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Large deviations in the van der Waals limit[1]

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Abstract

In this paper we extend the analysis in Benois et al. (Markov Process. Rel. Fields (1997) 175–198) by proving a strong large deviation principle for the empirical distribution of Ising spins in $d \geqslant 2$ dimensions when the interaction is determined by a Kac potential and the temperature is below the critical value. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Large deviations is the natural setup for studying structure and geometry of interfaces. At a phase transition the cost (i.e. the logarithm of the Gibbs probability) of a deviation from equilibrium in a region inside the system may be "only" proportional to its surface and not to its volume, as customary when there is no phase transition, because the deviation may just involve a change of the phase in that region. The process then looks atypical only in a neighborhood of the interface which thus characterizes the deviation and the rate function of the deviation, namely its cost, is proportional to the surface. Often in the applications the interface appears only implicitly in the problem through a constraint imposed on the state of the system. In the classical Wulff problem, for instance, the constraint consists in fixing the average magnetization m. Suppose that at the inverse temperature β there are just two phases symmetric under spin flip and call $\pm m_\beta$ the corresponding values of the magnetization, then, if the average magnetization $m \in (-m_\beta, m_\beta)$, both phases must be present and the Wulff problem is to determine the shape of the interface separating the two phases. If a strong large deviation principle (LDP) holds, a constraint problem for the spin system (for a suitable class of constraints) can be reduced to a variational problem with the rate function giving the cost of the interface.

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The $(d=2)$-dimensional, nearest-neighbor ferromagnetic Ising system is the most remarkable example where all this has been developed, see Dobrushin et al. (1992), Ioffe (1994), Ioffe (1995), Pfister (1991) Pfister and Velenik (1996). Unfortunately, not many other models have been worked out so thoroughly. Here we study and solve the problem under a simplifying feature, namely, we consider the ferromagnetic Ising system with Kac potentials in $d\geqslant 2$ dimensions. As proposed by Kac et al. (1963) and Lebowitz and Penrose (1966), we study the system by taking first the thermodynamic limit $(L\to\infty)$ and then the limit $\gamma\to 0$, where $\gamma>0$ is the scaling parameter of the Kac potential (Kac parameter). In Kac et al. (1963) and Lebowitz and Penrose (1966) this procedure is applied to the analysis of the free energy, yielding a rigorous derivation of the van der Waals theory. In Benois et al. (1997) the analysis is extended to the study of interfaces and a weak LDP is proved which shows that the rate function is the perimeter of the interface times the van der Waals surface tension. Here we prove a strong LDP which, as mentioned earlier, allows to characterize the optimal shape of the interface under a general class of constraints.

The order of the limits is very important, we emphasize that the scaling limit $\gamma\to 0$ is done here after the thermodynamic limit $L\to\infty$; the simultaneous limit with L and γ suitably related has been examined earlier in Alberti et al. (1996), Bellettini et al. (1996) and Alberti and Bellettini (1996) where it is solved together with the proof that the non-local van der Waals excess free energy functional Γ-converges to the perimeter functional (times the van der Waals surface tension). Our analysis is intermediate between this case and the other one with only $L\to\infty$ and $\gamma>0$ maybe very small but fixed, like in Cassandro and Presutti (1996), Bovier and Zahradnik (1996), Bodineau and Presutti (1996) and Buttà et al. (1997) where the goal was to prove phase transitions at fixed $\gamma>0$. Unfortunately, our techniques do not allow to extend the analysis to the large deviations at fixed $\gamma>0$ and we can only hope they may provide a step forward in this direction.

2. Basic notation and main results

We use the same notation as in Benois et al. (1997) that we recall briefly here for the reader's convenience.

2.1. Microscopic, mesoscopic and macroscopic representations of the system

We consider in this paper the Ising spin system with configuration space $\{-1,1\}^{\mathbb{Z}^d}$, $d\geqslant 2$, its elements being denoted by $\sigma=\{\sigma(i),\, i\in\mathbb{Z}^d\}$, $\sigma(i)$ the spin at the site i. As the spin configurations σ give a complete description of the state of the system we will refer to this as to the "microscopic representation" of the system. We will actually restrict to tori Λ of $\mathbb{Z}^d$ of side $L=2^n$, $n\in\mathbb{N}$, and use the following notation: for any subset Δ of $\mathbb{Z}^d$, $\sigma_\Delta\in\{-1,1\}^\Delta$ denotes the restriction of σ to Δ.

The macroscopic state of the system is instead determined by an order parameter which specifies the phase of the system (we will be working at a fixed temperature for which there are just two pure equilibrium phases, i.e. two extremal, translationally invariant Gibbs states, see below). It is convenient to choose the order parameter u in such a way that, at the two equilibrium phases, u has the values ± 1. The two pure phases are then represented by the two functions $u(r)$ constantly equal to 1 and to -1. We will suppose that the macroscopic region where our system is confined is the unit torus $\mathcal{T}$ in $\mathbb{R}^d$ with center the origin. Then $r \in \mathcal{T}$ and $u(r) = 1$ means that at r there is the phase $+1$. Our goal is to investigate the structure of macroscopic states $u(\cdot)$ where the order parameter takes both the value $+1$ and -1, but we will also consider states where it takes non-equilibrium values (not in $\{\pm 1\}$). As we will see, these states are much less probable than the others. Thus, the order parameter ranges in some interval $[-A, A]$, $A > 1$ and the macroscopic configurations are elements of

$$\mathcal{X} = L^1(\mathcal{T}; [-A, A]) \tag{2.1}$$

with $\|u\|$ denoting the $L^1(\mathcal{T})$ norm of u. The L^1 norm reflects the choice that two macroscopic configurations will be considered close to each other if their difference is small except possibly for a small fraction of the volume. The macroscopic observables are then elements of $C(\mathcal{X})$.

