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Heat kernels of non-symmetric jump processes with exponentially decaying jumping kernel

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Abstract

In this paper we study the transition densities for a large class of non-symmetric Markov processes whose jumping kernels decay exponentially or subexponentially. We obtain their upper bounds which also decay at the same rate as their jumping kernels. When the lower bounds of jumping kernels satisfy the weak upper scaling condition at zero, we also establish lower bounds for the transition densities, which are sharp.

MSC: primary 60J35; secondary 60J75

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1. Introduction

Let $d \in \mathbb{N}$, $\mathbb{R}^d$ be the d-dimensional Euclidian space and $\mathbb{R}_+ = \{x \in \mathbb{R}^1 : x > 0\}$. Define

$$\mathcal{L}^{\kappa} f(x) := \lim_{\varepsilon \downarrow 0} \mathcal{L}^{\kappa,\varepsilon} f(x) := \lim_{\varepsilon \downarrow 0} \int_{\{z \in \mathbb{R}^d : |z| > \varepsilon\}} (f(x+z) - f(x))\, \kappa(x,z) J(|z|) dz, \qquad (1.1)$$

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where $\kappa : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ is a Borel function satisfying the following conditions: there exist positive constants $\kappa_0, \kappa_1, \kappa_2$ and $\delta \in (0, 1)$ such that

$$\kappa_0 \le \kappa(x, z) \le \kappa_1, \quad \kappa(x, z) = \kappa(x, -z) \quad \text{for all } x, z \in \mathbb{R}^d \tag{1.2}$$

and

$$|\kappa(x, z) - \kappa(y, z)| \le \kappa_2 |x - y|^\delta \qquad \text{for all } x, y, z \in \mathbb{R}^d. \tag{1.3}$$

The operator $\mathcal{L}^\kappa$ can be regarded as the non-local counterpart of elliptic operators in non-divergence form. In this context, the Hölder continuity of $\kappa(\cdot, z)$ in (1.3) is a natural assumption.

In [5], Zhen-Qing Chen and Xicheng Zhang studied $\mathcal{L}^\kappa$ and its heat kernel when $J(r) = r^{-d-\alpha}$, $r > 0$ and $\alpha \in (0, 2)$. They proved the existence and uniqueness of the heat kernel and its sharp two-sided estimates, cf. [5, Theorem 1.1] for details. The methods in [5] are quite robust and have been applied to non-symmetric and non-convolution operators (see [2,3,6,7,13,12,10] and references therein). In particular, the first named author, jointly with Renming Song and Zoran Vondraček in [13], studied the operator $\mathcal{L}^\kappa$ and its heat kernel when J is comparable to jumping kernels of subordinate Brownian motions and its Lévy exponent satisfies a weak lower scaling condition at infinity. In this paper we consider the case that $J(r)$ decays exponentially or subexponentially when $r \to \infty$ and we obtain sharp two-sided estimates for the heat kernel of $\mathcal{L}^\kappa$.

Throughout this paper, we assume $d \in \mathbb{N}$, and that $J : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and non-increasing function satisfying that there exist a continuous and strictly increasing function $\phi : [0, 1] \to \mathbb{R}_+$ with $\phi(0) = 0$, and constants $b > 0$, $0 < \beta \le 1$ and $a \ge 1$ such that

$$\frac{a^{-1}}{r^d \phi(r)} \le J(r) \le \frac{a}{r^d \phi(r)}, \quad 0 < r \le 1 \quad \text{and} \quad J(r) \le a \exp(-b r^\beta), \quad r > 1. \tag{1.4}$$

In addition, we assume that J is differentiable in $\mathbb{R}_+$ and

$$r \longmapsto -\frac{J'(r)}{r} \quad \text{is non-increasing in } \mathbb{R}_+. \tag{1.5}$$

Our main assumption on ϕ is the following weak lower scaling condition at zero: there exist $\alpha_1 \in (0, 2]$ and $a_1 > 0$ such that

$$a_1 \Big(\frac{R}{r}\Big)^{\alpha_1} \le \frac{\phi(R)}{\phi(r)}, \quad 0 < r \le R \le 1. \tag{1.6}$$

Since we allow α_1 to be 2, to guarantee that J is to be a Lévy density, we also need the following integrability condition for ϕ near zero:

$$\int_0^1 \frac{s}{\phi(s)} ds := C_0 < \infty. \tag{1.7}$$

The monotonicity of $J(r)$ and (1.7) ensure the existence of an isotropic unimodal Lévy process in $\mathbb{R}^d$ with the Lévy measure $J(|x|)dx$, which is infinite because of (1.6) and the lower bound in (1.4).

Our goal is to obtain estimates of the heat kernel for $\mathcal{L}^\kappa$. First we introduce the function $\mathscr{G}(t, x)$ which plays an important role for the estimates of heat kernel. Let us define the function Φ and θ as

$$\Phi(r) := \begin{cases} \dfrac{r^2}{2\int_0^r \frac{s}{\phi(s)} ds}, & 0 < r \le 1, \\ \Phi(1) r^2, & r > 1, \end{cases} \tag{1.8}$$

and

$$\theta(r) := \begin{cases} \dfrac{1}{r^d \Phi(r)}, & r \le 1, \\ \exp(-br^\beta)\mathbf{1}_{\{0<\beta<1\}} + r^{-d-1}\exp(-\dfrac{b}{5}r)\mathbf{1}_{\{\beta=1\}}, & r > 1. \end{cases}$$

By (1.7), $\int_0^r \frac{s}{\phi(s)}ds$ is integrable so that Φ is well-defined. Note that $\Phi(1) = \left(2\int_0^1 \frac{s}{\phi(s)}ds\right)^{-1} = (2C_0)^{-1}$ is determined by C_0. Also, by Lemma 2.1 we will see that Φ is a strictly increasing function in $\mathbb{R}_+$ and $\lim_{r\downarrow 0}\Phi(r) = 0$, which imply that there exists an inverse function $\Phi^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$. For $t > 0$ and $r > 0$ define $\mathscr{G}(t,r)$ by

$$\mathscr{G}(t,r) = \mathscr{G}^{(d)}(t,r) := \frac{1}{t\,\Phi^{-1}(t)^d} \wedge \theta(r).$$

where $a \wedge b := \min\{a, b\}$. By an abuse of notation we also define

$$\mathscr{G}(t,x) = \mathscr{G}^{(d)}(t,x) := \frac{1}{t\,\Phi^{-1}(t)^d} \wedge \theta(|x|), \quad t > 0,\ x \in \mathbb{R}^d, \tag{1.9}$$

so $\mathscr{G}(t,x) = \mathscr{G}(t,|x|)$. Note that the definition of $\theta(r)$ for $\beta = 1$ is simply technical and it is harmless for readers to regard $\theta(r)$ as $\frac{1}{r^d\Phi(r)}\mathbf{1}_{\{r\le 1\}} + \exp(-\frac{b}{6}r)\mathbf{1}_{\{r>1\}}$ as the upper bound of heat kernel for $\beta = 1$ in Theorems 1.1–1.3.

Let us compare $\mathscr{G}$ with the following function defined by

$$\tilde{\mathscr{G}}(t,x) = \tilde{\mathscr{G}}(t,|x|) := \frac{1}{t\,\Phi^{-1}(t)^d} \wedge \frac{1}{|x|^d\,\Phi(|x|)}. \tag{1.10}$$

By [13, Proposition 2.1] and our Lemma 3.3 we see that $\tilde{\mathscr{G}}$ is the function used for the upper heat kernel estimate in [13] (see Remark 2.3 for details). It is easy to see that $\mathscr{G}(t,x) \le c\,\tilde{\mathscr{G}}(t,x)$ (see Lemma 2.2). Here is our main result.

