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The serial harness interacting with a wall

Pablo A. Ferrari[a,*], Luiz R.G. Fontes[a], Beat M. Niederhauser[a], Marina Vachkovskaia[b]

[a] *Universidade de São Paulo, Instituto de Matemática e Estatística, Caixa Postal 66281, 05311-970 São Paulo, Brazil*
[b] *Universidade Estadual de Campinas, IMECC UNICAMP, Caixa Postal 6065, 13081-970 Campinas SP, Brazil*

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Abstract

The serial harnesses introduced by Hammersley describe the motion of a hypersurface of dimension d embedded in a space of dimension $d+1$. The height assigned to each site i of $\mathbb{Z}^d$ is updated by taking a weighted average of the heights of some of the neighbors of i plus a "noise" (a centered random variable). The surface interacts by exclusion with a "wall" located at level zero: the updated heights are not allowed to go below zero. We show that for any distribution of the noise variables and in all dimensions, the surface delocalizes. This phenomenon is related to the so-called "entropic repulsion". For some classes of noise distributions, characterized by their tail, we give explicit bounds on the speed of the repulsion.

MSC: 60K35; 82B; 82C

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1. Introduction and results

Hammersley [13] introduced the *serial harness*, a discrete-time stochastic process that models the time evolution of a hypersurface of dimension d embedded in a $d+1$ dimensional space. A quantity $Y_n(i) \in \mathbb{R}$ stays for the height of the surface at site $i \in \mathbb{Z}^d$ at (integer) time $n \geqslant 0$. The initial configuration is the flat surface $Y_0(i) = 0$ for all i.

* Corresponding author. Tel.: +55-11-38157439; fax: +55-11-38144135.
E-mail addresses: pablo@ime.usp.br (P.A. Ferrari), lrenato@ime.usp.br (L.R.G. Fontes), beat@ime.usp.br (B.M. Niederhauser), marinav@ime.unicamp.br (M. Vachkovskaia).
URLs: http://www.ime.usp.br/∼pablo, http://www.ime.usp.br/∼beat

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Under the evolution, at each moment $n \geqslant 0$ the height at each site is substituted by a weighted average of the heights at the previous moment plus a symmetric random variable.

Let $\mathscr{P}=\{p(i,j)\}_{i,j\in\mathbb{Z}^d}$ be a stochastic matrix, i.e. $p(i,j)\geqslant 0$ and $\sum_j p(i,j)=1$, which satisfies $p(i,j)=p(0,j-i)=: p(j-i)$ (homogeneity), $\sum_j j\,p(j)=0$, and $p(j)=0$ for all $|j|>v$ for some v (finite range). Assume also that $\mathscr{P}$ is truly d-dimensional: $\{j\in\mathbb{Z}^d\ :\ p(j)\neq 0\}$ generates $\mathbb{Z}^d$.

Let $\mathscr{E}=(\varepsilon,(\varepsilon_n(i),\ i\in\mathbb{Z}^d),n\in\mathbb{Z})$ be a family of i.i.d. integrable symmetric random variables. Let $\mathbb{P}$ and $\mathbb{E}$ denote the probability and expectation in the probability space generated by $\mathscr{E}$. (We use preliminary $n\in\mathbb{N}$ in the definitions but later it will be useful to have $n\in\mathbb{Z}$.)

The *serial harness* $(Y_n,\ n\geqslant 0)$ is the discrete-time Markov process in $\mathbb{R}^{\mathbb{Z}^d}$ defined by

$$Y_n(i)=\begin{cases}0 & \text{if } n=0,\\ \displaystyle\sum_{j\in\mathbb{Z}^d} p(i,j)Y_{n-1}(j)+\varepsilon_n(i) & \text{if } n\geqslant 1.\end{cases} \tag{1.1}$$

Here $Y_n(i)$ denotes the height of the serial harness at site i at time n. In other words, the evolution is given by

$$Y_n=\mathscr{P}Y_{n-1}+\varepsilon_n, \tag{1.2}$$

where $\varepsilon_n=(\varepsilon_n(i),\ i\in\mathbb{Z}^d)$. Since the "noise variable" ε is symmetric and thus has zero mean, we have that $\mathbb{E}Y_n(i)=0$ for all i,n. We can interpret $p(i,j)$ as transition probabilities of a random walk on $\mathbb{Z}^d$; let $p_m(i,j)$ be its m-step transition probabilities. By homogeneity, $p_m(i,j)=p_m(0,j-i)=: p_m(j-i)$. Iterating (1.1)

$$Y_n(i)=\sum_{r=1}^{n}\sum_{j\in\mathbb{Z}^d} p_{n-r}(i,j)\varepsilon_r(j)\stackrel{d}{=}\sum_{r=0}^{n-1}\sum_{j\in\mathbb{Z}^d} p_r(j)\varepsilon_r(j) \tag{1.3}$$

for all $n\geqslant 1, i\in\mathbb{Z}^d$, where $\stackrel{d}{=}$ means equidistributed. Hammersley [13] obtained that

$$\mathbb{E}(Y_n(i))^2=\sigma^2 s(n), \tag{1.4}$$

where σ^2 is the variance of ε and

$$s(n):=\sum_{r=0}^{n-1}\sum_{j\in\mathbb{Z}^d} p_r(j)^2 \tag{1.5}$$

is the expected number of encounters up to time n of two independent copies of a random walk starting at 0 with transition probabilities $\mathscr{P}$. Equality (1.4) follows immediately from (1.3). Since $s(n)\sim\sqrt{n}$ for $d=1$, $s(n)\sim\log n$ for $d=2$ and $s(n)$ is uniformly bounded in n for $d\geqslant 3$ (see, for example, [18]), the surface delocalizes in dimensions $d\leqslant 2$ and stays localized in dimensions $d\geqslant 3$. Toom [19] studies localization of the surface and surface-differences in function of the decay of the distribution of ε.

