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Stability for multidimensional jump-diffusion processes [1]

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Dedicated to the memory of William Pruitt

Abstract

The aim of this work is to obtain sufficient conditions for stability of multidimensional jump-diffusion processes in the sense of stability in distribution and stability at the equilibrium solution. The technique employed is to construct appropriate Lyapunov functions. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Jump-diffusion; Stability in distribution; Stochastic stability; a.s. exponential stability

1. Introduction

We consider a n-dimensional jump-diffusion process $\{X_t^x\}$ satisfying

$$X_t^x = x + \int_0^t \mu(X_{s-}^x)\,\mathrm{d}s + \int_0^t \sigma(X_{s-}^x)\,\mathrm{d}B_s + \int_0^t \int c(X_{s-}^x, u)\tilde{\nu}(\mathrm{d}s, \mathrm{d}u), \tag{1}$$

where $\mu(x)$ and $c(x,u)$ are $\mathbb{R}^n$-valued and $\sigma(x)$ is $n \times m$-matrix valued for x, $u \in \mathbb{R}^n$. Here $\{B_t\}$ is a standard m-dimensional Brownian motion, and

$$\tilde{\nu}(\mathrm{d}s, \mathrm{d}y) = \nu(\mathrm{d}s, \mathrm{d}y) - \Pi(\mathrm{d}y)\,\mathrm{d}s$$

is a compensated Poisson random measure on $[0, \infty) \times \mathbb{R}^n$ which is independent of $\{B_t\}$. Furthermore, we assume that there exists a positive constant L such that for any $x, y \in \mathbb{R}^n$,

$$\begin{aligned} &|\mu(x) - \mu(y)|^2 + ||\sigma(x) - \sigma(y)||^2 + \int |c(x,u) - c(y,u)|^2 \Pi(\mathrm{d}u) \\ &\quad \leqslant L|x - y|^2, \end{aligned} \tag{2}$$

$$|\mu(x)|^2 + ||\sigma(x)||^2 + \int |c(x,u)|^2 \Pi(\mathrm{d}u) \leqslant L(1 + |x|^2). \tag{3}$$

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Here $|\ |$ denotes Euclidean norm, and $\|\ \|$ denotes matrix norm induced by Euclidean norm in $\mathbb{R}^n$. It is well-known that Eqs. (2) and (3) imply the existence of unique solution $\{X_t^x\}$ of Eq. (1) whose almost all sample paths are in $D_{\mathbb{R}^n}[0,\infty)$.

The jump-diffusion process can be considered as continuous Ito diffusions pertubated by random jumps. This is one of the useful stochastic models which appears frequently in many applications. In mathematical finance theory, one of the principal interest is focused on option pricing. In the classical Black–Sholes model, the security price is expressed as a geometric Brownian Motion which is a solution of the linear stochastic differential equation without jumps. But in practical situations, the prices contain possible unpredictable jumps due to external inaccessible shocks. So the general semimartingale has been chosen for the security price model (Harrison and Pliska, 1981) and turns out to be quite useful in many situations. In particular, jump-diffusion models described as Ito process disturbed by Poisson process or random measure are general enough to include most interesting cases that may arise. These models are discussed in Aase (1982,1984,1986,1988), Jeanblanc-Picque and Pontier (1990), Merton (1976), Merculio and Runggaldier (1993), and Mulinacci (1996) although the list is not the most inclusive. As other applications of this model, there are damage level processes of certain devices by environmental random stocks (Abdel-Hameed, 1984a,b; Esary et al., 1973; Drosen, 1986), and the content of a dam and the level of a storage process subject to input process and a release rule. (Moran, 1969; Cinlar and Pinsky, 1971; Brockwell et al., 1982; Zakusilo, 1989,1990).

The main purpose of this work is to obtain sufficient criteria for the stability of Eq. (1) in various senses. There is extensive literature concerning the stability theory for Ito equations (i.e. Eq. (1) with $c\equiv 0$). Bucy (1965) discovered for Ito equations that stochastic Lyapunov functions provide supermartingales and gave simple sufficient criteria for stochastic stability and for stability of moments. Has'minskii (1967) gave necessary and sufficient conditions for stability of linear Ito equations at the equlibrium solution using Lyapunov functions. In Arnold et al. (1984a,b), almost sure and moment exponential stability of linear Ito equations were studied. More recently Mao (1991,1994) and references therein) dealt with exponential stability theory for stochastic differential equations driven by continuous semimartingales. For a detailed account and further references concerning stability theory, the reader may consult the books by Has'minskii (1980) and Mao (1991,1994).

We also consider the question of stability in distribution of $\{X_t^x\}$. To be more precise, we ask whether $P(X_t^x\in \mathrm{d}y)$ converges weakly as $t\to\infty$ to a probability measure which is independent of x. For nondegenerate one-dimensional diffusion processes without jumps (i.e. $c\equiv 0$, and $\sigma\neq 0$) under weaker conditions than Eqs. (2) and (3), complete characterizations are known for positive recurrence and null recurrence. Moreover, positive recurrence is equivalent to stability in distribution and to the existence of unique invariant probability measure, respectively. Also for nondegenerate multidimensional diffusion processes without jumps, sufficient conditions for positive recurrence, null recurrence, and existence of invariant measure are obtained in Bhattacharya (1978) and Has'minskii (1967). Furthermore, for a class of degenerate multidimensional, diffusion processes without jumps, sufficient conditions for stability in distribution and existence of invariant probability measures are proved in Basak and

Bhattacharya (1992). In this article, we extend the previous results for continuous diffusions to multidimensional jump-diffusion processes (possibly degenerate), and obtain sufficient conditions for stability in distribution and stochastic and exponential stabilities at equilibrium using an appropriate stochastic Lyapunov function. A major difficulty in finding the right Lyapounov function here is that we have the integro-differential operator as the infinitesimal generator instead of the usual elliptic differential operator for a continuous diffusion.

The paper is organized as follows: In Section 2, the results for stability in distribution are given. Section 3 is devoted to the results for stabilities at the equilibrium solution such as stochastic asymptotic stability and a.s. exponential stability. Finally, some examples are added at the end of Section 3. Throughout this work, we denote a positive finite generic constant by C, whose value differs from line to line.

2. Stability in distribution

Recall that the standing hypotheses (2) and (3) hold. Before presenting the main result, we define stability in distribution which we deal with in this section.

Definition 1. $\{X_t^x\}$ is called stable in distribution if $P(X_t^x \in \mathrm{d}y)$ converges weakly as $t \to \infty$ to some probability measure which is independent of x.

