Available online at www.sciencedirect.com
SciVerse ScienceDirect
Stochastic Processes and their Applications 123 (2013) 3122–3131
stochastic processes and their applications
www.elsevier.com/locate/spa

Weak convergence of subordinators to extremal processes

Offer Kella[a,*], Andreas Löpker[b]

[a] Department of Statistics, The Hebrew University of Jerusalem, Mount Scopus, Jerusalem 91905, Israel
[b] Department of Economics and Social Sciences, Helmut Schmidt University Hamburg, 22043 Hamburg, Germany

Received 8 September 2012; received in revised form 14 March 2013; accepted 17 March 2013
Available online 22 March 2013

Abstract

For certain subordinators $(X_t)_{t\geq 0}$ it is shown that the process $(-t\log X_{ts})_{s>0}$ tends to an extremal process $(\widehat{\eta}_s)_{s>0}$ in the sense of convergence of the finite dimensional distributions. Additionally it is also shown that $(z\wedge(-t\log X_{ts}))_{s\geq 0}$ converges weakly to $(z\wedge\widehat{\eta}_s)_{s\geq 0}$ in $\mathcal{D}[0,\infty)$, the space of càdlàg functions equipped with Skorohod's J_1 metric.

MSC: 60G51; 60J35; 60F17

Keywords: Lévy process; Subordinator; Extremal process; Small-time convergence

1. Introduction

It was shown in [1] that if $(X_t)_{t>0}$ is a family of positive random variables and if X is a non-constant random variable with distribution function F, then X_t^{-t} converges weakly to X as $t\to 0$ if and only if $\psi_t(u^{1/t})\to 1-F(u)$ as $t\to 0$ at all continuity points u of F, where ψ_t is the Laplace transform of X_t. In [2] it was found that for the convolution family $\psi_t(u)=\varphi(u)^t$, where φ is the Laplace transform of an infinitely divisible random variable, i.e. if the process X_t is a subordinator, the limit distribution, if not concentrated on a single point, is always a Pareto distribution. Equivalently we can formulate the convergence in terms of the convergence of $-t\log X_t$ as t tends to zero, with the only possible limit distribution being the exponential distribution. We will apply and extend these results to show that in fact the process $(-t\log X_{st})_{s>0}$ converges to a, so called, *extremal process* $(\widehat{\eta}_s)_{s>0}$, to be reviewed in Section 3. We will first observe the convergence of the finite dimensional distributions and then establish weak convergence of a truncated

* Corresponding author. Tel.: +972 54 4403659; fax: +972 2 5882839.
E-mail addresses: offer.kella@huji.ac.il (O. Kella), lopker@hsu-hh.de (A. Löpker).

0304-4149/$ - see front matter
http://dx.doi.org/10.1016/j.spa.2013.03.010

version in $\mathcal{D}[0,\infty)$, the space of càdlàg functions equipped with Skorohod's J_1 metric. Since the prelimit and limit processes are Markovian, this will be done by proving uniform convergence of the associated generators and applying the necessary theory from [6] for this setup.

2. Setup, review and convergence of finite dimensional distributions

Let $(X_t)_{t\geq 0}$ be a pure jump subordinator, i.e. an increasing Lévy process with

$$\psi_t(u) = \mathbb{E}(e^{-uX_t}) = e^{-t\varphi(u)}, \tag{1}$$

where

$$\varphi(u) = \int_0^\infty (1 - e^{-ux})\, d\nu(x) \tag{2}$$

and the Lévy measure ν in this case must satisfy $\nu(-\infty, 0] = 0$, $\nu(1,\infty) < \infty$ and

$$\rho = \int_{[0,1]} u\, d\nu(u) = \int_0^1 \nu(x, 1] dx < \infty. \tag{3}$$

We recall that $G_t(x) = \mathbb{P}(X_t \leq x)$ is an infinitely divisible distribution.

In what follows $\wedge$ and $\vee$ denote minima and maxima (respectively), $\stackrel{d}{=}$ denotes equality in distribution, $\stackrel{d}{\Rightarrow}$ is for convergence in distribution and $x \downarrow x_0$ means $x \to x_0,\ x > x_0$. Finally, for finite $\gamma > 0$, denote by E_γ an exponential random variable with mean $1/\gamma$.