The order parameter as a function of the spin configurations will be defined later via a limit procedure which involves empirical averages. To this end, it is convenient to represent Ising configurations as functions on $\mathbb{R}^d$. Let $\hat{e} \in \mathbb{R}^d$ be the point with coordinates all equal to $\frac{1}{2}$ and $\mathcal{D}$ the partition into unit cubes C with centers the points $i + \hat{e}$, $i \in \mathbb{Z}^d$. A face in common to two cubes is attributed to the one with the largest center, so that the cube with center $i + \hat{e}$ contains i. We also use the notation $C(r)$ for the cube of $\mathcal{D}$ which contains r. Finally, $\mathcal{D}^{(\ell)}$, $\ell \in \{2^n, n \in \mathbb{Z}\}$, denotes the partition into cubes $C^{(\ell)}$ of side ℓ obtained by scaling $\mathcal{D}$ by ℓ and given a bounded function f on $\mathbb{R}^d$ we define the empirical averages (coarse graining) of f as

$$f^{(\ell)}(r) = \frac{1}{\ell^d} \int_{C^{(\ell)}(r)} \mathrm{d}r' f(r'). \tag{2.2}$$

The macroscopic region corresponding to the tori Λ of side L is always the unit torus $\mathcal{T}$. The spin configurations are then represented by functions $s \in L^\infty(\mathcal{T}; \{\pm 1\})$ that are $\mathcal{D}^{(1/L)}$-measurable, i.e. constant on the cubes $C^{(1/L)}$ of $\mathcal{D}^{(1/L)}$. The relation with the microscopic representation is then given by

$$s(r) = \sigma(i), \quad Lr \in C(i), \tag{2.3}$$

where $C(i)$ is the cube of $\mathcal{D}$ that contains i. In this way, the thermodynamic limit $L \to \infty$ is represented as a continuum limit with the mesh $1/L$ of the coarse graining going to 0.

In many instances it is convenient to work on an intermediate scale, the mesoscopic scale, whose units are chosen so that the range of the interaction becomes 1. As we will see, in microscopic units the range is γ^{-1}, where γ, the Kac parameter, takes values in $\{2^{-n}\}$, we will always restrict to the case $L_\gamma := \gamma L > 1$. The mesoscopic space is then the torus $L_\gamma \mathcal{T}$ of $\mathbb{R}^d$ and the mesoscopic spin configurations are the functions $S \in L^\infty(L_\gamma \mathcal{T}; \{\pm 1\})$ which are $\mathcal{D}^{(1/L_\gamma)}$-measurable, so that

$$S(x) = s(L_\gamma^{-1} x), \quad L_\gamma = \gamma L, \qquad S(x) = \sigma(i), \quad \gamma^{-1} x \in C(i). \tag{2.4}$$

To distinguish the points in the various spaces we write (when possible) r for macroscopic, x for mesoscopic and i for microscopic.

2.2. Kac interaction and Gibbs measures

For any $\gamma > 0$ and any bounded set Λ in $\mathbb{Z}^d$, we define the energy of σ_Λ in interaction with σ_{Λ^c} as

$$H_\gamma(\sigma_\Lambda|\sigma_{\Lambda^c}) = -\frac{1}{2}\sum_{i\neq j\in\Lambda} J_\gamma(i,j)\sigma(i)\sigma(j) - \sum_{i\in\Lambda, j\in\Lambda^c} J_\gamma(i,j)\sigma(i)\sigma(j), \tag{2.5}$$

where

$$J_\gamma(i,j) := \gamma^d J(\gamma|i-j|), \qquad \forall i,j\in\mathbb{Z}^d \tag{2.6}$$

and J is a non-negative, smooth function supported by $[0,1]$ and normalized so that

$$\int_{\mathbb{R}^d} \mathrm{d}r\, J(|r|) = 1. \tag{2.7}$$

The conditional Gibbs probability of σ_Λ given σ_{Λ^c} is

$$\mu_{\gamma,\Lambda}(\sigma_\Lambda|\sigma_{\Lambda^c}) = Z_{\gamma,\Lambda}(\sigma_{\Lambda^c})^{-1}\exp[-\beta H_\gamma(\sigma_\Lambda|\sigma_{\Lambda^c})], \tag{2.8}$$

where $Z_{\gamma,\Lambda}(\sigma_{\Lambda^c})$ being the partition function. The Gibbs measure on the torus Λ of side L will be denoted by $\mu_{\gamma,L}$.

The infinite volume Gibbs measures μ_γ are the probabilities on $\{-1,1\}^{\mathbb{Z}^d}$ whose conditional probabilities satisfy Eq. (2.8). In Cassandro and Presutti (1996), and Bovier and Zahradnik (1996) it is shown that if $\beta > 1$ there is $\gamma_\beta > 0$ so that for all $\gamma \leqslant \gamma_\beta$ there are two distinct, translationally invariant Gibbs states $\mu_\gamma^\pm$, limits of the finite volume Gibbs states with all $+1$ and, respectively, all -1 boundary conditions. In Buttà et al. (1996) it is shown that these are the only extremal, translationally invariant Gibbs states. Moreover, their magnetizations, $\pm m_{\beta,\gamma}$, converge when $\gamma\to 0^+$ to $\pm m_\beta$, where m_β is the positive root of

$$m_\beta = \tanh\{\beta m_\beta\}. \tag{2.9}$$

2.3. Large deviations

Our order parameter is the ratio of the magnetization density with its equilibrium value $m_{\beta,\gamma}$, and since the absolute value of the magnetization density cannot exceed 1, we take A in Eq. (2.1) so that $A > m_{\beta,\gamma}^{-1}$, for all $\gamma\leqslant\gamma_\beta$. We will define the order parameter as a function of the spin configurations by a limit procedure. Starting from a spin configuration σ, we first go to its macroscopic representation s and then, recalling the definition (2.2) of the empirical averages, we take as an approximation for the order parameter the (normalized) coarse grained configurations $s^{(\varepsilon)}/m_{\beta,\gamma}$. Our limit procedure is to first take the thermodynamic limit $L\to\infty$, then $\varepsilon\to 0$ and eventually $\gamma\to 0$ (it would be much nicer if we could avoid the last limit and keep $\gamma > 0$ fixed). In Theorem 1.2 of Benois et al. (1997) it is proved that for all γ small enough

$$\lim_{L\to\infty} \mu_{\gamma,L}(\|s^{(\varepsilon)}/m_{\beta,\gamma} \mp 1\| \leqslant \delta) = \tfrac{1}{2}$$

for all $\delta > 0$ and all $\varepsilon > 0$. In the thermodynamic limit therefore the probability concentrates on the two pure phases where the order parameter is constantly equal to 1 or to -1. Regarding the coarse grained configurations $s^{(\varepsilon)}/m_{\beta,\gamma}$ as elements of $\mathscr{X}$, see Eq. (2.1), we will prove in the next theorem a LDP in $\mathscr{X}$ for $s^{(\varepsilon)}/m_{\beta,\gamma}$. However, as the LDP holds unchanged for $s^{(\varepsilon)}/m_{\beta}$, we will rather state it for the latter, for notational simplicity.

