 1} (g_h(z) - z)\tilde{\mu}^h(ds,dz)|^{q_1}] \le C_{q_1}(h + h^{\frac{1}{\alpha}})^{q_1}$, $\forall q_1 \ge 2$. Using it and Holder inequality with q_1 big and q_2 close to 1 we have

$$\mathbb{E}[|g(\hat{L}_1)\|\int_0^1 \int_{|z|\le 1} (g_h(z) - z)\tilde{\mu}^h(ds,dz)\|\mathcal{H}_{L_1^h}(1) W^h|]$$

$$\le C_{q_1}(h + h^{\frac{1}{\alpha}})\mathbb{E}[|g(\hat{L}_1)|^{q_2} |\mathcal{H}_{L_1^h}(1)|^{q_2} W^h]^{\frac{1}{q_2}}$$

$$\le C_{q_1}(h + h^{\frac{1}{\alpha}})\mathbb{E}[|g(\hat{L}_1)|^{p_1 q_2} W^h]^{\frac{1}{q_2 p_1}} \mathbb{E}[|\mathcal{H}_{L_1^h}(1)|^{q_2 p_2}]^{\frac{1}{q_2 p_2}}, \tag{115}$$

where in the last inequality we have used again Holder inequality, with p_2 big and p_1 close to 1. Using the first point of Lemma 5, we know that $\mathbb{E}[|\mathcal{H}_{L_1^h}(1)|^{q_2 p_2}]^{\frac{1}{q_2 p_2}}$ is bounded, hence (115) is upper bounded by

$$C_{q_1 q_2 p_2} h \, \|g\|_\infty + C_{q_1 q_2 p_2} h^{\frac{1}{\alpha}} \mathbb{E}[|g(\hat{L}_1) W^h|^{p_1 q_2}]^{\frac{1}{q_2 p_1}}, \tag{116}$$

where we have bounded $|g(\hat{L}_1)|$ with its infinity norm and used that $0 \le W^h \le 1$. We remind that we are considering q_2 and p_1 next to 1, hence we can write $q_2 p_1$ as $1 + \epsilon$. We now introduce r in the following way:

$$\begin{aligned}\mathbb{E}[|g(\hat{L}_1)|^{1+\epsilon} W^h]^{\frac{1}{1+\epsilon}} &= \mathbb{E}[|g(\hat{L}_1)|^{(1+\epsilon)r} |g(\hat{L}_1)|^{(1+\epsilon)(1-r)} W^h]^{\frac{1}{1+\epsilon}} \\ &\le \|g\|_\infty^r \, \mathbb{E}[|g(\hat{L}_1)|^{(1+\epsilon)(1-r)} W^h]^{\frac{1}{1+\epsilon}} \le \\ &\quad \|g\|_\infty^r \, \|g\|_{pol}^{1-r} \, \mathbb{E}[(1 + |\hat{L}_1|^p)^{(1+\epsilon)(1-r)} W^h]^{\frac{1}{1+\epsilon}} \\ &\le c\, \|g\|_\infty^r \, \|g\|_{pol}^{1-r} + c\, \|g\|_\infty^r \, \|g\|_{pol}^{1-r} \, \mathbb{E}[|\hat{L}_1|^{p(1+\epsilon)(1-r)} W^h]^{\frac{1}{1+\epsilon}};\end{aligned} \tag{117}$$

where we have estimated g with its norm ∞ and we have used the property (9) of g and that $0 \le W^h \le 1$. We observe that $\hat{L}_1$ is between L_1^h and $L_1^{\alpha,h}$ hence $|\hat{L}_1| \le |L_1^h| + |L_1^{\alpha,h}|$. Moreover we choose r such that $p(1+\epsilon)(1-r) = \alpha$; therefore $r = 1 - \frac{\alpha}{p(1+\epsilon)}$. In this way we have that (117) is upper bounded by

$$c\, \|g\|_\infty^{1-\frac{\alpha}{p(1+\epsilon)}} \, \|g\|_{pol}^{\frac{\alpha}{p(1+\epsilon)}} \log(h^{-\frac{1}{\alpha}}), \tag{118}$$

where we have used that $\mathbb{E}[|\hat{L}_1|^\alpha W^h] \le c \log(h^{-\frac{1}{\alpha}})$, that we justify now. Indeed, using Lemma 2.1.5 in the appendix of [12] if $\alpha \in [1,2]$ and Jensen inequality if $\alpha \in [0,1)$, we have

$$\mathbb{E}[|\hat{L}_1|^\alpha W^h] \le c\mathbb{E}[(|L_1^h|^\alpha + |L_1^{\alpha,h}|^\alpha) W^h] \le c\mathbb{E}[|\int_0^1 \int_{|z|\le 1} z\tilde{\mu}^h(ds,dz)|]$$

$$+ c\mathbb{E}[|\int_0^1 \int_{|z|\le 1} g_h(z)\tilde{\mu}^h(ds,dz)|]$$

$$+ c\mathbb{E}[\int_0^1 \int_{1\le|z|\le\epsilon_0 h^{-\frac{1}{\alpha}}} |z|^\alpha \bar{\mu}^h(ds,dz)]$$

$$+ c\mathbb{E}[\int_0^1 \int_{1\le|z|\le\epsilon_0 h^{-\frac{1}{\alpha}}} |g_h(z)|^\alpha \bar{\mu}^h(ds,dz)].$$

We observe that, using Kunita inequality, the first term here above is bounded in L^p and, as a consequence of the second point of Lemma 6, the second term here above so does. Concerning the third term here above (and so, again, we act on the fourth in the same way), we have

$$c\mathbb{E}[\int_0^1 \int_{1\le|z|\le\epsilon_0 h^{-\frac{1}{\alpha}}} |z|^\alpha \bar{\mu}^h(ds,dz)] \le c\int_{1\le|z|\le\epsilon_0 h^{-\frac{1}{\alpha}}} |z|^{\alpha-1-\alpha}dz$$
$$\le c\, \log(h^{-\frac{1}{\alpha}}) \le c|\log(h)|, \tag{119}$$

where we have also used definition (102) of $\bar{\mu}^h$.

Replacing (118) in (116) we get

$$\mathbb{E}[|g(\hat{L}_1)\| \int_0^1 \int_{|z|\le 1} (g_h(z)-z)\tilde{\mu}^h(ds,dz)\|\mathcal{H}_{L_1^h}(1)W^h|] \le C_{q_1q_2p_2}h\,\|g\|_\infty$$
$$+ C_{q_1q_2p_2}h^{\frac{1}{\alpha}}\,\|g\|_\infty^{1-\frac{\alpha}{p}+\epsilon}\,\|g\|_{pol}^{\frac{\alpha}{p}-\epsilon}\,\log(h^{-\frac{1}{\alpha}}), \tag{120}$$

where we have taken another ϵ, using its arbitrariness. The constants depend also on it.

Let us now consider the large jumps term in (114). Using the second point of Lemma 6 and the following basic inequality

$$\int_0^1 \int_{1<|z|\le\epsilon_0 h^{-\frac{1}{\alpha}}} |z|^\delta \mu^h(ds,dz) \le \int_0^1 \int_{1<|z|\le\epsilon_0 h^{-\frac{1}{\alpha}}} |z|^{\delta-1} \mu^h(ds,dz)$$
$$\times \int_0^1 \int_{1<|z|\le\epsilon_0 h^{-\frac{1}{\alpha}}} |z| \mu^h(ds,dz)$$

for $\delta \ge 1$, we get it is upper bounded by

$$\mathbb{E}[|g(\hat{L}_1)| \int_0^1 \int_{1<|z|\le\epsilon_0 h^{-\frac{1}{\alpha}}} (h^{\frac{1}{\alpha}}|z| + h|z|^\alpha)\mu^h(ds,dz)$$
$$\times \int_0^1 \int_{1<|z|\le\epsilon_0 h^{-\frac{1}{\alpha}}} |z|\mu^h(ds,dz)|\mathcal{H}_{L_1^h}(1)|W^h]. \tag{121}$$

We now use Holder inequality with p_2 big and p_1 next to 1 and we observe that, from the second point of Lemma 5, it follows

$$\mathbb{E}[|\int_0^1 \int_{1<|z|\le\epsilon_0 h^{-\frac{1}{\alpha}}} |z|\mu^h(ds,dz)\mathcal{H}_{L_1^h}(1)|^{p_2}]^{\frac{1}{p_2}} \le C_{p_2}.$$

Hence (121) is upper bounded by

$$C_{p_2}\mathbb{E}[|g(\hat{L}_1)|^{p_1}| \int_0^1 \int_{1<|z|\le\epsilon_0 h^{-\frac{1}{\alpha}}} (h^{\frac{1}{\alpha}}|z| + h|z|^\alpha)\mu^h(ds,dz)|^{p_1} W^h]^{\frac{1}{p_1}} \tag{122}$$
$$\le C_{p_2}\,\|g\|_\infty\, h\mathbb{E}[|\int_0^1 \int_{1<|z|\le\epsilon_0 h^{-\frac{1}{\alpha}}} |z|^\alpha \mu^h(ds,dz)|^{p_1}]^{\frac{1}{p_1}}$$
$$+ C_{p_2}h^{\frac{1}{\alpha}}\mathbb{E}[|g(\hat{L}_1)|^{p_1}| \int_0^1 \int_{1<|z|\le\epsilon_0 h^{-\frac{1}{\alpha}}} |z|\mu^h(ds,dz)|^{p_1} W^h]^{\frac{1}{p_1}}. \tag{123}$$

Concerning the first term of (123), we use Lemma 2.1.5 in the appendix of [12] with $p_1 = (1+\epsilon) \in [1, 2]$ and the definition of F_h given in (103), getting

$$\mathbb{E}[| \int_0^1 \int_{1<|z|\le \epsilon_0 h^{-\frac{1}{\alpha}}} |z|^\alpha \mu^h(ds, dz)|^{1+\epsilon}]^{\frac{1}{1+\epsilon}} \le \mathbb{E}[\int_0^1 \int_{1<|z|\le \epsilon_0 h^{-\frac{1}{\alpha}}} |z|^{\alpha(1+\epsilon)} \bar{\mu}^h(ds, dz)]^{\frac{1}{1+\epsilon}}$$
$$\le c(\int_{1<|z|\le \epsilon_0 h^{-\frac{1}{\alpha}}} |z|^{\alpha(1+\epsilon)-1-\alpha} dz)^{\frac{1}{1+\epsilon}} \le c h^{-\frac{\epsilon}{1+\epsilon}} = c h^{-\epsilon}, \tag{124}$$

where we have used the arbitrariness of ϵ in the last equality.

