

Some theorems on harmonic renewal measures

A.J. Stam

Mathematisch Instituut, Rijksuniversiteit Groningen, P.O. Box 800, 9700 AV Groningen, Netherlands

Received 19 June 1990

Revised 1 October 1990

The harmonic renewal measure ν for the random walk S_n is defined by $\nu(A) = \sum_{n=1}^{\infty} n^{-1} P(S_n \in A)$. The paper gives weak asymptotic relations as $x \rightarrow \infty$ for $\nu([0, x])$ under weak conditions.

random walk * renewal theory * limit theorem * slow variation

1. Introduction

Let X_1, X_2, \dots be i.i.d. random variables with distribution function F , not degenerate at 0, and let $S_0 = 0$, $S_n = X_1 + \dots + X_n$, $n \geq 1$. We put $EX_1 = \mu$ if $E|X_1| < \infty$ and $\text{Var } X_1 = \sigma^2$ if $EX_1^2 < \infty$. The harmonic renewal measure associated with F is defined by

$$\nu(A) = \sum_{k=1}^{\infty} k^{-1} P(S_k \in A), \quad (1.1)$$

which is finite for bounded A , since $P(S_k \in A) \leq Ck^{-1/2}$, where C depends only on A , see Rosén (1962). We have

$$G(x) = \nu((-\infty, x]) = \sum_{k=1}^{\infty} k^{-1} P(S_k \leq x) < \infty \quad (1.2)$$

if and only if the descending ladder of the random walk is defect, see Feller (1971, Chapter XII.7).

When $X_1 \geq 0$ a.s., we put

$$f(s) = \int \exp(-sx) dF(x), \quad (1.3)$$

$$g(s) = \int \exp(-sx) dG(x) = -\log(1 - f(s)), \quad (1.4)$$

$$h(s) = \int_0^{\infty} (1 - F(x)) e^{-sx} dx = s^{-1}(1 - f(s)). \quad (1.5)$$

Harmonic renewal measures were studied by Greenwood, Omeij and Teugels (1982a,b) and by Grübel (1986, 1988). They play a rôle in the fluctuation theory of random walks, as is seen from the well-known distributions of ladder variables and entrance times into $(-\infty, 0)$, see Feller (1971). Other applications are to stable attraction, see Greenwood, Omeij and Teugels (1982b), where harmonic renewal measures on \mathbb{R}^2 are introduced.

Harmonic renewal sequences also occur in the theory of polynomials of binomial type, viz, of those having the renewal property defined in Stam (1987), where the discrete analogue of (1.4) occurs. Harmonic renewal measures are a special case of generalized renewal measures $\rho(A) = \sum_k c_k P(S_k \in A)$. Because of their simple form they have interesting analytic properties. The measure (1.1) is near the boundary between finite and infinite ρ and may indicate properties of those ρ being nearer to this boundary, e.g. when c_k varies regularly of index -1 . Greenwood, Omeij and Teugels (1982a) derive a number of asymptotic results for $G(x)$ and $\int_0^x t dG(t)$ as $x \rightarrow \infty$ by Abelian and Tauberian theorems under the assumption that $X_1 \geq 0$ a.s. Grübel (1986) proves asymptotic results for $G(x)$ and $\nu(A+x)$ as $x \rightarrow \infty$ or $|x| \rightarrow \infty$ by Banach algebra techniques. Our aim is to show that under weaker assumptions weaker versions of these theorems still hold. Theorems are stated in Section 2, and proofs, partly by probabilistic techniques, are given in Section 3.

Let

$$U(A) = \sum_{n=0}^{\infty} P(S_n \in A) \quad (1.6)$$

denote the (ordinary) renewal measure associated with F . When the random walk $\{S_n\}$ is transient, $U(A) < \infty$ for bounded A and then we have for ν with compact support

$$\begin{aligned} \int \nu(x) x d\nu(x) &= \sum_{k=1}^{\infty} k^{-1} E\{S_k \nu(S_k)\} = \sum_{k=1}^{\infty} E\{X_1 \nu(S_k)\} \\ &= \int \nu(x) d(Q * U)(x), \end{aligned}$$