In [2] the following result was proved.

Theorem 1. *Let Z be a positive random variable which is not concentrated at one point and let $F(x) = \mathbb{P}(Z \leq x)$. The following statements are equivalent.*

(S1) $-t \log X_t \stackrel{d}{\Rightarrow} Z$ *as* $t \downarrow 0$.
(S2) $t\varphi(u^{1/t}) \to -\log(1 - F(u))$ *as* $t \downarrow 0$, *for all continuity points* u *of* F.
(S3) $-t \log X_t \stackrel{d}{\Rightarrow} E_\gamma$ *as* $t \downarrow 0$ *for some finite* $\gamma > 0$.

Furthermore, for any finite $\gamma > 0$ the following statements are equivalent.

(S4) $-t \log X_t \stackrel{d}{\Rightarrow} E_\gamma$ *as* $t \downarrow 0$.
(S5) $\varphi(s)/\log s \to \gamma$ *as* $s \to \infty$.
(S6) $\log G_1(x)/\log x \to \gamma$ *as* $x \downarrow 0$.
(S7) $\nu(x,\infty)/\log x \to -\gamma$ *as* $x \downarrow 0$.

Note that since $\nu(\epsilon,\infty) < \infty$ for any $\epsilon > 0$, then (S7) is equivalent to

(S7') $\nu(x,\epsilon]/\log x \to -\gamma$ as $x \downarrow 0$.

Also note that this condition cannot hold for a compound Poisson process, so that when it does hold then necessarily $\nu(0,\epsilon] = \infty$, which in turn implies that $X_t > 0$ almost surely for each $t > 0$ and thus $-t \log X_t$ is well defined for all $t > 0$.

Several examples of subordinators fulfilling these conditions are given in [2]. A prominent member is the gamma process, where

$$G_t(x) = \frac{\lambda^{\gamma t}}{\Gamma(\gamma t)} \int_0^x u^{\gamma t - 1} e^{-\lambda u}\, du. \tag{4}$$

The following is a generalization of Proposition 2.2 of [2] to the multidimensional and dependent case.

Proposition 1. *For each $t > 0$, let $(X_{i,t})_{1\le i\le n}$ be a random vector with almost surely positive components and assume that for some random vector $(X_i)_{1\le i\le n}$,*

$$(-t\log X_{i,t})_{1\le i\le n} \overset{d}{\Rightarrow} (X_i)_{1\le i\le n}. \tag{5}$$

Then,

$$\left(-t\log\left(\sum_{i=1}^k X_{i,t}\right)\right)_{1\le k\le n} \overset{d}{\Rightarrow} \left(\bigwedge_{i=1}^k X_i\right)_{1\le k\le n}, \tag{6}$$

as $t\downarrow 0$.

Proof. It is well known that on a possibly different probability space we can take $(\tilde X_{i,t})_{1\le i\le n} \overset{d}{=} (X_{i,t})_{1\le i\le n}$ and $(\tilde X_i)_{1\le i\le n} \overset{d}{=} (X_i)_{1\le i\le n}$, where

$$(-t\log\tilde X_{i,t})_{1\le i\le n} \to (\tilde X_1)_{1\le i\le n} \tag{7}$$

almost surely. Since any (Borel) function of $(\tilde X_{i,t})_{1\le i\le n}$ is distributed like that of $(X_{i,t})_{1\le i\le n}$ (and similarly for the limits) this implies that it suffices to show the validity of this proposition for the deterministic case, where the multidimensional convergence in (5) is equivalent to the convergence of each coordinate separately. Observing each such coordinate, it is apparent that it suffices to show this for the case $n = 2$ and then proceed by induction. This can be concluded from Proposition 2.2 of [2], but we would also like to point out the straightforward alternative below.