The rate function in the LDP is the following one. Let $K = BV(\mathscr{T}; \{\pm 1\})$ and denote by $P(u)$, $u \in K$, the perimeter of the set $\{u = 1\}$ and by $\tau_\beta > 0$ the van der Waals surface tension, see Eq. (1.20) in Benois et al. (1997), we define the functional $\mathscr{I}$ on $\mathscr{X}$ as

$$\mathscr{I}(v) = \begin{cases} \tau_\beta P(v) & \text{if } v \in K, \\ +\infty & \text{else.} \end{cases} \tag{2.10}$$

Notice that $\mathscr{I}$ is a good rate function in the sense that it is lower semi-continuous and its level sets are compacts, as the sets

$$K_a = \{u \in K: P(u) \leqslant a\} \tag{2.11}$$

are compact in $\mathscr{X}$, see Dal Maso (1993).

Now, we can state the main result of the paper, that is a strong LDP for $s^{(\varepsilon)}/m_\beta$.

Theorem 2.1. *For any closed subset F of $\mathscr{X}$,*

$$\limsup_{\gamma\to 0} \limsup_{\varepsilon\to 0} \limsup_{L\to\infty} \frac{\gamma}{\beta L^{d-1}} \log \mu_{\gamma,L}(m_\beta^{-1} s^{(\varepsilon)} \in F) \leqslant -\inf_{u\in F} \mathscr{I}(u) \tag{2.12}$$

and for any open subset G of $\mathscr{X}$,

$$\liminf_{\gamma\to 0} \liminf_{\varepsilon\to 0} \liminf_{L\to\infty} \frac{\gamma}{\beta L^{d-1}} \log \mu_{\gamma,L}(m_\beta^{-1} s^{(\varepsilon)} \in G) \geqslant -\inf_{u\in G} \mathscr{I}(u). \tag{2.13}$$

Recalling that in our scheme the observables are elements f of $C(\mathscr{X})$ the physically most interesting questions concern the events

$$\{u \in \mathscr{X}: |f(u) - c| < \delta\},$$

$\delta > 0$, namely the probability that a measurement of f gives the value c with tolerance δ. By Theorem 2.1, using the lower semi-continuity and compactness of the rate function $\mathscr{I}(\cdot)$ we have, calling $g = |f - c|$,

$$\lim_{\delta\to 0} \lim_{\gamma\to 0} \lim_{\varepsilon\to 0} \lim_{L\to\infty} \frac{\gamma}{\beta L^{d-1}} \log \mu_{\gamma,L}(g(m_\beta^{-1} s^{(\varepsilon)}) < \delta) = -\inf_{g(u)=0} \mathscr{I}(u) \tag{2.14}$$

(this is a shorthand for the statement that the right-hand side is the limit both with all lim sup and all lim inf on the left-hand side). Eq. (2.14) thus states that the probability of having $g = 0$ is reduced to the variational problem about the minimizer of the rate function under the constraint $\{g=0\}$. Our proofs actually show that we can interchange the limits $\delta\to 0$ and $\gamma\to 0$, provided we change the normalization writing $m_{\beta,\gamma}^{-1}s^{(\varepsilon)}$ instead of $m_\beta^{-1}s^{(\varepsilon)}$. The special case where $g(v)=\|u-v\|$, $u\in K\equiv BV(\mathscr{T};\{\pm 1\})$, had already been worked out in Benois et al. (1997). The case

$$g(v)=\left|\int_{\mathscr{T}} \mathrm{d}r\, v(r)-s\right|, \qquad |s|<m_\beta$$

corresponds to the Wulff problem.

The lower bound (2.13) is a straight consequence of the weak LDP proved in Benois et al. (1997), its proof will be omitted. The upper bound follows from a proof that the coarse grained magnetization $s^{(\varepsilon)}$ is exponentially supported by neighborhoods of the compact sets K_a, see Proposition 3.1. This is the main technical point in the paper that will be proved in the next section, using contours and Peierls estimates. The upper bound (2.12) is proved in Section 4 using the exponential tightness and the upper bound of the weak LDP established in Benois et al. (1997), the proof is classical and it is reported for the sake of completeness.

3. Exponential tightness

For any set $A\subset L^1(\mathscr{T})$ and any $\delta>0$ we denote by A^δ the δ-neighborhood of A in the L^1-norm, that is

$$A^\delta=\left\{u\in L^1(\mathscr{T}):\ \inf_{v\in A}\|u-v\|\leqslant\delta\right\}. \tag{3.1}$$

In this section we will prove “weak exponential tightness” in the sense that:

Proposition 3.1. *There is a constant $c>0$ such that for any $a>0$ and $\delta>0$*

$$\limsup_{\gamma\to 0}\limsup_{\varepsilon\to 0}\limsup_{L\to\infty}\frac{\gamma}{\beta L^{d-1}}\log\mu_{\gamma,L}(m_\beta^{-1}s^{(\varepsilon)}\notin K_a^\delta)\leqslant -ca. \tag{3.2}$$

Outline of the proof. After recalling from Benois et al. (1997) the basic definitions of the block spin configurations η and of the corresponding contours Γ, we will use these notions to construct ± 1 valued, random variables $T(x)$, $x\in L_\gamma\mathscr{T}$, with the property that with large probability for a large $P(T)\leqslant aL_\gamma^{d-1}$ ($P(T)$ the perimeter of the boundary of the set $\{T=+1\}$). T will be obtained from η by “erasing the small contours” and by putting $T=\pm 1$ in the “large contours” in some careful way that will be specified below. In Lemma 3.2 we will then show that $P(T)\leqslant aL_\gamma^{d-1}$ with large probability for a large and in Lemma 3.3 that η is super-exponentially close in L^1-norm to T. With these ingredients we will then prove Proposition 3.1 at the end of the section.

