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Extinction properties of super-Brownian motions with additional spatially dependent mass production

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Abstract

Consider the finite measure-valued continuous super-Brownian motion X on $\mathbb{R}^d$ corresponding to the log-Laplace equation $(\partial/\partial t)u = \frac{1}{2}\Delta u + \beta u - u^2$, where the coefficient $\beta(x)$ for the additional mass production varies in space, is Hölder continuous, and bounded from above. We prove criteria for (finite time) extinction and local extinction of X in terms of β. There exists a threshold decay rate $k_d|x|^{-2}$ as $|x| \to \infty$ such that X does not become extinct if β is above this threshold, whereas it does below the threshold (where for this case β might have to be modified on a compact set). For local extinction one has the same criterion, but in dimensions $d > 6$ with the constant k_d replaced by $K_d > k_d$ (phase transition). h-transforms for measure-valued processes play an important role in the proofs. We also show that X does not exhibit local extinction in dimension 1 if β is no longer bounded from above and, in fact, degenerates to a single point source δ_0. In this case, its expectation grows exponentially as $t \to \infty$. © 2000 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction and statement of results

1.1. Motivation

In Pinsky (1996, Theorem 6) an abstract (spectral theoretical) criterion has been presented for the local extinction of supercritical superdiffusions with everywhere constant branching mechanism. In Pinsky (1996, Remark 1) and later in Engländer and Pinsky (1999) it was noted that the proof works just as well in the variable coefficient case, that is for so-called $(L, \beta, \alpha; D)$-superdiffusions X. In the latter paper also abstract conditions have been derived for extinction and for the compact support property of X. Here L is a

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diffusion operator on a domain $D \subset \mathbb{R}^d$, and, loosely speaking, $\beta(x)u(t,x) - \alpha(x)u^2(t,x)$ refers to the branching mechanism. These abstract theorems however do not yield a sort of "test" in terms of the coefficients (as α, β and the coefficients in L) to decide whether a superdiffusion becomes (locally) extinct or possesses the compact support property. (Note nevertheless that by Theorem 3.5 in Engländer and Pinsky (1999) a sufficient condition has already been established for having the compact support property; see also Theorem 3.6 there.) Recently [Engländer, 2000] this gap has been partially filled by giving concrete criteria for the compact support property in a simple setting, namely, when the underlying migration process is a time-changed Brownian motion (that is, $L = \varrho(x)\Delta$ with $\varrho > 0$) and the spatially constant branching mechanism is critical (that is $\beta(x) \equiv 0$). In particular, it has been shown that a phase transition occurs between $d = 1$ and $d \geqslant 2$.

In this paper we are going to derive similar concrete criteria for (finite time) extinction and local extinction, again in a relatively simple setup. In fact, we consider a continuous super-Brownian motion ($L = \frac{1}{2}\Delta$) in $D = \mathbb{R}^d$ with coefficient $\alpha(x) \equiv 1$, but with additional spatially dependent mass "production" β (local sub- and supercriticalities). It turns out that it is possible to have an additional mass production β decaying at infinity on the order $|x|^{-2}$, and still observe finite time extinction of X. Moreover, the order $|x|^{-2}$ is maximal in the sense that there exists a threshold decay rate $k_d|x|^{-2}$ such that above this rate extinction is excluded, while below this rate extinction occurs (except a possibly needed local change of β). Finally, for local extinction there is the same picture, except that in dimensions $d > 6$ the constant k_d has to be replaced by a larger one. See Theorems 1 and 2 in Section 1.3 below. Unfortunately, we do not have any intuitive explanation for this interesting phase transition. But in Example 7 below, we will consider a superdiffusion in dimensions $d \geqslant 7$, for which local extinction holds, but not extinction, giving some kind of insight. Another perhaps surprising effect is, that the constants k_d and K_d are *not* monotone in d, since they disappear if and only if $d = 2$. Also here it would be nice to find an intuitive explanation.

The proofs of these criteria are based on the discussion of the criticality of several differential operators. A very effective tool is the h-transform under which the support process of X is invariant. A number of results from Pinsky (1995) are exploited.

A second purpose is to begin studying what happens if this mass production coefficient β varies in space in an irregular way. Here we restrict our attention to the simplest case, namely, if it degenerates to a single point source δ_0. Here our inspiration comes from the so-called catalytic branching models (see Fleischmann (1994), Dawson et al. (1995), Klenke (2000) or Dawson and Fleischmann (1999) for surveys). Theorem 3 in Section 1.4 implies that the process survives all finite times with positive probability, whereas Theorem 4 deals with the growth of expected mass. The proof includes finding subsolutions to the related log-Laplace equation (reaction–diffusion equation).

1.2. Preparation

Let $\mathscr{M}_\mathrm{f} = \mathscr{M}_\mathrm{f}(\mathbb{R}^d)$ denote the set of finite measures μ on $\mathbb{R}^d$, equipped with the topology of weak convergence. $\mathscr{M}_\mathrm{c} = \mathscr{M}_\mathrm{c}(\mathbb{R}^d)$ refers to the subset of all compactly supported $\mu \in \mathcal{M}_{\mathrm{f}}$. Write $C^{k,\gamma} = C^{k,\gamma}(\mathbb{R}^d)$ for the usual Hölder spaces of index $\gamma \in (0,1]$ including derivatives of order $k \leqslant 2$, and set $C^\gamma := C^{0,\gamma}$.

Let L be an elliptic operator on $\mathbb{R}^d$ of the form

$$L = \tfrac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla \quad \text{on } \mathbb{R}^d, \tag{1}$$

where the coefficients $a_{i,j}$ and b_i belong to $C^{1,\gamma}$, $i,j = 1,\dots,d$, for some γ in $(0,1]$, and the symmetric matrix $a = \{a_{i,j}\}$ satisfies $\sum_{i,j=1}^d a_{ij}(x)v_i v_j > 0$, for all $v \in \mathbb{R}^d \backslash \{0\}$ and $x \in \mathbb{R}^d$. In addition, let α, β denote functions in C^γ satisfying

$$\alpha > 0 \quad \text{and} \quad \sup_{x\in\mathbb{R}^d} \beta(x) < \infty. \tag{2}$$

Now we will introduce our basic object of interest:

Notation 1 (*Superdiffusion*)**.** Let $(X, \boldsymbol{P}_\mu, \mu \in \mathcal{M}_{\mathrm{f}})$ denote the $(L,\beta,\alpha;\mathbb{R}^d)$-*superdiffusion*. That is, X is the unique $\mathcal{M}_{\mathrm{f}}$-valued (time-homogeneous) continuous Markov process which satisfies, for any bounded continuous $g : \mathbb{R}^d \mapsto \mathbb{R}_+$,

$$\boldsymbol{E}_\mu \exp\langle X_t, -g\rangle = \exp\left(- \int_{\mathbb{R}^d} \mu(\mathrm{d}x)\, u(t,x)\right), \tag{3}$$

where u is the minimal non-negative solution to

$$\begin{aligned} &\frac{\partial}{\partial t}u = Lu + \beta u - \alpha u^2 \quad \text{on } \mathbb{R}^d \times (0,\infty), \\ &\lim_{t\to 0+} u(t,\cdot) = g(\cdot) \end{aligned} \tag{4}$$

(see Engländer and Pinsky (1999), in particular for an approximation by branching particle systems). Here $\langle v, f\rangle$ denotes the integral $\int_{\mathbb{R}^d} v(\mathrm{d}x)\, f(x)$.

For convenience, we expose the notions of extinction we use in the present paper:

Definition 2 (*Extinction*)**.** A measure-valued path X *becomes extinct* (in finite time) if $X_t = 0$ for all sufficiently large t. It *exhibits local extinction* if $X_t(B) = 0$ for all sufficiently large t, for each ball $B \subset \mathbb{R}^d$. The measure-valued process X corresponding to $\boldsymbol{P}_\mu$ is said to possess any one of these properties if that property is true with $\boldsymbol{P}_\mu$-probability one.

Remark 3 (*Process properties*)**.** In Engländer and Pinsky (1999) it is shown that, for fixed L, β and α, if any one of the properties in Definition 2 holds for some $\boldsymbol{P}_\mu$, $\mu \in \mathcal{M}_{\mathrm{c}}$ with $\mu \neq 0$, then it in fact holds for every $\boldsymbol{P}_\mu$, $\mu \in \mathcal{M}_{\mathrm{c}}$.

1.3. Criteria for extinction

Local extinction can be characterized in terms of L and β (see Pinsky, 1996, Theorem 6, Remark 1):

Lemma 4 (Local extinction)**.** *The* $(L,\beta,\alpha;\mathbb{R}^d)$-*superdiffusion* X *exhibits local extinction if and only if there exists a* (*strictly*) *positive solution* u *to the equation* $(L+\beta)u = 0$ *on* $\mathbb{R}^d$.

The following sufficient condition for extinction will be proved in Section 4.2 below.