Throughout this work, we set, for simplicity,

$$a(x) = \sigma(x)\sigma(x)^{\mathrm{T}},$$

$$\tilde{a}(x,y) = (\sigma(x) - \sigma(y))(\sigma(x) - \sigma(y))^{\mathrm{T}},$$

$$\tilde{\mu}(x,y) = \mu(x) - \mu(y),$$

$$\tilde{c}(x,y,u) = c(x,u) - c(y,u).$$

We also define following integrals which will be used frequently:

$$J_1(x) \equiv \int \left(\ln \frac{|x + c(x,u)|}{|x|} \right)^2 \Pi(\mathrm{d}u),$$

$$I_1(x,y) \equiv \int \left(\ln \frac{|x - y + \tilde{c}(x,y,u)|}{|x - y|} \right)^2 \Pi(\mathrm{d}u).$$

For a symmetric positive-definite matrix P, we define

$$J_2^P(x) \equiv \frac{2x^{\mathrm{T}} P\mu(x)}{x^{\mathrm{T}} Px} - \frac{2(Px)^{\mathrm{T}} a(x) Px}{(x^{\mathrm{T}} Px)^2} + \frac{\mathrm{trace}(a(x)P)}{x^{\mathrm{T}} Px}$$
$$+ \int \left(\ln \frac{(x + c(x,u))^{\mathrm{T}} P (x + c(x,u))}{x^{\mathrm{T}} Px} - \frac{2x^{\mathrm{T}} Pc(x,u)}{x^{\mathrm{T}} Px} \right) \Pi(\mathrm{d}u),$$

$$I_2^P(x,y) \equiv \frac{2(x-y)^{\mathrm{T}} P\tilde{\mu}(x,y)}{(x-y)^{\mathrm{T}} P(x-y)} - \frac{2(P(x-y))^{\mathrm{T}} \tilde{a}(x,y) P(x-y)}{((x-y)^{\mathrm{T}} P(x-y))^2}$$

$$+\frac{\operatorname{trace}(\tilde{a}(x,y)P)}{(x-y)^{\mathrm{T}}P(x-y)}+\int\left\{\ln\frac{(x-y+\tilde{c}(x,y,u))^{\mathrm{T}}P(x-y+\tilde{c}(x,y,u))}{(x-y)^{\mathrm{T}}P(x-y)}\right.$$
$$\left.-\frac{2(x-y)^{\mathrm{T}}P\tilde{c}(x,y,u)}{(x-y)^{\mathrm{T}}P(x-y)}\right\}\Pi(\mathrm{d}u).$$

The main result of this section is as follows:

Theorem 1 (Stability in distribution)**.** (1) *Assume that there exist a positive-definite symmetric matrix P and positive constants M_1, N such that*

$$\sup_{|x|\geqslant N} J_1(x)<\infty, \tag{4}$$

$$\sup_{|x|\geqslant N} J_2^P(x)<-M_1. \tag{5}$$

Then $\{P(X_t^x\in \mathrm{d}y),\ t\geqslant 0\}$ is tight for each x, and there exists an invariant probability measure.

(2) *Assume that there exist a positive-definite symmetric matrix P and a positive constant M_2 such that*

$$\sup_{x\neq y} I_1(x,y)<\infty, \tag{6}$$

$$\sup_{x\neq y} I_2^P(x,y)<-M_2. \tag{7}$$

Then $\{P(X_t^x\in \mathrm{d}y),\ t\geqslant 0\}$ is stable in distribution, and there exists the unique invariant probability measure.

To establish the proof, we use a Lyapunov function of the type

$$V(t,x)=\mathrm{e}^{\alpha t}(x^{\mathrm{T}}Px)^{\delta}$$

for appropriate α and δ, and prove that $V(t,X_t^x)$ is a supermartingale. Basically, the proofs for parts (1) and (2) of Theorem 1 are similar in nature although they need slight modification from each other. To prove part (2) of Theorem 1, it suffices to show that $\{P(X_t^x\in \mathrm{d}y),\ t\geqslant 0\}$ is tight and there exists $\delta>0$ such that for every compact K

$$\limsup_{t\to\infty}\ \sup_{x,y\in K} E|X_t^x-X_t^y|^{2\delta}=0, \tag{8}$$

from which the stability in distribution follows. Moreover, it is easy to show that $\{X_t^{\bullet}\}$ is weak Feller under Eqs. (2) and (3). Therefore, the existence and uniqueness of the invariant probability measure follow from the tightness of $\{P(X_t^x\in \mathrm{d}y)\}$ and condition (8), respectively.

Lemma 1. *Suppose that* Eqs. (6) *and* (7) *hold. Then there exists $\delta>0$ such that for any compact K,* Eq. (8) *holds.*

Proof. For a compact set K and $x, y \in K$ with $x \neq y$, we write

$$X_t^x - X_t^y = x - y + \int_0^t (\mu(X_s^x) - \mu(X_s^y))\,\mathrm{d}s + \int_0^t (\sigma(X_s^x) - \sigma(X_s^y))\,\mathrm{d}B_s$$
$$+ \int_0^t \int (c(X_s^x, u) - c(X_s^y, u))\tilde{\nu}(\mathrm{d}s, \mathrm{d}u).$$

Let $\tau = \inf\{t \geqslant 0\colon X_t^x = X_t^y\}$, and $v(x) = (x^{\mathrm{T}}Px)^{\delta}$, where $\delta > 0$ will be chosen later. Let

$$\gamma(x, y, u) = \frac{(x - y + \tilde{c}(x, y, u))^{\mathrm{T}} P (x - y + \tilde{c}(x, y, u))}{(x - y)^{\mathrm{T}} P (x - y)}.$$