Note that if $-t\log a(t)\to a$ and $-t\log b(t)\to b$ then

$$-t\log(a(t)\wedge b(t)) = (-t\log a(t))\vee(-t\log b(t))\to a\vee b$$

and, similarly, $-t\log(a(t)\vee b(t))\to a\wedge b$, all as $t\downarrow 0$. Since $a(t)\wedge b(t)+a(t)\vee b(t) = a(t)+b(t)$ it therefore follows that it suffices to treat the case where $a(t)\ge b(t)$ for all $t>0$ and $a\le b$. For this case we have that

$$0\le\log(a(t)+b(t))-\log a(t) = \log\left(1+\frac{b(t)}{a(t)}\right)\le\log 2 \tag{8}$$

and thus $t\log(a(t)+b(t)) - t\log a(t)\to 0$ as $t\downarrow 0$ and the proof is complete. □

Remark 1. Of course, if we assume in Proposition 1 that $(X_{i,t})_{1\le i\le n}$ are independent, then $(-t\log X_{i,t})_{1\le i\le n} \overset{d}{\Rightarrow} (X_i)_{1\le i\le n}$ if and only if $-t\log X_{i,t}\overset{d}{\Rightarrow} X_i$ for each i and $(X_i)_{1\le i\le n}$ are independent as well (on an appropriate probability space). This will be needed in what follows.

We now recall that if, in Proposition 1, $(X_{i,t})_{t\ge 0}$ are independent subordinators, then X_i are independent and are either constant or necessarily exponential. Thus, when they are all exponential, the distribution of the kth coordinate on the right side of (6) is exponential as well, with parameter given by the sum of the first k parameters for the individual limits.

Now let $0 = s_0 < s_1 < s_2 < \cdots < s_n$ and, for $i = 1,\ldots,n$, let $(X_{i,t})_{t\ge 0}$ be i.i.d. copies of $(X_t)_{t\ge 0}$. It follows from the stationary and independent increment property of the Lévy process X_t that

$$X_{s_k t} = \sum_{i=1}^k (X_{s_i t} - X_{s_{i-1} t}) \overset{d}{=} \sum_{i=1}^k X_{i,(s_i - s_{i-1})t}. \tag{9}$$

Consequently, with $Z_1, Z_2, \ldots$ being i.i.d. exp(1) random variables (so that $Z_i/\beta \stackrel{d}{=} \exp(\beta)$), applying (6) it follows that, as $t \downarrow 0$,

$$\left(-t \log X_{s_1 t}, -t \log X_{s_2 t}, \ldots, -t \log X_{s_n t}\right) \stackrel{d}{\Rightarrow} \frac{1}{\gamma}\left(\bigwedge_{i=1}^{k} \frac{Z_i}{s_i - s_{i-1}}\right)_{1\le k\le n}. \tag{10}$$

Hence, we see that we have convergence of the finite dimensional distributions of $(-t \log X_{ts})_{s>0}$ to those of some process $(\widehat{\eta}_s)_{s>0}$, where $\left(\widehat{\eta}_{s_1}, \ldots, \widehat{\eta}_{s_n}\right)$ is distributed like the right hand side of (10).

In the next section we will identify this process, which turns out to be a known one and then show in the following section that the convergence of a truncated version of the process above holds in the sense of weak convergence in $\mathcal{D}[0, \infty)$.

3. The extremal process

Recall that $Z_1, Z_2, \ldots$ are i.i.d. exp(1) random variables and let $M_n = \frac{1}{\gamma} \bigwedge_{k=1}^{n} Z_k$. Then the process $(n \cdot M_{[tn]+1})_{t>0}$ converges as $n \to \infty$ weakly to a process $(\widehat{\eta}_t)_{t>0}$, the so called extremal process (see [4,7]). This process has the following properties (see Section 4.3 in [8]).