where $*$ denotes convolution and $dQ(x) = x dF(x)$. So the measure R with $dR(x) = x d\nu(x)$ is the convolution of Q and U . When $E|X_1| < \infty$ and $\mu > 0$, it follows that $\mu^{-1}R$ is the (delayed) renewal measure for the random walk with $P(S_0 \in A) = \mu^{-1}Q(A)$ and $P(X_i \leq x) = F(x)$, $i \geq 1$. It follows then that $\nu((x, x+h]) \sim hx^{-1}$ when X_1 is nonarithmetic and $\nu(\{n\}) \sim n^{-1}$ when X_1 is arithmetic with span 1. Here $a(x) \sim b(x)$ means $a(x)/b(x) \rightarrow 1$. Grübel (1988) even shows that in the latter case $\sum_{n=1}^{\infty} |\nu(\{n\}) - n^{-1}| < \infty$. A necessary and sufficient condition for $\nu((x, x+h]) \sim hx^{-1}$ or $\nu(\{n\}) \sim n^{-1}$ does not seem to be known. From (1.4) and (1.5) we have, when $X_1 \geq 0$ a.s.,

$$\int \exp(-sx) x d\nu(x) = -f'(s)/(1-f(s)) = s^{-1} - (h(s) + f'(s))/(1-f(s)).$$

When F has density F' ,

$$h(s) + f'(s) = \int_0^\infty (1 - F(x) - xF'(x)) e^{-sx} dx.$$

We may have $\mu = \infty$, whereas $1 - F(x) - xF'(x)$ is the density of a finite signed measure ρ , e.g. when $F(x) = 1 - x^{-1}$, $x \geq 1$. Then $R = \lambda_+ - \rho * U$, where λ_+ is Lebesgue measure on \mathbb{R}_+ and $U((x, x+h]) \rightarrow 0$ as $x \rightarrow \infty$.

A class of distributions for which $\nu(\{n\}) = n^{-1}$, $n \geq 1$, is given in Example 1.

Euler's constant is denoted by γ . Different constants in a proof will be denoted by C when no confusion is expected.

2. Results

Example 1. If $X_i \in \{\dots, -2, -1, 0, 1\}$ a.s., and $EX_i \geq 0$, then $\nu(\{n\}) = n^{-1}$, $n \geq 1$.

Theorem 1. Let $X_i \geq 0$ a.s. Then

$$G(x) \sim \log x, \quad x \rightarrow \infty, \quad (2.1)$$

if and only if

$$\lim_{x \rightarrow \infty} (\log H(x)) / \log x = 0, \quad (2.2)$$

where

$$H(x) = \int_0^x (1 - F(t)) dt. \quad (2.3)$$

Theorem 2. Let $X_i \geq 0$ a.s. Then

$$\int_0^x t d\nu(t) \sim x, \quad x \rightarrow \infty, \quad (2.4)$$

if and only if $H(x)$ varies slowly as $x \rightarrow \infty$.

Remark 1. It was shown by Greenwood, Omey and Teugels (1982a) that then

$$G(x) - \log x + \log H(x) \rightarrow \gamma.$$

Theorem 3. If $E|X_1| < \infty$ and $\mu > 0$,

$$G(x) - \log x + \log \mu \rightarrow \gamma, \quad x \rightarrow \infty. \quad (2.5)$$

Remark 2. Cf. Theorem 2 in Grübel (1986).

Remark 3. Greenwood, Omeij and Teugels (1982a) proved the following partial converse of Theorem 3. When $X_i \geq 0$ a.s., we have $G(x) - \log x \rightarrow L$ if and only if $EX_1 < \infty$ and then $L = \gamma - \log \mu$. Since

$$G(x) = \sum_{k \leq cx} k^{-1} - \sum_{k \leq cx} k^{-1} P(S_k > x) + \sum_{k > cx} k^{-1} P(S_k \leq x),$$

it follows that if $X_i \geq 0$ a.s., and for some $c > 0$,

$$\sum_{k > cx} k^{-1} P(S_k \leq x) - \sum_{k \leq cx} k^{-1} P(S_k > x) \rightarrow M,$$

as $x \rightarrow \infty$, then $\mu < \infty$ and $c e^M = \mu^{-1}$.

Theorem 4. Let X_1 be nondegenerate, $EX_1^2 < \infty$ and $\mu = 0$. Then as $x \rightarrow \infty$,

$$\nu([0, x]) - \log x \rightarrow -\log \sigma + \gamma + \frac{1}{2} \log 2 + \sum_{k=1}^{\infty} k^{-1} \{P(S_k \geq 0) - \frac{1}{2}\}. \quad (2.6)$$

Remark 4. Cf. Grübel (1986, Theorem 4). There is a printing error: a term $\log 2$ is missing. That the series in the right-hand side of (2.6) converges absolutely, was proved by Rosén (1962).

The theorems in this section do not distinguish between arithmetic and nonarithmetic X_1 . This shows that the approximations contained in them are not sharp.