We consider the serial harness interacting by exclusion with a wall located at the origin. The *wall process* $(W_n,\, n \geqslant 0)$ is the Markov process in $(\mathbb{R}^+)^{\mathbb{Z}^d}$ defined by

$$W_n(i) = \begin{cases} 0 & \text{if } n = 0, \\ \left(\sum_{j\in\mathbb{Z}^d} p(i,j) W_{n-1}(j) + \varepsilon_n(i) \right)^+ & \text{if } n \geqslant 1 \end{cases} \tag{1.6}$$

for $i \in \mathbb{Z}^d$, where for $a \in \mathbb{R}$, $a^+ = a \vee 0 = \max(a, 0)$; this can be reexpressed as

$$W_n = (\mathscr{P} W_{n-1} + \varepsilon_n)^+. \tag{1.7}$$

We say that the law of a random surface Z is an *invariant measure* for the wall process if $Z \stackrel{d}{=} (\varepsilon_0 + \mathscr{P}Z)^+$, with ε_0 and Z independent. We show in Section 2 that

$$W_n \leqslant W_{n+1} \text{ stochastically.} \tag{1.8}$$

This implies that W_n is stochastically nondecreasing and thus their laws converge to a limit (that could give positive weight to infinity). If the limit is nondegenerate, then it is an invariant measure for the wall process. Monotonicity (1.8) implies in particular

$$\mu_n := \mathbb{E} W_n(0)$$

is nondecreasing and thus converges either to a finite limit or to ∞. Our first result is general and rules out the former possibility, showing however that μ_n goes to infinity slower than n.

Theorem 1.1. (a) *There is no nondegenerate invariant measure for the wall process* (W_n); (b) $W_n \to \infty$ *in probability*; (c) $\mu_n \to \infty$ *as* $n \to \infty$; (d) $\mu_n/n \to 0$ *as* $n \to \infty$.

This theorem is proven in Section 2.

Let F be the law of ε, $\bar{F}(x) = \mathbb{P}(\varepsilon > x)$ and define

$$\mathscr{L}_\alpha^- := \{F : \bar{F}(x) \leqslant c\mathrm{e}^{-c' x^\alpha},\ x > 0,\ \text{for some positive } c, c'\} \tag{1.9}$$

$$\mathscr{L}_\alpha^+ := \{F : \bar{F}(x) \geqslant c\mathrm{e}^{-c' x^\alpha},\ x > 0,\ \text{for some positive } c, c'\} \tag{1.10}$$

and

$$\mathscr{L}_\alpha := \mathscr{L}_\alpha^- \cap \mathscr{L}_\alpha^+. \tag{1.11}$$

We next state our main result. It consists of upper and lower bounds for μ_n for different noise distributions.

Theorem 1.2. *There exist constants c and C that may depend on the dimension such that*

(i) *for $d = 1$ if $F \in \mathscr{L}_1^-$*

$$cn^{1/4} \leqslant \mu_n \leqslant Cn^{1/4}\sqrt{\log n}; \tag{1.12}$$

(ii) *for* $d=2$, *if* $F\in\mathscr{L}_\alpha$, *for some* $\alpha\geqslant 1$

$$c(\log n)^{1/\alpha\vee 1/2}\leqslant \mu_n\leqslant C\log n; \tag{1.13}$$

(iii) *for* $d\geqslant 3$, *if* $F\in\mathscr{L}_\alpha$, *for some* $1\leqslant\alpha\neq 1+d/2$

$$c(\log n)^{1/\alpha}\leqslant \mu_n\leqslant C(\log n)^{1/\alpha\vee 2/(2+d)}; \tag{1.14}$$

(iv) *for* $d\geqslant 3$ *if* $F\in\mathscr{L}_{1+d/2}$

$$c(\log n)^{2/(2+d)}\leqslant \mu_n\leqslant C(\log n)^{2/(2+d)}(\log\log n)^{d/(2+d)}. \tag{1.15}$$

Our upper bound in (1.15) can be slightly improved, see (6.4) and Remark 6.2 below. The lower bound in (i) can be shown to hold under weaker conditions; that is also the case for some cases of (ii); see (6.7) and Remark 6.7 below. If the noise distribution is in $\mathscr{L}_\alpha$ for some $\alpha\geqslant 1$, then our lower and upper bounds to μ_n are of the same order in the case that $d\geqslant 3$, $1\leqslant\alpha<1+d/2$ (which includes the Gaussian case $\alpha=2$ for all such dimensions), and also in the case that $d=2$, $\alpha=1$.

Theorems 1.1 and 1.2 catch the effect of the "entropic repulsion" in a stochastically moving surface interacting with a wall by exclusion.

Many papers deal with the problem of entropic repulsion in Equilibrium Statistical Mechanics. The role of the entropic repulsion in the Gaussian free field was studied by Lebowitz and Maes [15], Bolthausen et al. [3], Deuschel [8], Deuschel and Giacomin [9] and Bolthausen et al. [2]. In the Ising, SOS and related models the matter was discussed by Bricmont et al. [5], Bricmont [4], Cesi and Martinelli [6], Dinaburg and Mazel [10], Holický and Zahradník [14], and Ferrari and Martínez [12].

The exponent $\frac{1}{4}$ for dynamic entropic repulsion in $d=1$ was predicted by Lipowsky [16] using scaling arguments. This exponent was then found numerically by Mon et al. [17], Binder [1], De Coninck et al. [7]. Dunlop et al. [11] proved bounds (slightly worse than) (1.12) for a one-dimensional interface related to the phase separation line in the two-dimensional Ising model at zero temperature. Funaki and Olla [20] studied a one-dimensional model in a finite box rescaled as the square of the time.