1. $\widehat{\eta}$ is stochastically continuous and has a version in $\mathcal{D}[0, \infty)$ (from hereon this is the assumed version).
2. $\widehat{\eta}$ has non-increasing paths, is piecewise constant, almost surely $\lim_{s\to 0} \widehat{\eta}_s = \infty$ and $\lim_{s\to\infty} \widehat{\eta}_s = 0$.
3. the finite dimensional distributions are given by the right hand side of (10), in particular

$$\mathbb{P}(\widehat{\eta}_{s_i} > x_i, i = 1, \ldots, n) = \exp\left(-\gamma \sum_{i=1}^{n} (s_i - s_{i-1}) \bigvee_{j=i}^{n} x_j\right). \tag{11}$$

4. The holding times in x are exponential with rate γx.
5. If the process jumps at time t then $\widehat{\eta}_t = \widehat{\eta}_{t-} \cdot U$, where U is independent of $\{\widehat{\eta}_s, 0 \le s < t\}$ (in an appropriate sense) and has a uniform distribution in [0, 1].

Now let η_0 be a random variable with values in $(0, z]$ for some $z > 0$, independent of $\{\widehat{\eta}_t, t \ge 0\}$ and define

$$\eta_t := \widehat{\eta}_t \wedge \eta_0. \tag{12}$$

The process η_t is a Markov process that inherits the above properties 1–5 from $\widehat{\eta}$, except for the following.

2*. η has non-increasing paths, is piecewise constant, almost surely $\lim_{s\to\infty} \eta_s = 0$.
3*. The finite dimensional distributions are given by

$$\left(\eta_{s_1}, \eta_{s_2}, \ldots, \eta_{s_n}\right) \stackrel{d}{=} \left(\eta_0 \wedge \frac{1}{\gamma}\left(\bigwedge_{i=1}^{k} \frac{Z_i}{s_i - s_{i-1}}\right)\right)_{1\le k\le n}. \tag{13}$$

For a proof note that the first jump below η_0 of the process η will go uniformly into the interval $[0, \eta_0]$. Since from then on the process η will continue just like $\widehat{\eta}$, we only have to show that the holding time in η_0, given by $T = \inf\{t > 0 : \eta_t \le \eta_0\}$, has an exponential distribution with rate

$\gamma\eta_0$. Indeed, we have for all $s > 0$, $\mathbb{P}(T > s|\eta_0) = \mathbb{P}(\widehat{\eta}_s > \eta_0|\eta_0) = e^{-\gamma s \eta_0}$. The property 3* is obvious from the construction.

The Markov process η has state space $[0, z]$, but we extend it to $(-\infty, z]$ by letting $\eta_t = \eta_0$ whenever $\eta_0 \le 0$. The reason is, that the process $-t \log X_{ts}$ can also attain negative values.

It follows from the above properties that the transition probabilities of the Markov process η are given for $0 \le x < y$ by

$$\mathbb{P}(\eta_{s+t} > x|\eta_s = y) = \exp(-\gamma t x), \quad t, s \ge 0. \tag{14}$$

Let $\mathcal{C}_z$ denote the class of continuous functions $f : (-\infty, z]$ for which $f(x) \to 0$ as $x \to -\infty$. The transition semi-group of the process is, for functions $f \in \mathcal{C}_z$, given by

$$\mathscr{P}_t f(x) := \mathbb{E}_x[f(\eta_t)] = \begin{cases} e^{-\gamma t x} f(x) + \gamma t \int_0^x f(y) e^{-\gamma t y}\, dy; & x \ge 0 \\ f(x); & x < 0. \end{cases} \tag{15}$$

Consequently the limit

$$\lim_{t\to 0} \frac{\mathbb{E}_x[f(\eta_t)] - f(x)}{t} = -\gamma x f(x) + \gamma \int_0^x f(y)\, dy \tag{16}$$

for $x \ge 0$ and zero for $x < 0$ exists uniformly at least for $f \in \mathcal{C}_z$. Moreover, the Feller property holds, i.e. $\mathscr{P}_t \mathcal{C}_z \subset \mathcal{C}_z$ and $\mathscr{P}_t f(x) \to f(x)$ as $t \to 0$ for $f \in \mathcal{C}_z$.