By generalized Ito's formula, we have, for $\alpha > 0$

$$E(v(X_{t\wedge\tau}^x - X_{t\wedge\tau}^y)\mathrm{e}^{\alpha(t\wedge\tau)}) = v(x - y) + E\int_0^{t\wedge\tau} Lv(X_s^x, X_s^y)\mathrm{e}^{\alpha s}\,\mathrm{d}s$$
$$+ E\int_0^{t\wedge\tau} v(X_s^x - X_s^y)\alpha\mathrm{e}^{\alpha s}\,\mathrm{d}s,$$

where

$$Lv(x, y) \equiv (\nabla v(x - y))^{\mathrm{T}}\tilde{\mu}(x, y) + \frac{1}{2}\sum_{i,j=1}^{n} \tilde{a}_{ij}(x, y)\frac{\partial^2 v}{\partial x_i \partial x_j}(x - y)$$
$$+ \int \{v(x - y + \tilde{c}(x, y, u)) - v(x - y) - (\nabla v(x - y))^{\mathrm{T}}\tilde{c}(x, y, u)\}\Pi(\mathrm{d}u)$$
$$= \delta v(x - y)\Bigg[\frac{2(x - y)^{\mathrm{T}}P\tilde{\mu}(x, y)}{(x - y)^{\mathrm{T}}P(x - y)}$$
$$- \frac{2(1-\delta)(P(x - y))^{\mathrm{T}}\tilde{a}(x, y)P(x - y)}{((x - y)^{\mathrm{T}}P(x - y))^2} + \frac{\mathrm{trace}(\tilde{a}(x, y)P)}{(x - y)^{\mathrm{T}}P(x - y)}$$
$$+ \frac{1}{\delta}\int \left\{(\gamma(x, y, u))^{\delta} - 1 - \frac{2\delta(x - y)^{\mathrm{T}}P\tilde{c}(x, y, u)}{(x - y)^{\mathrm{T}}P(x - y)}\right\}\Pi(\mathrm{d}u)\Bigg].$$

Now we note that as $\delta \to 0$,

$$\frac{1}{\delta}((\gamma(x, y, u))^{\delta} - 1) = \ln\gamma(x, y, u) + \frac{\delta}{2}(\ln\gamma(x, y, u))^2\mathrm{e}^{\theta},$$

where $\theta \equiv \theta(\delta, u, x, y)$ lies between 0 and $\delta\ln\gamma(x, y, u)$. We show that for δ small,

$$\sup_{x\neq y\in K}\int (\ln\gamma(x, y, u))^2\mathrm{e}^{\theta}\Pi(\mathrm{d}u) < \infty. \tag{9}$$

For $x \neq y$, we set $\gamma(x, y, u) \equiv \gamma = 1 + \gamma_1 + \gamma_2$, where

$$\gamma_1 \equiv \gamma_1(x, y, u) = \frac{2\tilde{c}(x, y, u)^{\mathrm{T}}P(x - y)}{(x - y)^{\mathrm{T}}P\,(x - y)},$$

$$\gamma_2 \equiv \gamma_2(x, y, u) = \frac{\tilde{c}(x, y, u)^{\mathrm{T}}P\tilde{c}(x, y, u)}{(x - y)^{\mathrm{T}}P(x - y)}.$$

We note that

$$\begin{aligned}\int_{\gamma\geqslant 2}(\ln\gamma)^2\mathrm{e}^{\theta}\Pi(\mathrm{d}u) &\leqslant C\int_{\gamma_1+\gamma_2\geqslant 1}(\gamma_1(x,y,u)+\gamma_2(x,y,u))\Pi(\mathrm{d}u)\\ &\leqslant C\int_{\gamma_1\geqslant\gamma_2\vee 1/2}2\gamma_1\Pi(\mathrm{d}u)+C\int_{\gamma_2\geqslant\gamma_1\vee 1/2}2\gamma_2\Pi(\mathrm{d}u)\\ &\leqslant C\int_{\gamma_1\geqslant\gamma_2\vee 1/2}\gamma_1^2\Pi(\mathrm{d}u)+C\int_{\gamma_2\geqslant\gamma_1\vee 1/2}\gamma_2\Pi(\mathrm{d}u)\\ &\leqslant C\int\frac{|\tilde{c}(x,y,u)|^2}{|x-y|^2}\Pi(\mathrm{d}u)\end{aligned}$$

which, in conjuction with Eq. (6), yields Eq. (9). Therefore we have

$$Lv(x,y)=\delta v(x-y)\{I_2^P(x,y)+\delta\mathrm{O}(1)\},$$

where O(1) is uniformly bounded for any x and y. Using Eq. (7), we choose $0<\delta_0<1/2$ so that for any x and y,

$$Lv(x,y)\leqslant -M_2\delta_0 v(x-y)/2,$$

and for $0<\alpha<M_2\delta_0/2$, we have

$$E(v(X_{t\wedge\tau}^x-X_{t\wedge\tau}^y)\mathrm{e}^{\alpha(t\wedge\tau)})\leqslant v(x-y).$$

Also, by the uniqueness of the solution of Eq. (1), we have, for any $t>0$,

$$E(v(X_t^x-X_t^y))\leqslant \mathrm{e}^{-\alpha t}v(x-y),$$

from which the assertion follows. □

Proof of Theorem 1. Suppose that Eqs. (4) and (5) hold. By the similar argument as in Lemma 1, Eqs. (4) and (5) imply that there exists $0<\delta_1<1/2$ such that for $|x|\geqslant N$

$$\begin{aligned}&\frac{2x^{\mathrm{T}}P\mu(x)}{x^{\mathrm{T}}Px}-2(1-\delta_1)\frac{(Px)^{\mathrm{T}}a(x)Px}{(x^{\mathrm{T}}Px)^2}+\frac{\mathrm{trace}(a(x)P)}{x^{\mathrm{T}}Px}\\ &\quad+\frac{1}{\delta_1}\int\left[\left(\frac{(x+c(x,u))^{\mathrm{T}}P(x+c(x,u))}{x^{\mathrm{T}}Px}\right)^{\delta_1}-1\right.\\ &\quad\left.-\frac{2\delta_1x^{\mathrm{T}}Pc(x,u)}{x^{\mathrm{T}}Px}\right]\Pi(\mathrm{d}u)<-M_1/2. \qquad (10)\end{aligned}$$

Let $g\in C^2(0,\infty)$ be nonnegative and nondecreasing on $[0,\infty)$, $g(\lambda)=\lambda^{\delta_1}$ for λ large and $h(x)=g(x^{\mathrm{T}}Px)$. Define $Z_t=\mathrm{e}^{\beta t}h(X_t^x)$ where $\beta=M_1\delta_1/2$. Then by the generalized Ito's formula,

$$EZ_t=h(x)+E\int_0^t\mathrm{e}^{\beta s}(\beta h(X_s^x)+\tilde{L}h(X_s^x))\,\mathrm{d}s,$$

where

$$\tilde{L}h(x) \equiv (\nabla h(x))^{\mathrm{T}}\mu(x) + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 h}{\partial x_i \partial x_j}(x)$$
$$+\int (h(x+c(x,u)) - h(x) - (\nabla h(x))^{\mathrm{T}}c(x,u))\Pi(\mathrm{d}u).$$