The strategy to show part of Theorem 1.2 is to compare the wall process with a "free process"—in our case the serial harness—as proposed by Dunlop et al. [11]. The following lemmas are the basic ingredients in this approach. The first two concern moderate deviations of the serial harness Y_n; they are then extended to the wall process W_n in the last one.

Lemma 1.3. *If the distribution of* ε *is in* $\mathscr{L}_1^-$, *then in* $d\leqslant 2$ *there exist constants* $k,c,c'>0$ *such that for all* $K>0$ *and* $0\leqslant l\leqslant n$

$$\mathbb{P}[Y_l(0)\geqslant K\sqrt{s(n)\log n}]\leqslant kn^{c-c'K}. \tag{1.16}$$

Lemma 1.4. *If the distribution of* ε *is in* $\mathscr{L}_\alpha^-$ *for some* $\alpha\geqslant 1$, *then in* $d\geqslant 3$ *there exist constants* $k,c,c'>0$ *such that, for all* $K>0$ *and* $0\leqslant l\leqslant n$

(i) *if* $\alpha\neq 1+d/2$, *then*

$$\mathbb{P}[Y_l(0)\geqslant K(\log n)^{1/\alpha\vee 2/(2+d)}]\leqslant kn^{c-c'K}; \tag{1.17}$$

(ii) *if* $\alpha = 1 + d/2$, *then*

$$\mathbb{P}[Y_l(0) \geqslant KL_n(1 + 2/d)] \leqslant kn^{c-c'K}, \tag{1.18}$$

where $L_n(\cdot)$ *is defined in* (6.1) *below*.

Lemma 1.5. *The bounds of Lemmas* 1.4 *and* 1.3 *hold for* $l = n$ *if we replace* Y_n *with* W_n, *possibly with worse constants* k, c.

We conclude this introduction with a remark concerning the form (1.6) of the interaction with the wall. Two other choices are also natural. First, if the noise would push the process below zero, simply do nothing. Or, in the same case, only take the convex combination without a noise. Formally, these two cases are, respectively,

$$W_0'(i) = W_0''(i) \equiv 0$$

and for $n \geqslant 1$

$$W_n'(i) = \begin{cases} \sum_{j\in\mathbb{Z}^d} p(i,j)W_{n-1}'(j) + \varepsilon_n(i) & \text{if this is positive,} \\ W_{n-1}'(i) & \text{otherwise} \end{cases} \tag{1.19}$$

and

$$W_n''(i) = \begin{cases} \sum_{j\in\mathbb{Z}^d} p(i,j)W_{n-1}''(j) + \varepsilon_n(i) & \text{if this is positive,} \\ \sum_{j\in\mathbb{Z}^d} p(i,j)W_{n-1}''(j) & \text{otherwise.} \end{cases} \tag{1.20}$$

Coupling W, W', W'' by the same realization of the noise variables, one sees that, stochastically, both $W' \geqslant W$ and $W'' \geqslant W$. This implies immediately that any lower bound for μ_n (in particular the ones in this paper) hold for $\mu_n' := \mathbb{E}W_n'(0)$ and $\mu_n'' := \mathbb{E}W_n''(0)$ as well. These dominations also imply immediately the validity of the results of Theorem 1.1 (a)–(c) for W' and W''. For the analogue of Theorem 1.1 (d), domination does not help (it goes in the wrong direction). An argument along the same lines as the one for W can be made for W'' straightforwardly; see paragraphs containing (2.14) and (3.2). Under the assumption that $\mathscr{P}(0,0) > 0$, one can also make a similar argument for W'; otherwise, the matter is more delicate, and we do not have an argument.

As for upper bounds for μ_n', μ_n'', the ones we get for μ_n also hold for both of them, since the proof only relies on the free process started at some height r dominating stochastically the wall process started at the same height, and this holds for all three choices.

2. Delocalization

In this section we show Theorem 1.1. The wall process is *attractive*, i.e.

$$\text{if } W \leqslant W' \text{ then } (\mathscr{P}W + \varepsilon_0)^+ \leqslant (\mathscr{P}W' + \varepsilon_0)^+ \quad a.s. \tag{2.1}$$

coordinatewise, which implies

$$\text{if } W_n \leqslant W'_n \text{ stochastically, then } W_{n+1} \leqslant W'_{n+1} \text{ stochastically.} \tag{2.2}$$

Since for the process with initial flat surface $0 \equiv W_0 \leqslant W_1$ *a.s.* this implies (1.8).

Theorem 1.1 is a consequence of the following three lemmas:

Lemma 2.1. *There is no invariant measure for* (W_n) *with finite mean.*

Proof. Suppose there exists an invariant measure v_o with finite mean m_o. Let $I=[-c,c]$ be the support of the distribution of ε. Then there exists $0 < c' < c$ such that $\mathbb{P}[\varepsilon < -c'] > 0$ and, by Markov's inequality, for any n, $\mathbb{P}[\sum_j p_n(0,j)W(j) < 2m_o] > \frac{1}{2}$, where p_n are the n-step transition probabilities.

The preceding implies that the process started from the invariant measure v_o reaches the wall at the origin in $n' = 2m_o/c'$ steps with strictly positive probability. This yields a positive drift, contradicting the assumption. □

Lemma 2.2. *Every invariant measure for* (W_n) *dominates stochastically*

$$\lim_n \mathbb{P}(W_n \in \cdot).$$

Proof. Attractiveness (2.2) implies that the law of W_n is stochastically nondecreasing and hence converges to a limit. Since the initial flat configuration is dominated by any other, any invariant measure dominates stochastically that limit. □

Consider the family of processes $((W_n^k, n \geqslant k),\ k \in \mathbb{Z})$ defined by

$$W_n^k = \begin{cases} 0 & \text{if } n = k, \\ (\mathscr{P}W_{n-1}^k + \varepsilon_n)^+ & \text{if } n \geqslant k+1. \end{cases} \tag{2.3}$$

$(W_n^k, n \geqslant k)$ is the wall process evolving from time k on, having flat configuration at initial time k. It is clear that for $k \geqslant 0$

$$W_0^{-k} \stackrel{d}{=} W_k^0 (= W_k). \tag{2.4}$$

Since $0 = W_k^k \leqslant W_k^{k-1}$, by attractiveness (2.1), $W_n^k \leqslant W_n^{k-1}$ for all $n \geqslant k$, and in particular

$$W_0^k \leqslant W_0^{k-1} \tag{2.5}$$

so that $W_0^{-\infty} = \lim_{k\to\infty} W_0^{-k}$ is well defined (but could be infinity).