For $f \in \mathcal{C}_z$ the generator of the Markov process η is then given by

$$\mathscr{A} f(x) = \gamma x \textstyle\int_0^1 (f(xy) - f(x))\, dy = \gamma \int_0^x (f(y) - f(x))\, dy \tag{17}$$

for $x \ge 0$ and $\mathscr{A} f(x) = 0$ for $x < 0$, so $\mathscr{A} f \in \mathcal{C}_z$. We choose a smaller domain, namely those functions $f \in \mathcal{C}_z$ which are differentiable with derivative $f' \in \mathcal{C}_z$. Let $\mathcal{D}_{\mathscr{A}}$ denote this class. Using integration by parts or Fubini's theorem, we can write

$$\mathscr{A} f(x) = -\gamma \int_0^x \int_y^x f'(u)\, du\, dy = -\gamma \int_0^x u f'(u)\, du. \tag{18}$$

Note that if $f \in \mathcal{D}_{\mathscr{A}}$ then also $\mathscr{P}_t f \in \mathcal{D}_{\mathscr{A}}$ since for $x \ge 0$

$$(\mathscr{P}_t f)'(x) = e^{-\gamma t x}(f'(x) - \gamma t f(x)) + \gamma t e^{-\gamma t x} f(x) = e^{-\gamma t x} f'(x). \tag{19}$$

4. Convergence in $\mathcal{D}[0, \infty)$

Recalling (12), the following is the main result of this paper.

Theorem 2. *Suppose that the subordinator $(X)_{t\ge 0}$ satisfies one of the conditions of Theorem 1 and that $z \in (0, \infty)$. Then*

$$(z \wedge (-t \log X_{ts}))_{s\ge 0} \overset{d}{\Rightarrow} (\eta_s)_{s\ge 0} \tag{20}$$

as $t \to 0$ weakly in $(\mathcal{D}[0, \infty), J_1)$ and $\eta_0 = z$.

Proof. Let us write $X_t = X_t' + X_t''$, where X_t' has Lévy measure $\nu'(A) = \nu(A \cap (0, 1))$ and X_t'' has Lévy measure $\nu''(A) = \nu(A \cap [1, \infty))$. That is, X_t' captures the small jumps and X_t'' is a compound Poisson process with jumps of size at least one. It is well known that X_t' and X_t'' are independent. Moreover, $X''_t = 0$ for $t < \kappa$, where κ is an exponential random variable, so that $X_t = X'_t$ for $t < \kappa$. Since we are interested in the limiting behavior as $t \to 0$, we may assume that ν is concentrated on $(0, 1)$. Then X_t is a Markov process with generator [3] given by

$$\mathscr{L} f(x) = \int_0^1 (f(x+y) - f(x))\nu(dy), \quad x \geq 0 \tag{21}$$

for appropriate functions $f : [0, \infty) \to \mathbb{R}$. For fixed t the process $\eta_s^{(t)} = -t \log X_{ts} \wedge z$ is a Markov process with sample paths in $\mathcal{D}[0, \infty)$. The time-change $X_s \to X_{ts}$ transforms $\mathscr{L} f$ into $t\mathscr{L} f(x)$, while the subsequent state-space transformation $X_t \to g(X_t)$, with $g(x) = -t \log x$, changes $t\mathscr{L} f(x)$ to $t(\mathscr{L}(f \circ g))(g^{-1}(x))$; see e.g. [5]. Hence the generator of the process $\eta_s^{(t)}$, which has state space $(-\infty, z]$, is given by

$$\mathscr{A}^{(t)} f(x) = t \int_0^1 (f(-t \log(y + e^{-x/t})) - f(x))\, \nu(dy), \tag{22}$$

for $x \leq z$ and functions in $\mathcal{D}_{\mathscr{A}}$. For the transition semi-group of $\eta^{(t)}$ we obtain

$$\mathscr{P}_s^{(t)} f(x) = \mathbb{E}[f(-t \log X_{ts} \wedge x)], \quad x \leq z. \tag{23}$$

Hence, $\mathscr{P}_s^{(t)} f \in \mathcal{C}_z$ and $\mathscr{P}_s^{(t)} f(x) \to f(x)$ as $s \to 0$ by dominated convergence and the fact that $-t \log X_{ts} \to \infty$ as $s \to 0$. It follows that for every $t > 0$ the process $\eta^{(t)}$ has the Feller-property.