3. Proofs

Proof of Example 1. Let $\tau_n = \min\{k: S_k = n\} \leq \infty$. It was proved in Kemperman (1961) and Wendel (1975) that

$$P(\tau_n = j) = \frac{n}{j} P(S_j = n). \quad (3.1)$$

If $EX_i \geq 0$ we have $\tau_n < \infty$ a.s., which proves the assertion of the example. When $-\infty \leq EX_i < 0$ the relation (3.1) gives

$$\nu(\{n\}) = n^{-1} P\left(\max_{r \geq 1} S_r \geq n\right). \quad \square \quad (3.2)$$

Proof of Theorem 1. If $\mu < \infty$ both (2.1) and (2.2) are true, see Section 1. So we assume $\mu = \infty$. From (1.4) and (1.5),

$$g(s) = \log s^{-1} - \log h(s).$$

So by Karamata's Abel-Tauber theorem, see Bingham, Goldie and Teugels (1987, Chapter 1.7) the relation (2.1) holds if and only if $(\log s)^{-1} \log h(s) \rightarrow 0$ as $s \downarrow 0$. The theorem now follows from the following lemma. \square

Lemma 1. Let W be the distribution function of a measure on \mathbb{R}_+ with $W(\infty) > 1$, and let w be its Laplace–Stieltjes transform. Then the following relations are equivalent:

$$\lim_{x \rightarrow \infty} (\log W(x))/\log x = 0, \quad (3.3)$$

$$\lim_{s \downarrow 0} (\log w(s))/\log s = 0. \quad (3.4)$$

Proof. We show that (3.3) is equivalent with

$$\lim_{x \rightarrow \infty} x^{-\delta} W(x) = 0, \quad \delta > 0, \quad (3.5)$$

and (3.4) with

$$\lim_{s \downarrow 0} s^{\delta} w(s) = 0, \quad \delta > 0. \quad (3.6)$$

By Karamata's Abel–Tauber theorem, see Bingham, Goldie and Teugels (1987), Theorem 1.7.1 with $c = 0$, the relations (3.5) and (3.6) are equivalent, which proves Theorem 1.

That (3.3) implies (3.5) is easily seen. Now let (3.5) be given. Putting $\gamma(x) = (\log W(x))/\log x$ we have

$$\lim_{x \rightarrow \infty} (\gamma(x) - \delta) \log x = -\infty.$$

Since $W(\infty) > 1$ we have $\gamma(x) \geq 0$ for $x \geq x_1 > 1$, so that we must have $0 \leq \gamma(x) < \delta$ for $x > x(\delta)$. The equivalence of (3.4) and (3.6) is proved similarly. Note that $\log w(s) \geq 0$ for $0 < s < s_1$. \square

Proof of Theorem 2. From (1.4) and (1.5),

$$\int e^{-sx} x \, d\nu(x) = s^{-1} + h'(s)/h(s).$$

Therefore, by Karamata's Abel–Tauber theorem, see Bingham, goldie and Teugels (1987, Chapter 1.7), the relation (2.4) holds if and only if

$$sh'(s)/h(s) \rightarrow 0, \quad s \downarrow 0. \quad (3.7)$$

The relation (3.7) implies for some $\varepsilon(u) \rightarrow 0$, $u \downarrow 0$,

$$\log h(s) = \log h(s_0) - \int_s^{s_0} u^{-1} \varepsilon(u) \, du, \quad 0 < s < s_0,$$

so that $h(s)$ varies slowly as $s \downarrow 0$ by the Karamata representation theorem, see Bingham, Goldie and Teugels (1987, Chapter 1.3). This in turn implies by Karamata's Tauberian theorem that $H(x)$ varies slowly as $x \rightarrow \infty$.

Now let $H(x)$ vary slowly as $x \rightarrow \infty$. Then

$$h(s) \sim H(s^{-1}), \quad s \downarrow 0. \quad (3.8)$$

We have

$$\int_0^x t(1-F(t)) dt = \int_0^x t dH(t) = xH(x) - \int_0^x H(t) dt.$$

Since $\int_0^x H(t) dt \sim xH(x)$, see Bingham, Goldie and Teugels (1987, Chapter 1.5),

$$\int_0^x t(1-F(t)) dt / xH(x) \rightarrow 0, \quad x \rightarrow \infty,$$

and by Karamata's Abel theorem,

$$h'(s)/(s^{-1}H(s^{-1})) \rightarrow 0, \quad s \downarrow 0. \quad (3.9)$$

From (3.8) and (3.9) it follows that (3.7) holds. \square

Proof of Theorem 3. We have

$$G(x) = \sum_{k\mu \leq x} k^{-1} - \sum_{k\mu \leq x} k^{-1}P(S_k > x) + \sum_{k\mu > x} k^{-1}P(S_k \leq x), \quad (3.10)$$