Eq. (10) implies that for $|x| \geqslant N$,

$$\beta h(x) + \tilde{L}h(x) \leqslant 0.$$

Therefore, we have, for some finite constant C_N,

$$\sup_{t\geqslant 0} Eh(X_t^x) \leqslant h(x) + \frac{C_N}{\beta},$$

which implies the tightness in part (1). Existence of an invariant probability measure follows easily by the weak Feller property of $\{X_t^\bullet\}$ (see the comment made before Lemma 1). To complete the proof of part (2), it suffices to show that Eqs. (6) and (7) imply Eq. (5) for large $|x|$. By using a similar argument as in Lemma 1, we have that as $\delta \to 0$,

$$J_2^P(x) = I_2^P(x,0) + \frac{2x^{\mathrm{T}}P\mu(0)}{x^{\mathrm{T}}Px} - \frac{2(Px)^{\mathrm{T}}(\sigma(0)\sigma(x)^{\mathrm{T}} + \sigma(x)\sigma(0)^{\mathrm{T}} - \sigma(0)\sigma(0)^{\mathrm{T}})Px}{(x^{\mathrm{T}}Px)^2}$$
$$+\frac{\mathrm{trace}[(\sigma(0)\sigma(x)^{\mathrm{T}} + \sigma(x)\sigma(0)^{\mathrm{T}} - \sigma(0)\sigma(0)^{\mathrm{T}})P]}{x^{\mathrm{T}}Px}$$
$$+\frac{1}{\delta}\int\left[\left(\frac{(x+c(x,u))^{\mathrm{T}}P(x+c(x,u))}{x^{\mathrm{T}}Px}\right)^{\delta}\right.$$
$$\left.-\left(\frac{(x+\tilde{c}(x,0,u))^{\mathrm{T}}P(x+\tilde{c}(x,0,u))}{x^{\mathrm{T}}Px}\right)^{\delta} - \frac{2\delta x^{\mathrm{T}}Pc(0,u)}{x^{\mathrm{T}}Px}\right]\Pi(\mathrm{d}u)$$
$$+\delta\mathrm{O}(1),$$

where $\mathrm{O}(1)$ is uniformly bounded for any x. Fix $\delta_2 > 0$ so that

$$\sup_x(-M_2 + \delta_2\mathrm{O}(1)) < 0.$$

It remains to show that as $|x| \to \infty$, $J_2^P(x) - I_2^P(x,0) - \delta_2\mathrm{O}(1) \to 0$. We limit ourselves to prove that as $|x| \to \infty$

$$\frac{(Px)^{\mathrm{T}}(\sigma(0)\sigma(x)^{\mathrm{T}} + \sigma(x)\sigma(0)^{\mathrm{T}} - \sigma(0)\sigma(0)^{\mathrm{T}})Px}{(x^{\mathrm{T}}Px)^2} \to 0 \tag{11}$$

and

$$\int\left[\left(\frac{(x+c(x,u))^{\mathrm{T}}P(x+c(x,u))}{x^{\mathrm{T}}Px}\right)^{\delta_2} - \left(\frac{(x+\tilde{c}(x,0,u))^{\mathrm{T}}P(x+\tilde{c}(x,0,u))}{x^{\mathrm{T}}Px}\right)^{\delta_2}\right.$$
$$\left.-2\delta_2\frac{x^{\mathrm{T}}Pc(0,u)}{x^{\mathrm{T}}Px}\right]\Pi(\mathrm{d}u) \to 0, \tag{12}$$

since the other terms can be treated similarly. To prove Eq. (11), we observe that

$$
\begin{aligned}
&(x^{\mathrm{T}}Px)^{-2}(Px)^{\mathrm{T}}(\sigma(0)(\sigma(x)-\sigma(0))^{\mathrm{T}}+(\sigma(x)-\sigma(0))\sigma(0)^{\mathrm{T}}+\sigma(0)\sigma(0)^{\mathrm{T}})Px\\
&\quad\leqslant (x^{\mathrm{T}}Px)^{-2}|Px|^2(2||\sigma(0)||||\sigma(x)-\sigma(0)||+||\sigma(0)||^2)\\
&\quad\leqslant C(|x|^{-1}+|x|^{-2}).
\end{aligned}
$$

Let $f(x,u)$ denote the integrand in Eq. (12). Since for each u,

$$\lim_{|x|\to\infty} f(x,u)=0,$$

it suffices to show that

$$\lim_{l\to\infty}\sup_{|x|>1}\int_{|f(x,u)|>l}|f(x,u)|\Pi(\mathrm{d}u)=0.$$

Set

$$
\begin{aligned}
\alpha\equiv\alpha(x,u)&\equiv\frac{c(x,u)^{\mathrm{T}}Pc(x,u)+2x^{\mathrm{T}}Pc(x,u)}{x^{\mathrm{T}}Px},\\
\beta\equiv\beta(x,u)&\equiv\frac{\tilde{c}(x,0,u)^{\mathrm{T}}P\tilde{c}(x,0,u)+2x^{\mathrm{T}}P\tilde{c}(x,0,u)}{x^{\mathrm{T}}Px},
\end{aligned}
$$

and

$$B\equiv B(x)\equiv\{u: |f(x,u)|>l\}.$$

Note that

$$
\begin{aligned}
|\alpha|&\leqslant\frac{\Lambda}{\lambda}\left(\frac{|c(x,u)|^2}{|x|^2}+\frac{2|c(x,u)|}{|x|}\right),\\
|\beta|&\leqslant\frac{\Lambda}{\lambda}\left(\frac{|\tilde{c}(x,0,u)|^2}{|x|^2}+\frac{2|\tilde{c}(x,0,u)|}{|x|}\right),
\end{aligned}
$$

where $\lambda>0$ is the smallest eigenvalue and $\Lambda>0$ is the largest eigenvalue of P. Fix $\eta>0$ so that

$$\frac{\Lambda}{\lambda}(\eta^2+2\eta)=\frac{1}{2}.$$

Let

$$
\begin{aligned}
A_1&=\left\{u: \frac{|c(x,u)|}{|x|}\leqslant\eta,\ \frac{|\tilde{c}(x,0,u)|}{|x|}\leqslant\eta\right\},\\
A_2&=\left\{u: \frac{|c(x,u)|}{|x|}>\eta,\ \frac{|\tilde{c}(x,0,u)|}{|x|}\leqslant\eta\right\},\\
A_3&=\left\{u: \frac{|c(x,u)|}{|x|}\leqslant\eta,\ \frac{|\tilde{c}(x,0,u)|}{|x|}>\eta\right\},\\
A_4&=\left\{u: \frac{|c(x,u)|}{|x|}>\eta,\ \frac{|\tilde{c}(x,0,u)|}{|x|}>\eta\right\}.
\end{aligned}
$$