Lemma 2.3. $W_0^{-\infty}$ *(and hence* $W_n^{-\infty}$ *for all* n*) is almost surely identically infinity.*

Proof. The event $\{W_0^{-\infty} = \infty\}$ belongs to the tail σ-algebra of $\{\varepsilon_k : k \leqslant 0\}$ and is thus trivial. Write

$$W_0^{-\infty} = (\varepsilon_0 + \mathscr{P}W_{-1}^{-\infty})^+ = \cdots \tag{2.6}$$

$$= (\varepsilon_0 + \mathscr{P}(\varepsilon_{-1} + \cdots \mathscr{P}(\varepsilon_{-k+1} + \mathscr{P}W_{-k}^{-\infty})^+ \cdots)^+)^+ \tag{2.7}$$

$$\geqslant U_k + \mathscr{P}^k W_{-k}^{-\infty} \tag{2.8}$$

for $k>0$, where $U_k=\sum_{i=0}^{k-1}\mathscr{P}^i\varepsilon_{-i}$. Notice that U_k is symmetric and that U_k and $W_{-k}^{-\infty}$ are independent: U_k is a function of $(\varepsilon_i : -k+1 \leqslant i \leqslant 0)$ while $W_{-k}^{-\infty}$ is function of $(\varepsilon_i : i \leqslant -k)$. Since $W_{-k}^{-\infty}\overset{d}{=}W_0^{-\infty}$, for all $k\geqslant 0$

$$W_0^{-\infty} \geqslant V_k + \mathscr{P}^k W_0^{-\infty}, \quad \text{stochastically} \tag{2.9}$$

with $V_k\overset{d}{=}U_k$, V_k and $W_0^{-\infty}$ independent.

A key observation is that $W_0^{-\infty}$ is ergodic for spatial shifts. This follows from the fact that $W_0^{-\infty}$ is a function of $\varepsilon_n(i)$'s for a cone of indices (n,i) in $-\mathbb{N}\times\mathbb{Z}^d$ with vertex in $(0,x)$. Now, $\mathbb{E}(W_0^{-\infty})=\infty$, the Ergodic Theorem implies that $\mathscr{P}^k W_0^{-\infty}\to\infty$ almost surely as $k\to\infty$. Indeed,

$$\mathscr{P}^k W_0^{-\infty}(0) = \sum_{i\in\mathbb{Z}^d} p_k(i) W_0^{-\infty}(i) \geqslant \frac{c}{k^{d/2}}\sum_{|i|\leqslant k} W_0^{-\infty}(i)\to\infty \tag{2.10}$$

as $k\to\infty$, by the Ergodic Theorem. We have used the positivity of $W_0^{-\infty}$ and the well known Local Central Limit Theorem estimate to the effect that $\inf_{|i|\leqslant k} p_k(i)\geqslant c/k^{d/2}$ for some $c>0$. For this estimate, aperiodicity is required; we leave the necessary and straightforward adaptations for the periodic case to the reader.

Now, (2.9), (2.10) and the symmetry of V_k imply that for arbitrary $M>0$

$$\mathbb{P}(W_0^{-\infty}>M) \geqslant \liminf_{k\to\infty}\mathbb{P}(V_k+\mathscr{P}^k W_0^{-\infty}>M) \tag{2.11}$$

$$\geqslant \liminf_{k\to\infty}\mathbb{P}(V_k\geqslant 0)\mathbb{P}(\mathscr{P}^k W_0^{-\infty}>M) \tag{2.12}$$

$$\geqslant \frac{1}{2}\liminf_{k\to\infty}\mathbb{P}(\mathscr{P}^k W_0^{-\infty}>M)=\frac{1}{2}. \tag{2.13}$$

Thus $\mathbb{P}(W_0^{-\infty}=\infty)\geqslant\frac{1}{2}$ and triviality implies $\mathbb{P}(W_0^{-\infty}=\infty)=1$. □

Proof of Theorem 1.1. (a) is immediate consequence of Lemmas 2.2 and 2.3: any invariant surface dominates stochastically $W_0^{-\infty}$ and $W_0^{-\infty}$ is almost surely identically infinity. (b) follows from Lemma 2.3 and (2.4). (c) follows from the identity $\mu_n=\mathbb{E}W_n(0)=\mathbb{E}W_0^{-n}$ and the monotone convergence theorem. Finally, in (3.2) below it is shown that

$$\mu_n-\mu_{n-1}=\mathbb{E}\int_{\mathscr{P}W_{n-1}}^{\infty}\mathbb{P}(\varepsilon>x)\,\mathrm{d}x. \tag{2.14}$$

Since ε is integrable and $\mathscr{P}W_{n-1}$ increases to infinity in probability, (2.14) converges to zero, and we get (d). □

3. A generic lower bound

From (1.7),

$$W_n(i)=(\mathscr{P}W_{n-1}(i)+\varepsilon_n(i))^+$$

$$=\mathscr{P}W_{n-1}(i)+\varepsilon_n(i)+(-\mathscr{P}W_{n-1}(i)-\varepsilon_n(i))^+. \tag{3.1}$$