Theorem 1.1. *Let $\mathcal{L}^\kappa$ be the operator in (1.1). Assume that jumping kernel J satisfies (1.4) and (1.5), that ϕ satisfies (1.6) and (1.7), and that κ satisfies (1.2) and (1.3). Then, there exists a unique jointly continuous function $p^\kappa(t,x,y)$ on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ solving*

$$\partial_t p^\kappa(t,x,y) = \mathcal{L}^\kappa p^\kappa(t,\cdot,y)(x), \quad x \ne y, \tag{1.11}$$

and satisfying the following properties:

(i) (Upper bound) For every $T \ge 1$, there is a constant $c_1 > 0$ such that for all $t \in (0,T]$ and $x, y \in \mathbb{R}^d$,

$$p^\kappa(t,x,y) \le c_1 t\mathscr{G}(t,x-y). \tag{1.12}$$

(ii) (Fractional derivative) For any $x, y \in \mathbb{R}^d$ with $x \ne y$, the map $t \mapsto \mathcal{L}^\kappa p^\kappa(t,\cdot,y)(x)$ is continuous, and for each $T \ge 1$, there exists a constant $c_2 > 0$ such that for all $t \in (0,T]$, $\varepsilon \in [0,1]$ and $x, y \in \mathbb{R}^d$,

$$|\mathcal{L}^{\kappa,\varepsilon} p^\kappa(t,\cdot,y)(x)| \le c_2\tilde{\mathscr{G}}(t,x-y). \tag{1.13}$$

(iii) (Continuity) For any bounded and uniformly continuous function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d} p^\kappa(t,x,y)f(y)dy - f(x)\right| = 0. \tag{1.14}$$

Furthermore, such unique function $p^\kappa(t,x,y)$ satisfies the following lower bound: for every $T \ge 1$, there exists a constant $c_3, c_4 > 0$ such that for all $t \in (0,T]$,

$$p^\kappa(t,x,y) \ge c_3 \begin{cases} \Phi^{-1}(t)^{-d}, & |x-y| \le c_4 \Phi^{-1}(t) \\ tJ(|x-y|), & |x-y| > c_4 \Phi^{-1}(t) \end{cases} \tag{1.15}$$

The constants c_i, $i = 1,\dots,4$, depend only on $d, T, a, a_1, \alpha_1, b, \beta, C_0, \delta, \kappa_0, \kappa_1$ and κ_2.

The upper bound of the fractional derivative of p^κ in (1.13), which is a counterpart of [13, (1.12)], will be used to prove the uniqueness of heat kernel.

We emphasize here that unlike [13, (1.21)] we obtain (1.15) without any upper weak scaling condition on ϕ. The estimates in (1.12) and (1.15) in Theorem 1.1 are not sharp in general. However, when the jumping kernel J satisfies

$$J(r) \ge a_1 \exp(-b_1 r^{\beta_1}), \quad r > 1, \tag{1.16}$$

and ϕ satisfies weak scaling condition at zero, that is,

$$\frac{\phi(R)}{\phi(r)} \le a_2 \Big(\frac{R}{r}\Big)^{\alpha_2}, \quad 0 < r \le R \le 1 \tag{1.17}$$

for some $a_2 > 0$ and $\alpha_2 \in (0,2)$, then the lower bound in (1.15) is comparable to that in [9, Theorem 1.2], which is lower heat kernel estimates for symmetric Hunt process with exponentially decaying jumping kernel. We remark here that, under the assumption (1.17), the constant α_1 in (1.6) must be in $(0,2)$.

Note that ϕ is comparable to Φ under (1.6) and (1.17). Therefore, under additional assumptions (1.16) and (1.17) we have the following corollary.

Corollary 1.2. *Let $\mathcal{L}^\kappa$ be the operator in (1.1). Assume that jumping kernel J satisfies (1.4), (1.5) and (1.16), that ϕ satisfies (1.6)* and *(1.17), and that κ satisfies (1.2) and (1.3). Then, the heat kernel $p^\kappa(t,x,y)$ for $\mathcal{L}^\kappa$ satisfies the following estimates: for every $T \ge 1$, there is a constant $c > 0$ such that for all $t \in (0,T]$ and $x, y \in \mathbb{R}^d$,*

$$c^{-1}\left(\phi^{-1}(t)^{-d} \wedge \frac{t}{|x-y|^d \phi(|x-y|)}\right) \le p^\kappa(t,x,y) \le c\left(\phi^{-1}(t)^{-d} \wedge \frac{t}{|x-y|^d \phi(|x-y|)}\right), \quad |x-y| \le 1,$$

and

$$c^{-1} t \exp(-b_1 |x-y|^{\beta_1}) \le p^\kappa(t,x,y) \le ct\theta(|x-y|), \quad |x-y| > 1.$$

The constant c depends on $d, T, a, a_1, a_2, \alpha_1, \alpha_2, b, b_1, \beta, \beta_1, C_0, \delta, \kappa_0, \kappa_1$ and κ_2.

Comparing to [13], Corollary 1.2 provides further precise heat kernel estimates for the operator (1.1) with exponential decaying function J. We remark here that, when $\beta > 1$, the estimates of $p^\kappa(t,x,y)$ are different and so the result in Corollary 1.2 does not hold even for symmetric Lévy processes. See [4,15]. We will address this interesting case somewhere else.

More properties of the heat kernel $p^\kappa(t,x,y)$ are listed in the following theorems.

Theorem 1.3. *Suppose that the assumptions of Theorem 1.1 are satisfied.*

(1) (Conservativeness) For all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p^\kappa(t,x,y)\,dy = 1\,. \tag{1.18}$$

(2) (Chapman–Kolmogorov equation) For all $s, t > 0$ and $x, y \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p^\kappa(t, x, z) p^\kappa(s, z, y)\, dz = p^\kappa(t+s, x, y). \tag{1.19}$$

(3) (Hölder continuity) For every $T \geq 1$ and $\gamma \in (0, \alpha_1) \cap (0, 1]$, there is a constant $c_1 > 0$ such that for all $0 < t \leq T$ and $x, x', y \in \mathbb{R}^d$ with either $x \neq y$ or $x' \neq y$,

$$|p^\kappa(t, x, y) - p^\kappa(t, x', y)| \leq c_1 |x - x'|^\gamma t\, \Phi^{-1}(t)^\gamma (\mathscr{G}(t, x-y) \vee \mathscr{G}(t, x'-y)). \tag{1.20}$$

(4) (Gradient estimate) Further assume that $\alpha_1 \in (2/3, 2)$ and $\alpha_1 + \delta > 1$. Then, for every $T \geq 1$, there is a constant $c_2 > 0$ such that for all $0 < t \leq T$ and $x, y \in \mathbb{R}^d$ with $x \neq y$,

$$|\nabla_x p^\kappa(t, x, y)| \leq c_2 \Phi^{-1}(t) t \tilde{\mathscr{G}}(t, x-y). \tag{1.21}$$

The constants c_1 and c_2 depend on $d, T, a, a_1, \alpha_1, b, \beta, C_0, \gamma, \delta, \kappa_0, \kappa_1$ and κ_2.

For $t > 0$, define the operator P_t^κ by

$$P_t^\kappa f(x) = \int_{\mathbb{R}^d} p^\kappa(t, x, y) f(y)\, dy, \quad x \in \mathbb{R}^d, \tag{1.22}$$

where f is a nonnegative (or bounded) Borel function on $\mathbb{R}^d$, and let $P_0^\kappa = \mathrm{Id}$. Then by Theorem 1.3, $(P_t^\kappa)_{t \geq 0}$ is a Feller semigroup with the strong Feller property. Let $C_b^{2,\varepsilon}(\mathbb{R}^d)$ be the space of bounded twice differentiable functions in $\mathbb{R}^d$ whose second derivatives are uniformly Hölder continuous.