3.1. Block spins and contours

Given k and h in $\mathbb{N}$, $\zeta > 0$, we define the block spin $\eta \in L^\infty(L_\gamma \mathscr{T}; \{0, \pm 1\})$ as a function of the coarse grained configuration $S^{(2^{-k})} \in L^\infty(L_\gamma \mathscr{T}; [-1, 1])$

$$\eta(x) = \begin{cases} \pm 1 & \text{if } |S^{(2^{-k})}(y) \mp m_\beta| < \zeta \quad \text{for all } y \in C^{(2^h)}(x), \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

We also define the block spin η induced by a function $m \in L^\infty(\Delta; [-1, 1])$ using the analogous of Eq. (3.3)

The point x is called correct, or, equivalently, $\eta(x)$ is correct, if $\eta(x) \neq 0$ and $\eta(y) = \eta(x)$ on the cubes $C^{(2^h)}$ that are $\star$-connected to $C^{(2^h)}(x)$. x is incorrect if it is not correct.

Each maximal $\star$-connected component of the incorrect set is the support of a contour, the contour Γ is defined by its support and by the values of the block spins on its support. When there is no risk of confusion, we may denote by Γ only its support. We denote by $\#\Gamma$ the number of block cubes $C^{(2^h)}$ in the spatial support of Γ and by $|\Gamma|$ its length ($|\Gamma| = 2^{hd}\#\Gamma$). $\mathrm{Ext}(\Gamma)$ is the largest connected component of Γ^{c} and $\mathrm{Int}(\Gamma) = \mathrm{Ext}(\Gamma)^{\mathrm{c}}$; finally, $\mathrm{vol}(\Gamma)$ denotes the number of block cubes $C^{(2^h)}$ in $\mathrm{Int}(\Gamma)$. If Γ is a contour produced by a spin configuration σ, we write $\sigma \Rightarrow \Gamma$ and we say that $\{\Gamma_1, \ldots, \Gamma_k\}$ is a collection of compatible contours if there is a spin configuration which produces all of them. In the same way, we write $m \Rightarrow \Gamma$ when the block spin η is induced by $m \in L^\infty(\Delta; [-1, 1])$.

3.2. Non local excess free energy functional, Peierls estimates

Let Λ be a $\mathscr{D}^{(1)}$ measurable set in $\mathbb{R}^d$ (or in a torus) and $m \in L^\infty(\Lambda, [-1, 1])$. The excess free energy $\mathscr{F}_\Lambda(m)$ of m in Λ is given by

$$\mathscr{F}_\Lambda(m) = \int_\Lambda \mathrm{d}x\, [f(m(x)) - f(m_\beta)] + \frac{1}{4} \int\!\!\int_{\Lambda \times \Lambda} \mathrm{d}x\, \mathrm{d}y\, J(|x - y|)[m(x) - m(y)]^2, \tag{3.4}$$

where

$$f(m) = -\frac{m^2}{2} - \beta^{-1} i(m), \tag{3.5}$$

$$i(m) = -\frac{1-m}{2} \log \frac{1-m}{2} - \frac{1+m}{2} \log \frac{1+m}{2}. \tag{3.6}$$

The functional, that already appears in Lebowitz and Penrose (1966) has a clear interpretation in terms of free energy. The term $i(m)$ has the meaning of entropy (Eq. (3.6) is the entropy of a ± 1 Bernoulli scheme with average m); the double product obtained by expanding the square in Eq. (3.4) clearly corresponds to the interaction energy in Eq. (2.5); the squares $m(x)^2 + m(y)^2$ arising from the same term simplify with the contribution to the first integral in Eq. (3.4) obtained by taking for $f(m(x))$ the first term on the r.h.s. of Eq. (3.5). In the case we consider where $\beta>1$, f is a double well function with minima $\pm m_\beta$ and the minimizer of $\mathscr{F}_\Lambda(m)$ is the function constantly equal to m_β (or to $-m_\beta$). Consequently, the minimum of $\mathscr{F}_\Lambda$ is equal to 0 and $f(m_\beta)$ is the free energy density (after the Lebowitz–Penrose limit $\gamma\to 0$).

$\mathscr{F}_\Lambda(m)$ measures the excess free energy of m with respect to the equilibrium value of the magnetization and is therefore related to the rate function, in the scaling limit $\gamma\to 0^+$. In Lemma 6.5 of Benois et al. (1997) it is shown that if $\underline{\Gamma}=\{\Gamma_1,\dots,\Gamma_k\}$ is a collection of compatible contours, then

$$\mu_{\gamma,L}(\sigma\Rightarrow\underline{\Gamma})\leqslant \exp\left[-\beta\gamma^{-d}\sum_{i=1}^{\kappa}\left(\inf_{m\Rightarrow\Gamma_i}\mathscr{F}_{\Gamma_i}(m)-o_\gamma(1)|\Gamma_i|\right)\right], \tag{3.7}$$

where $o_\gamma(1)$ vanishes with γ. Moreover, by Theorem 6.2 of Benois et al. (1997), there is a constant $\alpha>0$ (α depends on ζ and k and can be chosen as $\alpha=c\zeta^2 2^{-kd}$) such that for any contour Γ

$$\inf_{m\Rightarrow\Gamma}\mathscr{F}_\Gamma(m)\geqslant\alpha\ \#\Gamma, \tag{3.8}$$

where $\#\Gamma$ was defined as the number of cubes of size 2^h in Γ.

Therefore,

$$\log\mu_{\gamma,L}(\sigma\Rightarrow\underline{\Gamma})\leqslant-\beta\gamma^{-d}\sum_{i=1}^{\kappa}\left(\frac{1}{2}\inf_{m\Rightarrow\Gamma_i}\mathscr{F}_{\Gamma_i}(m)+\left(\alpha/2-o_\gamma(1)2^{hd}\right)\#\Gamma_i\right). \tag{3.9}$$

We fix $\zeta'>0$ and for $\Lambda\subset L_\gamma\mathscr{T}$, $m\in L^\infty(\Lambda;[-1,1])$ we consider

$$\Phi_m(x)=\begin{cases}-1 & \text{if } S^{(1)}(x)\leqslant-m_\beta+\zeta',\\ 1 & \text{otherwise.}\end{cases} \tag{3.10}$$

We will prove in the appendix that there is $c>0$ depending only on ζ', such that

$$N^{\pm}(m)\leqslant c\mathscr{F}_\Lambda(m), \tag{3.11}$$

where $N^{\pm}(m)$ is the number of pairs of cubes C in Λ which are connected and where Φ_m has opposite signs.