$$\sum_{k\mu \leq x} k^{-1}P(S_k > x) = T_1 + T_2 + T_3, \quad (3.11)$$

where

$$T_1 = \sum_{k=1}^M k^{-1}P(S_k > x) \rightarrow 0, \quad x \rightarrow \infty, \quad (3.12)$$

$$T_2 = \sum_{M < k \leq \mu^{-1}x(1-\varepsilon)} k^{-1}P(S_k - k\mu > x - k\mu), \quad (3.13)$$

$$T_3 = \sum_{x(1-\varepsilon) < k\mu \leq x} k^{-1}P(S_k > x) \leq \frac{\varepsilon x}{\mu} \frac{\mu}{x(1-\varepsilon)} = \frac{\varepsilon}{1-\varepsilon}.$$

To estimate T_2 we note that for all $\delta > 0$,

$$\sum_{k=1}^{\infty} k^{-1}P\{|S_k - k\mu| > k\delta\} < \infty, \quad (3.14)$$

see Chow and Teicher (1978, Chapter 5.2). In T_2 we have $x \geq (1-\varepsilon)^{-1}k\mu$, so $x - k\mu \geq \varepsilon(1-\varepsilon)^{-1}k\mu$, so

$$T_2 \leq \sum_{k=M}^{\infty} k^{-1}P(S_k - k\mu > k\mu\varepsilon). \quad (3.15)$$

From (3.11)–(3.15),

$$\limsup_{x \rightarrow \infty} \sum_{k\mu \leq x} k^{-1}P(S_k > x) \leq \varepsilon(1-\varepsilon)^{-1} + \sum_{k=M}^{\infty} k^{-1}P(S_k - k\mu > k\mu\varepsilon).$$

By letting first $M \rightarrow \infty$ and then $\varepsilon \downarrow 0$ we see with (3.14),

$$\lim_{x \rightarrow \infty} \sum_{k\mu \leq x} k^{-1}P(S_k > x) = 0. \quad (3.16)$$

We have

$$\sum_{k\mu > x} k^{-1} P(S_k \leq x) = T_4 + T_5, \quad (3.17)$$

$$T_4 = \sum_{x < k\mu < x(1+\varepsilon)} k^{-1} P(S_k \leq x) \leq \frac{\varepsilon x}{\mu} \frac{\mu}{x} = \varepsilon, \quad (3.18)$$

$$T_5 = \sum_{k\mu \geq x(1+\varepsilon)} k^{-1} P(S_k - k\mu \leq x - k\mu).$$

In T_5 we have $x \leq (1+\varepsilon)^{-1}k\mu$, $x - k\mu \leq -\varepsilon(1+\varepsilon)^{-1}k\mu$. So with (3.14) we have $T_5 \rightarrow 0$ as $x \rightarrow \infty$ and then from (3.17) and (3.18) by letting first $x \rightarrow \infty$ and then $\varepsilon \downarrow 0$,

$$\lim_{x \rightarrow \infty} \sum_{k\mu > x} k^{-1} P(S_k \leq x) = 0. \quad (3.19)$$

The theorem now follows from (3.10), (3.16) and (3.19). \square

Proof of Theorem 4. Let Y_1, Y_2, \dots be i.i.d. with $EY_i = 0$ and $\text{Var } Y_i = \sigma^2$. Put $Z_n = Y_1 + \dots + Y_n$ and consider

$$\begin{aligned} D(x) &= \sum_{k=1}^{\infty} k^{-1} P(0 \leq S_k \leq x) - \sum_{k=1}^{\infty} k^{-1} P(0 \leq Z_k \leq x) = T_1 + T_2 + T_3 + T_4, \\ T_1 &= \sum_{k \leq ax^2} k^{-1} \{P(S_k \geq 0) - P(Z_k \geq 0)\}, \\ T_2 &= \sum_{k \leq ax^2} k^{-1} \{P(Z_k > x) - P(S_k > x)\}, \\ T_3 &= \sum_{ax^2 < k < bx^2} k^{-1} \{P(0 \leq S_k \leq x) - P(0 \leq Z_k \leq x)\}, \\ T_4 &= \sum_{k \geq bx^2} k^{-1} \{P(0 \leq S_k \leq x) - P(0 \leq Z_k \leq x)\}. \end{aligned} \quad (3.20)$$