Theorem 1.4. *(1) (Generator) Let $\varepsilon > 0$. For any $f \in C_b^{2,\varepsilon}(\mathbb{R}^d)$, we have*

$$\lim_{t \downarrow 0} \frac{1}{t} \left(P_t^\kappa f(x) - f(x) \right) = \mathcal{L}^\kappa f(x), \tag{1.23}$$

and the convergence is uniform. (2) (Analyticity) The semigroup $(P_t^\kappa)_{t \geq 0}$ of $\mathcal{L}^\kappa$ is analytic in $L^p(\mathbb{R}^d)$ for every $p \in [1, \infty)$.

In this paper, we defined the function $\mathscr{G}(t, x)$ from the conditions on J directly, while in [13] the function $\rho(t, x)$ is defined by the characteristic exponent of an isotropic unimodal Lévy process with jumping kernel $J(x)dx$. The reason is that, in our situation, it is more convenient than using characteristic exponent to describe exponential decaying jumping kernel. See Remark 2.3 for the connections between two definitions.

As [13], the approach in this paper is based on the method originally developed in [5]. In Section 2, we introduce basic setup and scaling inequalities. In addition, we obtain some convolution inequalities at Proposition 2.8 in Section 2.2. The results in Section 2.2 are similar to [13, Lemma 2.6], although our function $\mathscr{G}(t, x)$ is smaller than that in [13].

In Section 3, we discuss gradient estimates for the heat kernel of isotropic unimodal Lévy process with jumping kernel $J(|x|)dx$, which follow from the results in [11,12]. We only use Proposition 3.2 in the proof of our main theorem, but Proposition 3.1 itself is of independent interest.

In Section 4, we obtain some useful estimates on functions involving the heat kernel for the isotropic Lévy process whose jumping kernel is $J(|x|)dx$. In Section 4.1, we improve inequalities in Proposition 3.2 and 4.1–4.2 for the symmetric Lévy processes whose jumping kernel is $\mathfrak{K}(x)J(|x|)dx$, where $\mathfrak{K}(x)$ is symmetric and bounded between two positive constants. As [13, Section 3], we also observe the continuous dependency of the heat kernel $p^{\mathfrak{K}}$ with respect to the jumping kernel $\mathfrak{K}(x)J(|x|)dx$.

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In Section 5, we follow the Levi's construction in [13, Section 4]. Note that as in [13, Section 4], many results in Section 5 are derived from the estimates in Sections 2 and 4 so that we can follow [13] for the most of proofs. Finally we provide the proofs of Theorems 1.1, 1.3 and 1.4 in Section 6.

In this paper, we use the following notations. We will use ":=" to denote a definition, which is read as "is defined to be". For any two positive functions f and g, $f \asymp g$ means that there is a positive constant $c \geq 1$ such that $c^{-1} g \leq f \leq c\, g$ on their common domain of definition. Denote $\operatorname{diam}(A) = \sup\{|x-y| : x, y \in A\}$ and $\sigma(dz) = \sigma_d(dz)$ be a uniform measure in the sphere $\{z \in \mathbb{R}^d : |z| = 1\}$. For a function $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$, we define $f(t, x \pm z) = f(t, x+z) + f(t, x-z)$ and

$$\delta_f(t, x; z) := f(t, x+z) + f(t, x-z) - 2f(t,x) = f(t, x \pm z) - 2f(t,x). \tag{1.24}$$

Throughout the rest of this paper, the positive constants $T, a, a_1, \alpha_1, b, \beta, \delta, \kappa_0, \kappa_1, \kappa_2$ and C_i, $i = 0, 1, 2, \ldots$, can be regarded as fixed. In the statements of results and the proofs, the constants $c_i = c_i(a, b, c, \ldots)$, $i = 0, 1, 2, \ldots$, denote generic constants depending on $a, b, c, \ldots$, whose exact values are unimportant. They start anew in each statement and each proof.

2. Preliminaries

In this section we first study some elementary properties of Φ defined in (1.8).

Lemma 2.1. *Assume that ϕ satisfies (1.6) and (1.7). Then, Φ is continuous and strictly increasing in* $(0, 1]$, *and satisfies*

$$\Phi(r) \leq \phi(r), \qquad 0 < r \leq 1 \tag{2.1}$$

and

$$a_1 \left(\frac{R}{r}\right)^{\alpha_1} \leq \frac{\Phi(R)}{\Phi(r)} \leq \left(\frac{R}{r}\right)^2, \qquad 0 < r \leq R, \tag{2.2}$$

where $a_1 > 0$ and $\alpha_1 \in (0, 2]$ are constants in (1.6). In particular, (2.1) implies $\lim_{r \downarrow 0} \Phi(r) = 0$.

Proof. Since ϕ is continuous in $(0, 1]$, Φ is continuous in $\mathbb{R}_+$ by definition. Also, since ϕ is strictly increasing, we have

$$\Phi(r) = \frac{r^2}{2\int_0^r \frac{s}{\phi(s)} ds} \leq \frac{r^2}{2\int_0^r \frac{s}{\phi(r)} ds} = \phi(r).$$

To show that Φ is strictly increasing, it suffices to observe that for $0 < r < 1$,

$$\left(\frac{1}{2\Phi(r)}\right)' = \left(r^{-2} \int_0^r \frac{s}{\phi(s)} ds\right)' = 2r^{-3} \int_0^r s\left(\frac{1}{\phi(r)} - \frac{1}{\phi(s)}\right) ds < 0.$$

Now we prove (2.2). Clearly, by the definition of Φ, (2.2) holds for $1 \leq r \leq R$.

For $0 < r \leq R \leq 1$, we have $R^2/\Phi(R) = \int_0^R (s/\phi(s)) ds \geq \int_0^r (s/\phi(s)) ds = r^2/\Phi(r)$, which implies the second inequality in (2.2). Also, by change of variables and (1.6)

$$\begin{aligned}\frac{\Phi(R)}{\Phi(r)} &= \frac{2\Phi(R)}{r^2} \int_0^r \frac{s}{\phi(s)} ds = \frac{2\Phi(R)}{r^2} \int_0^R \frac{(r/R)t}{\phi((r/R)t)} (r/R) dt \\ &= \frac{2\Phi(R)}{R^2} \int_0^R \frac{t}{\phi((r/R)t)} dt \geq a_1 \left(\frac{R}{r}\right)^{\alpha_1} \frac{2\Phi(R)}{R^2} \int_0^R \frac{t}{\phi(t)} dt = a_1 \left(\frac{R}{r}\right)^{\alpha_1}.\end{aligned}$$

For $R \geq 1 \geq r > 0$, using $\Phi(R) = \Phi(1)R^2$ and above estimates we have

$$a_1 \left(\frac{R}{r}\right)^{\alpha_1} \leq a_1 \frac{R^2}{r^{\alpha_1}} \leq \frac{\Phi(R)}{\Phi(r)} = \frac{\Phi(R)}{\Phi(1)} \frac{\Phi(1)}{\Phi(r)} \leq \frac{R^2}{r^2}. \quad \Box$$

Note that our main results hold for all $t \leq T$, while the definition of $\mathscr{G}$ in (1.9) is independent of T. To make our proofs simpler, we introduce a family of auxiliary functions which will be used mostly in proofs.