On the second term of (123) we act differently depending on whether or not α is more than 1. If it does, we act as we did in (117), considering $p_1 = 1+\epsilon < \alpha$ and introducing r, this time we set it such that the following equality holds:

$$p(1+\epsilon)(1-r) + (1+\epsilon) = \alpha. \tag{125}$$

We also use the property (9) on g, hence it is upper bounded by

$$C_{p_2} h^{\frac{1}{\alpha}} \|g\|_\infty^r \|g\|_{pol}^{1-r} \mathbb{E}[(1+|\hat{L}_1|^{p(1+\epsilon)(1-r)})| \int_0^1 \int_{1<|z|\le \epsilon_0 h^{-\frac{1}{\alpha}}} |z| \mu^h(ds, dz)|^{1+\epsilon} W^h]^{\frac{1}{1+\epsilon}}. \tag{126}$$

Now on the first term here above we use that $0 \le W^h \le 1$ and Lemma 2.1.5 in the appendix of [12] as we did in (124) in order to get

$$\mathbb{E}[| \int_0^1 \int_{1<|z|\le \epsilon_0 h^{-\frac{1}{\alpha}}} |z| \mu^h(ds, dz)|^{1+\epsilon}]^{\frac{1}{1+\epsilon}} \le c. \tag{127}$$

Moreover we observe, as we have already done, that $|\hat{L}_1| \le |L_1^h| + |L_1^{\alpha,h}|$ and that, from the second point of Lemma 6, there exists $c > 0$ such that $|g_h(z)| \le c|z|$; so we get

$$\mathbb{E}[|\hat{L}_1|^{p(1+\epsilon)(1-r)} | \int_0^1 \int_{1<|z|\le \epsilon_0 h^{-\frac{1}{\alpha}}} |z| \mu^h(ds, dz)|^{1+\epsilon} W^h]^{\frac{1}{1+\epsilon}}$$
$$\le c + \mathbb{E}[| \int_0^1 \int_{1<|z|\le \epsilon_0 h^{-\frac{1}{\alpha}}} |z| \mu^h(ds, dz)|^{p(1+\epsilon)(1-r)+(1+\epsilon)}]^{\frac{1}{1+\epsilon}}$$
$$\le c(\int_{1<|z|\le \epsilon_0 h^{-\frac{1}{\alpha}}} |z|^\alpha |z|^{-1-\alpha} dz)^{\frac{1}{1+\epsilon}} \le c \frac{1}{1+\epsilon} \log(h^{-\frac{1}{\alpha}}) \le c|\log(h)|, \tag{128}$$

having chosen a particular r just in order to have the exponent here above equal to α and so having found out the same computation of (119). We have not considered the integral on $|z| \le 1$ because, as we have already seen above (119), the integral is bounded in L^p and so we simply get (127) again. From (125) we obtain $r = 1 + \frac{1}{p} - \frac{\alpha}{p(1+\epsilon)}$. Replacing it and using (127) and (128) we get (126) is upper bounded by

$$C_{p_2} h^{\frac{1}{\alpha}} \|g\|_\infty^{1+\frac{1}{p}-\frac{\alpha}{p(1+\epsilon)}} \|g\|_{pol}^{-\frac{1}{p}+\frac{\alpha}{p(1+\epsilon)}} (c + |\log(h)|)$$
$$= C_{p_2} h^{\frac{1}{\alpha}} \|g\|_\infty^{1+\frac{1}{p}-\frac{\alpha}{p(1+\epsilon)}} \|g\|_{pol}^{-\frac{1}{p}+\frac{\alpha}{p(1+\epsilon)}} |\log(h)|. \tag{129}$$

If otherwise α is less than 1, then the second term of (123) is upper bounded by

$$C_{p_2} h^{\frac{1}{\alpha}} \|g\|_\infty \mathbb{E}[|\int_0^1 \int_{1<|z|\leq \epsilon_0 h^{-\frac{1}{\alpha}}} |z|\mu^h(ds,dz)|^{p_1} W^h]^{\frac{1}{p_1}} \leq C_{p_2} h^{\frac{1}{\alpha}} \|g\|_\infty h^{\frac{1}{1+\epsilon}-\frac{1}{\alpha}}$$

$$= C_{p_2} h^{\frac{1}{1+\epsilon}} \|g\|_\infty , \tag{130}$$

where we have taken $p_1 = 1+\epsilon$ and we have used the fact that $0 \leq W^h \leq 1$ and that, for $\alpha < 1$,

$$\mathbb{E}[|\int_0^1 \int_{1<|z|\leq \epsilon_0 h^{-\frac{1}{\alpha}}} |z|\mu^h(ds,dz)|^{1+\epsilon}]^{\frac{1}{1+\epsilon}} \leq c h^{\frac{1}{1+\epsilon}-\frac{1}{\alpha}}.$$

Using (123), (124), (129) and (130) it follows

$$\mathbb{E}[|g(\hat{L}_1)\| \int_0^1 \int_{1\leq|z|\leq \epsilon_0 h^{-\frac{1}{\alpha}}} (g_h(z)-z)\mu^h(ds,dz)\|\mathcal{H}_{L_1^h}(1)W^h|]$$

$$\leq C_{p_2} h^{1-\epsilon} \|g\|_\infty + C_{p_2} h^{\frac{1}{\alpha}} \|g\|_\infty^{1+\frac{1}{p}-\frac{\alpha}{p(1+\epsilon)}} \|g\|_{pol}^{-\frac{1}{p}+\frac{\alpha}{p(1+\epsilon)}} |\log(h)| 1_{\alpha>1}. \tag{131}$$

Now from (114), (120), and (131) it follows

$$\tilde{T}_{2,1} \leq C_{q_1 q_2 p_2} h^{1-\epsilon} \|g\|_\infty + C_{q_1 q_2 p_2} h^{\frac{1}{\alpha}} \|g\|_\infty^{1-\frac{\alpha}{p}+\epsilon} \|g\|_{pol}^{\frac{\alpha}{p}-\epsilon} |\log(h)|$$

$$+ C_{q_1 q_2 p_2} h^{\frac{1}{\alpha}} \|g\|_\infty^{1+\frac{1}{p}-\frac{\alpha}{p}+\epsilon} \|g\|_{pol}^{-\frac{1}{p}+\frac{\alpha}{p}-\epsilon} |\log(h)| 1_{\alpha>1}. \tag{132}$$

Concerning $\tilde{T}_{2,2}$, it is already proved in Theorem 4.2 in [7] that

$$\tilde{T}_{2,2} \leq ch \|g\|_\infty . \tag{133}$$

Let us now consider $\tilde{T}_1$. Using (104) and (106) we can write

$$\mathcal{H}_{L_1^h}(W^h) = \frac{-W^h L(L_1^h)}{\Gamma(L_1^h, L_1^h)} + L(\frac{W^h}{\Gamma(L_1^h, L_1^h)})L_1^h - L(\frac{L_1^h W^h}{\Gamma(L_1^h, L_1^h)}).$$

With computations using that L is a self-adjoint operator we get

$$\tilde{T}_1 = |\mathbb{E}[g(L_1^{\alpha,h})W^h] - \mathbb{E}[g(L_1^{\alpha,h})\frac{\Gamma(L_1^{\alpha,h}, L_1^h)}{\Gamma(L_1^h, L_1^h)}W^h]| \leq \mathbb{E}[|g(\hat{L}_1)||\frac{\Gamma(L_1^h - L_1^{\alpha,h}, L_1^h)}{\Gamma(L_1^h, L_1^h)}|W^h]. \tag{134}$$

Using Eq. (104), we have

$$\Gamma(L_1^h - L_1^{\alpha,h}, L_1^h) = \int_0^1 \int_{|z|<\frac{\eta}{2}h^{-\frac{1}{\alpha}}} \rho(z)(1-g_h'(z))\mu^h(ds,dz).$$

Using the third point of Lemma 6 we deduce the following on the event $W^h \neq 0$:

$$|\Gamma(L_1^h - L_1^{\alpha,h}, L_1^h)| \leq c \int_0^1 \int_{|z|\leq \epsilon_0 h^{-\frac{1}{\alpha}}} \rho(z)(h^{\frac{1}{\alpha}}|z| + h|z|^\alpha)\mu^h(ds,dz)$$

$$\leq c \int_0^1 \int_{|z|\leq 1} \rho(z)(h^{\frac{1}{\alpha}}|z| + h|z|^\alpha)\mu^h(ds,dz)$$

$$
\begin{aligned}
&+ c \int_0^1 \int_{1<|z|\le \epsilon_0 h^{-\frac{1}{\alpha}}} \rho(z)\mu^h(ds,dz)\\
&\times \int_0^1 \int_{1<|z|\le \epsilon_0 h^{-\frac{1}{\alpha}}} (h^{\frac{1}{\alpha}}|z| + h|z|^{\alpha})\mu^h(ds,dz)\\
\le{}& c \int_0^1 \int_{\mathbb{R}} \rho(z)\mu^h(ds,dz)(h^{\frac{1}{\alpha}} + h) + c \int_0^1 \int_{\mathbb{R}} \rho(z)\mu^h(ds,dz)\\
&\times \int_0^1 \int_{1<|z|\le \epsilon_0 h^{-\frac{1}{\alpha}}} (h^{\frac{1}{\alpha}}|z| + h|z|^{\alpha})\mu^h(ds,dz)\\
={}& c(h^{\frac{1}{\alpha}} + h)\Gamma(L_1^h, L_1^h) + c\Gamma(L_1^h, L_1^h)\\
&\times \Big(\int_0^1 \int_{1<|z|\le \epsilon_0 h^{-\frac{1}{\alpha}}} (h^{\frac{1}{\alpha}}|z| + h|z|^{\alpha})\mu^h(ds,dz)\Big),
\end{aligned} \tag{135}
$$

where we have used that z is always less than 1 in the first integral and that, since ρ is a positive function, we can upper bound the integrals considering whole set $\mathbb{R}$. Moreover, we have used the definition of $\Gamma(L_1^h, L_1^h)$. Replacing (135) in (134) we get

$$
\begin{aligned}
\tilde{T}_1 &\le c(h^{\frac{1}{\alpha}} + h)\mathbb{E}[|g(\hat{L}_1)|] + c\mathbb{E}[|g(\hat{L}_1)| \int_0^1 \int_{1<|z|\le \epsilon_0 h^{-\frac{1}{\alpha}}} (h^{\frac{1}{\alpha}}|z| + h|z|^{\alpha})\mu^h(ds,dz)]\\
&=: \tilde{T}_{1,1} + \tilde{T}_{1,2}.
\end{aligned} \tag{136}
$$

Concerning $\tilde{T}_{1,1}$, we have

$$
\tilde{T}_{1,1} \le ch\,\|g\|_\infty + ch^{\frac{1}{\alpha}}\mathbb{E}[|g(\hat{L}_1)|] \le ch\,\|g\|_\infty + ch^{\frac{1}{\alpha}}\,\|g\|_\infty^{1-\frac{\alpha}{p}}\,\|g\|_{pol}^{\frac{\alpha}{p}}\,|\log(h)|, \tag{137}
$$

where in the last inequality we have acted exactly like we did in (117) and (118) with the exponent on g that is exactly equal to 1 instead of $1+\epsilon$ and so we have chosen r such that $p(1-r) = \alpha$. Let us now consider $\tilde{T}_{1,2}$. We observe that it is exactly like (122) but with $p_1 = 1$ instead of $p_1 = 1 + \epsilon$, with the only difference that computing (124) now we get $c\,\log(h^{-\frac{1}{\alpha}})$ instead of $ch^{-\epsilon}$ and in the definition (125) we choose r such that $p(1-r)+1 = \alpha$. Acting exactly like we did above it follows

$$
\tilde{T}_{1,2} \le C_{p_2} h|\log(h)|\,\|g\|_\infty + C_{p_2} h^{\frac{1}{\alpha}}\,\|g\|_\infty^{1+\frac{1}{p}-\frac{\alpha}{p}}\,\|g\|_{pol}^{-\frac{1}{p}+\frac{\alpha}{p}}\,|\log(h)|1_{\alpha>1}. \tag{138}
$$

Using (132), (133), (137) and (138), the lemma is proved. □

It follows Proposition 1, using also (108), (109) and (112). □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix

In this section we will prove the technical proposition and lemmas we have used.