Taking expectations, since ε is symmetric,

$$\mu_n = \mu_{n-1} + \mathbb{E}\int_{\mathscr{P}W_{n-1}}^{\infty} \mathbb{P}(\varepsilon > x)\,\mathrm{d}x. \tag{3.2}$$

As $\int_y^\infty \mathbb{P}(\varepsilon > x)\,\mathrm{d}x$ is a convex function of y,

$$\mu_n \geqslant \mu_{n-1} + \int_{\mathbb{E}(\mathscr{P}W_{n-1})}^{\infty} \mathbb{P}(\varepsilon > x)\,\mathrm{d}x = \mu_{n-1} + \mathbb{E}(\varepsilon - \mu_{n-1})^+. \tag{3.3}$$

For $s \geqslant 0$, let $G(s) = \mathbb{E}(\varepsilon - s)^+$, $H(s) = s + G(s)$, and $v(t)$ be such that $\int_0^{v(t)} [G(s)]^{-1}\,\mathrm{d}s = t$.

Theorem 3.1. *$\mu_n \geqslant v(n)$ for all $n \geqslant 0$.*

Remark 3.2. This general lower bound does not depend on the dimension.

Corollary 3.3. *If the distribution of ε belongs to $\mathscr{L}_\alpha^+$ for some $\alpha > 0$, then there exists $c_2 = c_2(\alpha) > 0$ such that*

$$\mu_n \geqslant c_2(\log n)^{1/\alpha}. \tag{3.4}$$

Corollary 3.4. *Suppose that the distribution of ε decays at most polynomially, i.e. $\mathbb{P}(\varepsilon > x) \geqslant c_0 x^{-\alpha}$ for all $x > 1$ and some positive constants c_0 and $\alpha > 1$. Then there exists $c_1 = c_1(\alpha) > 0$ such that*

$$\mu_n \geqslant c_1 n^{1/\alpha}. \tag{3.5}$$

Proof of Theorem 3.1. Notice first that $v(t)$ is a solution of

$$v(t) = \int_0^t G(v(s))\,\mathrm{d}s$$

and thus satisfies

$$v(n) = v(n-1) + \int_{n-1}^n G(v(s))\,\mathrm{d}s.$$

Notice also that $G(x)$ is decreasing and $H(x)$ is increasing. We prove the lemma by induction. First, $\mu_0 = v(0) = 0$. Suppose that $\mu_{n-1} \geqslant v(n-1)$. Then,

$$\begin{aligned} v(n) &= v(n-1) + \int_{n-1}^n G(v(s))\,\mathrm{d}s \leqslant v(n-1) + G(v(n-1)) \\ &= H(v(n-1)) \leqslant H(\mu_{n-1}) \leqslant \mu_n, \end{aligned}$$

where the last inequality is (3.3). □

Proof of Corollary 3.4. Note that

$$G(x) = \mathbb{E}(\varepsilon - x)^+ = -\int_x^{+\infty} (y - x)\,\mathrm{d}\mathbb{P}[\varepsilon \geqslant y] = \int_x^{+\infty} \mathbb{P}[\varepsilon \geqslant y]\,\mathrm{d}y$$

and thus

$$g(t) := \int_0^t \frac{\mathrm{d}s}{G(s)} = \int_0^t \frac{\mathrm{d}s}{\int_s^{+\infty} \mathbb{P}[\varepsilon \geqslant y]\,\mathrm{d}y}. \tag{3.6}$$

Thus, from the assumption in the statement of Corollary 3.4

$$g(t) \leqslant \frac{1}{c_0} \int_0^t \frac{\mathrm{d}s}{1/(\alpha-1)s^{1-\alpha}} = \frac{\alpha-1}{c_0\alpha} t^{\alpha} \tag{3.7}$$

and

$$v(t) \geqslant c_1 t^{1/\alpha}$$

follows immediately. □

Proof of Corollary 3.3. As above, we have

$$g(t) \leqslant \frac{1}{c} \int_0^t \frac{\mathrm{d}s}{\int_s^{+\infty} \mathrm{e}^{-c' y^{\alpha}}\,\mathrm{d}y} \leqslant c_1 \int_0^t \mathrm{e}^{c_2 s^{\alpha}}\,\mathrm{d}s \leqslant c_3 \mathrm{e}^{c_4 t^{\alpha}} \tag{3.8}$$

and the result follows. □

4. Moderate deviations for the serial harness

The proofs of Lemmas 1.3 and 1.4 are based on the behavior of $\mathbb{E}(\mathrm{e}^{\lambda Y_n(0)})$ for small and large λ, established in Lemmas 4.1 and 4.3 below.

Lemma 4.1. *Let λ_n be a sequence of positive numbers such that*

$$\bar{\lambda}_n := \lambda_n / \sqrt{s(n)} \leqslant 1. \tag{4.1}$$

Then there exists a constant c such that for all $0 \leqslant l \leqslant n$

$$\mathbb{E}[\mathrm{e}^{\bar{\lambda}_n Y_l(0)}] \leqslant \mathrm{e}^{c\lambda_n^2}. \tag{4.2}$$

Proof. For all $0 \leqslant l \leqslant n$

$$\mathbb{E}[\mathrm{e}^{\bar{\lambda}_n Y_l(0)}] = \prod_{r=0}^{l-1} \prod_{j\in\mathbb{Z}^d} \mathbb{E}[\mathrm{e}^{\bar{\lambda}_n p_r(j)\varepsilon}] \leqslant \prod_{r=0}^{l-1} \prod_{j\in\mathbb{Z}^d} \mathrm{e}^{c\bar{\lambda}_n^2 p_r(j)^2}$$

$$= \exp\{c\lambda_n^2 s(n)^{-1} s(l)\} \leqslant \mathrm{e}^{c\lambda_n^2},$$

where $c = \mathbb{E}(\mathrm{e}^{\varepsilon})$ and we have used that for a symmetric random variable W, if $|\lambda| \leqslant 1$, then

$$\mathbb{E}(\mathrm{e}^{\lambda W}) \leqslant 1 + \mathbb{E}(\mathrm{e}^{W})\lambda^2 \leqslant \mathrm{e}^{\mathbb{E}(\mathrm{e}^{W})\lambda^2} \tag{4.3}$$

and the fact that $s(\cdot)$ is nondecreasing. □

Proof of Lemma 1.3.