We have

$$\lim_{x \rightarrow \infty} T_1 = \sum_{k=1}^{\infty} k^{-1} \{P(S_k \geq 0) - P(Z_k \geq 0)\}, \quad (3.21)$$

where the series converges absolutely by Remark 4. With Chebychev's inequality,

$$|T_2| \leq \sum_{k \leq ax^2} 2\sigma^2 x^{-2} \leq 2a\sigma^2. \quad (3.22)$$

By the central limit theorem,

$$|T_3| \leq \sum_{ax^2 < k < bx^2} k^{-1} \varepsilon(k),$$

where $\varepsilon(k) \rightarrow 0$ as $k \rightarrow \infty$ so that

$$\lim_{x \rightarrow \infty} T_3 = 0. \quad (3.23)$$

Since $P(y \leq S_k \leq y+1) \leq Ck^{-1/2}$, uniformly in y , see Rosén (1962), and similarly for Z_k , we have for $x \geq 1$,

$$\begin{aligned} |T_4| &\leq Cx \sum_{k \geq bx^2} k^{-3/2} \leq Cb^{-1/2}, \\ \limsup_{x \rightarrow \infty} |T_4| &\leq Cb^{-1/2}. \end{aligned} \quad (3.24)$$

From (3.20), (3.22), (3.23) and (3.24),

$$\limsup_{x \rightarrow \infty} |D(x) - T_1| \leq 2a\sigma^2 + 2Cb^{-1/2},$$

so that with (3.21), by letting $a \rightarrow 0$ and $b \rightarrow \infty$,

$$\lim_{x \rightarrow \infty} D(x) = \sum_{k=1}^{\infty} k^{-1} \{P(S_k \geq 0) - P(Z_k \geq 0)\}. \quad (3.25)$$

We now take the Y_i to have probability density $\frac{1}{2}\lambda \exp(-\lambda|y|)$, $y \in \mathbb{R}$, with λ so that $\text{Var } Y_i = 2\lambda^{-2} = \sigma^2$. From Example 2 in Grübel (1986) we see that

$$\begin{aligned} \sum_{k=1}^{\infty} k^{-1} P(0 \leq Z_k \leq x) &= \int_0^x t^{-1} (1 - e^{-\lambda t}) dt \\ &= \int_0^{\lambda x} y^{-1} \left\{ \int_0^y e^{-u} du \right\} dy \\ &= \int_0^{\lambda x} e^{-u} (\log \lambda x - \log u) du \\ &= (1 - e^{-\lambda x}) \log \lambda x + \gamma + \int_{\lambda x}^{\infty} e^{-u} \log u du \\ &= \log x - \log \sigma + \frac{1}{2} \log 2 + \gamma + \varepsilon(x), \end{aligned}$$

with $\varepsilon(x) \rightarrow 0$ as $x \rightarrow \infty$, and (2.6) follows from (3.20) and (3.25), since $P(Z_k \geq 0) = \frac{1}{2}$. \square

Acknowledgment

The author thanks a referee for remarks leading to simplification of some proofs.

References

- N.H. Bingham, C.M. Goldie and J.L. Teugels, *Regular Variation* (Cambridge Univ. Press, Cambridge, 1987).
- Y.S. Chow and H. Teicher, *Probability Theory, Independence, Interchangeability, Martingales* (Springer, New York, 1978).

- W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. II (Wiley, New York, 1971, 2nd ed.).
- P. Greenwood, E. Omev and J.L. Teugels, Harmonic renewal measures, *Z. Wahrsch. Verw. Gebiete* 59 (1982a) 391–409.
- P. Greenwood, E. Omev and J.L. Teugels, Harmonic renewal measures and bivariate domains of attraction in fluctuation theory, *Z. Wahrsch. Verw. Gebiete* 61 (1982b) 527–539.
- R. Grübel, On harmonic renewal measures, *Probab. Theory Rel. Fields* 71 (1986) 393–404.
- R. Grübel, Harmonic renewal sequences and the first positive sum, *J. London Math. Soc. (2)* 38 (1988) 179–192.
- J.H.B. Kemperman, *The Passage Problem for a Stationary Markov Chain* (Univ. of Chicago Press, Chicago, IL, 1961).
- B. Rosén, On the asymptotic distribution of sums of independent identically distributed random variables, *Ark. Mat.* 4 (1962) 323–332.
- A.J. Stam, Polynomials of binomial type and renewal sequences, *Studies Appl. Math.* 77 (1987) 183–193.
- J.G. Wendel, Left-continuous random walk and the Lagrange expression, *Amer. Math. Monthly* 82 (1975) 494–499.