A.1. Proof of Lemma 1

Proof. We start proving 1. From the dynamic (3) of a it is

$$\mathbb{E}[|a_t - a_s|^p] \le \mathbb{E}[|\int_s^t \tilde{b}_u du|^p] + \mathbb{E}[|\int_s^t \tilde{a}_u dW_u|^p] + \mathbb{E}[|\int_s^t \hat{a}_u d\hat{W}_u|^p]$$
$$+ \mathbb{E}[|\int_s^t \int_{\mathbb{R}\setminus\{0\}} \tilde{\gamma}_u\, z\, \tilde{\mu}(du, dz)|^p]$$
$$+ \mathbb{E}[|\int_s^t \int_{\mathbb{R}\setminus\{0\}} \hat{\gamma}_u\, z\, \tilde{\mu}_2(du, dz)|^p] =: \sum_{j=1}^5 I_j.$$

In the following, since we will act on the two Brownian motions W and $\hat{W}$ in the same way, we will not report I_3 anymore. Also considering the Poisson random measures, we will deal only with I_4 in detail, underlining that on I_5 the same reasoning applies. We use Burkholder–Davis–Gundy inequalities on the stochastic integral and Kunita inequality on the jump part, in addition to a repeated use of Jensen inequality to get

$$I_1 + I_2 + I_4 \le |t-s|^{p-1} \int_s^t \mathbb{E}[|\tilde{b}_u|^p] du + \mathbb{E}[|\int_t^s (\tilde{a}_u)^2 du|^{\frac{p}{2}}]$$
$$+ \mathbb{E}[\int_t^s \int_{\mathbb{R}\setminus\{0\}} |\tilde{\gamma}_u|^p |z|^p \bar{\mu}(du, dz)]$$
$$+ \mathbb{E}[|\int_t^s \int_{\mathbb{R}\setminus\{0\}} (\tilde{\gamma}_u)^2 (z)^2 \bar{\mu}(du, dz)|^{\frac{p}{2}}] \le c|t-s|^p$$
$$+ |t-s|^{\frac{p}{2}-1} \int_t^s \mathbb{E}[|\tilde{a}_u|^p] du$$
$$+ \int_s^t \mathbb{E}[|\tilde{\gamma}_u|^p] ds (\int_{\mathbb{R}\setminus\{0\}} |z|^p F(z) dz)$$
$$+ |t-s|^{\frac{p}{2}-1} \int_s^t \mathbb{E}[|\tilde{\gamma}_u|^2] ds (\int_{\mathbb{R}\setminus\{0\}} |z|^2 F(z) dz)$$
$$\le c(|t-s|^p + |t-s|^{\frac{p}{2}} + |t-s| + |t-s|^{\frac{p}{2}}) \le c|t-s|,$$

with the inequalities above holding true also because all the coefficients in the dynamic of a are supposed to be bounded. The reasoning here above joint with A3 also yields that, for all $q > 0$, $\sup_{t\ge 0} \mathbb{E}[|a_t|^q] < \infty$.

The proof of 2 follows the same lines as the proof of 1 above.

As we proved in point 1 that the volatility has bounded moments, it is possible to get points 3 and 4 from Theorem 66 of [21] and Proposition 3.1 in [22]. The fifth point is showed in [1], below Lemma 1, and the last one in Section 8 of [10]. □

A.2. Proof of Proposition 3

Proof. In order to show (25), we reformulate $(\Delta X_i^J)^2 \varphi_{\Delta_n^\beta}(\Delta X_i)$ as

$$(\Delta X_i^J)^2 [\varphi_{\Delta_n^\beta}(\Delta X_i) - \varphi_{\Delta_n^\beta}(\Delta X_i^J)] + (\Delta X_i^J)^2 [\varphi_{\Delta_n^\beta}(\Delta X_i^J) - \varphi_{\Delta_n^\beta}(\Delta \tilde{X}_i^J)]$$
$$+ (\Delta X_i^J - \Delta \tilde{X}_i^J)^2 \varphi_{\Delta_n^\beta}(\Delta \tilde{X}_i^J) \tag{139}$$
$$+ 2\Delta \tilde{X}_i^J (\Delta X_i^J - \Delta \tilde{X}_i^J) \varphi_{\Delta_n^\beta}(\Delta \tilde{X}_i^J) + (\Delta \tilde{X}_i^J)^2 \varphi_{\Delta_n^\beta}(\Delta \tilde{X}_i^J) =: \sum_{k=1}^5 I_k^n(i).$$

Comparing (25) with (139) it turns out that our goal is to show that $\sum_{k=1}^4 I_k^n(i) = o_{L^1}(\Delta_n^{\beta(2-\alpha)+1})$. In the sequel will prove that $\sum_{k=1}^4 \mathbb{E}[|I_k^n(i)|] \le c\Delta_n^{\beta(2-\alpha)+1}$; the same reasoning applies to the conditional version, that is $\sum_{k=1}^4 \mathbb{E}_i[|I_k^n(i)|] \le R_i(\Delta_n^{\beta(2-\alpha)+1})$.

Let us start considering $I_1^n(i)$. We know that $\Delta X_i = \Delta X_i^c + \Delta X_i^J$, where we have denoted by ΔX_i^c the continuous part of the increments of the process X. We study

$$I_1^n(i) = I_{1,1}^n + I_{1,2}^n := I_1^n(i)1_{\left\{|\Delta X_i| \ge 3\Delta_n^\beta\right\}} + I_1^n(i)1_{\left\{|\Delta X_i| < 3\Delta_n^\beta\right\}}, \tag{140}$$

having omitted the dependence upon i in $I_{1,1}^n$ and $I_{1,2}^n$ in order to make the notation easier. Concerning $I_{1,1}^n$, we split again on the sets $\left\{|\Delta X_i^J| \ge 2\Delta_n^\beta\right\}$ and $\left\{|\Delta X_i^J| < 2\Delta_n^\beta\right\}$. Recalling that $\varphi(\zeta) = 0$ for $|\zeta| \ge 2\Delta_n^\beta$, we observe that if $|\Delta X_i^J| \ge 2\Delta_n^\beta$ then $I_{1,1}^n$ is just 0. Otherwise, if $|\Delta X_i^J| < 2\Delta_n^\beta$, then it means that $|\Delta X_i^c|$ must be more than Δ_n^β, so we can use (34). In the sequel the constant c may change value from line to line. Using the bound on $(\Delta X_i^J)^2$ and the boundedness of φ we get

$$\mathbb{E}[|I_{1,1}^n|] \le c\Delta_n^{2\beta}\mathbb{E}[1_{\left\{|\Delta X_i| \ge 3\Delta_n^\beta, |\Delta X_i^J| < 2\Delta_n^\beta\right\}}] \le c\Delta_n^{2\beta}\mathbb{P}(|\Delta X_i^c| \ge \Delta_n^\beta) \le c\Delta_n^{2\beta + (\frac{1}{2}-\beta)r}. \tag{141}$$

Hence

$$\frac{1}{\Delta_n^{1+\beta(2-\alpha)}}\mathbb{E}[|I_{1,1}^n|] \le c\Delta_n^{(\frac{1}{2}-\beta)r - 1 + \alpha\beta}, \tag{142}$$

that goes to 0 for $n \to \infty$ since for each choice of $\beta \in (0, \frac{1}{2})$ and $\alpha \in (0, 2)$ we can always find r big enough such that the exponent on Δ_n is positive.

We now consider $I_{1,2}^n$ on the sets $\left\{|\Delta X_i^J| \ge 4\Delta_n^\beta\right\}$ and $\left\{|\Delta X_i^J| < 4\Delta_n^\beta\right\}$. Using the boundedness of φ we have

$$\mathbb{E}[|I_{1,2}^n|1_{\left\{|\Delta X_i^J| \ge 4\Delta_n^\beta\right\}}] \le c\mathbb{E}[(\Delta X_i^J)^2 1_{\left\{|\Delta X_i| < 3\Delta_n^\beta, |\Delta X_i^J| \ge 4\Delta_n^\beta\right\}}].$$

We observe that also in this case $|\Delta X_i| < 3\Delta_n^\beta$ and $|\Delta X_i^J| \ge 4\Delta_n^\beta$ involve $|\Delta X_i^c| \ge \Delta_n^\beta$. Moreover $(\Delta X_i^J)^2 \le c(\Delta X_i)^2 + c(\Delta X_i^c)^2 \le c\Delta_n^{2\beta} + c(\Delta X_i^c)^2$, hence

$$\begin{aligned}\mathbb{E}[|I_{1,2}^n|1_{\left\{|\Delta X_i^J| \ge 4\Delta_n^\beta\right\}}] &\le c\Delta_n^{2\beta}\mathbb{P}(|\Delta X_i^c| \ge \Delta_n^\beta) + c\mathbb{E}[(\Delta X_i^c)^2 1_{\left\{|\Delta X_i^c| \ge \Delta_n^\beta\right\}}] \\ &\le c\Delta_n^{2\beta + r(\frac{1}{2}-\beta)} + c\mathbb{E}[(\Delta X_i^c)^4]^{\frac{1}{2}}\mathbb{P}(|\Delta X_i^c| \ge \Delta_n^\beta)^{\frac{1}{2}} \le c\Delta_n^{[2\beta + r(\frac{1}{2}-\beta)] \wedge [1 + \frac{r}{2}(\frac{1}{2}-\beta)]},\end{aligned} \tag{143}$$

where we have used Cauchy–Schwarz inequality, (34) and the sixth point of Lemma 1. Therefore we get

$$\frac{1}{\Delta_n^{1+\beta(2-\alpha)}}\mathbb{E}[|I_{1,2}^n|1_{\left\{|\Delta X_i^J| \ge 4\Delta_n^\beta\right\}}] \le c\Delta_n^{[r(\frac{1}{2}-\beta) - 1 + \alpha\beta] \wedge [\frac{r}{2}(\frac{1}{2}-\beta) - \beta(2-\alpha)]}, \tag{144}$$

that converges to 0 for $n \to \infty$ since we can always find $r \ge 1$ such that the exponent Δ_n is positive.

In order to conclude the study of $I_1^n(i)$, we study $I_{1,2}^n 1_{\left\{|\Delta X_i^J| < 4\Delta_n^\beta\right\}}$.

$$\mathbb{E}[|I_{1,2}^n|1_{\left\{|\Delta X_i^J| < 4\Delta_n^\beta\right\}}] \le c\left\|\varphi'\right\|_\infty \Delta_n^{-\beta}\mathbb{E}[(\Delta X_i^J)^2|\Delta X_i - \Delta X_i^J|1_{\left\{|\Delta X_i| \le 3\Delta_n^\beta, |\Delta X_i^J| \le 4\Delta_n^\beta\right\}}], \tag{145}$$

where we have used the smoothness of φ. Using Holder inequality and the sixth point of Lemma 1 it is upper bounded by

$$c\Delta_n^{-\beta}\mathbb{E}[|\Delta X_i^c|^p]^{\frac{1}{p}}\mathbb{E}[|(\Delta X_i^J)^{2q}1_{\left\{|\Delta X_i|\le 3\Delta_n^\beta, |\Delta X_i^J|\le 4\Delta_n^\beta\right\}}]^{\frac{1}{q}}$$
$$\le c\Delta_n^{\frac{1}{2}-\beta}\mathbb{E}[|(\Delta X_i^J)^{2q}1_{\left\{|\Delta X_i|\le 3\Delta_n^\beta, |\Delta X_i^J|\le 4\Delta_n^\beta\right\}}]^{\frac{1}{q}}. \tag{146}$$

Now, since our indicator function $1_{\left\{|\Delta X_i|\le 3\Delta_n^\beta, |\Delta X_i^J|\le 4\Delta_n^\beta\right\}}$ is less than $1_{\left\{|\Delta X_i^J|\le 4\Delta_n^\beta\right\}}$, we can use the first point of Lemma 3. Through the use of the conditional expectation we get

$$\mathbb{E}[|(\Delta X_i^J)^{2q}1_{\left\{|\Delta X_i|\le 3\Delta_n^\beta, |\Delta X_i^J|\le 4\Delta_n^\beta\right\}}]^{\frac{1}{q}} \le c\Delta_n^{\frac{1+\beta(2q-\alpha)}{q}}\mathbb{E}[R_i(1)] \le c\Delta_n^{\frac{1+\beta(2q-\alpha)}{q}}. \tag{147}$$

Replacing (147) in (146) and taking q small (next to 1), we obtain $\mathbb{E}[|I_{1,2}^n|1_{\left\{|\Delta X_i^J|<4\Delta_n^\beta\right\}}] \le c\Delta_n^{\frac{1}{2}+\beta+1-\alpha\beta-\epsilon}$. It follows

$$\frac{\mathbb{E}[|I_{1,2}^n|1_{\left\{|\Delta X_i^J|<4\Delta_n^\beta\right\}}]}{\Delta_n^{\beta(2-\alpha)+1}} \le c\Delta_n^{\frac{1}{2}-\beta-\epsilon}, \tag{148}$$

that goes to 0 for $n \to \infty$ since we can always find an ϵ as small as the exponent on Δ_n is positive, for $\beta \in (0, \frac{1}{2})$.