$$\mathbb{P}[Y_l(0) \geqslant K\sqrt{s(n)\log n}] = \mathbb{P}[\bar{\lambda}_n Y_n(0) \geqslant \log n^{c'K}] \leqslant n^{-c'K}\mathbb{E}[e^{\bar{\lambda}_n Y_n(0)}],$$

where $\lambda_n = c''\sqrt{\log n}$, for an appropriate constant c'', and Lemma 4.1 yields the result. □

For the proof of Lemma 1.4, we will use that in $d \geqslant 3$

$$s := \lim_{n\to\infty} s(n) < \infty. \tag{4.4}$$

We will also need the following converse of (4.3):

Lemma 4.2. *If the distribution of W is in $\mathscr{L}_\alpha^-$ for some $\alpha > 1$, then there exists a constant c such that*

$$\mathbb{E}(e^{\lambda W}) \leqslant e^{c\lambda^\beta} \tag{4.5}$$

for all $\lambda \geqslant 1$, where $\beta = \alpha/(\alpha-1)$.

Proof. We have that

$$\mathbb{E}e^{\lambda W} \leqslant 1 + c\int_0^\infty e^{\lambda x}e^{-c'x^\alpha}\,dx = 1 + c_1\int_0^\infty e^{\tilde{\lambda}x}e^{-x^\alpha}\,dx, \tag{4.6}$$

where $\tilde{\lambda} = \lambda/c^{1/\alpha}$. Now, we write the integral in (4.6) as

$$\int_0^{(2\tilde{\lambda})^{\beta-1}} e^{\tilde{\lambda}x}\,dx + \int_{(2\tilde{\lambda})^{\beta-1}}^\infty e^{\tilde{\lambda}x - x^\alpha}\,dx.$$

The former integral is bounded above by $e^{c'''\lambda^\beta}$. The latter one is bounded above by a uniform constant. □

Lemma 4.3. *In $d \geqslant 3$, if the distribution of ε is in $\mathscr{L}_\alpha^-$ for some $\alpha > 1$, then there exists a constant c such that for all large q*

$$\mathbb{E}(e^{qY_n(0)}) \leqslant \begin{cases} e^{cq^{\beta\vee(1+2/d)}} & \text{if } \alpha \neq 1 + d/2, \\ e^{cq^{1+2/d}\log q} & \text{if } \alpha = 1 + d/2, \end{cases} \tag{4.7}$$

where $\beta = \alpha/(\alpha-1)$ as before.

Proof.

$$\mathbb{E}(e^{qY_n(0)}) \leqslant \prod_{k=0}^\infty \prod_{x\in\mathbb{Z}^d} \mathbb{E}(e^{qp_k(x)\varepsilon}) \leqslant \prod_{k,x:qp_k(x)>1} e^{c(qp_k(x))^\beta} \prod_{k,x:qp_k(x)\leqslant 1} e^{c(qp_k(x))^2}$$

$$= \exp\left\{c\left[\sum_{k,x:qp_k(x)>1}(qp_k(x))^\beta + \sum_{k,x:qp_k(x)\leqslant 1}(qp_k(x))^2\right]\right\}. \tag{4.8}$$

We now estimate the expression within square brackets in (4.8). If $\beta \geqslant 2$ or, equivalently, $1 < \alpha \leqslant 2$, then that expression is bounded above by

$$q^\beta \sum_{k,x} p_k^2(x) = q^\beta s. \tag{4.9}$$

For the case $1 < \beta < 2$ (equivalently, $\alpha > 2$), we use the well known estimate on $p_k := \sup_{x \in \mathbb{Z}^d} p_k(x)$: there exists a constant C such that for all $k \geqslant 1$

$$p_k \leqslant C k^{-d/2} \tag{4.10}$$

(see e.g. [18]) to conclude that the expression within square brackets in (4.8) is bounded above by

$$q^\beta \sum_{k=0}^{(Cq)^{2/d}} p_k^{\beta-1} + q^2 \sum_{k=(Cq)^{2/d}}^{\infty} p_k \leqslant C' q^\beta \sum_{k=1}^{(Cq)^{2/d}} k^{-d(\beta-1)/2} + C'' q^{1+2/d} \tag{4.11}$$

for some constants C', C''. The result follows. □

Proof of Lemma 1.4. Let Q_n be a sequence of positive numbers such that $Q_n = o(\log n)$ and $q_n = (\log n)/Q_n$. Then

$$\mathbb{P}[Y_l(0) \geqslant KQ_n] \leqslant \mathbb{P}[q_n Y_l(0) \geqslant K(\log n)] \leqslant n^{-K} \mathbb{E}(e^{q_n Y_l(0)}). \tag{4.12}$$

We can thus use Lemma 4.3 for q_n. Therefore, if $1 < \alpha \neq 1 + d/2$, making $Q_n = (\log n)^{1/\alpha \vee 2/(2+d)}$, we have $q_n = (\log n)^{1-(1/\alpha \vee 2/(2+d))} = (\log n)^{1/\beta \wedge d/(d+2)}$ and thus, from (4.7)

$$\mathbb{P}[Y_l(0) \geqslant K(\log n)^{1/\alpha \vee 2/(2+d)}] \leqslant n^{c-K}. \tag{4.13}$$