In Lemma 1 to follow we will show that, for every $z > 0$, $\mathscr{A}^{(t)} f \to \mathscr{A} f$ uniformly on $(-\infty, z]$. As the process is non-increasing and thus, one does not need to consider uniform convergence on the entire state space $\mathbb{R}$, it will follow from Theorem 6.1, p. 28 in [6] that the respective transition operators converge too, provided that $\mathcal{D}_{\mathscr{A}}$ is a core for the generator. But this follows from Proposition 3.3, p. 17 in [6] since $\mathcal{D}_{\mathscr{A}}$ is dense in $\mathcal{C}_z$ and $\mathscr{P}_t f \in \mathcal{D}_{\mathscr{A}}$ if $f \in \mathcal{D}_{\mathscr{A}}$ (as was shown in (19)). From Theorem 2.5, p. 167 in [6] it then follows, using the Feller-property of $\eta^{(t)}$, that $\eta^{(t)}$ tends to η in $\mathcal{D}[0, \infty)$. Since $-t \log X_{ts}$ tends to ∞ as $s \to 0$, it is clear that $\eta_0 = z$. □

Lemma 1. *Suppose that condition (S7) of Theorem 1 holds, let $f \in \mathcal{C}_z$ be differentiable with $f' \in \mathcal{C}_z$ and recall*

$$\mathscr{A}^{(t)} f(x) = t \int_{(0,1]} (f(-t \log(y + e^{-x/t})) - f(x))\, \nu(dy), \tag{24}$$

as well as

$$\mathscr{A} f(x) = \gamma \int_0^x (f(y) - f(x)) 1_{[0,\infty)}(x)\, dy. \tag{25}$$

Then, for each $z > 0$,

$$\lim_{t \downarrow 0} \sup_{x \in (-\infty, z]} \left| \mathscr{A}^{(t)} f(x) - \mathscr{A} f(x) \right| = 0. \tag{26}$$

Proof. Denote $\|f'\| \equiv \sup_{x \in (-\infty, z]} |f'(x)|$ ($< \infty$ as $f' \in \mathcal{C}_z$). Since $|f(x) - f(y)| \leq \|f'\| |x - y|$ then, for $0 < y \leq 1$,

$$\begin{aligned} |f(-t \log(y + e^{-x/t})) - f(x)| &\leq \|f'\| |-t \log(y + e^{-x/t}) - x| \\ &= \|f'\| |\log(y + e^{-x/t}) + \log e^{x/t}| t \end{aligned}$$

$$= \|f'\| t \log(ye^{x/t} + 1)$$
$$\leq \|f'\| t y e^{x/t}. \tag{27}$$

Thus, recalling that

$$\rho \equiv \int_{(0,1]} y\nu(dy) \left(= \int_0^1 \nu(y,1]dy \right) < \infty\,, \tag{28}$$

we have that for $x \leq 0$

$$|\mathscr{A}^{(t)} f(x)| \leq \|f'\| \rho t^2 e^{x/t} \leq \|f'\| \rho t^2. \tag{29}$$

Since $\mathscr{A} f(x) = 0$ for $x \leq 0$, this implies that

$$\lim_{t \downarrow 0} \sup_{x \leq 0} |\mathscr{A}^{(t)} f(x) - \mathscr{A} f(x)| = 0 \tag{30}$$

as $t \to 0$.

Next, note that for $0 \leq x \leq z$,

$$\int_{(0,1]} (f(-t\log(y + e^{-x/t})) - f(x))\, \nu(dy)$$
$$= -\int_{(0,1]} \int_{-t\log(y+e^{-x/t})}^{x} f'(u) du \nu(dy)$$
$$= -\int_{-t\log(1+e^{-x/t})}^{x} f'(u)\nu\left(e^{-u/t} - e^{-x/t}, 1\right] du. \tag{31}$$