For $u\in A_1$,

$$|f(x,u)|=\left|(1+\alpha)^{\delta_2}-(1+\beta)^{\delta_2}-\frac{2\delta_2x^{\mathrm{T}}Pc(0,u)}{x^{\mathrm{T}}Px}\right|$$

$$= \left| 1 + \delta_2\alpha - \frac{\delta_2(1-\delta_2)}{2}(1+\alpha_1)^{\delta_2-2}\alpha^2 - (1+\delta_2\beta) \right.$$

$$\left. + \frac{\delta_2(1-\delta_2)}{2}(1+\beta_1)^{\delta_2-2}\beta^2 - \frac{2\delta_2 x^{\mathrm{T}}Pc(0,u)}{x^{\mathrm{T}}Px} \right|$$

$$\leqslant \delta_2 \left(\frac{c(x,u)^{\mathrm{T}}Pc(x,u)}{x^{\mathrm{T}}Px} + \frac{\tilde{c}(x,0,u)^{\mathrm{T}}P\tilde{c}(x,0,u)}{x^{\mathrm{T}}Px} \right)$$

$$+ \left(\frac{|\alpha|^2}{(1-|\alpha_1|)^{2-\delta_2}} + \frac{|\beta|^2}{(1-|\beta_1|)^{2-\delta_2}} \right) \frac{\delta_2(1-\delta_2)}{2}$$

$$\leqslant 2,$$

where $|\alpha_1| \equiv |\alpha_1(x,u)| \leqslant |\alpha(x,u)|$, $|\beta_1| \equiv |\beta_1(x,u)| \leqslant |\beta(x,u)|$. Hence

$$\int_{B\cap A_1} |f(x,u)| \Pi(\mathrm{d}u) = 0 \quad \text{if } l > 2.$$

For $u \in A_2$,

$$|f(x,u)| = \left| (1+\alpha)^{\delta_2} - ((1+\beta)^{\delta_2} - 1 - \delta_2\beta) - 1 - \delta_2\beta - \delta_2 \frac{2x^{\mathrm{T}}Pc(0,u)}{x^{\mathrm{T}}Px} \right|$$

$$\leqslant \left(\left(\frac{\Lambda}{\lambda} + \frac{1}{\eta} \right) \frac{|c(x,u)|}{|x|} \right)^{2\delta_2} + \frac{\delta_2(1-\delta_2)}{2} |1+\beta_1|^{\delta_2-2}|\beta|^2$$

$$+1 + \delta_2 \left| \frac{\tilde{c}(x,0,u)^{\mathrm{T}}P\tilde{c}(x,0,u) + 2x^{\mathrm{T}}Pc(x,u)}{x^{\mathrm{T}}Px} \right|$$

$$\leqslant \left(\frac{\Lambda}{\lambda} + \frac{1}{\eta} \right)^{2\delta_2} \left(\frac{|c(x,u)|}{|x|} \right)^{2\delta_2} + 2 + \frac{\delta_2\Lambda|\tilde{c}(x,0,u)|^2}{\lambda|x|^2} + \frac{2\delta_2\Lambda|c(x,u)|}{\lambda|x|}$$

$$\leqslant C\frac{|c(x,u)|}{|x|}.$$

Therefore

$$\int_{A_2\cap B} |f(x,u)| \Pi(\mathrm{d}u) \leqslant \frac{C}{l} \int \frac{|c(x,u)|^2}{|x|^2} \Pi(\mathrm{d}u).$$

For $u \in A_3$, using a similar argument, we have

$$\int_{A_3\cap B} |f(x,u)| \Pi(\mathrm{d}u) \leqslant \frac{C}{l} \int \frac{|\tilde{c}(x,0,u)|^2}{|x|^2} \Pi(\mathrm{d}u).$$

For $u \in A_4$, we observe that

$$|f(x,u)| \leqslant (1+|\alpha|)^{\delta_2} + (1+|\beta|)^{\delta_2} + \delta_2 \frac{|2x^{\mathrm{T}}P\tilde{c}(x,0,u)|}{x^{\mathrm{T}}Px} + \delta_2 \frac{|2x^{\mathrm{T}}Pc(x,u)|}{x^{\mathrm{T}}Px}$$

$$\leqslant \left(\left(\frac{\Lambda}{\lambda} + \frac{1}{\eta} \right) \frac{|c(x,u)|}{|x|} \right)^{2\delta_2} + \left(\left(\frac{\Lambda}{\lambda} + \frac{1}{\eta} \right) \frac{|\tilde{c}(x,0,u)|}{|x|} \right)^{2\delta_2}$$

$$+ \delta_2 \frac{2\Lambda|\tilde{c}(x,0,u)|}{\lambda|x|} + \delta_2 \frac{2\Lambda|c(x,u)|}{\lambda|x|}$$

$$\leqslant C \left(\frac{|c(x,u)|}{|x|} + \frac{|\tilde{c}(x,0,u)|}{|x|} \right).$$

Hence

$$\int_{A_4\cap B} |f(x,u)|\Pi(\mathrm{d}u) \leqslant \frac{C}{l} \int \left(\frac{|c(x,u)|^2}{|x|^2} + \frac{|\tilde{c}(x,0,u)|^2}{|x|^2} \right) \Pi(\mathrm{d}u).$$

The proof is completed by using Eqs. (2) and (3). □

3. Stability at the equilibrium solution

In this section we assume that $\mu(0)=\sigma(0)=c(0,u)\equiv 0$ so that Eq. (1) admits the zero solution as an equilibrium solution. We obtain asymptotic stochastic stability under some appropriate conditions in Theorem 2 and a.s. exponential stability under the stronger conditions in Theorem 3. The basic strategy used here is rather simple and well-known to specialists (see Has'minskii, 1980; Mao, 1991,1994). Before we establish the main results, we introduce the notion of stochastic asymptotic stability and a.s. exponential stability.

Definition 2. (1) The zero solution is called stochastically stable if there exists $r = r(\varepsilon_1, \varepsilon_2)$, for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that for any $|x| < r$,

$$P\left(\sup_{t\geqslant 0} |X_t^x| < \varepsilon_1 \right) > 1 - \varepsilon_2.$$

(2) The zero solution is called stochastically asymptotically stable if it is stochastically stable and for any $\varepsilon > 0$ there exists $r = r(\varepsilon)$ such that for any $|x| < r$,

$$P\left(\lim_{t\to\infty} |X_t^x| = 0 \right) > 1 - \varepsilon.$$

Definition 3. Eq. (1) is called almost surely exponential stable if there exists an $M > 0$ independent of x satisfying

$$\limsup_{t\to\infty} \frac{1}{t} \log|X_t^x| \leqslant -M \quad \text{a.s.}$$

Now we introduce stopping times: for $\varepsilon > 0$, let

$$\begin{aligned} T_\varepsilon &= \inf\{t \geqslant 0;\ |X_t^x| < \varepsilon\}, \\ T_0 &= \lim_{\varepsilon\downarrow 0} T_\varepsilon, \\ \tau_\varepsilon &= \inf\{t\geqslant 0;\ |X_t^x| > \varepsilon\}. \end{aligned} \tag{13}$$

We start with three lemmas for Theorem 2.