Let $T \geq \Phi(1)$ and define $\mathscr{G}_T : (0, T] \times (0, \infty) \to (0, \infty)$ by

$$\mathscr{G}_T(t, r) = \begin{cases} \dfrac{1}{t\,\Phi^{-1}(t)^d}, & r \leq \Phi^{-1}(t), \\ \dfrac{1}{r^d\,\Phi(r)}, & \Phi^{-1}(t) < r \leq \Phi^{-1}(T), \\ C_T \exp(-br^\beta)\mathbf{1}_{0<\beta<1} + \dfrac{C_T}{r^{d+1}} \exp(-\dfrac{b}{5}r)\mathbf{1}_{\beta=1}, & r > \Phi^{-1}(T), \end{cases} \tag{2.3}$$

where $C_T := T^{-1}\,\Phi^{-1}(T)^{-d} \exp(b\,\Phi^{-1}(T)^\beta)\mathbf{1}_{0<\beta<1} + T^{-1}\,\Phi^{-1}(T) \exp(\frac{b}{4}\,\Phi^{-1}(T))\mathbf{1}_{\beta=1}$. Note that $r \mapsto \mathscr{G}_T(t, r)$ is continuous and non-increasing (due to such choice of C_T).

Recall that $\tilde{\mathscr{G}}(t, r)$ is defined in (1.10). In the following lemma we show that $\mathscr{G}_T$ and $\mathscr{G}(t, x)$ are comparable and less than $\tilde{\mathscr{G}}(t, r)$.

Lemma 2.2. *(a) Let $T \geq \Phi(1)$. Then, there exists a constant $c_1 = c_1(T) > 0$ such that*

$$c_1^{-1}\mathscr{G}_T(t, r) \leq \mathscr{G}(t, r) \leq c_1\mathscr{G}_T(t, r) \tag{2.4}$$

for any $t \in (0, T]$ and $r > 0$.

(b) There exists a constant $c_2 > 0$ such that

$$\mathscr{G}(t, r) \leq c_2\tilde{\mathscr{G}}(t, r). \tag{2.5}$$

for any $t > 0$ and $r > 0$. The constant c_1 depends on $d, b, T, \Phi^{-1}(T), \beta$ and C_0, and c_2 depends on d, b, β and C_0.

Proof. (a) Define

$$\theta_T(r) := \begin{cases} r^{-d}\,\Phi(r)^{-1}, & r \leq \Phi^{-1}(T), \\ C_T \exp(-br^\beta)\mathbf{1}_{0<\beta<1} + C_T r^{-d-1} \exp(-\dfrac{b}{5}r)\mathbf{1}_{\beta=1}, & r > \Phi^{-1}(T). \end{cases}$$

Note that $r \mapsto \theta_T(r)$ is strictly decreasing and $\theta_T(\Phi^{-1}(t)) = \frac{1}{t\,\Phi^{-1}(t)^d}$ for any $0 < t \leq T$. Thus we can obtain

$$\theta_T(r) \leq \frac{1}{t\,\Phi^{-1}(t)^d} \quad \text{if and only if} \quad t \leq \Phi(r). \tag{2.6}$$

By (2.3) and (2.6) we have

$$\mathscr{G}_T(t, r) = \frac{1}{t\,\Phi^{-1}(t)^d} \wedge \theta_T(r). \tag{2.7}$$

Let

$$M_T := \begin{cases} \sup_{1\le r\le \Phi^{-1}(T)} \frac{1}{r^d\Phi(r)}\exp(br^\beta) & \text{for} \quad 0<\beta<1, \\ \sup_{1\le r\le \Phi^{-1}(T)} \frac{r}{\Phi(r)}\exp(\frac{b}{5}r) & \text{for} \quad \beta=1 \end{cases}$$

and

$$m_T := \begin{cases} \inf_{1\le r\le \Phi^{-1}(T)} \frac{1}{r^d\Phi(r)}\exp(br^\beta) & \text{for} \quad 0<\beta<1, \\ \inf_{1\le r\le \Phi^{-1}(T)} \frac{r}{\Phi(r)}\exp(\frac{b}{5}r) & \text{for} \quad \beta=1 \end{cases}$$

Then, for $0<\beta<1$,

$$\theta(r) = \begin{cases} \frac{1}{r^d\Phi(r)} = \theta_T(r), & r\le 1, \\ \exp(-br^\beta) \ge M_T^{-1}\frac{1}{r^d\Phi(r)} = M_T^{-1}\theta_T(r), & 1<r\le \Phi^{-1}(T), \\ \exp(-br^\beta) \le m_T^{-1}\theta_T(r), & 1<r\le \Phi^{-1}(T), \\ \exp(-br^\beta) = C_T^{-1}\theta_T(r), & r>\Phi^{-1}(T) \end{cases}$$

and for $\beta=1$,

$$\theta(r) = \begin{cases} \frac{1}{r^d\Phi(r)} = \theta_T(r), & r\le 1, \\ \frac{1}{r^{d+1}}\exp(-\frac{b}{5}r) \ge M_T^{-1}\frac{1}{r^d\Phi(r)} = M_T^{-1}\theta_T(r), & 1<r\le \Phi^{-1}(T), \\ \exp(-br^\beta) \le m_T^{-1}\theta_T(r), & 1<r\le \Phi^{-1}(T), \\ \frac{1}{r^{d+1}}\exp(-\frac{b}{5}r) = C_T^{-1}\theta_T(r), & r>\Phi^{-1}(T). \end{cases}$$

Thus, for any $0<\beta\le 1$ and $r>0$,

$$(1\wedge M_T^{-1}\wedge C_T^{-1})\theta_T(r) \le \theta(r) \le (1\vee m_T^{-1}\vee C_T^{-1})\theta_T(r).$$

Using this and (2.7) we arrive (2.4).

(b) Clearly we have $\tilde{\mathscr{G}}(t,r) = \mathscr{G}(t,r)$ for $r\le 1$. For any $r>1$ and $0<\beta<1$ we have

$$\tilde{\mathscr{G}}(t,r) = \frac{1}{r^d\Phi(r)} \ge \left(\sup_{s\ge 1} s^d\Phi(s)\exp(-bs^\beta)\right)^{-1}\exp(-br^\beta) = c(\beta)\mathscr{G}(t,r).$$

Similarly, for $r>1$ and $\beta=1$

$$\tilde{\mathscr{G}}(t,r) = \frac{1}{r^{d+2}\Phi(1)} \ge \left(\sup_{s\ge 1}\frac{\Phi(s)}{s}\exp(-\frac{b}{5}s)\right)^{-1}\frac{1}{r^{d+1}}\exp(-\frac{b}{5}r) = c(1)\mathscr{G}(t,r).$$

Combining above estimates with (2.4) we arrive (2.5) with $c_2 = c(\beta)\wedge c_1^{-1}$. □

In the following remark we will see that our $\tilde{\mathscr{G}}(t,x)$ and the function $\rho(t,x)$ in [13] are comparable.

Remark 2.3. Let $r(t,r) := \psi^{-1}(t^{-1})^d \wedge [t\psi(r^{-1})r^{-d}]$ as in [13], where ψ is the characteristic exponent with respect to the Lévy process whose jumping kernel is $J(|y|)dy$. By Lemma 3.3 we have $\psi(r^{-1})^{-1} \asymp \Phi(r)$ for all $r>0$, which implies that $r(t,r)/t \asymp \tilde{\mathscr{G}}(t,r)$ for all $r>0$. Thus, by [13, Proposition 2.1] we conclude that $\tilde{\mathscr{G}}(t,x)$ is comparable to the function $\rho(t,x)$ in [13].