Let us now consider $I_2^n(i)$.

$$I_2^n(i) = I_2^n(i)\, 1_{\left\{|\Delta X_i^J|\le 2\Delta_n^\beta\right\}} + I_2^n(i)\, 1_{\left\{|\Delta X_i^J|> 2\Delta_n^\beta\right\}} =: I_{2,1}^n + I_{2,2}^n. \tag{149}$$

Concerning the first term of (149), we have

$$\mathbb{E}[|I_{2,1}^n|] \le \Delta_n^{-\beta}\left\|\varphi'\right\|_\infty \mathbb{E}[(\Delta X_i^J)^2|\Delta X_i^J - \Delta\tilde{X}_i^J|1_{\left\{|\Delta X_i^J|\le 2\Delta_n^\beta\right\}}]$$
$$\le c\Delta_n^{-\beta}\mathbb{E}[(\Delta X_i^J)^4 1_{\left\{|\Delta X_i^J|\le 2\Delta_n^\beta\right\}}]^{\frac{1}{2}}\mathbb{E}[|\Delta X_i^J - \Delta\tilde{X}_i^J|^2]^{\frac{1}{2}}, \tag{150}$$

where we have used the smoothness of φ and Cauchy–Schwarz inequality. Using again the first point of Lemma 3, we have that

$$\mathbb{E}[(\Delta X_i^J)^4 1_{\left\{|\Delta X_i^J|\le 2\Delta_n^\beta\right\}}]^{\frac{1}{2}} = \mathbb{E}[\mathbb{E}_i[(\Delta X_i^J)^4 1_{\left\{|\Delta X_i^J|\le 2\Delta_n^\beta\right\}}]]^{\frac{1}{2}}$$
$$\le \Delta_n^{\frac{1+\beta(4-\alpha)}{2}}\mathbb{E}[R_i(1)] \le c\Delta_n^{\frac{1}{2}+2\beta-\frac{\alpha\beta}{2}}. \tag{151}$$

We now introduce a lemma that will be proved later:

Lemma 8. *Suppose that A1–A4 hold. Then*

1. *$\forall q \ge 2$ we have*
$$\mathbb{E}[|\Delta X_i^J - \Delta\tilde{X}_i^J|^q] \le c\Delta_n^2, \tag{152}$$
2. *for $q \in [1, 2]$ and $\alpha < 1$, we have*
$$\mathbb{E}[|\Delta X_i^J - \Delta\tilde{X}_i^J|^q]^{\frac{1}{q}} \le c\Delta_n^{\frac{1}{2}+\frac{1}{q}}. \tag{153}$$

Replacing (151) and (152) in (150) we get

$$\mathbb{E}[|I_{2,1}^n|] \le c\Delta_n^{-\beta+\frac{1}{2}+2\beta-\frac{\alpha\beta}{2}+1} = c\Delta_n^{\frac{3}{2}+\beta-\frac{\alpha\beta}{2}}. \tag{154}$$

Hence

$$\frac{\mathbb{E}[|I^n_{2,1}|]}{\Delta_n^{1+\beta(2-\alpha)}} \leq c\Delta_n^{\frac{1}{2}-\beta+\frac{\alpha\beta}{2}}, \tag{155}$$

that goes to 0 for $n \to \infty$ since the exponent on Δ_n is positive for $\beta < \frac{1}{2(1-\frac{\alpha}{2})}$, that is always true with α and β in the intervals chosen.

We now want to show that also $I^n_{2,2}$ is $o_{L^1}(\Delta_n^{\beta(2-\alpha)+1})$. We split $I^n_{2,2}$ on the sets $\left\{|\Delta\tilde{X}^J_i| \leq 2\Delta_n^\beta\right\}$ and $\left\{|\Delta\tilde{X}^J_i| > 2\Delta_n^\beta\right\}$. We observe that, by the definition of φ, $I^n_{2,2}$ is null on the second set. Adding and subtracting $\Delta\tilde{X}^J_i$ in $I^n_{2,2}1_{\left\{|\Delta\tilde{X}^J_i| \leq 2\Delta_n^\beta\right\}}$ we have

$$\begin{aligned}\mathbb{E}[|I^n_{2,2}|1_{\left\{|\Delta\tilde{X}^J_i| \leq 2\Delta_n^\beta\right\}}] \leq c\mathbb{E}[(\Delta X^J_i - \Delta\tilde{X}^J_i)^2|\varphi_{\Delta_n^\beta}(\Delta X^J_i) \\ - \varphi_{\Delta_n^\beta}(\Delta\tilde{X}^J_i)|1_{\left\{|\Delta\tilde{X}^J_i| \leq 2\Delta_n^\beta, |\Delta X^J_i| > 2\Delta_n^\beta\right\}}] \\ + c\mathbb{E}[(\Delta\tilde{X}^J_i)^2|\varphi_{\Delta_n^\beta}(\Delta X^J_i) - \varphi_{\Delta_n^\beta}(\Delta\tilde{X}^J_i)|1_{\left\{|\Delta\tilde{X}^J_i| \leq 2\Delta_n^\beta\right\}}].\end{aligned} \tag{156}$$

On the second term of (156) we can act exactly as we have done in $I^n_{2,1}$, with $\Delta\tilde{X}^J_i$ instead of ΔX^J_i (and so using (30) instead of (29)). We get

$$\mathbb{E}[(\Delta\tilde{X}^J_i)^2|\varphi_{\Delta_n^\beta}(\Delta X^J_i) - \varphi_{\Delta_n^\beta}(\Delta\tilde{X}^J_i)|1_{\left\{|\Delta\tilde{X}^J_i| \leq 2\Delta_n^\beta\right\}}] \leq c\Delta_n^{\frac{3}{2}+\beta-\frac{\alpha\beta}{2}}. \tag{157}$$

Concerning the first term of (156), by the definition of φ we know it is

$$\mathbb{E}[(\Delta X^J_i - \Delta\tilde{X}^J_i)^2| - \varphi_{\Delta_n^\beta}(\Delta\tilde{X}^J_i)|1_{\left\{|\Delta\tilde{X}^J_i| \leq 2\Delta_n^\beta, |\Delta X^J_i| > 2\Delta_n^\beta\right\}}] \leq c\mathbb{E}[(\Delta X^J_i - \Delta\tilde{X}^J_i)^2] \leq c\Delta_n^2, \tag{158}$$

where in the last inequality we have used (152). Using (156)–(158) it follows

$$\mathbb{E}[|I^n_{2,2}|] = \mathbb{E}[|I^n_{2,2}|1_{\left\{|\Delta\tilde{X}^J_i| \leq 2\Delta_n^\beta\right\}}] \leq c\Delta_n^{\frac{3}{2}+\beta-\frac{\alpha\beta}{2}} + c\Delta_n^2 = c\Delta_n^{\frac{3}{2}+\beta-\frac{\alpha\beta}{2}}, \tag{159}$$

considering that Δ_n^2 is negligible compared to $\Delta_n^{\frac{3}{2}+\beta-\frac{\alpha\beta}{2}}$ since $\beta < \frac{1}{2(1-\frac{\alpha}{2})}$. Hence

$$\frac{\mathbb{E}[|I^n_{2,2}|]}{\Delta_n^{1+\beta(2-\alpha)}} \leq c\Delta_n^{\frac{1}{2}-\beta+\frac{\alpha\beta}{2}}, \tag{160}$$

that goes to 0 for $n \to \infty$.

Concerning $I^n_3(i)$, we have

$$\mathbb{E}[|I^n_3(i)|] \leq c\mathbb{E}[(\Delta X^J_i - \Delta\tilde{X}^J_i)^2] \leq c\Delta_n^2, \tag{161}$$

where the last inequality follows from (152). Hence $I^n_3(i) = o_{L^1}(\Delta_n^{\beta(2-\alpha)+1})$, indeed

$$\frac{\mathbb{E}[|I^n_3(i)|]}{\Delta_n^{1+\beta(2-\alpha)}} \leq c\Delta_n^{1-2\beta+\alpha\beta}, \tag{162}$$

that goes to 0 for $n \to \infty$ considering that the exponent on Δ_n is positive for $\beta < \frac{1}{2-\alpha}$, condition that is always satisfied for $\beta \in (0, \frac{1}{2})$ and $\alpha \in (0, 2)$.

Let us now consider $I_4^n(i)$. Using Cauchy–Schwarz inequality it is

$$\mathbb{E}[|I_4^n(i)|] \le c\mathbb{E}[(\Delta X_i^J - \Delta \tilde{X}_i^J)^2]^{\frac{1}{2}}\mathbb{E}[(\Delta \tilde{X}_i^J)^2\varphi^2_{\Delta_n^\beta}(\Delta \tilde{X}_i^J)]^{\frac{1}{2}} \le c\Delta_n \Delta_n^{\frac{1}{2}+\frac{\beta}{2}(2-\alpha)}$$

$$= c\Delta_n^{\frac{3}{2}+\beta-\frac{\alpha\beta}{2}}, \tag{163}$$

where we have used (152) and the first point of Lemma 3. It follows

$$\frac{\mathbb{E}[|I_4^n(i)|]}{\Delta_n^{1+\beta(2-\alpha)}} \le c\Delta_n^{\frac{1}{2}-\beta+\frac{\alpha\beta}{2}}, \tag{164}$$

that goes to 0 for $n \to \infty$ since the exponent on Δ_n is more than 0 if $\beta < \frac{1}{2(1-\frac{\alpha}{2})}$, that is always true. Using (139), (142), (144), (148), (155), (160), (162) and (164) we obtain (25).

In order to prove (27), we use again reformulation (139). Replacing it in the left hand side of (27) it turns out that our goal is to show that

$$\sum_{i=0}^{n-1}(\sum_{k=1}^{4} I_k^n(i))f(X_{t_i}) = o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}). \tag{165}$$

Using a conditional on $\mathcal{F}_{t_i}$ version of (149), (154) and (159) we have

$$\sum_{i=0}^{n-1}\mathbb{E}_i[|I_2^n(i)f(X_{t_i})|] \le \frac{1}{n}\sum_{i=0}^{n-1}R_i(\Delta_n^{\frac{3}{2}+\beta-\frac{\alpha\beta}{2}-1-\epsilon}) = \frac{1}{n}\sum_{i=0}^{n-1}R_i(\Delta_n^{\frac{1}{2}+\beta-\frac{\alpha\beta}{2}-\epsilon}).$$

Since $\beta(1-\frac{\alpha}{2})$ is always more than zero and, $\forall\tilde{\epsilon} > 0$ we can always find ϵ smaller than it, we get

$$\sum_{i=0}^{n-1} I_2^n(i)f(X_{t_i}) = o_{L^1}(\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}) = o_{L^1}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}). \tag{166}$$

From a conditional version of (161) we get that $\sum_{i=0}^{n-1} I_3^n(i)f(X_{t_i})$ is upper bounded in L^1 norm by the L^1 norm of $\frac{1}{n}\sum_{i=0}^{n-1}R_i(\Delta_n^{2-1-\epsilon}) = \frac{1}{n}\sum_{i=0}^{n-1}R_i(\Delta_n^{1-\epsilon})$ and so

$$\sum_{i=0}^{n-1} I_3^n(i)f(X_{t_i}) = o_{L^1}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}). \tag{167}$$

Using a conditional version of (163) we get that $\sum_{i=0}^{n-1} I_4^n(i)f(X_{t_i})$ is upper bounded in L^1 norm by the L^1 norm of $\frac{1}{n}\sum_{i=0}^{n-1}R_i(\Delta_n^{\frac{3}{2}+\beta-\frac{\alpha\beta}{2}-1-\epsilon}) = \frac{1}{n}\sum_{i=0}^{n-1}R_i(\Delta_n^{\frac{1}{2}+\beta-\frac{\alpha\beta}{2}-\epsilon})$, hence

$$\sum_{i=0}^{n-1} I_4^n(i)f(X_{t_i}) = o_{L^1}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}). \tag{168}$$