Lemma 2. *Suppose that* $\sup_{0<|x|\leqslant n} J_1(x) < \infty$ *for each* n. *Then* $X_t^x \neq 0$ *and* $X_{t-}^x \neq 0$ *a.s. for any* $t \geqslant 0$ *if* $x \neq 0$.

Proof. We observe that $T_0 = \inf\{t \geqslant 0; X_{t-} = 0 \text{ or } X_t = 0\}$ a.s. Assume that there exist t_0 and n such that $|x| \leqslant n-1$, and

$$P(G) \equiv P\left\{T_0 \leqslant t_0,\ \sup_{t \leqslant t_0} |X_t^x| \leqslant n-1\right\} > 0.$$

Define

$$g(r) = |\ln r| \quad \text{for } r > 1/\eta \quad \text{or} \quad 0 < r < \eta,$$

$g \in C^2(0,\infty)$, and $\inf_{r>0} g(r) > 0$, where $0 < \eta < 1$ is chosen sufficiently small. Let $h(x) = g(x^{\mathrm{T}}Px)$ and

$$\tilde{L}h(x) = Ah(x) + Bh(x),$$

where

$$\begin{aligned}
Ah(x) &\equiv (\nabla h(x))^{\mathrm{T}}\mu(x) + \frac{1}{2}\Sigma_{i,j}a_{ij}(x)\frac{\partial^2 h}{\partial x_i \partial x_j}(x) \\
&= 2g'(x^{\mathrm{T}}Px)x^{\mathrm{T}}P\mu(x) + 2g''(x^{\mathrm{T}}Px)(Px)^{\mathrm{T}}a(x)Px + g'(x^{\mathrm{T}}Px)\mathrm{trace}(a(x)P), \\
Bh(x) &\equiv \int \left\{h(x + c(x,u)) - h(x) - (\nabla h(x))^{\mathrm{T}}c(x,u)\right\} \Pi(\mathrm{d}u) \\
&= \int \left\{g\left((x + c(x,u))^{\mathrm{T}}P(x + c(x,u))\right) - g(x^{\mathrm{T}}Px)\right. \\
&\quad \left. -2g'(x^{\mathrm{T}}Px)x^{\mathrm{T}}Pc(x,u)\right\} \Pi(\mathrm{d}u).
\end{aligned}$$

Define

$$V(t,x) = \mathrm{e}^{-\alpha t}h(x),$$

for some $\alpha > 0$. By the generalized Ito's formula,

$$E(\mathrm{e}^{-\alpha(t_0 \wedge T_\varepsilon \wedge \tau_n)}h(X^x_{t_0 \wedge T_\varepsilon \wedge \tau_n})) = h(x) + E\int_0^{t_0 \wedge T_\varepsilon \wedge \tau_n} \mathrm{e}^{-\alpha s}(-\alpha h(X_s^x) + \tilde{L}h(X_s^x))\mathrm{d}s.$$

Now we claim that for $0 < |x| \leqslant n$,

$$\tilde{L}h(x) \leqslant C_n \quad \text{for some } C_n, \tag{14}$$

which implies that for $0 < |x| \leqslant n$,

$$\tilde{L}h(x) - \alpha_n h(x) \leqslant 0$$

for some $\alpha_n > 0$. Then we obtain for ε sufficiently small

$$\begin{aligned}
h(x) &\geqslant E(\mathrm{e}^{-\alpha_n(t_0 \wedge T_\varepsilon \wedge \tau_n)}h(X^x_{t_0 \wedge T_\varepsilon \wedge \tau_n})) \\
&\geqslant E(\mathrm{e}^{-\alpha_n T_\varepsilon}h(X^x_{T_\varepsilon})\chi_G) \\
&\geqslant \mathrm{e}^{-\alpha_n t_0}\ln(\Lambda\varepsilon^2)^{-1}P(G),
\end{aligned}$$

where Λ is the largest eigenvalue of P. Therefore letting $\varepsilon \to 0$ in

$$P(G) \leqslant \frac{h(x)\mathrm{e}^{\alpha_n t_0}}{\ln(\Lambda\varepsilon^2)^{-1}},$$

we get a contradiction. Finally, it remains to show Eq. (14). From the definition of h, it is easy to see that $\sup_x Ah(x) < \infty$. To complete the argument, let $F(x,u)$ denote the integrand in the expression for $Bh(x)$. We show that for any x and u,

$$|F(x,u)| \leqslant C_1 \left(\ln \frac{|x+c(x,u)|}{|x|} \right)^2 + C_2 \frac{|c(x,u)|^2}{|x|^2} \tag{15}$$

for some C_1 and C_2. Instead of carrying out tedious calculations depending on the different values of x and $x+c(x,u)$, we demonstrate them in a few cases. For example, if $x^{\mathrm{T}}Px < \eta$, and $(x+c(x,u))^{\mathrm{T}}P(x+c(x,u)) < \eta$, then we write

$$F(x,u) = -\ln(1+\alpha(x,u)) + \alpha(x,u) - \frac{c(x,u)^{\mathrm{T}}Pc(x,u)}{x^{\mathrm{T}}Px},$$

where

$$\alpha(x,u) = \frac{2x^{\mathrm{T}}Pc(x,u) + c(x,u)^{\mathrm{T}}Pc(x,u)}{x^{\mathrm{T}}Px}.$$

It is not hard to show that Eq. (15) holds. If $x^{\mathrm{T}}Px < \eta$ and $(x+c(x,u))^{\mathrm{T}}P(x+c(x,u)) > 1/\eta$, then

$$\begin{aligned} |F(x,u)| &= \left| \ln((x+c(x,u))^{\mathrm{T}}P(x+c(x,u)) + \ln x^{\mathrm{T}}Px + \frac{2x^{\mathrm{T}}Pc(x,u)}{x^{\mathrm{T}}Px} \right| \\ &\leqslant |\ln(1+\alpha(x,u))| + |\alpha(x,u)| + 2|\ln x^{\mathrm{T}}Px| + \frac{c(x,u)^{\mathrm{T}}Pc(x,u)}{x^{\mathrm{T}}Px} \\ &\leqslant C_2 \left(\frac{|c(x,u)|}{|x|} \right)^2 \end{aligned}$$

since $|\alpha(x,u)| > 1/\eta^2 - 1$ is large for sufficiently small η. □

Lemma 3. *Suppose that* $\sup_{0 \neq |x| \leqslant n} J_1(x) < \infty$ *for each n, and there exist a symmetric positive-definite matrix P and positive constants M_3 and r_0 such that*

$$\sup_{0 \neq |x| < r_0} J_2^P(x) < -M_3.$$

Then the zero solution is stochastically stable.