Proposition 5 (Extinction via local extinction). *Assume the* $(L,\beta,\alpha;\mathbb{R}^d)$*-superdiffusion* X *exhibits local extinction. If there exists a function* $h\in C^{2,\gamma}$ *and a (non-empty) open ball* $B\subset\mathbb{R}^d$ *such that*

$$\inf_{\mathbb{R}^d}\alpha h>0 \quad \text{and} \quad (L+\beta)h\leqslant 0 \quad \text{on } \mathbb{R}^d\backslash\bar{B}, \tag{5}$$

then X *becomes extinct.*[3]

In the remaining part of Section 1, we specialize to

$$L=\tfrac{1}{2}\Delta \quad \text{and} \quad \alpha(x)\equiv 1; \tag{6}$$

that is, X is the superdiffusion (*super-Brownian motion*) corresponding to the quadruple $(\frac{1}{2}\Delta,\beta,1;\mathbb{R}^d)$.

It is well-known that if β is a constant, this super-Brownian motion X becomes extinct if and only if $\beta\leqslant 0$. Using Lemma 4 one can show that for constant $\beta>0$ there is even no local extinction. If however β is spatially dependent, then the local branching mechanism might be supercritical (that is $\beta(x)>0$) in certain x-regions and critical or subcritical ($\beta(x)\leqslant 0$) in others. We are interested in obtaining specific criteria for extinction and local extinction of the super-Brownian motion X in terms of $\beta\in C^{\gamma}$. In the following subsection we will consider a non-regular β as well.

Already here we point out that one should not expect criteria for local extinction simply in terms of the growth rate of β at infinity. The reason for this is as follows. It is well-known that for any given ball $B\subset\mathbb{R}^d$ (with positive radius), β can be chosen large enough on B in order to guarantee non-existence of positive solutions to the equation $(\frac{1}{2}\Delta+\beta)u=0$ on B, or, equivalently, the positivity of the principal eigenvalue for $\frac{1}{2}\Delta+\beta$ on B (see Pinsky, 1995, Chapter 4, for more elaboration). For such β, a fortiori, there is no positive solution u to the equation $(\frac{1}{2}\Delta+\beta)u=0$ on $\mathbb{R}^d$. Hence, by Lemma 4, in this case X does not exhibit local extinction. This shows that a small "tail" for β alone would never guarantee local extinction. On the other hand, if we allow that β can be modified on a compact set, then we will get a criterion for local extinction in terms of a *threshold decay rate* $K_d/|x|^2$ (as $|x|\to\infty$) for (possibly modified) $\beta\in C^{\gamma}$. To make this precise in our first theorem, we will exploit the notation $r\gg 1$ for the phrase "r large enough", and $r\ll -1$ is defined similarly.

Theorem 1 (Threshold decay rate for local extinction). *Consider the* $(\frac{1}{2}\Delta,\beta,1;\mathbb{R}^d)$*-superdiffusion* X.

(a) *If*

$$\beta(x)\leqslant\frac{K_d}{|x|^2} \quad \text{for } |x|\gg 1, \text{ where } K_d:=\frac{(d-2)^2}{8}, \tag{7}$$

then there exists a $\beta^*\in C^{\gamma}$ *satisfying* $\beta^*=\beta$ *outside some (sufficiently large) compact set such that the* $(\frac{1}{2}\Delta,\beta^*,1;\mathbb{R}^d)$*-superdiffusion* X^* *exhibits local extinction.*

[3] $\bar{B}$ denotes the closure of B.

(b) *On the other hand, if*

$$\beta(x) \geqslant \frac{K}{|x|^2} \quad \textit{for } |x| \gg 1 \textit{ and some } K > K_d, \tag{8}$$

then X does not exhibit local extinction.

Remark 6 (*One-dimensional case*). In one dimension, Theorem 1(b) can be replaced by a stronger statement: If

$$\beta(x) \geqslant \frac{K}{x^2} \quad \text{for } x \gg 1 \text{ or } x \ll -1, \text{ and some } K > K_1 = \frac{1}{8}, \tag{9}$$

then X does not exhibit local extinction. See Section 4.2 for a proof.

Since, by Lemma 4, local extinction is completely determined by a property of the *linear* operator $L + \beta$, it is relatively easy to get conditions on local extinction (as, for instance, in Theorem 1) using techniques from linear pde. Characterizing extinction of the $(\frac{1}{2}\Delta, \beta, 1; \mathbb{R}^d)$-superdiffusion X however is a subtler question. Nevertheless, we will show that if $d \leqslant 2$ or if β is below a threshold decay rate $k_d/|x|^2$ at infinity then local extinction of X implies extinction, while, on the other hand, extinction does not hold for any β above this threshold. If $d \leqslant 6$, then $k_d = K_d$ where K_d is defined in (8). However, if $d > 6$, a phase transition occurs: $k_d < K_d$. In fact, our *main result* reads as follows. (For a superdiffusion for which local extinction occurs despite extinction does not hold, see Example 7 below.)

Theorem 2 (Extinction versus local extinction). *The $(\frac{1}{2}\Delta, \beta, 1; \mathbb{R}^d)$-superdiffusion X has the following properties*:

(a) *Let $d \leqslant 2$. Then local extinction implies extinction.*

(b) *If*

$$\beta(x) \leqslant \frac{k_d}{|x|^2} \quad \textit{for } |x| \gg 1, \textit{ where } k_d := \begin{cases} K_d & \textit{if } d \leqslant 6, \\ d-4 & \textit{if } d > 6, \end{cases} \tag{10}$$

then local extinction implies extinction.

(c) *However, if*

$$\beta(x) \geqslant \frac{k}{|x|^2} \quad \textit{for } |x| \gg 1 \textit{ and some } k > k_d, \tag{11}$$

then extinction does not hold.

Example 7 (*Local extinction but no extinction*). Let $d \geqslant 7$ and h be a positive $C^{2,\gamma}$-function satisfying

$$h(x) = |x|^{-(d-2)/2} \quad \text{for } |x| \gg 1. \tag{12}$$

Note that

$$\beta^*(x) := -\frac{\frac{1}{2}\Delta h(x)}{h(x)} = K_d \frac{1}{|x|^2} \quad \text{for } |x| \gg 1. \tag{13}$$

Consider the $(\frac{1}{2}\Delta, \beta^*, 1; \mathbb{R}^d)$-superdiffusion X^*. Since $K_d > k_d$, by Theorem 2(c), extinction does not hold for X^*. On the other hand, as we will see in the proof of Theorem 1(a) below (special case $\hat{\beta} = 0$), this X^* exhibits local extinction.

Remark 8 (*Generalization*). The claim in Theorem 2(a) remains true for any $(L, \beta, \alpha; \mathbb{R}^d)$-superdiffusion whenever L corresponds to a *recurrent* diffusion on $\mathbb{R}^d$, and α (> 0) is bounded away from zero. This can easily be seen from the proof in Section 4.3 below.

Remark 9 (*Non-negative* β). In the case $\beta \geqslant 0$ but $\beta(x) \not\equiv 0$, by using Lemma 4 one can show that X does not exhibit local extinction (and consequently extinction does not hold for X) if $d \leqslant 2$, while in some cases extinction will hold for $d \geqslant 3$. See the end of Section 4.3 for a proof. In particular, if $d \leqslant 2$ and β (with $\beta \geqslant 0$ as well as $\beta(x) \not\equiv 0$) satisfies (7), then its modification β^* in Theorem 1(a) must change the sign: In order to get local extinction, a local supercriticality has to be compensated by some local subcriticality.

1.4. A single point source

In the light of Remark 9, it seems to be interesting to ask what happens when β degenerates to a *single point source*, that is, when the additional mass production is zero everywhere except at a single point (the origin, say) where the mass production is infinite (in a δ-function sense). In other words, we drop now our requirement that β is bounded from above and even consider the one-dimensional superdiffusion X corresponding to the quadruple $(\frac{1}{2}\Delta, \delta_0, 1; \mathbb{R})$, where δ_0 denotes the Dirac δ-function at zero. More precisely, from the partial differential equation (4) we pass to the *integral equation*

$$\begin{aligned} u(t,\cdot) = & \int_{-\infty}^{\infty} \mathrm{d}y\, p(t,\cdot,y) g(y) + \int_0^t \mathrm{d}s\, p(t-s,\cdot,0)\, u(s,0) \\ & - \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\, p(t-s,\cdot,y) u^2(s,y), \quad t > 0, \end{aligned} \tag{14}$$

with

$$p(t,x,y) = p(t,y-x) = \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{(y-x)^2}{2t} \right], \quad t > 0,\ x,y \in \mathbb{R}, \tag{15}$$

the standard Brownian transition density. The construction of this continuous $\mathcal{M}_\mathrm{f}$-valued process X having again the Laplace transition functionals (3) [but with the new log-Laplace function u from (14)] goes along standard lines via regularization of δ_0; in particular, the limiting log-Laplace equation (14) makes sense and enjoys the needed continuity properties. (See, e.g. Dawson and Fleischmann (1997) and references therein.) The corresponding laws will be denoted by $\boldsymbol{P}_\mu^{\sin}$, $\mu \in \mathcal{M}_\mathrm{f}$.