3.3. *The set of small contours and the random variable $T(x)$*

We denote by Ω^b, $b\in(0,1/d)$, the set of all the contours with length less than L_γ^b and we define $T(x)$, $x\in L_\gamma\mathscr{T}$, as follows. If x belongs to Int(Γ), where $\Gamma\in\Omega^b$ is maximal in Ω^b (it is not contained in the interior of any other contour of Ω^b), then $T(x)=\pm 1$ according to the sign of the cubes in Ext(Γ) $\star$-connected to the boundary of Γ. If x is not in a contour, we set $T(x)=\eta(x)$ and finally, if x belongs to a contour $\Gamma\notin\Omega^b$, we consider a minimizer $m^\star$ of $\inf_{m\Rightarrow\Gamma}\mathscr{F}_\Gamma(m)$ and put $T(x)=\Phi_{m^\star}(x)$. We also define $t(r)=T(L_\gamma r)$.

Lemma 3.2. *There is a constant $c>0$ such that for any $a>0$*

$$\limsup_{\gamma\to 0}\limsup_{L\to\infty}\frac{\gamma}{\beta L^{d-1}}\log\mu_{\gamma,L}(P(T)>aL_\gamma^{d-1})\leqslant -ca. \tag{3.12}$$

Proof. Let $\underline{\Gamma}=\{\Gamma_1,\dots,\Gamma_\kappa\}$ be the collection of long contours produced by a spin configuration. We first remark from the definition of the variable T that we can bound its perimeter $P(T)$ proportionally to $\sum_{i=1}^{\kappa}N_i^{\pm}$. $N_i^{\pm}$ is the number of couples of connected cubes $C^{(1)}$ in the support of Γ_i where $\Phi_{m_i^\star}$ has opposite signs, $m_i^\star$ being the minimizer of $\inf_{m\Rightarrow\Gamma_i}\mathscr{F}_{\Gamma_i}(m)$. So using Eq. (3.11), there is a constant $c'>0$ (depending only on ζ') such that

$$\mu_{\gamma,L}(P(T)>aL_\gamma^{d-1})\leqslant\sum_{\underline{\Gamma}\in\mathscr{G}_{c'a}}\mu_{\gamma,L}\ (\sigma\Rightarrow\underline{\Gamma}), \tag{3.13}$$

where $\mathscr{G}_{c'a}$ is the set of all the collections of compatible contours $\underline{\Gamma}=\{\Gamma_1,\dots,\Gamma_\kappa\}$ such that $\Gamma_i\notin\Omega^b$, $\sum_i\mathscr{F}_{\Gamma_i}(m_i^\star)\geqslant c'aL_\gamma^{d-1}$. Notice that $\kappa\leqslant L_\gamma^{d-b}$ since the total length of the contours cannot exceed L_γ^d. Then applying Eq. (3.9) for $\underline{\Gamma}\in\mathscr{G}_{c'a}$

$$\log\mu_{\gamma,L}(\sigma\Rightarrow\underline{\Gamma})\leqslant-\beta\gamma^{-d}\left[c'aL_\gamma^{d-1}/2+(c\zeta^2 2^{-kd}-o_\gamma(1)2^{hd})\sum_{i=1}^{\kappa}\#\Gamma_i\right]. \tag{3.14}$$

Thus, for γ small enough the r.h.s. of Eq. (3.13) is bounded above by

$$\exp(-c'a\beta\gamma^{-d}L_\gamma^{d-1}/2)\left[1+\sum_{\Gamma:|\Gamma|\geqslant L_\gamma^b}\exp(-c\beta\gamma^{-d}\zeta^2 2^{-(k+h)d}|\Gamma|/2)\right]^{L_\gamma^{d-b}}. \tag{3.15}$$

Moreover, using a well-known combinatorial argument (see, for instance, Theorem 6.3 of Benois et al. (1997)), if γ is sufficiently small, then the previous term is less than

$$\exp(-c'a\beta\gamma^{-1}L^{d-1}/2)[1+L_\gamma^d\exp(-c\beta\zeta^2 2^{-(k+h)d}\gamma^{b-d}L^b/8)]^{L_\gamma^{d-b}} \tag{3.16}$$

and the lemma follows. □

Lemma 3.3. *For any $\delta>0$ and for γ small enough,*

$$\limsup_{L\to\infty}\frac{\gamma}{\beta L^{d-1}}\log\mu_{\gamma,L}(\|T-\eta\|>\delta L_\gamma^d)=-\infty. \tag{3.17}$$

Proof. From the definition of T, we get that

$$\|T-\eta\|\leqslant 2^{hd+1}\sum_{\Gamma\in\Omega^b}\mathrm{vol}(\Gamma)+2\sum_{\Gamma\notin\Omega^b}|\Gamma|. \tag{3.18}$$

By the Peierls estimates 3.9, for any $\delta>0$

$$\limsup_{L\to\infty}\frac{\gamma}{\beta L^{d-1}}\log\mu_{\gamma,L}\left(\sum_{\Gamma\notin\Omega^b}|\Gamma|>\delta L_\gamma^d\right)=-\infty, \tag{3.19}$$

provided γ is small enough. So we are reduced to study the cost of the event

$$\mathscr{B}(\delta) := \left\{ \sum_{\Gamma \in \Omega^b} \mathrm{vol}(\Gamma) \geqslant \delta |L_\gamma|^d \right\}. \tag{3.20}$$

Let $\mathscr{D}^{(\ell)}$ be the partition of $L_\gamma \mathscr{T}$ into cubes A_i of side $\ell = 10(L_\gamma)^b$ and let $N = \ell^{-d} L_\gamma^d$ be the number of these cubes. We call $d_a B$, $a \in \mathbb{R}^d$, $B \subset \mathbb{R}^d$, the translate by a of B.