Proof. Let $\varepsilon < |x| < r < r_0$, and $v(y) = (y^{\mathrm{T}}Py)^{\delta}$ where $\delta > 0$ will be determined later. Then by the generalized Ito's formula, in conjuction with a similar argument used in the proof of Theorem 1, we have

$$Ev(X^x_{t \wedge T_\varepsilon \wedge \tau_r}) = v(x) + E \int_0^{t \wedge T_\varepsilon \wedge \tau_r} \tilde{L}v(X^x_s)\,\mathrm{d}s,$$

where

$$\tilde{L}v(y) = \delta(y^{\mathrm{T}}Py)^{\delta}(J_2^P(y) + \delta \mathrm{O}(1))$$

and O(1) is uniformly bounded for $|y| < r_0$. Recall the notion $\tilde{L}v$ from the proof of Theorem 1. Hence for some $\delta_3 > 0$,

$$E(|X^x_{T_\varepsilon}|^{2\delta_3}\colon \{T_\varepsilon < t \wedge \tau_r\}) + E(|X^x_{t\wedge\tau_r}|^{2\delta_3}\colon \{t\wedge\tau_r \leqslant T_\varepsilon\})$$

$$= E(|X^x_{t\wedge T_\varepsilon\wedge\tau_r}|^{2\delta_3}) \leqslant \frac{\Lambda}{\lambda}|x|^{2\delta_3},$$

where λ and Λ are the smallest and the largest eigenvalue of P respectively. Letting $\varepsilon \downarrow 0$, we have

$$E(|X^x_{t\wedge\tau_r}|^{2\delta_3}\colon \{t\wedge\tau_r \leqslant T_0\}) \leqslant \frac{\Lambda}{\lambda}|x|^{2\delta_3},$$

hence

$$E(|X^x_{\tau_r}|^{2\delta_3}\colon \{\tau_r \leqslant t\}) \leqslant E(|X^x_{\tau_r\wedge t}|^{2\delta_3}) \leqslant \frac{\Lambda}{\lambda}|x|^{2\delta_3}$$

since $T_0 = \infty$ a.s. by Lemma 2. By letting $t \to \infty$ in

$$P(\tau_r \leqslant t) \leqslant \frac{\Lambda}{\lambda}\left(\frac{|x|}{r}\right)^{2\delta_3},$$

we have

$$P\left(\sup_{t\geqslant 0}|X^x_t| \leqslant r\right) \geqslant 1 - \frac{\Lambda}{\lambda}\left(\frac{|x|}{r}\right)^{2\delta_3}. \qquad \square$$

Lemma 4. *Suppose that the conditions in Lemma 3 hold. Then for* $\{X^x_t\}$ *with* $\varepsilon < |x| < r < r_0$,

$$E(T_\varepsilon \wedge \tau_r) < \infty.$$

Proof. For $\varepsilon < |x| < r < r_0$, let $v(x) = (y^{\mathrm{T}}Py)^{\delta_3}$ where δ_3 is chosen in the proof of Lemma 3. Then we have

$$Ev(X^x_{T_\varepsilon\wedge\tau_r\wedge t}) - v(x) = E\int_0^{T_\varepsilon\wedge\tau_r\wedge t} \tilde{L}v(X^x_s)\,\mathrm{d}s,$$

where for $\varepsilon < |y| < r$,

$$\begin{aligned}\tilde{L}v(y) &\leqslant -M_3\delta_3(y^{\mathrm{T}}Py)^{\delta_3}/2\\ &\leqslant -M_3\delta_3(\lambda\varepsilon^2)^{\delta_3}/2\\ &\equiv -\gamma(\varepsilon).\end{aligned}$$

Hence we let $t \to \infty$ in

$$E(T_\varepsilon \wedge \tau_r \wedge t) \leqslant \frac{v(x)}{\gamma(\varepsilon)}$$

and obtain the desired result. $\square$

Theorem 2 (Stochastic asymptotic stability)**.** *Suppose that the conditions in Lemma 3 hold. Then the zero solution is stochastically asymptotically stable.*

Proof. By Lemma 3, for $0 < r < r_0$ and $\xi > 0$, there exists η such that for $|x| < \eta$,

$$P\left(\sup_t |X_t^x| < r\right) > 1 - \xi. \tag{16}$$

Furthermore for $r_1 < \eta$, there exists η_1 such that for $|x| < \eta_1$,

$$P\left(\sup_t |X_t^x| < r_1\right) > 1 - \xi.$$

Then by Lemma 4 and Eq. (16) we have for $|x| < \eta$ and $0 < r_1 < \eta$,

$$\begin{aligned} P\left(\limsup_{t\to\infty} |X_t^x| \leqslant r_1\right) &\geqslant P\left(T_{\eta_1} < \infty, \sup_{t > T_{\eta_1}} |X_t^x| < r_1\right) \\ &> (1-\xi)^2. \end{aligned}$$

Letting $r_1 \to 0$, the conclusion follows. □

Now we shall discuss the almost sure exponential stability of $\{X_t^x\}$. Again the technique is fairly standard and is based on the use of Lyapounov functions. Distinct Lyapounov functions will be employed depending on different purposes.