Concerning $I_1^n(i)$, we consider $I_{1,1}^n(i)$ and $I_{1,2}^n(i)$ as defined in (140). Using a conditional version of (141) on $I_{1,1}^n(i)$ it follows that $n^{\frac{1}{2}-\tilde{\epsilon}}\sum_{i=0}^{n-1} I_{1,1}^n(i)f(X_{t_i})$ is upper bounded in L^1 norm by the L^1 norm of $\frac{1}{n}\sum_{i=0}^{n-1}R_i(\Delta_n^{(\frac{1}{2}-\beta)r+2\beta-1-\frac{1}{2}+\tilde{\epsilon}}) = \frac{1}{n}\sum_{i=0}^{n-1}R_i(\Delta_n^{(\frac{1}{2}-\beta)r+2\beta-\frac{3}{2}+\tilde{\epsilon}})$, that goes to zero because we can find r big enough such that the exponent on Δ_n is positive, hence

$$\sum_{i=0}^{n-1} I_{1,1}^n(i)f(X_{t_i}) = o_{L^1}(\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}) = o_{L^1}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}). \tag{169}$$

Acting as we did in the proof of (25), we consider $I^n_{1,2}(i)$ on the sets $\left\{|\Delta X^J_i| \geq 4\Delta^\beta_n\right\}$ and $\left\{|\Delta X^J_i| < 4\Delta^\beta_n\right\}$. Again, from (143) and the arbitrariness of $r > 0$ it follows

$$\sum_{i=0}^{n-1} I^n_{1,2}(i)1_{\left\{|\Delta X^J_i|\geq 4\Delta^\beta_{n,i}\right\}} f(X_{t_i}) = o_{L^1}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}). \tag{170}$$

When $|\Delta X^J_i| < 4\Delta^\beta_n$ we act in a different way, considering the development up to second order of $\varphi_{\Delta^\beta_n}$, centered in ΔX^J_i:

$$\begin{aligned} I^n_{1,2}(i)1_{\left\{|\Delta X^J_i|<4\Delta^\beta_n\right\}} &= [(\Delta X^J_i)^2 \Delta X^c_i \varphi'_{\Delta^\beta_n}(\Delta X^J_i)\Delta_n^{-\beta} \\ &\quad + (\Delta X^J_i)^2(\Delta X^c_i)^2 \varphi''_{\Delta^\beta_n}(X_u)\Delta_n^{-2\beta}]1_{\left\{|\Delta X_i|\leq 3\Delta^\beta_n,\, |\Delta X^J_i|<4\Delta^\beta_n\right\}} = \\ &=: \hat{I}^n_1(i)1_{\left\{|\Delta X_i|\leq 3\Delta^\beta_n,\, |\Delta X^J_i|<4\Delta^\beta_n\right\}} + \hat{I}^n_2(i)1_{\left\{|\Delta X_i|\leq 3\Delta^\beta_n,\, |\Delta X^J_i|<4\Delta^\beta_n\right\}}, \end{aligned}$$

where $X_u \in [\Delta X^J_i, \Delta X_i]$. Now, acting like we did in (145), (146) and (147), taking q next to 1 we get

$$\mathbb{E}_i[|\hat{I}^n_2(i)1_{\left\{|\Delta X_i|\leq 3\Delta^\beta_n,\, |\Delta X^J_i|<4\Delta^\beta_n\right\}}|] \leq R_i(\Delta_n^{1+\beta(2-\alpha)-\epsilon+1-2\beta}) = R_i(\Delta_n^{2-\alpha\beta-\epsilon}).$$

Since for each $\tilde{\epsilon} > 0$ we can find an ϵ such that $\tilde{\epsilon} - \epsilon > 0$ it follows, taking the conditional expectation

$$\sum_{i=0}^{n-1} \hat{I}^n_2(i)1_{\left\{|\Delta X_i|\leq 3\Delta^\beta_n,\, |\Delta X^J_i|<4\Delta^\beta_n\right\}} f(X_{t_i}) = o_{L^1}(\Delta_n^{1-\alpha\beta-\tilde{\epsilon}}) = o_{L^1}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}). \tag{171}$$

Concerning $\hat{I}^n_1(i)1_{\left\{|\Delta X_i|\leq 3\Delta^\beta_n,\, |\Delta X^J_i|<4\Delta^\beta_n\right\}}$, we no longer consider the indicator function because it is

$$(\Delta X^J_i)^2 \Delta X^c_i \varphi'_{\Delta^\beta_n}(\Delta X^J_i)\Delta_n^{-\beta} + (\Delta X^J_i)^2 \Delta X^c_i \varphi'_{\Delta^\beta_n}(\Delta X^J_i)\Delta_n^{-\beta}(1_{\left\{|\Delta X_i|\leq 3\Delta^\beta_n,\, |\Delta X^J_i|<4\Delta^\beta_n\right\}} - 1)$$

and the second term here above is different from zero only on a set smaller that $\left\{|\Delta X_i| \geq 3\Delta^\beta_n\right\}$ or $\left\{|\Delta X^J_i| \geq 4\Delta^\beta_n\right\}$, on which we have already proved the result (see the study of $I^n_{1,1}(i)$ in (169) and $I^n_{1,2}(i)$ in (170)). We want to show that

$$\sum_{i=0}^{n-1} \hat{I}^n_1(i) f(X_{t_i}) = o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}). \tag{172}$$

We start from the reformulation

$$\begin{aligned} \hat{I}^n_1(i) &= \Delta X^c_i \Delta_n^{-\beta}[(\Delta X^J_i)^2(\varphi'_{\Delta^\beta_n}(\Delta X^J_i) - \varphi'_{\Delta^\beta_n}(\Delta \tilde{X}^J_i)) + (\Delta X^J_i - \Delta \tilde{X}^J_i)^2 \varphi'_{\Delta^\beta_n}(\Delta \tilde{X}^J_i) \\ &\quad + 2\Delta \tilde{X}^J_i(\Delta X^J_i - \Delta \tilde{X}^J_i)\varphi'_{\Delta^\beta_n}(\Delta \tilde{X}^J_i) + (\Delta \tilde{X}^J_i)^2 \varphi'_{\Delta^\beta_n}(\Delta \tilde{X}^J_i)] = \sum_{j=1}^4 \hat{I}^n_{1,j}(i). \end{aligned}$$

and we observe that, after have used Holder inequality and have remarked that $\varphi'_{\Delta^\beta_n}$ acts like $\varphi_{\Delta^\beta_n}$, we can act on $\hat{I}^n_{1,1}$ as we did on I^n_2, on $\hat{I}^n_{1,2}$ as on I^n_3 and on $\hat{I}^n_{1,3}$ as on I^n_4. So we get, using also Holder inequality and the sixth point of Lemma 1,

$$\mathbb{E}_i[|\hat{I}^n_{1,1}(i)+\hat{I}^n_{1,2}(i)+\hat{I}^n_{1,3}(i)|] \le R_i(\Delta_n^{\frac{1}{2}-\beta})(\mathbb{E}_i[|I^n_2(i)|^q]^{\frac{1}{q}}+\mathbb{E}[|I^n_3(i)|^q]^{\frac{1}{q}}+\mathbb{E}[|I^n_4(i)|^q]^{\frac{1}{q}}). \tag{173}$$

Now, taking q next to 1, we need the following lemma that we will prove later:

Lemma 9. *Suppose that A1–A4 hold. Then, $\forall \epsilon > 0$,*

$$\mathbb{E}_i[|I^n_2(i)|^{1+\epsilon}+|I^n_3(i)|^{1+\epsilon}+|I^n_4(i)|^{1+\epsilon}]^{\frac{1}{1+\epsilon}} \le R_i(\Delta_n^{\frac{3}{2}+\beta-\frac{\alpha\beta}{2}-\epsilon}), \tag{174}$$

with $I^n_2(i)$, $I^n_3(i)$ and $I^n_4(i)$ as defined in (139).

From (173) and (174) it follows

$$\sum_{i=0}^{n-1}[\hat{I}^n_{1,1}(i)+\hat{I}^n_{1,2}(i)+\hat{I}^n_{1,3}(i)]f(X_{t_i}) = o_{L^1}(\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}) = o_{L^1}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}). \tag{175}$$

On $\sum_{i=0}^{n-1}\hat{I}^n_{1,4}f(X_{t_i}) =: \sum_{i=0}^{n-1}\zeta_{n,i}$ we want to use Lemma 9 in [9]. By the independence between L and W we get

$$\frac{1}{\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}}\sum_{i=0}^{n-1}\mathbb{E}_i[\zeta_{n,i}] = \frac{1}{\Delta_n^{\frac{1}{2}-\tilde{\epsilon}}}\sum_{i=0}^{n-1} f(X_{t_i})\Delta_{n,i}^{-\beta}\mathbb{E}_i[(\Delta\tilde{X}^J_i)^2\varphi'_{\Delta_n^\beta}(\Delta\tilde{X}^J_i)]\mathbb{E}_i[\Delta X^c_i] = 0 \tag{176}$$

and

$$\Delta_n^{-2(\frac{1}{2}-\tilde{\epsilon})}\sum_{i=0}^{n-1} f^2(X_{t_i})\Delta_{n,i}^{-2\beta}\mathbb{E}_i[(\Delta\tilde{X}^J_i)^4\varphi'^2_{\Delta_n^\beta}(\Delta\tilde{X}^J_i)]\mathbb{E}_i[(\Delta X^c_i)^2] \le c\Delta_n^{2\tilde{\epsilon}+2\beta-\alpha\beta}, \tag{177}$$

where we have also used the sixth point of Lemma 1 and the first point of Lemma 3. Using (176) and (177) we have

$$\sum_{i=0}^{n-1}\hat{I}^n_{1,4}f(X_{t_i}) = o_{\mathbb{P}}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})})$$

that, joint with (175) and the fact that the convergence in norm 1 implies the convergence in probability, give us (172). Using also (166)–(171) we get (165) and so (27).

In order to prove (26), we reformulate $\Delta X^J_i\varphi_{\Delta_n^\beta}(\Delta X_i)$ as we have already done in (139) getting

$$\begin{aligned}(\int_{t_i}^{t_{i+1}} a_s dW_s)\Delta X^J_i\varphi_{\Delta_n^\beta}(\Delta X_i) &= (\int_{t_i}^{t_{i+1}} a_s dW_s)(\Delta X^J_i)[\varphi_{\Delta_n^\beta}(\Delta X_i)-\varphi_{\Delta_n^\beta}(\Delta X^J_i)] \\ &+ (\int_{t_i}^{t_{i+1}} a_s dW_s)(\Delta X^J_i)[\varphi_{\Delta_n^\beta}(\Delta X^J_i)-\varphi_{\Delta_n^\beta}(\Delta\tilde{X}^J_i)] \\ &+ (\int_{t_i}^{t_{i+1}} a_s dW_s)(\Delta X^J_i-\Delta\tilde{X}^J_i)\varphi_{\Delta_n^\beta}(\Delta\tilde{X}^J_i) \\ &+ (\int_{t_i}^{t_{i+1}} a_s dW_s)(\Delta\tilde{X}^J_i)\varphi_{\Delta_n^\beta}(\Delta\tilde{X}^J_i) =: \sum_{j=1}^4 \tilde{I}^n_j(i).\end{aligned} \tag{178}$$

Comparing (178) with (26) it turns out that our goal is to prove that $\frac{1}{\Delta_{n,i}^{\beta(2-\alpha)+1}} \sum_{j=1}^3 \mathbb{E}[|\tilde{I}_j^n(i)|] \to 0$, for $n \to \infty$ (again, acting as we do in the sequel it is also possible to show that $\sum_{j=1}^3 \mathbb{E}_i[|\tilde{I}_j^n(i)|] \le R_i(\Delta_{n,i}^{\beta(2-\alpha)+1})$. Let us start considering $\tilde{I}_1^n(i)$. Using Holder inequality, its expected value is upper bounded by

$$\mathbb{E}[|\int_{t_i}^{t_{i+1}} a_s dW_s|^{p_1}]^{\frac{1}{p_1}} \mathbb{E}[|\Delta X_i^J|^{p_2} |\varphi_{\Delta_n^\beta}(\Delta X_i) - \varphi_{\Delta_n^\beta}(\Delta X_i^J)|^{p_2}]^{\frac{1}{p_2}}. \tag{179}$$