It turns out that in this one-dimensional situation the (additional) mass production at a single point is enough to guarantee that the process does not exhibit local extinction (and consequently extinction does not hold):

Theorem 3 (Single point source). *For any $\mu \in \mathcal{M}_{\mathrm{f}} \setminus \{0\}$, the super-Brownian motion X with law $\boldsymbol{P}_{\mu}^{\sin}$ does not exhibit local extinction.*

We mention that for the case when $\beta = 0$ and $\alpha = \delta_0$ instead, it is known, that

$$\boldsymbol{P}_{\mu}(\|X_t\| > 0,\ \forall t > 0,\ \text{but}\ \|X_t\| \to 0 \text{ as } t \to \infty) = 1 \tag{16}$$

for all $\mu \in \mathcal{M}_{\mathrm{f}} \setminus \{0\}$; see Fleischmann and Le Gall (1995) or Dawson et al. (1995, Corollary 5). (Here $\|\nu\|$ denotes the total mass of a measure ν.) Furthermore,

$$X_t(B) \to 0 \quad \text{in probability, for any ball } B \subset \mathbb{R}, \tag{17}$$

even if the starting measure μ is Lebesgue (see Dawson and Fleischmann, 1994).

Next, we will present an explicit formula for the expected total mass of the super-Brownian motion corresponding to $\boldsymbol{P}_{\mu}^{\sin}$, and show that the mass of any open subset $B \neq \emptyset$ of $\mathbb{R}$ grows exponentially in expectation.

Theorem 4 (Expectation of X). *Let $\mu \in \mathcal{M}_{\mathrm{f}}$. Consider the super-Brownian motion X with distribution $\boldsymbol{P}_{\mu}^{\sin}$.*

(a) (Expected total mass) *For all $t \geqslant 0$,*

$$\boldsymbol{E}_{\mu}^{\sin}\|X_t\| = \|\mu\| + \frac{2}{\sqrt{\pi}} \int_{\mathbb{R}} \mu(\mathrm{d}x) \int_0^t \mathrm{d}s\, p(t-s,x)\,\mathrm{e}^{s/2} \int_{-\sqrt{s/2}}^{\infty} \mathrm{d}y\,\mathrm{e}^{-y^2} \tag{18}$$

[*with the Brownian transition density p from* (15)]. *In particular*,

$$\boldsymbol{E}_{\delta_0}^{\sin}\|X_t\| = \frac{2}{\sqrt{\pi}}\,\mathrm{e}^{t/2} \int_{-\sqrt{t/2}}^{\infty} \mathrm{d}y\,\mathrm{e}^{-y^2}. \tag{19}$$

(b) (Exponential growth) *For all bounded continuous $g : \mathbb{R} \mapsto \mathbb{R}_+$,*

$$\lim_{t\to\infty} \mathrm{e}^{-t/2}\boldsymbol{E}_{\mu}^{\sin}\langle X_t, g\rangle = \int_{\mathbb{R}} \mu(\mathrm{d}x)\,\mathrm{e}^{-|x|} \int_{\mathbb{R}} \mathrm{d}y\, g(y)\,\mathrm{e}^{-|y|}. \tag{20}$$

Thus,

$$\lim_{t\to\infty} \frac{1}{t} \log \boldsymbol{E}_{\mu}^{\sin}\langle X_t, g\rangle = \frac{1}{2}, \tag{21}$$

provided that $\mu \neq 0$ and $g \neq 0$.

Remark 10 (*Spectral-theoretical connection*). Statement (20) is *formally* in line with the last displayed formula in Theorem 7(b)(ii) of Pinsky (1996), if one takes into account that in a weak sense $\frac{1}{2}$ is the principal eigenvalue of the self-adjoint operator $\frac{1}{2}\Delta + \delta_0$, and $x \mapsto \mathrm{e}^{-|x|}$ is the corresponding normalized positive L^2-eigenfunction. (Note that in the setup there, $\lambda_{\mathrm{c}} + \beta$ is the principal eigenvalue of $L + \beta$.)

Remark 11 (*Generalizations*). Our results on the model with a single point source suggest to deal with the following further question: Verify that the rescaled process $\mathrm{e}^{-t/2}X_t$ *itself has a limit* in law as $t \to \infty$ (instead of considering only its expectation). Also, deal with more general *non-regular* coefficients β.

1.5. Outline

The remainder of this paper is organized as follows. In Section 2 we present some auxiliary material. Section 3 gives a pde interpretation of some of the results stated in Section 1.3. Finally, the last section is devoted to the proofs.

For standard facts on superprocesses in general, we refer to Dawson (1993), Dynkin (1993), and for pde to Pinsky (1995).

2. Auxiliary definitions and tools

First we give a short review of some definitions and results for $(L, \beta, \alpha; \mathbb{R}^d)$-superdiffusions which we will need and which can be found in Engländer and Pinsky (1999).

Definition 12 (*Long-term properties*). Consider the $(L, \beta, \alpha; \mathbb{R}^d)$-superdiffusion X with law $\boldsymbol{P}_\mu$ where $\mu \in \mathscr{M}_\mathrm{c} \setminus \{0\}$.

(a) (Compact support property) X possesses the *compact support property if*

$$\boldsymbol{P}_\mu\left(\bigcup_{0 \leqslant s \leqslant t} \mathrm{supp}(X_s) \text{ is bounded}\right) = 1, \quad \text{for all } t \geqslant 0. \tag{22}$$

(b) (Recurrence) X is said to be *recurrent* if

$$\boldsymbol{P}_\mu(X_t(B) > 0 \text{ for some } t \geqslant 0 \mid E^\mathrm{c}) = 1 \tag{23}$$

for every (non-empty) open ball $B \subset \mathbb{R}^d$. Here E^c denotes the complement of the event that X becomes extinct. (Roughly speaking, each ball is charged, given survival.)

(c) (Transience) X is called *transient* if

$$\boldsymbol{P}_\mu(X_t(B) > 0 \text{ for some } t \geqslant 0 \mid E^\mathrm{c}) < 1 \tag{24}$$

for certain sets B which are specified as follows:

- if $d \geqslant 2$: all open balls $B \subset \mathbb{R}^d$ such that $\bar{B} \cap \mathrm{supp}(\mu) = \emptyset$;
- if $d = 1$: all finite intervals $B \subset \mathbb{R}$ satisfying $\sup B < \inf \mathrm{supp}(\mu)$, or all finite intervals $B \subset \mathbb{R}$ satisfying $\inf B > \sup \mathrm{supp}(\mu)$.

In Engländer and Pinsky (1999) it is shown that the $(L, \beta, \alpha; \mathbb{R}^d)$-superdiffusion X with law $\boldsymbol{P}_\mu$ is either recurrent or transient, and that if any one of the properties in Definition 12 holds for some $\boldsymbol{P}_\mu, \mu \in \mathscr{M}_\mathrm{c} \setminus \{0\}$, then it in fact holds for every $\boldsymbol{P}_\mu$, $\mu \in \mathscr{M}_\mathrm{c} \setminus \{0\}$.

We mention that recurrence and transience for superdiffusions were first defined and studied in Pinsky (1996) in the case when α and β are positive constants. But note that in Pinsky (1996), Engländer and Pinsky (1999) and Engländer (2000) the terminology is actually slightly different: Instead of calling X recurrent/transient, the *support* of X is called recurrent/transient respectively.

Definition 13 (*h-Transformed superdiffusion* X^h). Let $0 < h \in C^{2,\gamma}$ and consider the $(L,\beta,\alpha;\mathbb{R}^d)$-superdiffusion X. Define

$$X_t^h := hX_t \quad \left(\text{that is, } \frac{\mathrm{d}X_t^h}{\mathrm{d}X_t} = h\right), \qquad t \geqslant 0. \tag{25}$$

Then X^h is the $(L_0^h, \beta^h, \alpha^h; \mathbb{R}^d)$-superdiffusion, where

$$L_0^h := L + a\,\frac{\nabla h}{h}\cdot\nabla, \quad \beta^h := \frac{(L+\beta)h}{h}, \quad \text{and} \quad \alpha^h := \alpha h. \tag{26}$$

X^h makes sense even if β^h is unbounded from above (see Engländer and Pinsky (1999, Section 2) for more elaboration). X^h is called the h-transformed superdiffusion.

Remark 14 (*h-Transforms*). (i) L_0^h is just the diffusion part of the usual linear h-transformed operator L^h (see Pinsky, 1995, Chapter 4).

(ii) The operators $\mathscr{A}(u) := Lu + \beta u - \alpha u^2$ and $\mathscr{A}^h(u) := L_0^h u + \beta^h u - \alpha^h u^2$ are related by $\mathscr{A}^h(u) = (1/h)\mathscr{A}(hu)$.