If $\alpha = 1 + d/2$, we make $Q_n = L_n(1+2/d)$, and thus $q_n = (\log n)/L_n(1+2/d) = \ell_n(1+2/d)$. From (4.7) and the definition of $\ell_n(1+2/d)$ (above (6.1) below)

$$\mathbb{P}[Y_l(0) \geqslant KL_n(1+2/d)] \leqslant n^{c-K}. \tag{4.14}$$

For $\alpha = 1$, we have

$$\mathbb{E}e^{Y_n(0)} = \prod_{k,x} \mathbb{E}e^{p_k(x)\varepsilon} \leqslant e^{c \sum_{k,x} p_k^2(x)} = e^{cs}, \tag{4.15}$$

where we have used (4.3). Thus, we obtain that

$$\mathbb{P}[Y_n(0) > K \log n] \leqslant C n^{-K}. \qquad \square$$

5. Moderate deviations for the wall process

In this section we show Lemma 1.5. Introduce new processes $W_n^{0,r}$ and $Y_n^{0,r}$, which have the same evolution as W_n, respectively Y_n, but are started at time zero at height $r \in \mathbb{N}$. That is, $W_0^{0,r}(i) = Y_0^{0,r}(i) = r$, for all $i \in \mathbb{Z}^d$.

Let

$$a_n = \begin{cases} 2K(\log n)^{1/\alpha\vee 2/(2+d)} & \text{for the extension of (1.17)},\\ 2KL_n(1+2/d) & \text{for the extension of (1.18)},\\ 2K\sqrt{s(n)\log n}, & \text{for the extension of (1.16)}. \end{cases} \tag{5.1}$$

Then

$$\begin{aligned}\mathbb{P}[W_n(0)\geqslant a_n] &\leqslant \mathbb{P}[W_n^{0,r}(0)\geqslant a_n]\\ &= \mathbb{P}[W_n^{0,r}(0)\geqslant a_n, W_n^{0,r}(0)=Y_n^{0,r}(0)]\\ &\quad+\mathbb{P}[W_n^{0,r}(0)\geqslant a_n, W_n^{0,r}(0)\neq Y_n^{0,r}(0)]\\ &\leqslant \mathbb{P}[Y_n^{0,r}(0)\geqslant a_n] \qquad (5.2)\\ &\quad+\mathbb{P}[W_n^{0,r}(0)\neq Y_n^{0,r}(0)]. \qquad (5.3)\end{aligned}$$

To get a bound for the probability in (5.2) of the form (1.16)–(1.18), we take $r=a_n/2$ and use (1.16)–(1.18).

The probability in (5.3) is treated as follows. Note that $W_n^{0,r}(0)$ and $Y_n^{0,r}(0)$ differ if a discrepancy occurs in the cone (v is the maximal speed of a discrepancy)

$$\{(l,j)\in\mathbb{N}_0\times\mathbb{Z}^d : l\leqslant n,\ |j|\leqslant v(n-l)\}, \tag{5.4}$$

i.e.,

$$\{Y_n^{0,r}(0)\neq W_n^{0,r}(0)\}=\{Y_l^{0,r}(j)<0 \text{ for some } (l,j) \text{ with } l\leqslant n, |j|\leqslant v(n-l)\}.$$

Since $Y_n^{0,r}(0)$ has the same law as $Y_n(0)+r$ and by symmetry, we have

$$\mathbb{P}[Y_l^{0,r}(j)<0]=\mathbb{P}[Y_l(j)<-r]=\mathbb{P}[Y_l(j)>r]. \tag{5.5}$$

Hence,

$$\begin{aligned}\mathbb{P}[Y_n^{0,r}(0)\leqslant W_n^{0,r}(0)] &= \mathbb{P}[\exists(l,j) \text{ with } l\leqslant n, |j|\leqslant v(n-l) : Y_l(j)>r]\\ &\leqslant \sum_{l=0}^{n}\sum_{|j|\leqslant v(n-l)}\mathbb{P}[Y_l(j)>r].\end{aligned}$$

Taking $r=a_n/2$ as before and using (1.16)–(1.18), we obtain

$$\mathbb{P}[Y_n^{0,r}(0)\neq W_n^{0,r}(0)]\leqslant kn^{c-c'K}\sum_{l=0}^{n}\sum_{|j|\leqslant v(n-l)}1\leqslant k'n^{c''-c'K} \tag{5.6}$$

for some k', c''.

6. Bounds for the wall process

For $\gamma>1$, define $\ell_n(\gamma)$ as the solution of $x^\gamma\log x=\log n$, and let

$$L_n(\gamma)=(\log n)/\ell_n(\gamma). \tag{6.1}$$

Note that

$$(\log n)^{1-1/\gamma} \leqslant L_n(\gamma) \leqslant (\log n)^{1-1/\gamma}(\log\log n)^{1/\gamma} \quad \text{for all } n. \tag{6.2}$$

Theorem 6.1. *Suppose that the distribution of* ε *belongs to* $\mathscr{L}_\alpha^-$ *for some* $\alpha \geqslant 1$. *If* $d \geqslant 3$, *then there exists* $c_3 = c_3(\alpha, d) > 0$ *such that*

(i) *if* $1 \leqslant \alpha \neq 1 + d/2$, *then*

$$\mu_n \leqslant c_3(\log n)^{1/\alpha \vee 2/(2+d)}; \tag{6.3}$$

(ii) *if* $\alpha = 1 + d/2$, *then for all* $\delta > 0$ *we have*

$$\mu_n \leqslant c_3 L_n(1 + 2/d); \tag{6.4}$$

If $d = 2$, *then there exists* c_3 *such that*

$$\mu_n \leqslant c_3 \log n. \tag{6.5}$$

Remark 6.2. From (6.4) and (6.2), a slightly weaker alternative to (6.4) is

$$\mu_n \leqslant c_3(\log n)^{2/(2+d)}(\log\log n)^{d/(2+d)}. \tag{6.6}$$

We now restrict attention to the class of exponentially decaying noise distributions. When the noise distribution is in $\mathscr{L}_\alpha$, $\alpha \geqslant 1$, the results in Corollary 3.3 and Theorem 6.1 are our best explicit bounds (to leading order) for $d \geqslant 3$ and $d = 2$, $1 \leqslant \alpha \leqslant 2$. For $d = 1, \alpha \geqslant 1$ and $d = 2$, $\alpha > 2$, we have better bounds, which we discuss now.