2.1. Basic scaling inequalities

We start with weak scaling condition for the inverse function of Φ. In this subsection we assume that ϕ satisfies (1.6).

Lemma 2.4. *For any $0 < r \le R$,*

$$\left(\frac{R}{r}\right)^{1/2} \le \frac{\Phi^{-1}(R)}{\Phi^{-1}(r)} \le a_1^{-1/\alpha_1}\left(\frac{R}{r}\right)^{1/\alpha_1} \tag{2.8}$$

where a_1 and α_1 are constants in (1.6).

Proof. Letting $(r, R) = (\Phi^{-1}(r), \Phi^{-1}(R))$ in (2.2), we have that for $0 < r \le R$,

$$a_1\left(\frac{\Phi^{-1}(R)}{\Phi^{-1}(r)}\right)^{\alpha_1} \le \frac{R}{r} = \frac{\Phi(\Phi^{-1}(R))}{\Phi(\Phi^{-1}(r))} \le \left(\frac{\Phi^{-1}(R)}{\Phi^{-1}(r)}\right)^2,$$

which implies (2.8). □

Now we introduce some scaling properties of $\mathscr{G}$ which will be used throughout this paper.

Lemma 2.5. *Let $T \ge 1$ and $0 < \varepsilon$. Then, there exist constants $c_1, c_2 > 0$ such that for any $0 < t \le T$, $x \in \mathbb{R}^d$ and $z \in \mathbb{R}^d$ satisfying $\Phi(|z|) \le t$,*

$$\mathscr{G}(\varepsilon t, x) \le c_1 \mathscr{G}(t, x) \tag{2.9}$$

and

$$\mathscr{G}(t, x+z) \le c_2 \mathscr{G}(t, x), \tag{2.10}$$

where c_1 depends only on $d, a_1, \alpha_1, \varepsilon$, and c_2 depends only on $d, T, a_1, \alpha_1, b, \beta$ and C_0.

Proof. (a) Since $t \mapsto \mathscr{G}(t, x)$ is non-increasing, we can assume $\varepsilon < 1$ without loss of generality. By (2.8), there is a constant $c_1(\varepsilon) > 1$ satisfying

$$\frac{1}{\varepsilon t\, \Phi^{-1}(\varepsilon t)^d} \le \frac{c_1}{t\, \Phi^{-1}(t)^d}$$

for $t \le T$. Thus, we arrive

$$\mathscr{G}(\varepsilon t, x) = \frac{1}{\varepsilon t\, \Phi^{-1}(\varepsilon t)^d} \wedge \theta(|x|) \le \frac{c_1}{t\, \Phi^{-1}(t)^d} \wedge \theta(|x|) \le c_1 \mathscr{G}(t, x).$$

(b) We claim that (2.10) holds with the function $\mathscr{G}_T(t, x)$. In other words, there exists a constant $c_2 > 0$ such that

$$\mathscr{G}_T(t, x+z) \le c_2 \mathscr{G}_T(t, x), \quad x \in \mathbb{R}^d, \ \Phi(|z|) \le t.$$

Since $r \mapsto \mathscr{G}_T(t, r)$ is non-increasing, it suffices to show that there exists $c_3 > 0$ such that for any $0 < t \le T$ and $r > 0$,

$$\mathscr{G}_T(t, r) \le c_3 \exp(b\, \Phi^{-1}(t)^\beta) \mathscr{G}_T(t, r + \Phi^{-1}(t)). \tag{2.11}$$

Indeed, since $0 < t \le T$, (2.11) implies our claim with $c_2 = c_3 \exp(b\, \Phi^{-1}(T)^\beta)$.

We prove (2.11) by considering several cases separately. Firstly when $r \le \Phi^{-1}(T) - \Phi^{-1}(t)$, using (2.3) we have

$$\begin{aligned}\mathscr{G}_T(t, r + \Phi^{-1}(t)) &= \frac{1}{(r + \Phi^{-1}(t))^d\, \Phi(r + \Phi^{-1}(t))}\\ &\ge \frac{1}{(2\Phi^{-1}(t))^d\, \Phi(2\Phi^{-1}(t))} \wedge \frac{1}{(2r)^d\, \Phi(2r)}\\ &\ge c_4 \left(\frac{1}{t\,\Phi^{-1}(t)^d} \wedge \frac{1}{r^d\, \Phi(r)}\right) = c_4 \mathscr{G}_T(t, r),\end{aligned}$$

The second line above follows from (2.2).

When $r \ge \Phi^{-1}(T)$ and $0 < \beta < 1$, using (2.3) and triangular inequality $r^\beta + \Phi^{-1}(t)^\beta \ge (r + \Phi^{-1}(t))^\beta$ we get

$$\begin{aligned}\mathscr{G}_T(t, r + \Phi^{-1}(t)) &= \exp(-b(r + \Phi^{-1}(t))^\beta) \ge \exp(-b\Phi^{-1}(t)^\beta - br^\beta)\\ &= \exp(-b\Phi^{-1}(t)^\beta)\mathscr{G}_T(t, r).\end{aligned}$$

Similarly, for $r \ge \Phi^{-1}(T)$ and $\beta = 1$ we have

$$\begin{aligned}\mathscr{G}_T(t, r + \Phi^{-1}(t)) &= C_T \frac{1}{(r + \Phi^{-1}(t))^{d+1}} \exp(-\frac{b}{5}(r + \Phi^{-1}(t)))\\ &\ge C_T \frac{1}{(2r)^{d+1}} \exp(-\frac{b}{5}\Phi^{-1}(t) - \frac{b}{5}r)\\ &= 2^{-d-1} \exp(-\frac{b}{5}\Phi^{-1}(t))\mathscr{G}_T(t, r) \ge 2^{-d-1} \exp(-b\Phi^{-1}(t))\mathscr{G}_T(t, r).\end{aligned}$$

When $\Phi^{-1}(T) - \Phi^{-1}(t) \le r \le \Phi^{-1}(T)$, combining above estimates we arrive

$$\mathscr{G}_T(t, r + \Phi^{-1}(t)) \ge 2^{-d-1} \exp(-b\Phi^{-1}(t)^\beta)\mathscr{G}_T(t, \Phi^{-1}(T)) \ge c_5 \exp(-b\Phi^{-1}(t)^\beta)\mathscr{G}_T(t, r).$$

Note that $r \mapsto \mathscr{G}_T(t, r)$ is continuous at $r = \Phi^{-1}(T)$. Therefore, we conclude (2.11). Applying (2.4) for (2.11) we arrive our desired estimate (2.10). □

2.2. *Convolution inequalities*

In this section, we obtain some convolution inequalities for $\mathscr{G}(t, x)$ which will be used for Levi's method in Section 5. To get these inequalities we will use some estimates in [13, Section 2]. Note that by Remark 2.3 we already have convolution inequalities for $\tilde{\mathscr{G}}(t, x)$ (e.g. [13, Proposition 2.8]). For $a, b > 0$, let $B(a, b) = \int_0^1 s^{a-1}(1 - s)^{b-1} ds = \frac{(a+b-1)!}{(a-1)!(b-1)!}$ be the beta function.

Using (2.8), the proof of the following lemma is same as the one in [13, Lemma 2.3]. Thus we skip the proof.