We now act on $\mathbb{E}[|\Delta X_i^J|^{p_2} |\varphi_{\Delta_n^\beta}(\Delta X_i) - \varphi_{\Delta_n^\beta}(\Delta X_i^J)|^{p_2}]^{\frac{1}{p_2}}$ as we did in the study of $I_1^n(i)$:

$$\begin{aligned}|\Delta X_i^J|^{p_2} |\varphi_{\Delta_n^\beta}(\Delta X_i) - \varphi_{\Delta_n^\beta}(\Delta X_i^J)|^{p_2} &= |\Delta X_i^J|^{p_2} |\varphi_{\Delta_n^\beta}(\Delta X_i) \\ &- \varphi_{\Delta_n^\beta}(\Delta X_i^J)|^{p_2} 1_{\left\{|\Delta X_i| \ge 3\Delta_n^\beta\right\}} \\ &+ |\Delta X_i^J|^{p_2} |\varphi_{\Delta_n^\beta}(\Delta X_i) - \varphi_{\Delta_n^\beta}(\Delta X_i^J)|^{p_2} 1_{\left\{|\Delta X_i| < 3\Delta_n^\beta\right\}} =: \tilde{I}_{1,1}^n + \tilde{I}_{1,2}^n.\end{aligned}$$

Concerning $\tilde{I}_{1,1}^n$, if $|\Delta X_i^J| \ge 2\Delta_n^\beta$ it is just 0, otherwise we can act exactly as we have done on $I_{1,1}^n$, taking $p_2 = 2$. Hence, $\forall r \ge 1$,

$$\mathbb{E}[|\tilde{I}_{1,1}^n|]^{\frac{1}{2}} \le (c\Delta_n^{2\beta + r(\frac{1}{2}-\beta)})^{\frac{1}{2}} = c\Delta_n^{\beta + \frac{r}{2}(\frac{1}{2}-\beta)}. \tag{180}$$

Let us now consider $\tilde{I}_{1,2}^n$. If $|\Delta X_i^J| \ge 4\Delta_n^\beta$, we act again like we did on $I_{1,2}^n$, taking $p_2 = 2$. It yields again

$$\mathbb{E}[|\tilde{I}_{1,2}^n| 1_{\left\{|\Delta X_i^J| \ge 4\Delta_n^\beta\right\}}]^{\frac{1}{2}} \le c\Delta_n^{\beta + \frac{r}{2}(\frac{1}{2}-\beta)}. \tag{181}$$

If $|\Delta X_i^J| < 4\Delta_n^\beta$ we use the smoothness of φ and Holder inequality getting

$$\begin{aligned}\mathbb{E}[|\tilde{I}_{1,2}^n| 1_{\left\{|\Delta X_i^J| < 4\Delta_n^\beta\right\}}] &\le \Delta_n^{-\beta} \mathbb{E}[|\Delta X_i^J|^{p_2} |\varphi'(\zeta)|^{p_2} |\Delta X_i^c|^{p_2} 1_{\left\{|\Delta X_i| < 3\Delta_n^\beta, |\Delta X_i^J| < 4\Delta_n^\beta\right\}}]^{\frac{1}{p_2}} \\ &\le \Delta_n^{-\beta} \mathbb{E}[|\Delta X_i^c|^{p_2 p}]^{\frac{1}{p_2 p}} \mathbb{E}[|\varphi'(\zeta)|^{p_2 q} |\Delta X_i^J|^{p_2 q} 1_{\left\{|\Delta X_i| < 3\Delta_n^\beta, |\Delta X_i^J| < 4\Delta_n^\beta\right\}}]^{\frac{1}{p_2 q}},\end{aligned} \tag{182}$$

with ζ a point between ΔX_i^J and ΔX_i.

Now we observe that, if $|\Delta X_i^c| \ge \frac{\Delta_n^\beta}{4}$, then taking $p_2 q = 1 + \epsilon$ we have

$$\mathbb{E}[|\varphi'(\zeta)|^{1+\epsilon} |\Delta X_i^J|^{1+\epsilon} 1_{\left\{|\Delta X_i| < 3\Delta_n^\beta, |\Delta X_i^J| < 4\Delta_n^\beta, |\Delta X_i^c| \ge \frac{\Delta_n^\beta}{4}\right\}}]^{\frac{1}{1+\epsilon}} \le c\Delta_n^{\beta + r(\frac{1}{2}-\beta)\frac{1}{1+\epsilon}}$$

where we have used the bound on $|\Delta X_i^J|$ given by the indicator function, the boundedness of φ' and (34). Otherwise, by the definition of φ, we know that $|\varphi'(\zeta)| \ne 0$ only if $|\zeta| \in (\Delta_n^\beta, 2\Delta_n^\beta)$. Then $\Delta_n^\beta \le |\zeta| \le |\Delta X_i| + |\Delta X_i^J| \le 2|\Delta X_i^J| + |\Delta X_i^c| \le 2|\Delta X_i^J| + \frac{\Delta_n^\beta}{4}$, hence $|\Delta X_i^J| \ge \frac{3}{8}\Delta_n^\beta \ge \frac{\Delta_n^\beta}{4}$ and so we can say it is

$$\begin{aligned}&\mathbb{E}[|\varphi'(\zeta)|^{1+\epsilon} |\Delta X_i^J|^{1+\epsilon} 1_{\left\{|\Delta X_i| < 3\Delta_n^\beta, |\Delta X_i^J| < 4\Delta_n^\beta, |\Delta X_i^c| < \frac{\Delta_n^\beta}{4}\right\}}]^{\frac{1}{1+\epsilon}} \\ &\le c\mathbb{E}[|\Delta X_i^J|^{1+\epsilon} 1_{\left\{\frac{\Delta_n^\beta}{4} \le |\Delta X_i^J| < 4\Delta_n^\beta,\right\}}].\end{aligned}$$

Using the second point of Lemma 3, passing through the conditional expected value we get it is upper bounded by

$$\Delta_n^{1+\beta(1+\epsilon-\alpha)}\mathbb{E}[R_i(1)] \le c\Delta_n^{1+\beta(1+\epsilon-\alpha)}.$$

Hence

$$\mathbb{E}[|\varphi'(\zeta)|^{1+\epsilon}|\Delta X_i^J|^{1+\epsilon}1_{\left\{|\Delta X_i|<3\Delta_n^{\beta},|\Delta X_i^J|<4\Delta_n^{\beta}\right\}}]^{\frac{1}{1+\epsilon}} \le c\Delta_n^{[\beta+r(\frac{1}{2}-\beta)-\epsilon]\wedge[1+\beta(1+\epsilon-\alpha)]\frac{1}{1+\epsilon}}$$
$$= c\Delta_n^{[1+\beta(1+\epsilon-\alpha)]\frac{1}{1+\epsilon}}. \tag{183}$$

The last equality follows from the fact that, for each choice of $\beta \in (0, \frac{1}{2})$ and $\alpha \in (0, 2)$, we can always find $r \ge 1$ and $\epsilon > 0$ such that $\beta + r(\frac{1}{2} - \beta) - \epsilon > 1 + \beta(1 + \epsilon - \alpha)$.

Replacing (183) in (182) and using the sixth point of Lemma 1 we have that

$$\mathbb{E}[|\tilde{I}_{1,2}^n|1_{\left\{|\Delta X_i^J|<4\Delta_n^{\beta}\right\}}]^{\frac{1}{p_2}} \le c\Delta_n^{[\frac{1}{2}-\beta+1+\beta(1+\epsilon-\alpha)]\frac{1}{p_2}} = c\Delta_n^{(\frac{3}{2}-\alpha\beta-\epsilon)\frac{1}{p_2}} = c\Delta_n^{\frac{3}{2}-\alpha\beta-\epsilon}, \tag{184}$$

the last equality follows from the choice of both p_2 and q next to 1. Using (180), (181) and (184) we get

$$\mathbb{E}[|\Delta X_i^J|^{p_2}|\varphi_{\Delta_n^{\beta}}(\Delta X_i) - \varphi_{\Delta_n^{\beta}}(\Delta X_i^J)|^{p_2}]^{\frac{1}{p_2}} \le c\Delta_n^{[\beta+\frac{r}{2}(\frac{1}{2}-\beta)]\wedge[\frac{3}{2}-\alpha\beta-\epsilon]} = c\Delta_n^{\frac{3}{2}-\alpha\beta-\epsilon}. \tag{185}$$

Replacing (33) and (185) in (179) it follows

$$\mathbb{E}[|\tilde{I}_1^n(i)|] \le c\Delta_n^{2-\alpha\beta-\epsilon}, \tag{186}$$

hence

$$\frac{\mathbb{E}[|\tilde{I}_1^n(i)|]}{\Delta_n^{1+\beta(2-\alpha)}} \le c\Delta_n^{1-2\beta-\epsilon}. \tag{187}$$

Since we can always find an $\epsilon > 0$ such that $1 - 2\beta - \epsilon > 0$, the expected value above goes to 0 for $n \to \infty$.

Concerning $\tilde{I}_2^n(i)$, we split again on $\tilde{I}_{2,1}^n := \tilde{I}_2^n(i)1_{\left\{|\Delta X_i^J|\le 2\Delta_n^{\beta}\right\}}$ and $\tilde{I}_{2,2}^n := \tilde{I}_2^n(i)1_{\left\{|\Delta X_i^J|>2\Delta_n^{\beta}\right\}}$.

$$\mathbb{E}[|\tilde{I}_{2,1}^n|] = \mathbb{E}[|\tilde{I}_2^n(i)|1_{\left\{|\Delta X_i^J|\le 2\Delta_n^{\beta}\right\}}] \le c\Delta_n^{-\beta}\mathbb{E}[|\int_{t_i}^{t_{i+1}} a_s dW_s||\Delta X_i^J||\Delta X_i^J$$
$$- \Delta\tilde{X}_i^J|1_{\left\{|\Delta X_i^J|\le 2\Delta_n^{\beta}\right\}}]$$
$$\le c\Delta_n^{-\beta}\mathbb{E}[|\int_{t_i}^{t_{i+1}} a_s dW_s|^2|\Delta X_i^J|^2 1_{\left\{|\Delta X_i^J|\le 2\Delta_n^{\beta}\right\}}]^{\frac{1}{2}}\mathbb{E}[|\Delta X_i^J - \Delta\tilde{X}_i^J|^2]^{\frac{1}{2}}$$
$$\le c\Delta_{n,i}^{1-\beta}\mathbb{E}[|\int_{t_i}^{t_{i+1}} a_s dW_s|^{2p}]^{\frac{1}{2p}}\mathbb{E}[|\Delta X_i^J|^{2q}1_{\left\{|\Delta X_i^J|\le 2\Delta_n^{\beta}\right\}}]^{\frac{1}{2q}},$$

where we have used Cauchy–Schwarz inequality, (152) and Holder inequality. Now we take p big and q next to 1, using (33) and the first point of Lemma 3 we get

$$\mathbb{E}[|\tilde{I}_{2,1}^n|] \le c\Delta_n^{1-\beta+\frac{1}{2}+\frac{1}{2}+\frac{\beta}{2}(2-\alpha)-\epsilon} \tag{188}$$

and so

$$\frac{1}{\Delta_n^{1+\beta(2-\alpha)}}\mathbb{E}[|\tilde{I}_{2,1}^n|] \leq \Delta_n^{1-2\beta+\frac{\alpha\beta}{2}-\epsilon}. \tag{189}$$

It goes to 0 for $n \to \infty$ because we can always find an $\epsilon > 0$ such that the exponent in Δ_n is positive. Let us now consider $\tilde{I}_{2,2}^n = \tilde{I}_{2,2}^n 1_{\left\{|\Delta \tilde{X}_i^J| \leq 2\Delta_n^\beta\right\}} + \tilde{I}_{2,2}^n 1_{\left\{|\Delta \tilde{X}_i^J| > 2\Delta_n^\beta\right\}}$. From the definition of φ, $\tilde{I}_{2,2}^n 1_{\left\{|\Delta \tilde{X}_i^J| > 2\Delta_n^\beta\right\}} = 0$.