A geometric remark. There are n vectors $\{e_j\}$ such that the following holds. Let Γ be a contour in Ω^b and $\Gamma \cap A_i \neq \emptyset$. Then there is $j \in \{1, \dots, n\}$ so that $\Gamma < d_{\ell e_j} A_i$, by this meaning that Γ is strictly contained in $d_{\ell e_j} A_i$ and the distance from the complement of $d_{\ell e_j} A_i$ is $> 2^{h+10}$, 2^h the side of the cubes in the definition of the block spins. As a consequence,

$$\sum_{A_i} \sum_{\{e_j\}} \sum_{\Gamma \in \Omega^b} \mathbf{1}_{\{\Gamma < d_{\ell e_j} A_i\}} \mathrm{vol}(\Gamma) \geqslant \sum_{\Gamma \in \Omega^b} \mathrm{vol}(\Gamma). \tag{3.21}$$

If we define for any vector $e \in \mathbb{R}^d$

$$\mathscr{B}_e(\delta) := \left\{ \sum_{A_i} \sum_{\Gamma \in \Omega^b} \mathbf{1}_{\{\Gamma < d_{\ell e} A_i\}} \mathrm{vol}(\Gamma) \geqslant \delta |\gamma L|^d \right\}, \tag{3.22}$$

then

$$\mathscr{B}(\alpha) \subset \bigcup_{j=1}^{n} \mathscr{B}_{e_j}(\alpha/n). \tag{3.23}$$

It is therefore enough to prove that for any $\delta > 0$ and any $e \in \mathbb{R}^d$

$$\limsup_{L \to \infty} \frac{\gamma}{\beta L^{d-1}} \log \mu_{\gamma,L}(\mathscr{B}_e(\delta)) = -\infty. \tag{3.24}$$

For notational simplicity we take in the following $e = 0$, dropping when possible the subscript 0 ($e = 0$).

For each cube A_i we define a random variable ξ_i with values in $\{0, 1\}$ as follows. We set $\xi_i = 1$ if

$$\sum_{\Gamma \in \Omega^b} \mathbf{1}_{\{\Gamma < A_i\}} \frac{\mathrm{vol}(\Gamma)}{|A_i|} \geqslant \delta', \quad \delta' = \frac{\delta}{2}. \tag{3.25}$$

Otherwise we set $\xi_i = 0$.

We want to prove that

$$\mathscr{B}_0(\delta) \subset \left\{ \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{\{\xi_i = 1\}} \geqslant \delta' \right\}. \tag{3.26}$$

Calling M the number of i's such that $\xi_i = 1$, we suppose, by contradiction, that $M < \delta' N$. Then

$$\frac{1}{|\gamma L|^d} \sum_{i=1}^{N} \sum_{\Gamma \in \Omega^b} \mathbf{1}_{\{\Gamma < A_i\}} \mathrm{vol}(\Gamma) \leqslant \frac{1}{N}(\delta'(N-M)+M) < \delta \tag{3.27}$$

and Eq. (3.26) is proved.

Let ∂A_i be the union of all the blocks in A_i connected to A_i^{c} and $\partial \mathscr{D}$ the union of all ∂A_i. By conditioning on $S_{\partial \mathscr{D}}$ we get

$$\mu_{\gamma,L}(\mathscr{B}_0(\delta)) \leqslant E_{\mu_{\gamma,L}} \left(\sum_{\{a_i\} \in \{0,1\}^N} \mathbf{1}_{\{\sum a_i > \delta' N\}} \prod_{i=1}^{N} \mu_{\gamma,L}(\xi_i = a_i | S_{\partial \mathscr{D}}) \right). \tag{3.28}$$

Thus,

$$\mu_{\gamma,L}(\mathscr{B}_0(\delta)) \leqslant 2^N \sup_{S_{\partial \mathscr{D}}} \sup_{\{\sum a_i > \delta' N\}} \prod_{\{a_i = 1\}} \mu_{\gamma,L}\left(\xi_i = 1 | S_{\partial \mathscr{D}}\right). \tag{3.29}$$

Let B_i be the intersection of A_i and the union of all the contours that intersect ∂A_i. The set of spin configurations that give rise to B_i is not in the σ-algebra generated by the spins in B_i itself. We then define $\bar{B}_i$ which is obtained as follows. We first add to B_i all the block cubes that are $\star$-connected to B_i and then repeat the operation starting from this new set, call $B_i^\star$ this second set. We next consider all the block cubes that are $\star$-connected to $\partial \mathscr{D}$, the union of this set and $B_i^\star$ is the set $\bar{B}_i$.

The set of spin configurations that give rise to B_i is in the σ-algebra generated by the spins in $\bar{B}_i$. Moreover, if $\Gamma < A_i$ then $\Gamma \cap B_i = \emptyset$.

After conditioning in Eq. (3.29) on $S_{\bar{B}_i}$ we use the Chebishev inequality and get

$$\mu_{\gamma,L}(\xi_i = 1 | S_{\bar{B}_i}) \leqslant \frac{1}{\delta'} \sum_{C \subset A_i \setminus B_i} \frac{|C|}{|A_i|} \sum_{\Gamma} \mathbf{1}_{\{C \subset \mathrm{Int}(\Gamma)\}} \mu_{\gamma,L}(\Gamma | S_{\bar{B}_i}), \tag{3.30}$$

where by an abuse of notation the sum is over all $\star$-connected sets Γ with $|\Gamma| \leqslant L_\gamma^b$ and such that $\Gamma \subset A_i \setminus B_i$; C is a block cube and $\mu_{\gamma,L}(\Gamma | S_{\bar{B}_i})$ is the probability that Γ is the spatial support of a contour.

Then

$$\mu_{\gamma,L}(\xi_i = 1 | S_{\bar{B}_i}) \leqslant \frac{1}{\delta'} \frac{|C|}{|A_i|} \sum_{l=4}^{L_\gamma^b} \sum_{\Gamma \subset A_i \setminus B_i, |\Gamma| = l} l^d \mu_{\gamma,L}(\Gamma | S_{\bar{B}_i}). \tag{3.31}$$

By the Peierls estimate

$$\mu_{\gamma,L}(\xi_i = 1 | S_{\bar{B}_i}) \leqslant \frac{1}{\delta'} c \mathrm{e}^{-c' \gamma^{-d}}. \tag{3.32}$$

Then, from Eq. (3.29)

$$\mu_{\gamma,L}(\mathscr{B}_0(\delta)) \leqslant \left(\frac{2c}{\delta'} \right)^N \mathrm{e}^{-c' \gamma^{-d} \delta' N}. \tag{3.33}$$

recalling $N = \ell^{-d} L_\gamma^d$, $\ell = 10 L_\gamma^b$ and $0 < b < 1/d$, we get Eq. (3.24). $\square$