Lemma 2.6. *Assume that ϕ satisfies (1.6) and $\gamma, \delta \ge 0$, $\eta, \theta \in \mathbb{R}$ are constants satisfying $\mathbf{1}_{\gamma \ge 0}(\gamma/2) + \mathbf{1}_{\gamma < 0}(\gamma/\alpha_1) + \delta/2 + 1 - \eta > 0$. Then for every $t > 0$, we have*

$$\begin{aligned}&\int_0^t s^{-\eta}\, \Phi^{-1}(s)^\gamma (t - s)^{-\theta}\, \Phi^{-1}(t - s)^\delta\, ds\\ &\quad \le B(\delta/2 + 1 - \theta, \gamma/2 + 1 - \eta) t^{1-\eta-\theta}\, \Phi^{-1}(t)^{\gamma+\delta}. \qquad (2.12)\end{aligned}$$

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For $0 \le s \le t$, let $g(s) := t^\beta + (2^\beta - 1)s^\beta - (t+s)^\beta$. Then we can easily check that $g(0) = g(t) = 0$ and

$$g'(s) = \beta\left((2^\beta - 1)s^{\beta-1} - (t+s)^{\beta-1}\right)\begin{cases}\ge 0, & s \in [0, kt],\\ \le 0, & s \in [kt, t],\end{cases}$$

where $k := ((2^\beta - 1)^{\frac{1}{\beta-1}} - 1)^{-1} \in (0,1)$ is the constant satisfying $g'(kt) = 0$. Thus, we conclude that $g(s) \ge 0$ for any $0 \le s \le t$, which implies

$$t^\beta + s^\beta \ge (t+s)^\beta + (2 - 2^\beta)(t^\beta \wedge s^\beta), \qquad \text{for all } 0 < \beta < 1 \text{ and } t, s > 0. \tag{2.13}$$

Using (2.13) we prove the following lemma, which we need for our convolution inequalities.

Lemma 2.7. *(a) Let $0 < \beta < 1$ and $b > 0$. Then, there exists a constant $c_1 > 0$ such that for any $x \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} \exp(-b|x-z|^\beta - b|z|^\beta)dz \le c_1 \exp(-b|x|^\beta). \tag{2.14}$$

(b) There exists a constant $c_2 > 0$ such that for any $x \in \mathbb{R}^d$ with $|x| \ge 1$,

$$\int_{\mathbb{R}^d} (|x-z|^{-d-1} \wedge 1)(|z|^{-d-1} \wedge 1)dz \le c_2|x|^{-d-1}. \tag{2.15}$$

The constant c_1 depends only on b, d and β, and c_2 depends only on d.

Proof. (a) Let

$$c_1 = 2\int_{\mathbb{R}^d} \exp(-b(2-2^\beta)|z|^\beta)dz < \infty.$$

Using (2.13) for the second line, we arrive

$$\begin{aligned}\int_{\mathbb{R}^d} \exp(-b|x-z|^\beta - b|z|^\beta)dz &\le \int_{\mathbb{R}^d} \exp(-b|x|^\beta)\exp\big(-b(2-2^\beta)(|z|^\beta \wedge |x-z|^\beta)\big)dz\\ &\le \exp(-b|x|^\beta)\left(\int_{\mathbb{R}^d} \exp(-b(2-2^\beta)|z|^\beta)dz\right.\\ &\quad \left.+ \int_{\mathbb{R}^d} \exp(-b(2-2^\beta)|x-z|^\beta)dz\right)\\ &= c_1\exp(-b|x|^\beta).\end{aligned}$$

This proves (2.14).
(b) Using $|x-z|^{-1} \wedge |z|^{-1} \le 2|x|^{-1}$, we have

$$\begin{aligned}&\int_{\mathbb{R}^d} (|x-z|^{-d-1} \wedge 1)(|z|^{-d-1} \wedge 1)dz\\ &\le (\frac{2}{|x|})^{d+1}\left(\int_{|x-z|\ge|z|} (|z|^{-d-1} \wedge 1)dz + \int_{|x-z|<|z|} (|x-z|^{-d-1} \wedge 1)dz\right)\\ &\le (\frac{2}{|x|})^{d+1}\left(\int_{\mathbb{R}^d} (|z|^{-d-1} \wedge 1)dz + \int_{\mathbb{R}^d} (|x-z|^{-d-1} \wedge 1)dz\right) := c_2|x|^{-d-1}.\end{aligned}$$

This concludes the lemma. □

For $\gamma, \delta \in \mathbb{R}$, $t > 0$ and $x \in \mathbb{R}^d$ we define

$$\mathscr{G}^\delta_\gamma(t,x) := \Phi^{-1}(t)^\gamma(|x|^\delta \wedge 1)\mathscr{G}(t,x) \quad \text{and} \quad \tilde{\mathscr{G}}^\delta_\gamma(t,x) := \Phi^{-1}(t)^\gamma(|x|^\delta \wedge 1)\tilde{\mathscr{G}}(t,x).$$

Note that $\mathscr{G}_0^0(t,x) = \mathscr{G}(t,x)$, and $\tilde{\mathscr{G}}_\gamma^\delta(t,x)$ is comparable to the function $\rho_\gamma^\delta(t,x)$ in [13] by Remark 2.3. Also, we can easily check that for $T \ge \Phi(1)$,

$$\mathscr{G}_{\gamma_1}^\delta(t,x) \le \Phi^{-1}(T)^{\gamma_1-\gamma_2}\mathscr{G}_{\gamma_2}^\delta(t,x), \qquad (t,x) \in (0,T]\times\mathbb{R}^d, \quad \gamma_2 \le \gamma_1, \tag{2.16}$$

$$\mathscr{G}_\gamma^{\delta_1}(t,x) \le \mathscr{G}_\gamma^{\delta_2}(t,x), \qquad (t,x) \in (0,\infty)\times\mathbb{R}^d, \quad 0 \le \delta_2 \le \delta_1. \tag{2.17}$$

We record the following inequality which immediately follows from (2.16) and (2.17): for any $T \ge \Phi(1)$, $\delta \ge 0$ and $(t,x) \in (0,T]\times\mathbb{R}^d$,

$$\big(\mathscr{G}_0^\delta + \mathscr{G}_\delta^0\big)(t,x) \le \big(\Phi^{-1}(T)^\delta + 1\big)\mathscr{G}(t,x) \le 2\Phi^{-1}(T)^\delta\mathscr{G}(t,x). \tag{2.18}$$

Now we are ready to introduce convolution inequalities for $\mathscr{G}(t,x)$.

Proposition 2.8. *Assume that ϕ satisfies (1.6). Let $T \ge 1$ and $0 < \alpha < \alpha_1$.*
(a) There exists a constant $c = c(d,T,a_1,\alpha,\alpha_1) > 0$ such that for any $0 < t \le T$, $\delta \in [0,\alpha]$ and $\gamma \in \mathbb{R}$,

$$\int_{\mathbb{R}^d} \tilde{\mathscr{G}}_\gamma^\delta(t,x)\,dx \le ct^{-1}\Phi^{-1}(t)^{\gamma+\delta}. \tag{2.19}$$

(b) There exists $C = C(\alpha,T) = C(d,T,a_1,\alpha,\alpha_1,b,\beta) > 0$ such that for all $x \in \mathbb{R}^d$, $\delta_1,\delta_2 \ge 0$ with $\delta_1+\delta_2 \le \alpha$, $\gamma_1,\gamma_2 \in \mathbb{R}$ and $0 < s < t \le T$,

$$\begin{aligned}\int_{\mathbb{R}^d} \mathscr{G}_{\gamma_1}^{\delta_1}(t-s,x-z)\mathscr{G}_{\gamma_2}^{\delta_2}(s,z)\,dz \le C\Big(&(t-s)^{-1}\Phi^{-1}(t-s)^{\gamma_1+\delta_1+\delta_2}\Phi^{-1}(s)^{\gamma_2}\mathscr{G}(t,x)\\ &+\Phi^{-1}(t-s)^{\gamma_1}s^{-1}\Phi^{-1}(s)^{\gamma_2+\delta_1+\delta_2}\mathscr{G}(t,x)\\ &+(t-s)^{-1}\Phi^{-1}(t-s)^{\gamma_1+\delta_1}\Phi^{-1}(s)^{\gamma_2}\mathscr{G}_0^{\delta_2}(t,x)\\ &+\Phi^{-1}(t-s)^{\gamma_1}s^{-1}\Phi^{-1}(s)^{\gamma_2+\delta_2}\mathscr{G}_0^{\delta_1}(t,x)\Big).\end{aligned} \tag{2.20}$$