$$\begin{aligned}
\mathbb{E}[|\tilde{I}_{2,2}^n| 1_{\left\{|\Delta \tilde{X}_i^J| \leq 2\Delta_n^\beta\right\}}] &= \mathbb{E}[|\int_{t_i}^{t_{i+1}} a_s dW_s \| \Delta \tilde{X}_i^J \| \varphi_{\Delta_n^\beta}(\Delta X_i^J) \\
&\quad - \varphi_{\Delta_n^\beta}(\Delta \tilde{X}_i^J)| 1_{\left\{|\Delta \tilde{X}_i^J| \leq 2\Delta_n^\beta, |\Delta X_i^J| > 2\Delta_n^\beta\right\}}] \\
&\quad + \mathbb{E}[|\int_{t_i}^{t_{i+1}} a_s dW_s \| \Delta X_i^J - \Delta \tilde{X}_i^J \| \varphi_{\Delta_n^\beta}(\Delta X_i^J) \\
&\quad - \varphi_{\Delta_n^\beta}(\Delta \tilde{X}_i^J)| 1_{\left\{|\Delta \tilde{X}_i^J| \leq 2\Delta_n^\beta, |\Delta X_i^J| > 2\Delta_{n,i}^\beta\right\}}] \\
&\leq c\Delta_n^{2-\frac{\alpha\beta}{2}-\epsilon} + \mathbb{E}[|\int_{t_i}^{t_{i+1}} a_s dW_s \| \Delta X_i^J - \Delta \tilde{X}_i^J \| - \varphi_{\Delta_n^\beta}(\Delta \tilde{X}_i^J)|],
\end{aligned}$$

where we have acted exactly like we did in $\tilde{I}_{2,1}^n$, using that $\Delta \tilde{X}_i^J$ is less than $2\Delta_n^\beta$. We have also used that, by the definition of φ, evaluated in ΔX_i^J it is zero. Now we use Holder inequality, (33) and the boundedness of φ to get

$$\begin{aligned}
\mathbb{E}[|\tilde{I}_{2,2}^n|] &\leq c\Delta_n^{2-\frac{\alpha\beta}{2}-\epsilon} + \mathbb{E}[|\int_{t_i}^{t_{i+1}} a_s dW_s|^p]^{\frac{1}{p}} \mathbb{E}[|\Delta X_i^J - \Delta \tilde{X}_i^J|^q]^{\frac{1}{q}} \\
&\leq c\Delta_n^{2-\frac{\alpha\beta}{2}-\epsilon} + c\Delta_n^{\frac{1}{2}} \mathbb{E}[|\Delta X_i^J - \Delta \tilde{X}_i^J|^q]^{\frac{1}{q}}.
\end{aligned}$$

Now, if $\alpha < 1$ we use (153), with $q = 1 + \epsilon$, getting

$$\mathbb{E}[|\tilde{I}_{2,2}^n|] \leq c\Delta_n^{2-\frac{\alpha\beta}{2}-\epsilon} + c\Delta_n^{\frac{1}{2}+\frac{1}{2}+\frac{1}{1+\epsilon}} = c\Delta_n^{2-\frac{\alpha\beta}{2}-\epsilon}. \tag{190}$$

Therefore, for $\alpha < 1$, we have

$$\frac{1}{\Delta_n^{1+\beta(2-\alpha)}}\mathbb{E}[|I_{2,2}^n|] \leq c\Delta_n^{1-2\beta+\frac{\alpha\beta}{2}-\epsilon}. \tag{191}$$

We can find an $\epsilon > 0$ such that the exponent on Δ_n is positive hence, if $\alpha < 1$, then $I_{2,2}^n = o_{L^1}(\Delta_n^{1+\beta(2-\alpha)})$. Otherwise, if $\alpha \geq 1$, we use (152) having taken $q = 2$. We get

$$\mathbb{E}[|\tilde{I}_{2,2}^n|] \leq c\Delta_n^{2-\frac{\alpha\beta}{2}-\epsilon} + c\Delta_n^{\frac{1}{2}+1} = c\Delta_n^{\frac{3}{2}}.$$

It follows that, for $\alpha \geq 1$, it is

$$\frac{1}{\Delta_n^{1+\beta(2-\alpha)}}\mathbb{E}[|I_{2,2}^n|] \leq c\Delta_n^{\frac{1}{2}-\beta(2-\alpha)}. \tag{192}$$

We observe that the exponent on Δ_n is more than 0 if $\beta < \frac{1}{2}\frac{1}{(2-\alpha)}$, that is always true for $\beta \in (0, \frac{1}{2})$ and $\alpha \in [1, 2)$.

To conclude, we use on $\tilde{I}_3(i)$ Holder inequality, (33), the boundedness of φ and then we act as we did on $\tilde{I}^n_{2,2}$, using (153) or (152), depending on whether or not α is less than 1. In the case $\alpha < 1$ we get

$$\frac{1}{\Delta_n^{1+\beta(2-\alpha)}}\mathbb{E}[|\tilde{I}_3^n(i)|] \le \frac{1}{\Delta_n^{1+\beta(2-\alpha)}} c\Delta_n^{\frac{1}{2}+\frac{1}{2}+\frac{1}{1+\epsilon}} = c\Delta_n^{1-\beta(2-\alpha)-\epsilon}, \tag{193}$$

that goes to 0 for $n \to \infty$ since we can always find $\epsilon > 0$ such that the exponent on Δ_n is positive. Otherwise it follows

$$\frac{1}{\Delta_n^{1+\beta(2-\alpha)}}\mathbb{E}[|\tilde{I}_3^n(i)|] \le \frac{1}{\Delta_n^{1+\beta(2-\alpha)}} c\Delta_n^{\frac{3}{2}} = c\Delta_{n,i}^{\frac{1}{2}-\beta(2-\alpha)}. \tag{194}$$

The exponent on Δ_n is positive if $\beta < \frac{1}{2}\frac{1}{(2-\alpha)}$, that is always true since we are in the case $\alpha \ge 1$. Hence $\tilde{I}_3^n(i) = o_{L^1}(\Delta_n^{1+\beta(2-\alpha)})$.

From (187)–(194) and the reformulation (178), it follows (26).

Replacing reformulation (178) in the left hand side of (28), it turns out that the theorem is proved if

$$\sum_{i=0}^{n-1}(\sum_{k=1}^{3}\tilde{I}_k^n(i))f(X_{t_i}) = o_{L^1}(\Delta_n^{(\frac{1}{2}-\tilde{\epsilon})\wedge(1-\alpha\beta-\tilde{\epsilon})}). \tag{195}$$

Using a conditional version of Eqs. (186), (188), (190), (193) and (194) (adding in the last two $\beta(2-\alpha)$ in the exponent of Δ_n) we easily get (195) and so (28). □

A.3. *Proof of Lemma 4*

Proof. By the definition of $d(\zeta_n)$, as in law we have that $S_1^\alpha = -S_1^\alpha$, we get $d(\zeta_n) = d(|\zeta_n|)$ and thus we can assume that $\zeta_n > 0$. Using a change of variable we obtain

$$d(\zeta_n) = \mathbb{E}[(S_1^\alpha)^2\varphi(S_1^\alpha\zeta_n)] = \int_{\mathbb{R}} z^2\varphi(z\zeta_n)f_\alpha(z)dz = (\zeta_n)^{-3}\int_{\mathbb{R}} u^2\varphi(u)f_\alpha(\frac{u}{\zeta_n})du. \tag{196}$$

We want to use an asymptotic expansion of the density (see Theorem 7.22 in [15], with $d = 1$ and $\sigma = 1$) which states that, if z is big enough, then a development up to order N of $f_\alpha(z)$ is

$$\frac{c_\alpha}{|z|^{1+\alpha}} + \frac{1}{\pi}\frac{1}{|z|}\sum_{k=2}^{N}\frac{a_k}{k!}(|z|^{-\alpha})^k + o(|z|^{-\alpha N}), \tag{197}$$

for some coefficients a_k. We therefore take $M > 0$ big enough such that, for $\frac{u}{\zeta_n} > M$, we can use (197). Hence the right hand side of (196) can be seen as

$$(\zeta_n)^{-3}\int_{|u|\le\zeta_n M} u^2\varphi(u)f_\alpha(\frac{u}{\zeta_n})du + (\zeta_n)^{-3}\int_{|u|>\zeta_n M} u^2\varphi(u)f_\alpha(\frac{u}{\zeta_n})du =: I_1^n + I_2^n. \tag{198}$$

We have that, $\forall\hat{\epsilon} > 0$, $I_1^n = o(\zeta_n^{-\hat{\epsilon}})$. Indeed, using that φ and f_α are both bounded, we get

$$\frac{I_1^n}{\zeta_n^{-\hat{\epsilon}}} \le \zeta_n^{-3+\hat{\epsilon}}\int_{|u|\le\zeta_n M} u^2 du \le c\zeta_n^{\hat{\epsilon}}, \tag{199}$$

that goes to zero because we have assumed that $\zeta_n \to 0$. I_2^n is

$$(\zeta_n)^{-3} \int_{|u|>\zeta_n M} u^2 \varphi(u) c_\alpha (\zeta_n)^{1+\alpha} |u|^{-1-\alpha} du + (\zeta_n)^{-3} \times \int_{|u|>\zeta_n M} u^2 \varphi(u) [f_\alpha(\frac{u}{\zeta_n}) - \frac{c_\alpha}{|u|^{1+\alpha}} |\zeta_n|^{1+\alpha}] du. \tag{200}$$

The first term here above can be seen as

$$(\zeta_n)^{\alpha-2} c_\alpha \int_{\mathbb{R}} |u|^{1-\alpha} \varphi(u) du - (\zeta_n)^{\alpha-2} c_\alpha \int_{|u|\le\zeta_n M} |u|^{1-\alpha} \varphi(u) du$$
$$= (\zeta_n)^{\alpha-2} c_\alpha \int_{\mathbb{R}} |u|^{1-\alpha} \varphi(u) du + o((\zeta_n)^{-\hat{\epsilon}}).$$

Indeed, using that φ is bounded, we have

$$\frac{1}{(\zeta_n)^{-\hat{\epsilon}}} |(\zeta_n)^{\alpha-2} c_\alpha \int_{|u|\le\zeta_n M} |u|^{1-\alpha} \varphi(u) du| \le c(\zeta_n)^{\hat{\epsilon}+\alpha-2} \int_{|u|\le\zeta_n M} |u|^{1-\alpha} du \le c(\zeta_n)^{\hat{\epsilon}}. \tag{201}$$

that goes to zero for $n \to \infty$.

Replacing (199), (200) and (201) in (198) and comparing it with (90), it turns out that our goal is to show that the second term of (200) is $o(\zeta_n^{(-\hat{\epsilon})\wedge(2\alpha-2-\hat{\epsilon})})$. Using on it (197) with $N = 2$, which implies $|f_\alpha(z) - \frac{c_\alpha}{|z|^{1+\alpha}}| \le \frac{c}{|z|^{1+2\alpha}}$ for $|z| > M$ and some $c > 0$, we can upper bound it with $c(\zeta_n)^{2\alpha-2} \int_{|u|\le\zeta_n M} |u|^{1-2\alpha} du$. By the definition of φ we have

$$\int_{|u|>\zeta_n M} |u|^{1-2\alpha} \varphi(u) du = \int_{-2}^{-\zeta_n M} (-u)^{1-2\alpha} \varphi(u) du + \int_2^{\zeta_n M} u^{1-2\alpha} \varphi(u) du \le c + c(\zeta_n)^{2-2\alpha}. \tag{202}$$

Therefore we get that the second term of (200) is upper bounded by

$$c\zeta_n^{2\alpha-2} + c.$$

The first term here above is clearly $o(\zeta_n^{2\alpha-2-\hat{\epsilon}})$ while the second is $o(\zeta_n^{-\hat{\epsilon}})$, hence the sum is $o(\zeta_n^{(-\hat{\epsilon})\wedge(2\alpha-2-\hat{\epsilon})})$. The lemma is therefore proved. □

A.4. Proof of Lemma 8

Proof. We observe that, $\forall \alpha \in [0, 2]$, we have

$$\begin{aligned}
\mathbb{E}[|\Delta X_i^J - \Delta \tilde{X}_i^J|^2] &= \mathbb{E}[(\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}} [\gamma(X_{s-}) - \gamma(X_{t_i})] z \tilde{\mu}(ds, dz))^2] \\
&= \mathbb{E}[\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}} [\gamma(X_{s-}) - \gamma(X_{t_i})]^2 |z|^2 \bar{\mu}(ds, dz)] \\
&\le c \int_{t_i}^{t_{i+1}} \mathbb{E}[|X_s - X_{t_i}|^2] ds \int_{\mathbb{R}} |z|^2 F(z) dz \le c \int_{t_i}^{t_{i+1}} \Delta_n ds \le c\Delta_n^2,
\end{aligned} \tag{203}$$

where we have used Ito isometry, the regularity of γ and the third point of Lemma 1.