Lemma 3.4. *For any $a>0$ and $\delta>0$,*

$$\limsup_{\varepsilon\to 0}\ \sup_{u\in K_a}\ \sup_{\|u-v\|\leqslant\delta} \|u-v^{(\varepsilon)}\|\leqslant\delta. \tag{3.34}$$

Proof. Let u be a function in K_a, then for any $\alpha>0$, there exists $w_\alpha\in BV(\mathscr{T};\{\pm 1\})$ such that the boundary of the set $\{w_\alpha=+1\}$ is a C^∞ surface, $\|u-w_\alpha\|\leqslant\alpha$ and $|P(u)-P(w_\alpha)|\leqslant\alpha$, see Giusti (1984). We define the $\mathscr{D}^{(\varepsilon)}$-measurable function $\tilde{w}_\alpha^{(\varepsilon)}$ as ± 1 according to the sign of the coarse grained $w_\alpha^{(\varepsilon)}$. Remark that since w_α has a regular boundary, the volume of the cubes $C^{(\varepsilon)}\in\mathscr{D}^{(\varepsilon)}$ where $\tilde{w}_\alpha^{(\varepsilon)}\neq w_\alpha$ is going to 0 with ε and as a consequence we have

$$\limsup_{\varepsilon\to 0}\|\tilde{w}_\alpha^{(\varepsilon)}-u\|\leqslant\alpha. \tag{3.35}$$

Let $v\in L^1(\mathscr{T})$ such that $\|u-v\|\leqslant\delta$. As $\tilde{w}_\alpha^{(\varepsilon)}$ is $\mathscr{D}^{(\varepsilon)}$-measurable

$$\|v^{(\varepsilon)}-\tilde{w}_\alpha^{(\varepsilon)}\|\leqslant\|v-\tilde{w}_\alpha^{(\varepsilon)}\|. \tag{3.36}$$

We deduce from this inequality that

$$\|u-v^{(\varepsilon)}\|\leqslant\|u-v\|+2\|\tilde{w}_\alpha^{(\varepsilon)}-u\| \tag{3.37}$$

and from Eq. (3.35) that for any $\alpha>0$

$$\limsup_{\varepsilon\to 0}\ \sup_{\|u-v\|\leqslant\delta}\|u-v^{(\varepsilon)}\|\leqslant\delta+2\alpha. \tag{3.38}$$

The compactness of K_a implies that the supremum over $u\in K_a$ in Eq. (3.34) can be written as a maximum over a finite number of elements of K_a. Thus, the lemma follows from Eq. (3.38). $\square$

Proof of Proposition 3.1. We first relate the mesoscopic coarse grained configuration $S^{(2^h)}$ to the variable T: we observe that

$$\|S^{(2^h)}-m_\beta T\|\leqslant\zeta L_\gamma^d+\int \mathrm{d}r\,\mathbf{1}_{\{|S^{(2^h)}-m_\beta T|\geqslant\zeta\}}\leqslant\zeta L_\gamma^d+\|\eta-T\|. \tag{3.39}$$

Fix $a>0$ and $\delta>0$, then, recalling that $t(r)=T(L_\gamma r)$,

$$\mu_{\gamma,L}(m_\beta^{-1}s^{(\varepsilon)}\notin K_a^\delta)\leqslant\mu_{\gamma,L}(P(t)>a)+\mu_{\gamma,L}(\|m_\beta^{-1}s^{(\varepsilon)}-t\|>\delta,P(t)\leqslant a). \tag{3.40}$$

From Lemma 3.4, there exists $\varepsilon(\delta)$ such that for any $0<\varepsilon<\varepsilon(\delta)$, the last term of the r.h.s. of the previous inequality is bounded above by

$$\mu_{\gamma,L}(\|m_\beta^{-1}S^{(2^h)}-T\|>\delta L_\gamma^d/2) \tag{3.41}$$

and using Eq. (3.39) with $\zeta<m_\beta\delta/4$, we obtain

$$\mu_{\gamma,L}(\|m_\beta^{-1}s^{(\varepsilon)}-t\|>\delta,P(t)\leqslant a)\leqslant\mu_{\gamma,L}(\|\eta-T\|>\delta m_\beta L_\gamma^d/4). \tag{3.42}$$

Finally, by Eq. (3.40) and Lemma 3.3,

$$\limsup_{\gamma\to 0}\limsup_{\varepsilon\to 0}\limsup_{L\to\infty}\frac{\gamma}{\beta L^{d-1}}\log\mu_{\gamma,L}(m_\beta^{-1}s^{(\varepsilon)}\notin K_a^\delta)$$
$$\leqslant \limsup_{\gamma\to 0}\limsup_{L\to\infty}\frac{\gamma}{\beta L^{d-1}}\log\mu_{\gamma,L}(P(T)>aL_\gamma^{d-1}) \tag{3.43}$$

and Lemma 3.2 concludes the proof. □

4. Upper bound

Upper bound 2.12 will follow from the exponential tightness (see Proposition 3.1) if for any closed subset F of $L^1(\mathscr{T})$ and for any $a>0$,

$$\lim_{\delta\to 0}\limsup_{\gamma\to 0}\limsup_{\varepsilon\to 0}\limsup_{L\to\infty}\frac{\gamma}{\beta L^{d-1}}\log\mu_{\gamma,L}(m_\beta^{-1}s^{(\varepsilon)}\in(F\cap K_a)^\delta)$$
$$\leqslant -\inf_{u\in F}\mathscr{I}(u). \tag{4.1}$$

From the compactness of the level set K_a, there exists a finite subset $F(a,\delta)$ of $F\cap K_a$ such that

$$(F\cap K_a)^\delta\subset\bigcup_{u\in F(a,\delta)}B(u,2\delta), \tag{4.2}$$

where $B(u,\delta)$ is the ball with center u and radius δ for the L^1-norm. Therefore,

$$\limsup_{\varepsilon\to 0}\limsup_{L\to\infty}\frac{\gamma}{\beta L^{d-1}}\log\mu_{\gamma,L}(m_\beta^{-1}s^{(\varepsilon)}\in(F\cap K_a)^\delta)$$
$$\leqslant \max_{u\in F(a,\delta)}\limsup_{\varepsilon\to 0}\limsup_{L\to\infty}\frac{\gamma}{\beta L^{d-1}}\log\mu_{\gamma,L}(m_\beta^{-1}s^{(\varepsilon)}\in B(u,2\delta)). \tag{4.3}$$