Remark 15 (*Invariance under h-transforms*). An obvious but important property of the h-transform is that it leaves invariant the *support process* $t \mapsto \operatorname{supp}(X_t)$ of X. It is also important to point out that extinction, local extinction, recurrence/transience, as well as the compact support property are in fact properties of the support process, and that these properties are therefore invariant under h-transforms.

Remark 16 (*Additive h-transforms*). In the particular case when h satisfies the equation $(L+\beta)h = 0$ on $\mathbb{R}^d$, the superdiffusion X^h coincides with Overbeck's (1994) additive h-transform of X in a time-independent case.

The following lemma collects some more detailed facts taken from Engländer and Pinsky (1999, Theorems 3.1–3.3 and 4.1–4.2). Recall that a diffusion in $\mathbb{R}^d$ is called *conservative*, if (loosely speaking) it has an infinite lifetime in $\mathbb{R}^d$, whereas in the opposite case it can leave $\mathbb{R}^d$ in finite time with positive probability, and one speaks of *explosion*.

Lemma 17 (Details). *Consider the* $(L,\beta,\alpha;\mathbb{R}^d)$*-superdiffusion* X.

(a) (w-Function and extinction) *There exists a function* $w: \mathbb{R}^d \mapsto \mathbb{R}_+$ *which solves the "stationary" equation*

$$Lu + \beta u - \alpha u^2 = 0 \quad \text{on } \mathbb{R}^d, \tag{27}$$

and for which

$$\boldsymbol{P}_\mu(X \text{ becomes extinct}) = \mathrm{e}^{-\langle \mu, w\rangle}, \quad \mu \in \mathscr{M}_\mathrm{c}. \tag{28}$$

If $\inf_{\mathbb{R}^d}\alpha > 0$ *and* $\beta \leqslant 0$ *then* $w = 0$. *On the other hand, if* $w \neq 0$, *then* w *is actually positive. Also, if* L *corresponds to a conservative diffusion on* $\mathbb{R}^d$, *and* α *and* β *are constants, then* $w = (\beta \vee 0)/\alpha$.

(b) ($w_{\max}$ and the compact support property) *There exists a maximal non-negative solution* $w_{\max}$ *to* (27). *Furthermore,* $w_{\max} = w$ *with the function* w *from* (a) *if* X

has the compact support property. Finally, if $w = 0$, *then* $w_{\max} = 0$ *if and only if* X *has the compact support property.*

(c) ($\varphi_{\min}$ and recurrence/transience) *Take an open ball* $B \subset \mathbb{R}^d$. *There exists a minimal positive solution* $\varphi_{\min}$ *to*

$$Lu + \beta u - \alpha u^2 = 0 \quad \text{on } \mathbb{R}^d \setminus \bar{B},$$
$$\lim_{x \to \partial B} u(x) = \infty. \tag{29}$$

Moreover, exactly one of the following two possibilities occurs:

(c1) $\varphi_{\min} > w$ *on* $\mathbb{R}^d \setminus \bar{B}$ *for any open ball* B, *and* X *is recurrent.*

(c2) $\liminf_{|x| \to \infty} (\varphi_{\min}/w)(x) = \inf_{x \in \mathbb{R}^d \setminus \bar{B}} (\varphi_{\min}/w)(x) = 0$ *for any open ball* B, *and* X *is transient.*

Remark 18 (*Construction of* $\varphi_{\min}$). Take balls $B_n \supset \bar{B}$ centered at the origin and with (sufficiently large) radius n, where B is from Lemma 17(c). Moreover, let φ_n be the unique solution to

$$Lu + \beta u - \alpha u^2 = 0 \quad \text{on } B_n \setminus \bar{B},$$
$$u = n \quad \text{on } \partial B,$$
$$u = 0 \quad \text{on } \partial B_n. \tag{30}$$

Then $\varphi_{\min} = \lim_{n \to \infty} \varphi_n$ (see Pinsky, 1996, p. 250).

For relations between extinction and the compactness of the range of super-Brownian motions with constant β but otherwise general branching mechanism, see Sheu (1997).

3. A pde interpretation of some of our results

Recall that $\beta \in C^\gamma$ is assumed to be bounded from above. Consider the following two possibilities:

(I) There is no positive solution to $(\frac{1}{2}\Delta + \beta)u = 0$ on $\mathbb{R}^d$.

(II) There exists a positive solution to $\frac{1}{2}\Delta u + \beta u - u^2 = 0$ on $\mathbb{R}^d$.

By Lemma 4, case (I) is equivalent to exhibiting no local extinction for the $(\frac{1}{2}\Delta, \beta, 1, \mathbb{R}^d)$-superdiffusion X. In the light of this correspondence we point out that conditions for (I) like the ones appearing in Theorem 1 and Remark 6 are, of course, well-known from standard pde literature. By Engländer and Pinsky (1999, Theorem 3.5), the compact support property holds for X, and thus, by Lemma 17(b), $w = w_{\max}$, where w and $w_{\max}$ are defined in (a) and (b) of Lemma 17 respectively. Putting this together with the first sentence in Lemma 17(a) gives the following result:

Lemma 19 (No extinction). *Statement* (II) *is satisfied if and only if extinction does not hold for* X.

Using this together with Theorem 2, we immediately obtain the following relations between (I) and (II), respectively condition on (II); we omit the trivial proof.

Corollary 20 (Relations between (I) and (II)).
(a) (I) *implies* (II).
(b) *Statements* (I) *and* (II) *are equivalent if* $d \leqslant 2$, *or if* $\beta(x) \leqslant k_d/|x|^2$ *for* $|x| \geqslant 1$ [*with* k_d *defined in* (10)].
(c) (II) *holds, if* $\beta(x) \geqslant k/|x|^2$ *for* $|x| \geqslant 1$ *and some* $k > k_d$.

4. Proofs

4.1. Preparation

We will utilize the following two lemmata.

Lemma 21 (Condition for extinction). *X becomes extinct if all of the following conditions are true*:
(i) *the* $(L, \beta, \alpha; \mathbb{R}^d)$*-superdiffusion* X *exhibits local extinction*,
(ii) $\beta \leqslant 0$ *outside a compact set*, *and*
(iii) $\inf_{\mathbb{R}^d} \alpha > 0$.

Lemma 22 (Condition for non-extinction). *Let* X^i *be the* $(L_i, \beta_i, \alpha_i; \mathbb{R}^d)$*-superdiffusions*, $i = 1, 2$, *and assume that*, *outside a compact set*, α_1, β_1, *and the coefficients of* L_1 *coincide with* α_2, β_2, *and the coefficients of* L_2 *respectively. Furthermore*, *assume that*
(i) X^2 *does not become extinct*, *and*
(ii) X^2 *is transient.*
Then X^1 *does not become extinct either.*

For the proofs of the Lemmas 21 and 22, we refer to Engländer (2000, Theorem 1.1), more precisely, to the proof of part (a) and to the end of the proof of part (b) there respectively.

4.2. Proof of Proposition 5 and Theorem 1

Proof of Proposition 5. Take h and B as in the proposition, and consider the h-transformed superdiffusion X^h according to Definition 13. Then, by assumption, $\beta^h \leqslant 0$ on $\mathbb{R}^d \backslash \bar{B}$. Note that $\alpha^h = \alpha h$, and thus α^h is bounded away from 0, also by assumption. Since X exhibits local extinction, also X^h does, and from Lemma 21 it follows that X^h becomes extinct. Then the same is true for X. □

Remark 23 (*Monotonicity*). We will use the following *comparison*, for simplicity we refer to this as "monotonicity": If $\beta_1 \leqslant \beta_2$ and there is no positive solution for the equation $(\frac{1}{2}\Delta + \beta_1)v = 0$ on $\mathbb{R}^d$, then there is no positive solution to $(\frac{1}{2}\Delta + \beta_2)v = 0$ on $\mathbb{R}^d$ either. In fact, similar to the discussion preceding Theorem 1, the non-existence of positive solutions for $(\frac{1}{2}\Delta + \beta)u = 0$ on $\mathbb{R}^d$ is equivalent to $\lambda_c^{(\beta)} > 0$, where $\lambda_c^{(\beta)}$ denotes the so-called generalized principal eigenvalue of $\frac{1}{2}\Delta + \beta$ on $\mathbb{R}^d$. Use next the following probabilistic characterization of $\lambda_c^{(\beta)}$ (Pinsky, 1995, Theorem 4.4.6): For all $x \in \mathbb{R}^d$,

$$\lambda_c^{(\beta)} = \sup_{A \subset\subset \mathbb{R}^d} \lim_{t\to\infty} \frac{1}{t} \log E_x \left\{ \exp\left[\int_0^t \mathrm{d}s\, \beta(Y_s) \right]; \tau_A > t \right\}, \tag{31}$$

where $A \subset\subset \mathbb{R}^d$ means that $A \subset \mathbb{R}^d$ is a smooth bounded domain $(Y, P_x, x \in \mathbb{R}^d)$ denotes a standard Brownian motion, as well as $\tau_A := \inf\{t \geqslant 0\colon Y_t \notin A\}$. From (31) it is immediate that $\lambda_c^{(\beta)}$ is monotone non-decreasing in β. This implies the mentioned monotonicity.