In particular, letting $\gamma_1 = \gamma_2 = \delta_1 = \delta_2 = 0$ in (2.20) we have

$$\int_{\mathbb{R}^d} \mathscr{G}(t-s,x-z)\mathscr{G}(s,z)dz \le 2C\big(s^{-1}+(t-s)^{-1}\big)\mathscr{G}(t,x). \tag{2.21}$$

(c) For all $x \in \mathbb{R}^d$, $0 < t \le T$, $\delta_1,\delta_2 \ge 0$ and $\theta,\eta \in [0,1]$ satisfying $\delta_1+\delta_2 \le \alpha$, $\mathbf{1}_{\gamma_1\ge 0}(\gamma_1/2)+\mathbf{1}_{\gamma_1<0}(\gamma_1/\alpha_1)+\delta_1/2+1-\theta > 0$ and $\mathbf{1}_{\gamma_2\ge 0}(\gamma_2/2)+\mathbf{1}_{\gamma_2<0}(\gamma_2/\alpha_1)+\delta_2/2+1-\eta > 0$, we have a constant $C_2 > 0$ satisfying

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^d}(t-s)^{1-\theta}\mathscr{G}_{\gamma_1}^{\delta_1}(t-s,x-z)s^{1-\eta}\mathscr{G}_{\gamma_2}^{\delta_2}(s,z)\,dz\,ds\\ &\le C_2t^{2-\theta-\eta}\left(\mathscr{G}_{\gamma_1+\gamma_2+\delta_1+\delta_2}^0+\mathscr{G}_{\gamma_1+\gamma_2+\delta_2}^{\delta_1}+\mathscr{G}_{\gamma_1+\gamma_2+\delta_1}^{\delta_2}\right)(t,x)\end{aligned} \tag{2.22}$$

for any $0 < t \le T$ and $x \in \mathbb{R}^d$. Moreover, when $\gamma_1,\gamma_2 \ge 0$ we further have

$$C_2 = 4CB\left(\frac{\gamma_1+\delta_1}{2}+1-\theta, \frac{\gamma_2+\delta_2}{2}+1-\eta\right). \tag{2.23}$$

Proof. (a) See [13, Lemma 2.6(a)].

(b) By (2.4), it suffices to show (2.20) with the function $(\mathscr{G}_T)_\gamma^\delta(t,x) := \Phi^{-1}(t)^\gamma(|x|^\delta \wedge 1)\mathscr{G}_T(t,x)$. Without loss of generality we assume $T \ge \Phi(1)$ and for notational convenience we drop T in the notations so we use $\mathscr{G}(t,x)$ and $\mathscr{G}_\gamma^\delta(t,x)$ instead of $\mathscr{G}_T(t,x)$ and $(\mathscr{G}_T)_\gamma^\delta(t,x)$ respectively.

First let $|x| \le \Phi^{-1}(T)$. By Remark 2.3 and [13, Lemma 2.6(b)], we already have that there exists $c_1 > 0$ satisfying

$$\int_{\mathbb{R}^d} \tilde{\mathscr{G}}^{\delta_1}_{\gamma_1}(t-s,x-z)\tilde{\mathscr{G}}^{\delta_2}_{\gamma_2}(s,z)\,dz \le c_1\Big((t-s)^{-1}\Phi^{-1}(t-s)^{\gamma_1+\delta_1+\delta_2}\Phi^{-1}(s)^{\gamma_2}\tilde{\mathscr{G}}(t,x) + \Phi^{-1}(t-s)^{\gamma_1}s^{-1}\Phi^{-1}(s)^{\gamma_2+\delta_1+\delta_2}\tilde{\mathscr{G}}(t,x) + (t-s)^{-1}\Phi^{-1}(t-s)^{\gamma_1+\delta_1}\Phi^{-1}(s)^{\gamma_2}\tilde{\mathscr{G}}^{\delta_2}_0(t,x) + \Phi^{-1}(t-s)^{\gamma_1}s^{-1}\Phi^{-1}(s)^{\gamma_2+\delta_2}\tilde{\mathscr{G}}^{\delta_1}_0(t,x)\Big).$$

Note that $\mathscr{G}(t,x) = \tilde{\mathscr{G}}(t,x)$ by (2.3) since $|x| \le \Phi^{-1}(T)$. Using (2.5) for the left-hand side and $\mathscr{G}(t,x) = \tilde{\mathscr{G}}(t,x)$ for the right-hand side, we obtain (2.20) for $|x| \le \Phi^{-1}(T)$.

Now assume $|x| > \Phi^{-1}(T)$ and observe that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathscr{G}^{\delta_1}_{\gamma_1}(t-s,x-z)\mathscr{G}^{\delta_2}_{\gamma_2}(s,z)dz \\
&= \Bigg(\int_{\substack{|z|>\Phi^{-1}(T),\\ |x-z|>\Phi^{-1}(T)}} + \int_{\substack{|z|>\Phi^{-1}(T),\\ |x-z|\le\Phi^{-1}(T)}} + \int_{\substack{|z|\le\Phi^{-1}(T),\\ |x-z|>\Phi^{-1}(T)}} + \int_{\substack{|z|\le\Phi^{-1}(T),\\ |x-z|\le\Phi^{-1}(T)}}\Bigg)\mathscr{G}^{\delta_1}_{\gamma_1}(t-s,x-z) \\
&\quad\times \mathscr{G}^{\delta_2}_{\gamma_2}(s,z)dz \\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

First we assume $0 < \beta < 1$ and obtain upper bounds for I_i, $i = 1,\dots 4$. For I_1, using $\Phi^{-1}(T) \ge 1$ we have

$$\begin{aligned}
I_1 &= \int_{|x-z|>\Phi^{-1}(T),|z|>\Phi^{-1}(T)} \mathscr{G}^{\delta_1}_{\gamma_1}(t-s,x-z)\mathscr{G}^{\delta_2}_{\gamma_2}(s,z)dz \\
&= \int_{|x-z|>\Phi^{-1}(T),|z|>\Phi^{-1}(T)} \Phi^{-1}(t-s)^{\gamma_1}\big(|x-z|^{\delta_1}\wedge 1\big)\mathscr{G}(t-s,x-z) \\
&\quad\times \Phi^{-1}(s)^{\gamma_2}\big(|z|^{\delta_2}\wedge 1\big)\mathscr{G}(s,z)dz \\
&= \Phi^{-1}(t-s)^{\gamma_1}\Phi^{-1}(s)^{\gamma_2}\int_{|x-z|>\Phi^{-1}(T),|z|>\Phi^{-1}(T)} \exp(-b|x-z|^{\beta} - b|z|^{\beta})dz. \qquad (2.24)
\end{aligned}$$

By (2.14) we obtain

$$\begin{aligned}
I_1 &\le c_1\Phi^{-1}(t-s)^{\gamma_1}\Phi^{-1}(s)^{\gamma_2}\exp(-b|x|^{\beta}) = c_1\Phi^{-1}(t-s)^{\gamma_1}\Phi^{-1}(s)^{\gamma_2}\mathscr{G}(t,x) \\
&\le c_2(t-s)^{-1}\Phi^{-1}(t-s)^{\gamma_1+\delta_1+\delta_2}\Phi^{-1}(s)^{\gamma_2}\mathscr{G}(t,x).
\end{aligned}$$

where we used $\delta_1, \delta_2 \ge 0$ and $t - s \le T$ for the last line. For the estimates of I_2, I_3 and I_4 we omit counterpart of the last line above.