Theorem 3 (Almost sure exponential stability). *Suppose that there exist a symmetric positive-definite matrix P and a constant M_4 such that*

$$\sup_{x\neq 0} J_1(x) < \infty, \qquad \sup_{x\neq 0} J_2^P(x) < M_4.$$

Then for any $x \neq 0$, $X_t^x \neq 0$ and $X_{t-}^x \neq 0$ a.s. for any t and

$$\limsup_{t\to\infty} \frac{\ln|X_t^x|}{t} \leqslant \frac{M_4}{2} \quad \textit{a.s.}$$

Proof. The first part is proved in Lemma 2. Let $v(x) = \ln(x^{\mathrm{T}}Px)^\delta$ where $\delta > 0$ will be chosen later. Fix $x \neq 0$ and write $X_t^x = X_t$. Then by the generalized Ito's formula,

$$v(X_t) = v(x) + \int_0^t \tilde{L}v(X_s)\,\mathrm{d}s + \int_0^t \phi(s)\,\mathrm{d}B_s + \int_0^t \int \psi(s,u)\tilde{\nu}(\mathrm{d}s,\mathrm{d}u)$$

where

$$\begin{aligned} \phi(s) &= 2\delta(X_s^{\mathrm{T}}PX_s)^{-1}X_s^{\mathrm{T}}P\sigma(X_s), \\ \psi(s,u) &= v(X_s + c(X_s,u)) - v(X_s), \\ \tilde{L}v(y) &= \delta J_2^P(y). \end{aligned}$$

We set

$$M_t = \int_0^t \phi(s)\,\mathrm{d}B_s + \int_0^t \int \psi(s,u)\tilde{\nu}(\mathrm{d}s,\mathrm{d}u)$$

and

$$A_t = \int_0^t |\phi(s)|^2/2\,\mathrm{d}s + \int_0^t \int \left(\mathrm{e}^{\psi(s,u)} - 1 - \psi(s,u)\right) \Pi(\mathrm{d}u)\,\mathrm{d}s.$$

Then $\{\exp(M_t - A_t)\}$ forms a martingale, hence for any t and $\lambda > 0$,

$$P\left(\sup_{0\leqslant s\leqslant t}(M_s - A_s)\geqslant\lambda\right)\leqslant e^{-\lambda}.$$

Set $\lambda = 2\log n$, and $t = n$. Then by Borel–Cantelli lemma, there exists $n_0(w)$ such that for $n\geqslant n_0(w)$, and $t\leqslant n$,

$$M_t\leqslant 2\log n + A_t.$$

Therefore for $t\leqslant n$, and $n\geqslant n_0(w)$,

$$\begin{aligned}
v(X_t) &\leqslant v(x) + \int_0^t \tilde{L}v(X_s)\,ds + 2\log n + A_t \\
&= v(x) + 2\log n + \delta\int_0^t (X_s^{\mathrm{T}}PX_s)^{-1}\left\{2X_s^{\mathrm{T}}P\mu(X_s)\right. \\
&\quad \left. - 2(1-\delta)\frac{X_s^{\mathrm{T}}Pa(X_s)PX_s}{X_s^{\mathrm{T}}PX_s} + \mathrm{trace}(a(X_s)P)\right\}\,ds \\
&\quad + \int_0^t\int\left\{\left(\frac{(X_s + c(X_s,u))^{\mathrm{T}}(X_s + c(X_s,u))}{X_s^{\mathrm{T}}PX_s}\right)^{\delta} - 1\right. \\
&\quad \left. - \frac{2\delta X_s^{\mathrm{T}}Pc(X_s,u)}{X_s^{\mathrm{T}}PX_s}\right\}\Pi(du)\,ds \\
&= v(x) + 2\log n + \delta\int_0^t J_2^P(X_s)\,ds + \delta^2 t\,\mathrm{O}(1) \\
&\leqslant v(x) + 2\log n + \delta M_4 + \delta^2 t\,\mathrm{O}(1)
\end{aligned}$$

for sufficiently small $\delta > 0$ where $\mathrm{O}(1)$ is uniformly bounded for any t and w. For given $\varepsilon > 0$, fix $\delta > 0$ so that $\delta\,\mathrm{O}(1) < \varepsilon$. Then for $n-1\leqslant t < n$, and $n > n_0(w)$,

$$\frac{\ln(X_t^{\mathrm{T}}PX_t)}{t}\leqslant\frac{v(x)}{\delta t} + \frac{2\log n}{\delta(n-1)} + M_4 + \varepsilon,$$

which implies that

$$\limsup_{t\to\infty}\frac{\ln(X_t^{\mathrm{T}}PX_t)}{t}\leqslant M_4 + \varepsilon.$$

Then the conclusion follows easily. □

Finally we provide some examples.

Example 1. Let $\mu(x) = \mu x$, $\sigma(x)\sigma(x)^{\mathrm{T}} = a(x) = \sigma^2|x|^2 I$, and $c(x,u) = \delta(u)x$, where $\delta(u)$ is real-valued. Suppose $\int\delta(u)^2\Pi(du) < \infty$. Set

$$M \equiv \sigma^2(n-2) + 2\mu + 2\int(\ln|1+\delta(u)| - \delta(u))\,\Pi(du).$$

Then $M < 0$ implies almost sure exponential stability of Eq. (1). In particular, if $n = m = 1$, $M < 0$ provides a necessary and sufficient condition for almost sure exponential stability of Eq. (1), since then the solution of Eq. (1) in this case can be written explicitly as

$$X_t^x = x\exp\left\{t\left[\mu - \frac{\sigma^2}{2} + \int (\ln|1+\delta(u)| - \delta(u))\,\Pi(\mathrm{d}u)\right]\right\}$$
$$\times\exp\left(\sigma W_t + \int_0^t \int \ln|1+\delta(u)|\tilde{\nu}(\,\mathrm{d}t,\,\mathrm{d}u)\right).$$

Example 2. Let

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + \sigma\begin{pmatrix} X_2(t) \\ -X_1(t)\end{pmatrix}\mathrm{d}B_t + \int c\begin{pmatrix} u_1X_1(t) \\ u_2X_2(t)\end{pmatrix}\tilde{\nu}(\,\mathrm{d}t,\,\mathrm{d}u),$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}.$$

It is known that if $c = 0$, and $x \neq 0$,

$$\liminf_{t\to\infty} \frac{\ln|X_t^x|}{t} \geqslant \frac{\sigma^2}{2} - 1 \text{ a.s.} \quad (\text{Mao}, 1994,\ \text{p.}170)$$

Suppose that

$$\int |u|^2 \wedge |u|\Pi(\mathrm{d}u) < \infty.$$

Then taking $P = I$, it is not hard to show that

$$2 + \int f(u)\Pi(\mathrm{d}u) < 0$$

provides a sufficient condition for almost sure exponential stability, where

$$f(u) = \begin{cases} \ln\left(1 + \dfrac{c(u_1+u_2)}{2}\right) - \dfrac{cu_1 + cu_2 + 2c^2u_1u_2}{2 + c(u_1+u_2)} & \text{if } u_1 \neq u_2, \\ \ln(1+cu_1)^2 - 2cu_1 & \text{if } u_1 = u_2. \end{cases}$$

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