Proof of Remark 6. Let $d = 1$. By Lemma 4 it is sufficient to show that there is no positive solution to the equation $(\frac{1}{2}\Delta + \beta)u = 0$ on $\mathbb{R}$. We may assume, that $\beta(x) \geqslant K/x^2$, $x \gg 1$, where $K > \frac{1}{8}$. By monotonicity (Remark 23), it is enough to verify the statement for

$$\beta(x) = K/x^2, \quad x \gg 1. \tag{32}$$

Suppose on the contrary that there exists a function $f > 0$ satisfying

$$\frac{1}{2} f'' + \beta f = 0. \tag{33}$$

Then

$$\frac{1}{2} f'' + (K/x^2) f = 0 \quad \text{for } x \gg 1. \tag{34}$$

But the two-dimensional space of complex solutions to this equation is spanned by the power functions x^{ϱ_+} and x^{ϱ_-}, where $\varrho_\pm = \frac{1}{2}(1 \pm \sqrt{1 - 8K})$. Since $\mathrm{Im}(\varrho_\pm) \neq 0$, there is no positive solution, getting a contradiction. This already finishes the proof. □

Proof of Theorem 1. We will prove both parts of the theorem in the reversed order.

(b) Because of the proof of Remark 6, we can assume that $d \geqslant 2$. Recall that it suffices to show that there is no positive solution to the equation $(\frac{1}{2}\Delta + \beta)u = 0$ on $\mathbb{R}^d$. Again, by monotonicity, it is enough to verify the statement for

$$\beta(x) = K/|x|^2, \quad x \gg 1. \tag{35}$$

Suppose that there exists a function $f > 0$ satisfying $\frac{1}{2}\Delta f + \beta f = 0$ in $\mathbb{R}^d$. Then $\frac{1}{2}\Delta f + (K/|x|^2) f = 0$ on some annulus of the form $\{x \in \mathbb{R}^d\colon |x| > c\}$, $c > 0$. Using a scaling argument, it then follows that there exists a positive solution to $\frac{1}{2}\Delta f + (K/|x|^2) f = 0$ on any annulus of the above form. Then, by a compactness argument, there exists a positive solution on $\mathbb{R}^d \setminus \{0\}$ as well. [For compactness arguments, see Pinsky (1995, Chapter 4), in particular the proofs of Theorems 4.2.5, 4.3.1 and 4.3.2(c)]. But this is known to be false (see Pinsky, 1995, Example 4.3.12). Consequently, part (b) of Theorem 1 is proved.

(a) Assume that

$$\beta(x) \leqslant K_d/|x|^2 \quad \text{for } |x| \gg 1, \tag{36}$$

and let h be a positive $C^{2,\gamma}$-function satisfying

$$h(x) = |x|^{-(d-2)/2} \quad \text{for } |x| \geqslant 1. \tag{37}$$

Note that

$$\frac{\frac{1}{2}\Delta h}{h} = -K_d \frac{1}{|x|^2} \quad \text{for } |x| \geqslant 1. \tag{38}$$

Moreover, let $\hat{\beta} \leqslant 0$ be a C^{γ}-function satisfying

$$\hat{\beta}(x) = \beta(x) - K_d/|x|^2, \qquad |x| \geqslant 1. \tag{39}$$

(The existence of such a $\hat{\beta}$ is guaranteed by the growth rate assumption on β). Define

$$\beta^* := \hat{\beta} - \frac{1}{2}\frac{\Delta h}{h}. \tag{40}$$

It is easy to see that β^* belongs to C^{γ}, and moreover, using (38) and (39) we have

$$\beta^*(x) = \beta(x) \quad \text{for } |x| \geqslant 1. \tag{41}$$

Taking the linear h-transform (see Pinsky, 1995, Chapter 4) of the operator

$$\tfrac{1}{2}\Delta + \beta^*, \tag{42}$$

we get

$$\frac{1}{2}\Delta + \frac{\nabla h}{h} \cdot \nabla + \hat{\beta}. \tag{43}$$

Since $\hat{\beta} \leqslant 0$, it is well-known (see, e.g. Pinsky, 1995, Theorem 4.3.3(iii)) that there exists a positive solution for

$$\left(\frac{1}{2}\Delta + \frac{\nabla h}{h} \cdot \nabla + \hat{\beta}\right) u = 0 \quad \text{on } \mathbb{R}^d. \tag{44}$$

Therefore,

$$(\tfrac{1}{2}\Delta + \beta^*)(hu) = 0 \tag{45}$$

[recall Remark 14(ii)], and thus, by Lemma 4, the $(\frac{1}{2}\Delta, \beta^*, 1; \mathbb{R}^d)$-superdiffusion X^* exhibits local extinction, finishing the proof. □

4.3. *Proof of Theorem 2*

(a) Let $d \leqslant 2$, and suppose to the contrary that X does not become extinct but exhibits local extinction. Since β is bounded from above, using the recurrence of the Brownian motion and Theorem 4.5(a) of Engländer and Pinsky (1999), it follows that X is recurrent. But this contradicts the local extinction (see the remark after Theorem 4.2 in Engländer and Pinsky (1999)), giving the claim (a).

(b) If $d \leqslant 2$, then the statement follows from (a).

Assume now that $3 \leqslant d \leqslant 6$ and that X exhibits local extinction. Take h as in the proof of Theorem 1(a). Similarly to (42) and (43), the operator $\frac{1}{2}\Delta + \beta$ transforms into

$$\frac{1}{2}\Delta + \frac{\nabla h}{h} \cdot \nabla + \beta^h \quad \text{where } \beta^h = \hat{\beta} \leqslant 0 \quad \text{for } |x| \geqslant 1 \tag{46}$$

with $\hat{\beta}$ as in (39). Moreover, the whole quadruple $(\frac{1}{2}\Delta, \beta, 1; \mathbb{R}^d)$ corresponding to X is transformed into

$$\left(\frac{1}{2}\Delta + \frac{\nabla h}{h}\cdot\nabla, \beta^h, h; \mathbb{R}^d\right) \tag{47}$$

associated with X^h.

According to Engländer and Pinsky (1999, Theorem 3.5), the compact support property holds for X, thus the same is true for X^h. Therefore, using Lemma 17(b), it follows that the extinction of X^h is equivalent to the non-existence of positive solutions for the semi-linear elliptic equation (27). Dividing by h, we see that X^h (and also X) becomes extinct if and only if there is no positive solution to

$$\frac{1}{2h}\Delta u + \frac{\nabla h}{h^2}\cdot\nabla u + \frac{\beta^h}{h}u - u^2 = 0 \quad \text{on } \mathbb{R}^d, \tag{48}$$

that is, if and only if the corresponding maximal solution $w_{\max}$ is zero. Hence, our goal is to verify that $w_{\max} = 0$. We will do this in two steps.

Let $\tilde{X}$ denote the superdiffusion corresponding to the quadruple

$$\left(\frac{1}{2h}\Delta + \frac{\nabla h}{h^2}\cdot\nabla, \frac{\beta^h}{h}, 1, \mathbb{R}^d\right). \tag{49}$$

First we will show that $\tilde{X}$ becomes extinct, that, is the w-function of Lemma 17(a) is zero. In the second step we prove that $w = w_{\max}$, giving then the required $w_{\max} = 0$.

For the first statement, note that by the local extinction assumption on X and Lemma 4,

$$(\tfrac{1}{2}\Delta + \beta)u = 0 \quad \text{with some } u > 0. \tag{50}$$

By Remark 14(ii) then

$$\left(\frac{1}{2h}\Delta + \frac{\nabla h}{h^2}\cdot\nabla + \frac{\beta^h}{h}\right)\frac{u}{h} = 0, \tag{51}$$

and therefore by Lemma 4, also $\tilde{X}$ exhibits local extinction. Since $\beta^h \leqslant 0$ for $|x| \gg 1$, and $\alpha = 1$, Lemma 21 yields that $\tilde{X}$ becomes extinct.

For the present $3 \leqslant d \leqslant 6$ part, it remains to show that $w = w_{\max}$. By Lemma 17(b), it is enough to verify that the compact support property holds for $\tilde{X}$. Since in particular $d \leqslant 6$, for the diffusion coefficient in (49) we have

$$\frac{1}{2h(x)} = \mathrm{O}(|x|^2) \quad \text{as } |x| \to \infty. \tag{52}$$

Using this, the fact that the gradient vector $(\nabla h/h^2)(x)$ has non-positive coordinates for $|x| \gg 1$, and that β^h/h is bounded from above (non-positive outside a compact set), the compact support property is implied by Engländer and Pinsky (1999, Theorem 3.5).