We have in this way proved (152) and showed that (153) holds with $q = 2$. For $q > 2$, using Kunita inequality and acting like we did here above we get

$$\begin{aligned}\mathbb{E}[|\Delta X_i^J - \Delta \tilde{X}_i^J|^q] &\le \mathbb{E}[\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}} [\gamma(X_{s-}) - \gamma(X_{t_i})]^q |z|^q \bar{\mu}(ds, dz)] \\ &\quad + \mathbb{E}[(\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}} [\gamma(X_{s-}) - \gamma(X_{t_i})]^2 |z|^2 \bar{\mu}(ds, dz))^{\frac{q}{2}}] \\ &\le c \int_{t_i}^{t_{i+1}} \mathbb{E}[|X_s - X_{t_i}|^q] ds + \mathbb{E}[(\int_{t_i}^{t_{i+1}} |X_s - X_{t_i}|^2 ds)^{\frac{q}{2}}] \le c\Delta_n^2 \\ &\quad + c\Delta_n^{\frac{q}{2}-1} \int_{t_i}^{t_{i+1}} \mathbb{E}[|X_s - X_{t_i}|^q] ds = c\Delta_n^2 + c\Delta_n^{\frac{q}{2}-1} \le c\Delta_n^2,\end{aligned}$$

where we have also used Jensen inequality.

In order to prove (153) we observe that, if $\alpha < 1$, then we have

$$\begin{aligned}\mathbb{E}[|\Delta X_i^J - \Delta \tilde{X}_i^J|] &\le \mathbb{E}[|\int_{t_i}^{t_{i+1}} \int_{|z| \ge 2\Delta_n^\beta} [\gamma(X_{s-}) - \gamma(X_{t_i})] z \tilde{\mu}(ds, dz)|] \\ &\quad + \mathbb{E}[|\int_{t_i}^{t_{i+1}} \int_{|z| \le 2\Delta_n^\beta} [\gamma(X_{s-}) - \gamma(X_{t_i})] z \tilde{\mu}(ds, dz)|]. \end{aligned} \tag{204}$$

The first term in the right hand side of (204) is upper bounded by

$$\begin{aligned}&\|\gamma'\|_\infty \mathbb{E}[\int_{t_i}^{t_{i+1}} \int_{|z| \ge 2\Delta_n^\beta} |X_{s-} - X_{t_i}| |z| F(z) dz ds] \\ &\le c \int_{t_i}^{t_{i+1}} \int_{|z| \ge 2\Delta_n^\beta} \mathbb{E}[|X_{s-} - X_{t_i}|^2]^{\frac{1}{2}} ds |z| F(z) dz \\ &\le c \int_{t_i}^{t_{i+1}} \Delta_n^{\frac{1}{2}} (\int_{|z| \ge 2\Delta_n^\beta} |z| F(z) dz) ds \le c\Delta_n^{\frac{3}{2}},\end{aligned} \tag{205}$$

where we have used the compensation formula, the regularity of γ, Cauchy–Schwarz inequality in order to use the third point of Lemma 1 and the boundedness of the integral for $|z| \ge 2\Delta_n^\beta$. Moreover, acting in the same way, the second term in the right hand side of (204) is upper bounded by

$$\begin{aligned}&\|\gamma'\|_\infty \mathbb{E}[\int_{t_i}^{t_{i+1}} \int_{|z| \le 2\Delta_n^\beta} |X_{s-} - X_{t_i}| |z| F(z) dz ds] \\ &\le c \int_{t_i}^{t_{i+1}} \Delta_n^{\frac{1}{2}} (\int_{|z| \ge 2\Delta_n^\beta} |z|^{-\alpha} dz) ds \le c\Delta_n^{\frac{3}{2} + \beta(1-\alpha)},\end{aligned} \tag{206}$$

using again compensation formula, the regularity of γ and Cauchy–Schwarz inequality in order to use the third point of Lemma 1. We have also used the third point of A4 and computed the integral on z. Using (204)–(206) we get

$$\mathbb{E}[|\Delta X_i^J - \Delta \tilde{X}_i^J|] \le c\Delta_n^{\frac{3}{2} \wedge [\frac{3}{2} + \beta(1-\alpha)]} = c\Delta_n^{\frac{3}{2}}, \tag{207}$$

since $\alpha < 1$ and so $(1 - \alpha) > 0$. We now use interpolation theorem (see below Theorem 1.7 in Chapter 4 of [5]) getting

$$\mathbb{E}[|\Delta X_i^J - \Delta \tilde{X}_i^J|^q]^{\frac{1}{q}} \le \mathbb{E}[|\Delta X_i^J - \Delta \tilde{X}_i^J|]^\theta (\mathbb{E}[|\Delta X_i^J - \Delta \tilde{X}_i^J|^2]^{\frac{1}{2}})^{1-\theta},$$

with $\frac{1}{q} = \theta + \frac{1-\theta}{2}$, hence $\theta = \frac{2}{q} - 1$. Using (203) and (207) it follows

$$\mathbb{E}[|\Delta X_i^J - \Delta \tilde{X}_i^J|^q]^{\frac{1}{q}} \leq c\Delta_n^{\frac{3}{2}\theta} \Delta_{n,i}^{1-\theta} = c\Delta_n^{\frac{1}{2}\theta+1} = c\Delta_n^{\frac{1}{q}+\frac{1}{2}},$$

where we have also replaced θ. $\square$

A.5. *Proof of Lemma 9*

Proof. We want to use a conditional version of the interpolation theorem, therefore we have to estimate the norm 2 of $I_2^n(i)$, $I_3^n(i)$ and $I_4^n(i)$. Observing that φ is a bounded function and using Kunita inequality we get

$$\begin{aligned}
\mathbb{E}_i[|I_2^n(i)|^2] &\leq \mathbb{E}_i[|\Delta X_i^J|^4] \leq c\mathbb{E}_i[\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}} |\gamma(X_{s-})|^4 |z|^4 \bar{\mu}(ds, dz)] \\
&\quad + c\mathbb{E}_i[(\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}} |\gamma(X_{s-})|^2 |z|^2 \bar{\mu}(ds, dz))^2] \\
&\leq c(\int_{\mathbb{R}} |z|^4 F(z)dz)\mathbb{E}_i[\int_{t_i}^{t_{i+1}} |\gamma(X_{s-})|^4 ds] \\
&\quad + c\mathbb{E}_i[(\int_{\mathbb{R}} |z|^2 F(z)dz)^2 (\int_{t_i}^{t_{i+1}} |\gamma(X_{s-})|^2 ds)^2] \\
&\leq R_i(\Delta_n) + R_i(\Delta_n^2) = R_i(\Delta_n),
\end{aligned} \tag{208}$$

where in the last inequality we have also used the polynomial growth of γ and the fifth point of Lemma 1.

Concerning the norm 2 of $I_3^n(i)$, we use the conditional version of the first point of Lemma 8 for $q = 2$ to get

$$\mathbb{E}_i[|I_3^n(i)|^2] \leq \mathbb{E}_i[|\Delta X_i^J - \Delta \tilde{X}_i^J|^4] \leq R_i(\Delta_n^2). \tag{209}$$

We now consider $I_4^n(i)$. Using Cauchy–Schwarz inequality and a conditional version of both the first point of Lemma 8 for $q = 2$ and (30) in Lemma 3, where φ acts like the indicator function, we have

$$\mathbb{E}_i[|I_4^n(i)|^2]^{\frac{1}{2}} \leq c\mathbb{E}_i[|\Delta X_i^J - \Delta \tilde{X}_i^J|^4]^{\frac{1}{2}} \mathbb{E}_i[|\Delta \tilde{X}_i^J \varphi_{\Delta_n^\beta}(\Delta \tilde{X}_i^J)|^4]^{\frac{1}{2}} \leq R_i(\Delta_n^{\frac{3}{2}+\frac{\beta}{2}(4-\alpha)}). \tag{210}$$

Using interpolation theorem it follows, $\forall j \in \{2, 3, 4\}$,

$$\mathbb{E}_i[|I_j^n(i)|^{1+\epsilon}]^{\frac{1}{1+\epsilon}} \leq \mathbb{E}_i[|I_j^n(i)|]^\theta (\mathbb{E}_i[|I_j^n(i)|^2]^{\frac{1}{2}})^{1-\theta}, \tag{211}$$

with θ such that $\frac{1}{1+\epsilon} = \theta + \frac{1-\theta}{2}$, hence $\theta = \frac{2}{1+\epsilon} - 1 = 1 - \frac{2\epsilon}{1+\epsilon}$.

From a conditional version of (149), (154), (159) and Eqs. (208) and (211) it follows

$$\begin{aligned}
\mathbb{E}_i[|I_2^n(i)|^{1+\epsilon}]^{\frac{1}{1+\epsilon}} &\leq R_i(\Delta_n^{\frac{3}{2}+\beta-\frac{\alpha\beta}{2}})^\theta R_i(\Delta_n^{\frac{1}{2}})^{1-\theta} = R_i(\Delta_n^{(\frac{3}{2}+\beta-\frac{\alpha\beta}{2})(1-\frac{2\epsilon}{1+\epsilon})+\frac{\epsilon}{1+\epsilon}}) \\
&= R_i(\Delta_{n,i}^{\frac{3}{2}+\beta-\frac{\alpha\beta}{2}-\frac{\epsilon}{1+\epsilon}(2+2\beta-\alpha\beta)}).
\end{aligned} \tag{212}$$

Since $2+2\beta-\alpha\beta$ is always more than zero we can just see the exponent on $\Delta_{n,i}$ as $\frac{3}{2}+\beta-\frac{\alpha\beta}{2}-\epsilon$.

From a conditional version of (161), (209) and (211) it follows

$$\mathbb{E}_i[|I_3^n(i)|^{1+\epsilon}]^{\frac{1}{1+\epsilon}} \leq R_i(\Delta_n^2)^\theta R_i(\Delta_n)^{1-\theta} = R_i(\Delta_n^{1+\theta}) = R_i(\Delta_n^{2-\frac{2\epsilon}{1+\epsilon}}). \tag{213}$$

In the same way, using a conditional version of (163), (210) and (211) it follows

$$\mathbb{E}_i[|I_4^n(i)|^{1+\epsilon}]^{\frac{1}{1+\epsilon}} \leq R_i(\Delta_n^{(\frac{3}{2}+\beta-\frac{\alpha\beta}{2})(1-\frac{2\epsilon}{1+\epsilon})+\frac{2\epsilon}{1+\epsilon}(\frac{3}{2}+2\beta-\frac{\alpha\beta}{2})}) = R_i(\Delta_n^{\frac{3}{2}+\beta-\frac{\alpha\beta}{2}+\frac{2\beta\epsilon}{1+\epsilon}}). \qquad (214)$$

The result (174) is a consequence of (212), (213), (214) and that 2 is always more than $\frac{3}{2}+\beta-\frac{\alpha\beta}{2}$. □

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