In particular, upon substituting $y = e^{-u/t} - e^{-x/t}$, so that

$$dy = -e^{-u/t} du/t = -(y + e^{-x/t}) du/t,$$

we have that

$$\left| \int_{-t\log(1+e^{-x/t})}^{0} f'(u)\nu(e^{-u/t} - e^{-x/t}, 1] du \right|$$
$$\leq \|f'\| \int_{-t\log(1+e^{-x/t})}^{0} \nu(e^{-u/t} - e^{-x/t}, 1] du$$
$$= t\|f'\| \int_{1-e^{-x/t}}^{1} \frac{\nu(y,1]}{y + e^{-x/t}} dy$$
$$\leq t\|f'\| \int_{1-e^{-x/t}}^{1} \nu(y,1] dy$$
$$\leq t\|f'\| \int_0^1 \nu(y,1] dy = t\|f'\|\rho. \tag{32}$$

The last expression clearly vanishes as $t \downarrow 0$ and in particular when multiplying it by t. Thus the left side converges to zero uniformly on $x \in [0, z]$.

From (30), (32) and

$$\mathscr{A} f(x) = \gamma \int_0^x (f(y) - f(x)) dx = -\gamma \int_0^x f'(u) u du, \tag{33}$$

it remains to show that for each $z > 0$

$$\lim_{t\downarrow 0} \sup_{x\in[0,z]} \left| \int_0^x f'(u) \left(\gamma u - t\nu \left(e^{-u/t} - e^{-x/t}, 1\right]\right) du \right| = 0. \tag{34}$$

We clearly have that

$$\begin{aligned} &\left| \int_0^x f'(u) \left(\gamma u - t\nu \left(e^{-u/t} - e^{-x/t}, 1\right]\right) du \right| \\ &\quad \le \|f'\| \int_0^x \left|\gamma u - t\nu \left(e^{-u/t} - e^{-x/t}, 1\right]\right| du. \end{aligned} \tag{35}$$

Substituting $y = e^{-u/t} - e^{-x/t}$, adding and subtracting $\gamma \log y$ in the second line of the following equation and rearranging terms give

$$\begin{aligned} &\int_0^x \left|\gamma u - t\nu \left(e^{-u/t} - e^{-x/t}, 1\right]\right| du \\ &= t^2 \int_0^{1-e^{-x/t}} \left|-\gamma \log(y + e^{-x/t}) - \nu(y, 1]\right| \frac{dy}{y + e^{-x/t}} \\ &\le \gamma t^2 \int_0^1 \left|1 - \frac{\nu(y,1]}{-\gamma \log y}\right| \frac{-\log y}{y + e^{-x/t}}\, dy + \gamma t^2 \int_0^1 \frac{\log(y + e^{-x/t}) - \log y}{y + e^{-x/t}} dy. \end{aligned} \tag{36}$$

Substituting $y = e^{-x/t} v$ gives

$$\begin{aligned} \int_0^1 \frac{\log(y + e^{-x/t}) - \log y}{y + e^{-x/t}} dy &= \int_0^{e^{x/t}} \frac{\log(v+1) - \log v}{v+1} dv \\ &\le \int_0^{\infty} \frac{\log(v+1) - \log v}{v+1} dv. \end{aligned} \tag{37}$$

Since $\int_0^{\epsilon} (-\log v) dv = \epsilon(1 - \log \epsilon) < \infty$ and since

$$\frac{\log(v+1) - \log(v)}{v+1} = \frac{1}{v+1} \int_v^{v+1} \frac{1}{u} du \le \frac{1}{v^2} \tag{38}$$

it follows that the right hand side of (37) is finite and thus the second term of the right hand side of (36) converges to zero uniformly on $x \in [0, \infty)$. Therefore, as the first term on the right hand side of (36) is bounded above (on $x \in [0, z]$) by

$$\gamma t^2 \int_0^1 \left|1 - \frac{\nu(y,1]}{-\gamma \log y}\right| \frac{-\log y}{y + e^{-z/t}}\, dy \tag{39}$$

it remains to show that (39) vanishes as $t \downarrow 0$.