Assume now that $d > 6$. Take an $h \in C^{2,\gamma}$ satisfying $h(x) = |x|^{-2}$ for $|x| \gg 1$. Resolving the Laplacian in radial form, an elementary computation shows that if $\beta(x)|x|^2 \leqslant d - 4$ is satisfied for $|x| \gg 1$, then

(i) $(\frac{1}{2}\Delta + \beta)h(x) \leqslant 0$ and

(ii) the gradient vector $\nabla h(x)$ has non-positive coordinates for $|x| \geqslant 1$. Then the rest of the proof works similarly as in the case $3 \leqslant d \leqslant 6$. Indeed, reading carefully the proof, one can see that it relies only on the fact that the h chosen there satisfies (i) and (ii) of the present case as well as the asymptotics (52). Indeed, we replaced the previous h by the present one in order to guarantee (52) for $d > 6$. This completes the proof of (b).

(c) Obviously, we can assume that $d > 6$, otherwise the assertion follows from Theorem 1(b). Also, by comparison, we can set

$$\beta(x)|x|^2 = d - 4 + \varepsilon_0 \quad \text{for } |x| \geqslant 1, \tag{53}$$

with some $0 < \varepsilon_0 \leqslant \frac{1}{8}$. In fact, for the comparison one has to check that for larger β we have a larger w-function, that is, less chance for extinction. This can easily be seen from the construction of the w-function and the parabolic maximum principle (see Engländer and Pinsky (1999), Theorem 3.1 and Proposition 7.2 respectively).

Let h be a radially symmetric positive $C^{2,\gamma}$-function satisfying

$$h(x) = |x|^{-2} \quad \text{for } |x| \geqslant 1. \tag{54}$$

Making the h-transform and dividing by h in the quadruple corresponding to X, we obtain the quadruple (49) [but now with h as in (54)]. Let X^1 denote the corresponding superdiffusion. Note, that by a simple computation, $\beta^h/h = \varepsilon_0$ outside a large closed ball $B \subset \mathbb{R}^d$.

Similarly to the argument preceding (48), the extinction of X is equivalent to the non-existence of a positive solution to Eq. (48) [but now with h as in (54)]. Our goal is to prove that extinction does not hold for X^1. In fact, then by Lemma 17(a), the corresponding w-function is a positive solution to (48).

Next recall that for diffusions corresponding to

$$L := \frac{1}{2} a(x) \frac{\mathrm{d}^2}{\mathrm{d}x^2} + b(x) \frac{\mathrm{d}}{\mathrm{d}x} \quad \text{on } \mathbb{R}, \tag{55}$$

where $a > 0$ and b are smooth, one can decide whether or not explosion occurs by checking the finiteness of two integrals involving $a(x)$ and $b(x)$ for $|x| \geqslant 1$. This is known as *Feller's test for explosion*, see, e.g. Pinsky (1995, Theorem 5.1.5)

By (54), we have

$$\frac{\nabla h}{h^2}(x) = -2x, \quad |x| \geqslant 1. \tag{56}$$

Using this and polar coordinates along with the just mentioned Feller's test for explosion, we conclude that the operator

$$\frac{1}{2h}\Delta + \frac{\nabla h}{h^2} \cdot \nabla$$

corresponds to a conservative diffusion on $\mathbb{R}^d$. Thus, by the last part of Lemma 17(a) applied to X^2, which denotes the superdiffusion corresponding to the quadruple

$$\left(\frac{1}{2h}\Delta + \frac{\nabla h}{h^2} \cdot \nabla, \varepsilon_0, 1, \mathbb{R}^d \right), \tag{57}$$

we obtain $w(x) \equiv \varepsilon_0$. In particular, X^2 does not become extinct.

Applying Lemma 22 to X^1 and X^2 it will suffice to show that the latter process is transient. Then non-extinction of X^1 will follow.

Consider the $\varphi_{\min}$-function according to Lemma 17(c) applied to X and with the ball B introduced above. Resolving the Laplacian in radial form, and recalling that $\varepsilon_0 \leqslant \frac{1}{8}$, a simple computation reveals that if $\varepsilon > 0$ satisfies

$$\varepsilon^2 + (6-d)\varepsilon + 2\varepsilon_0 \leqslant 0, \tag{58}$$

(this is possible because $d \geqslant 7$) then $u(x) = |x|^{-2-\varepsilon}$ satisfies

$$\frac{1}{2}\Delta u + (d-4+\varepsilon_0)u - u^2 \leqslant 0, \quad |x| \gg 1. \tag{59}$$

Thus, by the elliptic maximum principle (Engländer and Pinsky, 1999, Proposition 7.1) and Remark 18, there exists a constant $c > 0$ such that

$$\varphi_{\min}(x) \leqslant cu(x), \quad |x| \gg 1. \tag{60}$$

(Cf. the end of the proof of Theorem 4.2 in Engländer and Pinsky (1999)). Since $\varphi^h_{\min} = \varphi_{\min}/h$ by Remark 14(ii), the $\varphi_{\min}$-function for X^1 (and also for X^2) on $\mathbb{R}^d \backslash \bar{B}$ is $\varphi_{\min}/h$. Putting this together with (54) and (60), the $\varphi_{\min}$-function for X^2 tends to zero as $|x| \to \infty$. Therefore

$$\lim_{|x|\to\infty} \frac{\varphi_{\min}(x)}{w(x)} = 0 \quad \text{for } X^2. \tag{61}$$

Thus, X^2 is transient, by Lemma 17(c2). This completes the proof of (c) and of Theorem 2 altogether. □

Before giving the proof of Remark 6, we recall some standard facts. First of all, the Laplacian $\frac{1}{2}\Delta$ on $\mathbb{R}^d$ corresponds to a recurrent or transient Brownian motion according to whether $d \leqslant 2$ or $d > 2$. The former case is a special case of a so-called *critical* operator, the latter of a *subcritical* one. The subcriticality (criticality) of the Laplacian means that it has a positive Green's function (or it does not). For more elaboration see Pinsky (1995, Chapter 4). The operator $\frac{1}{2}\Delta + \beta$ is called a perturbation of the Laplacian. For a general second-order elliptic operator L (instead of the Laplacian), it is known that the perturbed operator $L + \beta$ exhibits different qualitative behavior for critical or subcritical L. In the Laplacian case, this fact will be explained in more detail and utilized in the following proof.

Proof of Remark 9. First, let $d \leqslant 2$. By Pinsky (1995, Theorem 4.6.3(i)), there is no positive solution to the equation $(\frac{1}{2}\Delta + \beta)u = 0$ on $\mathbb{R}^d$. Thus, the statement is true by Lemma 4. On the other hand, if $d \geqslant 3$, $\beta \geqslant 0$, $\beta \neq 0$, and β is compactly supported, then by Pinsky (1995, Theorem 4.6.2), there exists an $\varepsilon > 0$ and a function $u > 0$ such that $(\frac{1}{2}\Delta + \varepsilon\beta)u = 0$ on $\mathbb{R}^d$. Then, by Lemma 4, the $(\frac{1}{2}\Delta, \varepsilon\beta, 1, \mathbb{R}^d)$-superdiffusion X exhibits local extinction, hence by Lemma 21 it even becomes extinct.

4.4. Proof of Theorem 3

We need a lemma. Define the δ_0-regularization

$$\beta_\varepsilon(x) := \frac{1}{\varepsilon}\beta\left(\frac{x}{\varepsilon}\right), \quad \varepsilon > 0,\ x \in \mathbb{R}, \tag{62}$$

where $\beta \geqslant 0$ is a compactly supported non-vanishing smooth symmetric function with $\beta'(x) \leqslant 0$ for $x \geqslant 0$.

Lemma 24 (Subsolutions for approximating equations). *There is a number $\ell > 0$ and there are functions $v_\varepsilon^- = v_{\varepsilon,\ell}^-$, $\varepsilon > 0$, defined on the interval $D_\ell := (-\ell, \ell)$, such that, for ε sufficiently small,*

(i) $v_\varepsilon^- \geqslant 0$, *and* $v_\varepsilon^- = 0$ *on* $\partial D_\ell := \{\pm \ell\}$,
(ii) $\frac{1}{2}(v_\varepsilon^-)'' + \beta_\varepsilon v_\varepsilon^- - (v_\varepsilon^-)^2 \geqslant 0$ *on* D_ℓ,
(iii) $\sup_{D_\ell} v_\varepsilon^- = v_\varepsilon^-(0)$,

and that $v_\varepsilon^-(0)$ is bounded away from zero as $\varepsilon \downarrow 0$.