Let $u_{a,\delta,\gamma}\in F(a,\delta)$ be the function for which the above maximum is obtained. Then using again the compactness of K_a, there are sequences of positive numbers δ_n and γ_k going to 0 such that u_{a,δ_n,γ_k} is converging in L^1 to some function $u_a\in F\cap K_a$ as k and then n go to infinity. So, for any $\alpha>0$,

$$\lim_{\delta\to 0}\limsup_{\gamma\to 0}\limsup_{\varepsilon\to 0}\limsup_{L\to\infty}\frac{\gamma}{\beta L^{d-1}}\log\mu_{\gamma,L}(m_\beta^{-1}s^{(\varepsilon)}\in(F\cap K_a)^\delta)$$
$$\leqslant \limsup_{\gamma\to 0}\limsup_{\varepsilon\to 0}\limsup_{L\to\infty}\frac{\gamma}{\beta L^{d-1}}\log\mu_{\gamma,L}(m_\beta^{-1}s^{(\varepsilon)}\in B(u_a,\alpha)). \tag{4.4}$$

Now, from the proof of the upper bound of the weak large deviation principle in Benois et al. (1997)

$$\lim_{\alpha\to 0}\limsup_{\gamma\to 0}\limsup_{\varepsilon\to 0}\limsup_{L\to\infty}\frac{\gamma}{\beta L^{d-1}}\log\mu_{\gamma,L}(m_\beta^{-1}s^{(\varepsilon)}\in B(u_a,\alpha))$$
$$\leqslant -\mathscr{I}(u_a)\leqslant -\inf_{u\in F}\mathscr{I}(u). \tag{4.5}$$

This inequality together with Eq. (4.4) implies Eq. (4.1).

Appendix

In this appendix we will prove inequality (3.11), the proof is similar to one in Bodineau and Presutti (1996). Observe that, in definition (3.4) of the excess free-energy functional, $f(m_\beta)$ is the minimum of $f(m)$ so that Eq. (3.4) is the sum of two-non negative terms.

We fix $\zeta'>0$ and for any function $m\in L^\infty(\Lambda,[-1,1])$ we consider

$$\Psi_m(x)=\begin{cases} 1 & \text{if } S^{(1)}(x)\geqslant m_\beta-\zeta', \\ -1 & \text{if } S^{(1)}(x)\leqslant -m_\beta+\zeta', \\ 0 & \text{otherwise.}\end{cases} \tag{A.1}$$

Notice that the function Φ_m defined in Eq. (3.10) satifies $\Phi_m=1$ if $\Psi_m\geqslant 0$ and $\Phi_m=-1$ if $\Psi_m=-1$. We denote by $N^0(m)$ the number of cubes C in Λ where $\Psi_m=0$ and by $N^\pm(m)$ the number of pairs of cubes C in Λ which are connected and where Ψ_m has opposite signs. Then Eq. (3.11) is a straight consequence of the following lemma

Lemma A.1. *There is a constant $c>0$ (depending on ζ') such that for any $m\in L^\infty(\Lambda,$ $[-1,1])$*

$$N^0(m)+N^\pm(m)\leqslant c\mathscr{F}_\Lambda(m). \tag{A.2}$$

Proof. We start from a geometric remark. Let $e_1,\dots,e_n$ be the unit coordinate vectors of $\mathbb{R}^d$, $e_0=0$ and $d_e\mathscr{D}$ be the translate of the partition $\mathscr{D}$ by the vector e. If C_1 and C_2 are two connected cubes in $\mathscr{D}^{(1)}$, then there exists $0\leqslant i\leqslant n$ and $C\in d_{e_i}\mathscr{D}^{(2)}$ such that $C_1\cup C_2\subset C$ and $C\subset\Lambda$. We denote by $\mathscr{D}_{(j)}$, $0\leqslant j\leqslant d$, the collection of all the cubes of $d_{e_j}\mathscr{D}^{(2)}$ that are in Λ. We also denote by $\mathscr{D}_{(-1)}$ the unit cubes in Λ. Finally, we let $N_j^\pm(m)$, $0\leqslant j\leqslant d$, be the number of cubes in $\mathscr{D}_{(j)}$ where Ψ_m takes both values 1 and -1. So

$$N^\pm(m)\leqslant\sum_{j=0}^{d}N_j^\pm(m). \tag{A.3}$$

Then dropping out the interaction between cubes,

$$\mathscr{F}_\Lambda(m)\geqslant\frac{1}{d+2}\sum_{j=-1}^{d}\sum_{C\in\mathscr{D}_{(j)}}\mathscr{F}_C(m_C), \tag{A.4}$$

where m_C is the restriction of m to C. We define

$$\chi_C(x)=\int_C \mathrm{d}y\,J(x-y) \tag{A.5}$$

and

$$\overline{\mathscr{F}}_C(m)=\int_C \mathrm{d}x\,\chi_C(x)[f(m(x))-f(m_\beta)] + \frac{1}{4}\int\!\!\int_{C\times C}\mathrm{d}x\,\mathrm{d}y\,J(|x-y|)[m(x)-m(y)]^2. \tag{A.6}$$

Since $\chi_C \leqslant 1$, $\overline{\mathscr{F}}_C \leqslant \mathscr{F}_C$. Moreover, $\overline{\mathscr{F}}_C$ is a lower semi-continuous functional for the weak topology because

$$\overline{\mathscr{F}}_C(m) = -\beta^{-1} \int_C \mathrm{d}x\, \chi_C(x) i(m(x)) - \frac{1}{2} \int\int_{C\times C} \mathrm{d}x\, \mathrm{d}y\, J(|x-y|) m(x) m(y) - |C| f(m_\beta). \tag{A.7}$$

By convexity, the first term is lower semi-continuous while the second one is continuous. Therefore, there is $c' > 0$ depending only on ζ' such that $\overline{\mathscr{F}}_C(m) \geqslant c'$ for any cube C in $\mathscr{D}_{(-1)}$ where $\Psi_m = 0$ and any cube C in $\mathscr{D}_{(j)}$ where Ψ_m takes both values 1 and -1. $\square$

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