Theorem 6.3. *If the distribution of* ε *is in* $\mathscr{L}_1^-$, *then for* $d \leqslant 2$, *there exist constants* $c, C > 0$ *such that*

$$c\sqrt{s(n)} \leqslant \mu_n \leqslant C\sqrt{s(n)\log n}, \tag{6.7}$$

where $s(n)$ *is defined in* (1.5). *In particular*

(i) *for* $d = 1$

$$cn^{1/4} \leqslant \mu_n \leqslant Cn^{1/4}\sqrt{\log n}; \tag{6.8}$$

(ii) *for* $d = 2$

$$c\sqrt{\log n} \leqslant \mu_n \leqslant C \log n. \tag{6.9}$$

Remark 6.4. The lower bound in (6.7) actually holds under the weaker assumption that $\mathbb{E}(\varepsilon^2) < \infty$. See Remark 6.7 below.

We prove first the lower bound (6.7). The first step is to calculate the variance of the serial harness, which will give us the proper scaling. From (1.3) we get (this is already contained in Hammersley [13]) $\mathbb{E}\, Y_n(0) = 0$ and $\mathbb{E}\, Y_n(0)^2 = \sigma^2 s(n)$.

The correct scaling for the serial harness is therefore $s(n)^{1/2}$, and we define accordingly

$$\tilde{Y}_n(0) \equiv s(n)^{-(1/2)} Y_n(0). \tag{6.10}$$

Analogously we define $\tilde{W}_n(0)$ for the wall process. We now show that $\tilde{Y}_n(0)$ is uniformly integrable (with respect to n).

Lemma 6.5. *The process* $(\tilde{Y}_n(0))_n$ *satisfies* $\sup_n \mathbb{E}(e^{|\tilde{Y}_n(0)|}) < \infty$.

Proof. By symmetry of the ε, $\mathbb{E}(e^{|\tilde{Y}_n(0)|}) \leqslant 2\mathbb{E}(e^{\tilde{Y}_n(0)}) \leqslant 2e^c$, where the last inequality follows from Lemma 4.1 with $\lambda_n \equiv 1$. □

From Lemma 6.5 it follows immediately that $s(n)^{-1} Y_n(0)^2$ is uniformly integrable.

Lemma 6.6. *There exists a constant* $c > 0$ *such that for all* n

$$\mathbb{E}|\tilde{Y}_n(0)| > c. \tag{6.11}$$

Proof. Clearly, for any positive M

$$\begin{aligned}\mathbb{E}[\tilde{Y}_n(0)^2] &= \mathbb{E}[\tilde{Y}_n(0)^2 \mathbf{1}\{|\tilde{Y}_n(0)| > M\}] + \mathbb{E}[\tilde{Y}_n(0)^2 \mathbf{1}\{\{|\tilde{Y}_n(0)| \leqslant M\}\}] \\ &\leqslant \mathbb{E}[\tilde{Y}_n(0)^2 \mathbf{1}\{|\tilde{Y}_n(0)| > M\}] + M\mathbb{E}[|\tilde{Y}_n(0)|].\end{aligned} \tag{6.12}$$

Since $\tilde{Y}_n(0)^2$ is uniformly integrable, for each $\delta > 0$ we can choose $M > 0$ such that

$$\mathbb{E}[\tilde{Y}_n(0)^2 \mathbf{1}\{|\tilde{Y}_n(0)| > M\}] < \delta \tag{6.13}$$

uniformly in n. Thus

$$\mathbb{E}[|\tilde{Y}_n(0)|] \geqslant \frac{\mathbb{E}[\tilde{Y}_n(0)^2] - \delta}{M} = \frac{\sigma^2 - \delta}{M} > c > 0 \tag{6.14}$$

for some $\delta > 0$. □

We finally prove the result about the wall process by coupling it with the serial harness using the same disorder variables $\mathscr{E}$. By symmetry

$$\mathbb{E}[|\tilde{Y}_n(0)|] = \mathbb{E}[(\tilde{Y}_n(0))^+] + \mathbb{E}[(-\tilde{Y}_n(0))^+] = 2\mathbb{E}[(\tilde{Y}_n(0))^+]. \tag{6.15}$$

On the other hand, by construction, $\tilde{W}_n(0) \geqslant (\tilde{Y}_n(0))^+$, and therefore,

$$\mathbb{E}[\tilde{W}_n(0)] \geqslant \mathbb{E}[(\tilde{Y}_n(0))^+] \geqslant \frac{1}{2}\mathbb{E}[|\tilde{Y}_n(0)|] \geqslant c' > 0. \tag{6.16}$$

This proves the lower bound (6.7).

The upper bounds (6.3)–(6.5) and (6.9) follow from Lemma 1.5 in the same, following way. Let a_n be as in (5.1) and $b_n = a_n/(2K)$. Then

$$\begin{aligned}\mu_n/b_n &= \mathbb{E}[W_n(0)/b_n] = \int_0^\infty \mathbb{P}(W_n(0) > Kb_n)\,\mathrm{d}K \\ &\leqslant c/c' + k\int_{c/c'}^\infty n^{c-c'K}\,\mathrm{d}K \leqslant C\end{aligned}$$

for some constant C.

Remark 6.7. The lower bound in (6.7) actually holds under the weaker assumption that $\mathbb{E}(\varepsilon^2) < \infty$, since this is enough to have $\tilde{Y}_n(0)^2$ uniformly integrable.

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