Proof. Denote by λ_ε^ℓ the principal eigenvalue for $\frac{1}{2}\Delta + \beta_\varepsilon$ on D_ℓ with zero boundary condition and with corresponding eigenfunction $\psi_\varepsilon^\ell > 0$. Furthermore, denote by λ^ℓ the principal eigenvalue for $\frac{1}{2}\Delta$ on D_ℓ with zero boundary condition and with corresponding eigenfunction $\psi_\ell > 0$, where ψ_ℓ has been normalized by $\int_{D_\ell} \mathrm{d}x\, \psi_\ell^2(x) = 1$. In other words,

$$\psi_\ell(x) = \frac{1}{\sqrt{\ell}} \cos\left(\frac{\pi x}{2\ell}\right) \quad \text{and} \quad \lambda^\ell = -\frac{\pi^2}{8\ell^2}. \tag{63}$$

Define

$$v_{\varepsilon,\ell}^- := \frac{\lambda_\varepsilon^\ell}{\sup_{D_\ell} \psi_\varepsilon^\ell} \psi_\varepsilon^\ell \quad \text{on } D_\ell. \tag{64}$$

Then $v_{\varepsilon,\ell}^-$ satisfies the boundary condition in (i), and a simple computation shows that (ii) also holds. We are going to show that there exists an $\ell > 0$ such that $\liminf_{\varepsilon \downarrow 0} \lambda_\varepsilon^\ell > 0$. This will prove that $v_{\varepsilon,\ell}^- \geqslant 0$ for ε sufficiently small and that $\sup_{D_\ell} v_{\varepsilon,\ell}^-$ is bounded away from zero as $\varepsilon \downarrow 0$. In order to do this, we invoke the following minimax representation of λ_ε^ℓ (see Pinsky, 1995, Theorem 3.7.1):

$$\lambda_\varepsilon^\ell = \sup_\mu \inf_{\substack{u > 0 \text{ on } D_\ell \\ u \in C^2(D_\ell)}} \int_{D_\ell} \mu(\mathrm{d}x) \left(\frac{1}{2}\frac{u''}{u} + \beta_\varepsilon\right)(x), \tag{65}$$

where the supremum is taken over all probability measures μ on D_ℓ with densities f satisfying $\sqrt{f} \in C^1(\overline{D_\ell})$ and $f(\pm\ell) \equiv 0$. (Of course, C^m, $m \geqslant 1$, refers to the set of all m-times continuously differentiable functions.) Take $\mu(\mathrm{d}x) = \psi_\ell^2(x)\,\mathrm{d}x$ in (65). Then,

$$\lambda_\varepsilon^\ell \geqslant \inf_{0 < u \in C^2(D_\ell)} \int_{D_\ell} \mathrm{d}x\, \frac{1}{2}\frac{u''}{u} \psi_\ell^2 + \int_{D_\ell} \mathrm{d}x\, \beta_\varepsilon \psi_\ell^2 =: I + II \tag{66}$$

(with the obvious correspondence). Using Pinsky (1995, Theorem 3.7.1) again, we get $I = \lambda^\ell$. Thus

$$\lambda_\varepsilon^\ell \geqslant \lambda^\ell + \int_{D_\ell} \mathrm{d}x\, \beta_\varepsilon \psi_\ell^2 = -\frac{\pi^2}{8\ell^2} + \int_{D_\ell} \mathrm{d}x\, \frac{1}{\ell} \cos^2\left(\frac{\pi x}{2\ell}\right) \beta_\varepsilon(x). \tag{67}$$

Since

$$\beta_\varepsilon(x)\,\mathrm{d}x \to \delta_0(\mathrm{d}x) \quad \text{weakly as } \varepsilon \downarrow 0, \tag{68}$$

the latter inequality yields that $\liminf_{\varepsilon \downarrow 0} \lambda_\varepsilon^\ell > 0$, provided that ℓ is sufficiently large.

It remains to show that $\sup_{D_\ell}\psi_\varepsilon^\ell = \psi_\varepsilon^\ell(0)$ and consequently $\sup_{D_\ell} v_{\varepsilon,\ell}^- = v_{\varepsilon,\ell}^-(0)$. For this purpose, we consider the equation

$$\tfrac{1}{2}(\psi_\varepsilon^\ell)'' = (\lambda_\varepsilon^\ell - \beta_\varepsilon)\psi_\varepsilon^\ell. \tag{69}$$

Clearly, $(\psi_\varepsilon^\ell)''(x) \geqslant 0$ if and only if $\beta_\varepsilon(x) \leqslant \lambda_\varepsilon^\ell$, and consequently

$$\lambda_\varepsilon^\ell \leqslant \sup_{D_\ell} \beta_\varepsilon = \beta_\varepsilon(0). \tag{70}$$

Putting this together with the positivity, symmetry and compact support of ψ_ε^ℓ, we conclude that $\sup_{D_\ell}\psi_\varepsilon^\ell = \psi_\varepsilon^\ell(0)$. This completes the proof of the lemma. □

Proof of Theorem 3.

Step 1: Let $\ell > 0$ and $v_\varepsilon^- = v_{\varepsilon,\ell}^-$ be as in Lemma 24. By that lemma, one can pick a constant $c > 0$ such that

$$\sup_{D_\ell} v_\varepsilon^- = v_\varepsilon^-(0) > c \quad \text{for all small } \varepsilon > 0. \tag{71}$$

Fix a non-negative continuous function g satisfying

$$g = c \quad \text{on } D_\ell \quad \text{and} \quad g = 0 \quad \text{on } \mathbb{R}\backslash D_{2\ell}. \tag{72}$$

Put

$$u_\varepsilon^- := \frac{c\cdot v_\varepsilon^-}{\sup_{D_\ell} v_\varepsilon^-}. \tag{73}$$

Note, that $u_\varepsilon^-(0)=c$ by Lemma 24(iii). Using (i)–(ii) of the same lemma and statement (71), an easy computation shows that, for $\varepsilon > 0$ sufficiently small, u_ε^- satisfies

$$\begin{aligned}
&\tfrac{1}{2}(u_\varepsilon^-)'' + \beta_\varepsilon u_\varepsilon^- - (u_\varepsilon^-)^2 \geqslant 0 \quad \text{on } D_\ell,\\
&u_\varepsilon^-(x) \leqslant g(x) \quad \text{on } D_\ell,\\
&u_\varepsilon^- = 0 \quad \text{on } \partial D_\ell.
\end{aligned} \tag{74}$$

Then, by the parabolic maximum principle [Engländer and Pinsky, 1999, Proposition 7.2], for all $\varepsilon > 0$ small enough,

$$u_\varepsilon^-(\cdot) \leqslant u_\varepsilon^g(t,\cdot), \quad t \geqslant 0, \tag{75}$$

where u_ε^g denotes the minimal non-negative solution to the evolution equation (4) with $d = 1$, $L = \frac{1}{2}\Delta$, β replaced by β_ε, $\alpha = 1$, and g from (72).

Step 2: First we verify the claim in the special case $\mu = r\delta_0$ with $r > 0$. Let $\boldsymbol{E}^\varepsilon$ denote the expectations corresponding to the $(\frac{1}{2}\Delta, \beta_\varepsilon, 1; \mathbb{R})$-superdiffusion. By (3) specialized to the present case, (75), and using

$$u_\varepsilon^-(0) \equiv c > 0, \tag{76}$$

we obtain for all $\varepsilon > 0$ small enough and $t > 0$,

$$\boldsymbol{E}_{r\delta_0}^\varepsilon \exp\langle X_t, -g\rangle = \exp[-ru_\varepsilon^g(t,0)] \leqslant \exp[-ru_\varepsilon^-(0)] = \mathrm{e}^{-rc}. \tag{77}$$

Since this holds for all $\varepsilon > 0$ small and $t > 0$, letting $\varepsilon \downarrow 0$ we get

$$\boldsymbol{E}_{r\delta_0}^{\sin} \exp\langle X_t, -g\rangle \leqslant \mathrm{e}^{-rc} < 1, \quad t > 0. \tag{78}$$

Assume for the moment that

$$\boldsymbol{P}_{r\delta_0}^{\sin}(X_t(D_{2\ell}) = 0 \text{ for all large } t) = 1, \tag{79}$$

then the left-hand side of (78) tends to one as $t \to \infty$, and this is a contradiction. Consequently, the super-Brownian motion X with law $\boldsymbol{P}_{r\delta_0}^{\sin}$ does not exhibit local extinction.