Clearly, for any $\delta \in (0, 1)$,

$$\int_\delta^1 \left|1 - \frac{\nu(y,1]}{-\gamma \log y}\right| \frac{-\log y}{y + e^{-z/t}}\, dy \le \int_\delta^1 \left|1 - \frac{\nu(y,1]}{-\gamma \log y}\right| \frac{-\log y}{y}\, dy < \infty, \tag{40}$$

so that upon multiplying by t^2 the left side converges to zero. Also, note that

$$\int_0^{e^{-z/t}} \frac{-\log y}{y+e^{-z/t}}\,dy \le e^{z/t}\int_0^{e^{-z/t}}(-\log y)dy$$
$$= e^{z/t}\cdot e^{-z/t}\left(1-\log e^{-z/t}\right) = 1+\frac{z}{t} \tag{41}$$

which, upon multiplication by t^2, vanishes as $t \downarrow 0$. Therefore, also

$$\gamma t^2 \int_0^{e^{-z/t}} \left|1-\frac{\nu(y,1]}{-\gamma \log y}\right| \frac{-\log y}{y+e^{-z/t}}\,dy \tag{42}$$

vanishes as $t \downarrow 0$, since by the assumptions $\left|1-\frac{\nu(y,1]}{-\gamma\log y}\right|$ is bounded on $[0, z]$.

To complete the proof, in view of (40) and (42), it remains to show that for any $\epsilon > 0$ there is some $\delta > 0$ and some $T > 0$, such that for all $0 < t < T$

$$t^2 \int_{e^{-z/t}}^{\delta} \left|1-\frac{\nu(y,1]}{-\gamma \log y}\right| \frac{-\log y}{y+e^{-z/t}}\,dy < \epsilon. \tag{43}$$

By the assumption we can pick some $0 < \delta < 1$ such that, for all $0 < y < \delta$,

$$\left|1-\frac{\nu(y,1]}{-\gamma \log y}\right| < \frac{\epsilon}{z^2}. \tag{44}$$

Then, take $T = \frac{z}{-\log\delta}$ and note that $t < T$ if and only if $e^{-z/t} < \delta$. We now have that for all $0 < t < T$,

$$\begin{aligned} t^2 \int_{e^{-z/t}}^{\delta} \left|1-\frac{\nu(y,1]}{-\gamma \log y}\right| \frac{-\log y}{y}dy &< \frac{\epsilon t^2}{z^2}\int_{e^{-z/t}}^{\delta}\frac{-\log y}{y}dy \\ &\le \frac{\epsilon t^2}{z^2}(-\log e^{-z/t})\int_{e^{-z/t}}^{\delta}\frac{1}{y}dy \\ &= \frac{\epsilon t}{z}\left(\log\delta - \log e^{-z/t}\right) \\ &= \epsilon\left(t\frac{\log\delta}{z}+1\right) < \epsilon \end{aligned}$$

and the proof is complete. □

Acknowledgments

The authors thank Shaul Bar Lev and Thomas Kurtz for useful discussions. The first author was supported in part by grant No. 434/09 from the Israel Science Foundation and the Vigevani Chair in Statistics.

References

[1] S.K. Bar-Lev, P. Enis, Existence of moments and an asymptotic result based on a mixture of exponential distributions, Statist. Probab. Lett. 5 (1987) 273–277 (in English).

[2] S.K. Bar-Lev, A. Löpker, W. Stadje, On the small-time behavior of subordinators, Bernoulli 18 (3) (2012) 823–835 (in English).

[3] J. Bertoin, Lévy Processes, Cambridge Univ. Press, Cambridge, 1998 (in English).

[4] M. Dwass, Extremal processes, Ann. Math. Stat. 35 (1964) 1718–1725 (in English).
[5] E.B. Dynkin, Markov Processes. Vols. I, II, Springer, 1965 (in English).
[6] S.N. Ethier, T.G. Kurtz, Markov Processes. Characterization and Convergence, John Wiley & Sons, Hoboken, NJ, 2005 (in English).
[7] J. Lamperti, On extreme order statistics, Ann. Math. Stat. 35 (1964) 1726–1737 (in English).
[8] S.I. Resnick, Extreme Values, Regular Variation, and Point Processes, Springer, 1987 (in English).