Step 3: Before turning to general starting measures, we need a slight generalization of (78). To this end, we modify the super-Brownian motion X with law $\boldsymbol{P}_{r\delta_0}^{\sin}$ a bit: Instead of starting at time 0 with the measure $r\delta_0$, we choose a starting time s according to a non-vanishing finite measure $\eta(\mathrm{d}s)$ on $\mathbb{R}_+$. Then, by definition (see, for instance Dynkin, 1991a, Theorem 1.1),

$$\boldsymbol{E}_{\eta}^{\sin} \exp\langle X_t, -g\rangle = \exp\left[-\int_{[0,t]} \eta(\mathrm{d}s) u(0, t-s)\right], \quad t \geqslant 0, \tag{80}$$

with u satisfying the integral equation (14) with g from (72). Moreover, by the estimate (75) and by (76), instead of (78) we then get

$$\boldsymbol{E}_{\eta}^{\sin} \exp\langle X_t, -g\rangle \leqslant \exp[-\eta([0,t])c] < 1, \quad t \geqslant 1. \tag{81}$$

Step 4: Finally, consider our original super-Brownian motion X with general starting measure $\mu \in \mathcal{M}_{\mathrm{f}} \backslash \{0\}$ (at time 0). Intuitively, the claim follows from the previous step, since one-dimensional Brownian motion (the underlying particles' motion law) reaches the state zero in finite time a.s. To be a bit more formal, we use Dynkin's *stopped* (*or exit*) *measures* X_τ and their so-called *special Markov property* (see Dynkin, 1991a). In the present case, τ is the Brownian (first) hitting time of 0, where the additional mass source is sitting. Having in mind a historical setting of the super-Brownian motion X (see, for instance Dawson and Perkins, 1991 or Dynkin, 1991b), then intuitively the present $X_\tau(\mathrm{d}s)$ is a measure on $\mathbb{R}_+$ which describes the mass distribution of all super-Brownian motion's "particles" which hit 0 the first time in the moment $\tau = s$. Of course, the formal description of stopped measures as X_τ along the historical setting and their special Markov property requires some technicalities, but we skip such details here and in the sequel, and trust the readers imagination. Now,

$$\boldsymbol{E}_{\mu}^{\sin} \exp\langle X_t, -g\rangle = \boldsymbol{E}_{\mu}^{\sin} \boldsymbol{E}_{\mu}^{\sin}\{\exp\langle X_t, -g\rangle \mid \mathcal{G}_{\tau\wedge t}\} \tag{82}$$

where $\mathcal{G}_{\tau\wedge t}$ denotes the pre-$(\tau \wedge t)$ σ-field (concerning the stopped historical super-Brownian motion and the Brownian stopping time $\tau \wedge t$). By the special Markov property, given the "history" $\mathcal{G}_{\tau\wedge t}$, the process starts anew with the measure $X_{\tau\wedge t}$ concentrated on $[0,t]$. That is, (82) can be continued with

$$= \boldsymbol{E}_{\mu}^{\sin} \boldsymbol{E}_{X_{\tau\wedge t}}^{\sin} \exp\langle X_t, -g\rangle. \tag{83}$$

But now we can apply (81) to continue with

$$\leqslant \boldsymbol{E}_{\mu}^{\sin} \exp[-X_{\tau\wedge t}([0,t])c]. \tag{84}$$

However, as $t \to \infty$, the right-hand side converges to

$$\boldsymbol{E}_{\mu}^{\sin} \exp[-\|X_\tau\|c] \leqslant \boldsymbol{E}_{\mu} \exp[-\|X_\tau\|c], \tag{85}$$

where $\boldsymbol{E}_\mu$ refers to the $(\frac{1}{2}\Delta, 0, 1, \mathbb{R})$-superdiffusion, the ordinary critical super-Brownian motion. (Indeed, dropping the additional mass source δ_0, we may loose some population mass.) However,

$$\boldsymbol{P}_\mu(\|X_\tau\| \neq 0) > 0 \tag{86}$$

since

$$\boldsymbol{E}_\mu \|X_\tau\| = \|\mu\| > 0 \tag{87}$$

by the expectation formula for X_τ-measures; see, e.g. (1.50a) in Dynkin (1991a) (with $F = 1$). Hence,

$$\boldsymbol{E}_\mu \exp[-\|X_\tau\| c] < 1, \tag{88}$$

and therefore altogether

$$\limsup_{t\to\infty} \boldsymbol{E}_\mu^{\sin} \exp\langle X_t, -g\rangle < 1. \tag{89}$$

Again arguments as in the end of Step 2 will finish the proof. □

4.5. *Proof of Theorem 4*

(a) Fix a bounded continuous g, and set

$$u(t,x) := \boldsymbol{E}_{\delta_x}^{\sin}\langle X_t, g\rangle, \quad x \in \mathbb{R}, \ t \geqslant 0. \tag{90}$$

Using Eq. (14), it is standard to verify the following integral equation for the expectations:

$$u(t,x) = \int_{\mathbb{R}} \mathrm{d}y\, p(t, y-x) g(y) + \int_0^t \mathrm{d}s\, p(t-s, x) u(s, 0), \tag{91}$$

$x \in \mathbb{R}$, $t \geqslant 0$. (Symbolically, $(\partial/\partial t)u = \frac{1}{2}\Delta u + \delta_0 u$ with $u(0,x) = g$.) Setting $g = 1$ and exploiting the notations $u_x(t) := u(t,x)$ and $p_x(t) := p(t,x)$, we realize that u_0 satisfies

$$u_0(t) = 1 + \int_0^t \mathrm{d}s\, p_0(t-s) u_0(s), \quad t \geqslant 0. \tag{92}$$

Taking Laplace transforms on both sides (where the Laplace transform of a function f is denoted by $\hat{f}$), the convolution on the right-hand side transforms into a product. Thus,

$$\widehat{u_0}(\lambda) = \frac{\hat{1}}{1 - \widehat{p_0}(\lambda)} = \frac{1}{\lambda\left(1 - \frac{1}{\sqrt{2\lambda}}\right)}, \quad \lambda > 0, \tag{93}$$

(see, for instance, McCollum and Brown, 1965). By an inverse Laplace transform, we get the formula for $u_0(t)$ as claimed in (19). (To verify this, proceed for instance conversely: Split the integral in (19) at $y = 0$, and use McCollum and Brown (1965), Eqs. (10) and (89).) For $\mu = \delta_x$, $x \in \mathbb{R}$, plug the expression obtained for $u_0(t)$ into (91) to get $u_x(t)$ as needed for (18) in the special case $\mu = \delta_x$. Finally, for general $\mu \in \mathscr{M}_{\mathrm{f}}$,

$$\boldsymbol{E}_\mu^{\sin}\langle X_t, g\rangle = \int_{\mathbb{R}} \mu(\mathrm{d}x) u_x(t), \tag{94}$$

and (18) follows.

(b) Recalling (90), set $F_x(t) := \mathrm{e}^{-t/2}u_x(t)$. Again first we prove the statement for $\mu=\delta_0$. For this purpose, let $C(g) := \int_{\mathbb{R}} \mathrm{d}y\, g(y)\mathrm{e}^{-|y|}$. Our goal is to verify that

$$F_0(t) \to C(g) \quad \text{as } t \to \infty. \tag{95}$$

By a well-known Tauberian theorem [Feller, 1971, formula (13.5.22)], it is enough to show that

$$\widehat{F_0}(\lambda) \sim C(g)\frac{1}{\lambda}, \quad \text{as } \lambda \downarrow 0. \tag{96}$$

Set $k(t) := \int_{\mathbb{R}} \mathrm{d}y\, p_y(t)g(y)$. By a similar computation as in (a), one obtains,

$$\widehat{F_0}(\lambda) = \widehat{u_0}\left(\lambda+\frac{1}{2}\right) = \hat{k}\left(\lambda+\frac{1}{2}\right)\frac{1}{1-\frac{1}{\sqrt{2\lambda+1}}}, \quad \lambda>0. \tag{97}$$

Using Fubini's Theorem,

$$\lim_{\lambda\to 0}\hat{k}\left(\lambda+\frac{1}{2}\right) = \hat{k}\left(\frac{1}{2}\right) = \int_{\mathbb{R}} \mathrm{d}y\, \widehat{p_y}\left(\frac{1}{2}\right)g(y). \tag{98}$$

Since $\widehat{p_y}(1/2) = \mathrm{e}^{-|y|}$ (use, for instance, McCollum and Brown, 1965, formula 507), we get

$$\hat{k}\left(\frac{1}{2}\right) = C(g). \tag{99}$$

Furthermore, an elementary computation shows that

$$\frac{1}{1-\frac{1}{\sqrt{2\lambda+1}}} \sim \frac{1}{\lambda} \quad \text{as } \lambda \downarrow 0. \tag{100}$$

This completes the proof of (b) in the case $\mu=\delta_0$.

For $\mu=\delta_x$, $x\in\mathbb{R}$, use again Eq. (91) and a similar argument as for the former case $x=0$ to obtain

$$\widehat{F_x}(\lambda) \sim \mathrm{e}^{-|x|}\widehat{F_0}(\lambda) \quad \text{as } \lambda \downarrow 0.$$

Finally, apply (96) to arrive at

$$\widehat{F_x}(\lambda) \sim \mathrm{e}^{-|x|}C(g)\frac{1}{\lambda} \quad \text{as } \lambda \downarrow 0$$

instead of (96). By the same Tauberian theorem, we get (20) in the case $\mu=\delta_x$.

Aimed to a general μ, first note that $u_x(t) \leqslant K\boldsymbol{E}^{\sin}_{\delta_x}\|X_t\|$ where K is a bound for g. By (18), at the right-hand side we can pass from x to 0. Finally, $\mathrm{e}^{-t/2}K\boldsymbol{E}^{\sin}_{\delta_0}\|X_t\|$ has a limit as $t\to\infty$ in virtue of (20) in the already proved case. Therefore, $\mathrm{e}^{-t/2}u_x(t)$ is bounded in (t,x). By bounded convergence we obtain (20), which immediately gives (21). This completes the proof of (b) consequently of Theorem 4 altogether. □

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