For I_2, using (2.3) we have

$$\begin{aligned}
I_2 &= \int_{|x-z|\le\Phi^{-1}(T),|z|>\Phi^{-1}(T)} \mathscr{G}^{\delta_1}_{\gamma_1}(t-s,x-z)\mathscr{G}^{\delta_2}_{\gamma_2}(s,z)dz \\
&= \Phi^{-1}(t-s)^{\gamma_1}\Phi^{-1}(s)^{\gamma_2}\int_{|x-z|\le\Phi^{-1}(T),|z|>\Phi^{-1}(T)} \tilde{\mathscr{G}}^{\delta_1}_0(t-s,x-z)\exp(-b|z|^{\beta})dz.
\end{aligned}$$

Since $|x-z| \le \Phi^{-1}(T)$, using triangular inequality we have

$$\exp(-b|z|^{\beta}) \le \exp(-b|x|^{\beta})\exp(b|x-z|^{\beta}) \le \exp(b\Phi^{-1}(T)^{\beta})\exp(-b|x|^{\beta}). \qquad (2.25)$$

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Thus by (2.19),

$$I_2 \le \Phi^{-1}(t-s)^{\gamma_1}\Phi^{-1}(s)^{\gamma_2}\exp(-b|x|^\beta)\int_{\mathbb{R}^d}\tilde{\mathscr{G}}_0^{\delta_1}(t-s,x-z)dz$$
$$\le c_3(t-s)^{-1}\Phi^{-1}(t-s)^{\gamma_1+\delta_1}\Phi^{-1}(s)^{\gamma_2}\mathscr{G}(t,x).$$

By the similar way, we obtain

$$I_3 \le c_3 s^{-1}\Phi^{-1}(t-s)^{\gamma_1}\Phi^{-1}(s)^{\gamma_2+\delta_2}\mathscr{G}(t,x).$$

When $|x| \ge 2\Phi^{-1}(T)$, we have $I_4 = 0$. So we can assume $|x| < 2\Phi^{-1}(T)$ without loss of generality for the estimate of I_4. By (2.5) We have

$$I_4 \le \int_{\mathbb{R}^d}\tilde{\mathscr{G}}_{\gamma_1}^{\delta_1}(t-s,x-z)\tilde{\mathscr{G}}_{\gamma_2}^{\delta_2}(s,z)dz \le c_4\tilde{\mathscr{G}}(t,x).$$

Using $\tilde{\mathscr{G}}(t,x) \le \tilde{\mathscr{G}}(t,\Phi^{-1}(T)) = \mathscr{G}(t,\Phi^{-1}(T)) \le e^{b\Phi^{-1}(T)^\beta}\mathscr{G}(t,2\Phi^{-1}(T)) \le e^{b\Phi^{-1}(T)^\beta}\mathscr{G}(t,x)$, we can obtain desired estimates. Combining estimates for I_1, I_2, I_3 and I_4, we arrive (2.20) for $0<\beta<1$.

For the case $\beta = 1$, estimate for I_4 is same as above. For I_2 and I_3, instead of (2.25) we argue as the following: using $|x-z| \le \Phi^{-1}(T)$ and $|x|, |z| \ge \Phi^{-1}(T)$, we have

$$\frac{1}{|z|^{d+1}}\exp(-\frac{b}{5}|z|) \le \frac{2^{d+1}}{|x|^{d+1}}\exp(\frac{b}{5}\Phi^{-1}(T))\exp(-\frac{b}{5}|x|).$$

For I_1, following (2.24) and using (2.15) for the fourth line and (2.8) for the fifth line we have

$$\begin{aligned}
I_1 &= \Phi^{-1}(t-s)^{\gamma_1}\Phi^{-1}(s)^{\gamma_2}\int_{|x-z|>\Phi^{-1}(T),|z|>\Phi^{-1}(T)}\frac{1}{|x-z|^{d+1}|z|^{d+1}}\\
&\quad\times\exp(-\frac{b}{5}|x-z|-\frac{b}{5}|z|)dz\\
&\le c_1\Phi^{-1}(t-s)^{\gamma_1}\Phi^{-1}(s)^{\gamma_2}\exp(-\frac{b}{5}|x|)\int_{|x-z|>1,|z|>1}\frac{1}{|x-z|^{d+1}|z|^{d+1}}dz\\
&\le c_1\Phi^{-1}(t-s)^{\gamma_1}\Phi^{-1}(s)^{\gamma_2}\exp(-\frac{b}{5}|x|)\int_{\mathbb{R}^d}\big(1\wedge|x-z|^{-d-1}\big)\big(1\wedge|z|^{-d-1}\big)dz\\
&\le c_2\Phi^{-1}(t-s)^{\gamma_1}\Phi^{-1}(s)^{\gamma_2}\frac{1}{|x|^{d+1}}\exp(-\frac{b}{5}|x|) = c_2\Phi^{-1}(t-s)^{\gamma_1}\Phi^{-1}(s)^{\gamma_2}\mathscr{G}(t,x)\\
&\le c_3(t-s)^{-1}\Phi^{-1}(t-s)^{\gamma_1+\delta_1+\delta_2}\Phi^{-1}(s)^{\gamma_2}\mathscr{G}(t,x).
\end{aligned}$$

(c) Integrating (2.20) with respect to s from 0 to t. With (2.12), we can follow the proof of [13, Lemma 2.6(c)]. □

3. Heat kernel estimates for Lévy processes

Following the framework of [5,13], we need estimates of derivatives of the heat kernel for the symmetric Lévy process whose jumping kernel is $J(|y|)$ (see, for example, [13, Proposition 3.2]). To be more precise, in our case, to get the upper bound of heat kernel for non-symmetric operator of the form (1.1), we need correct upper bounds of the first and second order derivatives of the heat kernel for unimodal Lévy processes. In this section, we will prove that (1.4) and (1.5) are sufficient condition for the estimates of the second order derivatives in Proposition 3.2, which decay exponentially or subexponentially.

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3.1. Settings

In this section, we fix $T \le [1, \infty)$ and let $\nu(dy) = \nu(|y|)dy$ be an isotropic measure in $\mathbb{R}^d$ satisfying $\int_{\mathbb{R}^d} (1 \wedge |y|^2)\nu(dy) < \infty$. Throughout this section we further assume that $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ is non-increasing, differentiable function.

Here are our goals in this section.

Proposition 3.1. *Let X be an isotropic unimodal Lévy process in $\mathbb{R}^d$ with Lévy measure $\nu(|y|)dy$ satisfying the following assumptions: ϕ is a nondecreasing function with $\phi(0) = 0$ satisfying (1.6) and (1.7), and there exist constants $a > 0$ and $0 < \beta \le 1$ such that*

$$\frac{a^{-1}}{r^d \phi(r)} \le \nu(r) \le \frac{a}{r^d \phi(r)}, \quad 0 < r \le 1 \quad \text{and} \quad \nu(r) \le a \exp(-br^\beta), \quad r > 1. \tag{3.1}$$

Then its transition density $x \mapsto p_t(x)$ is in $C_b^\infty(\mathbb{R}^d)$ and satisfies gradient estimates

$$|\nabla_x^k p_t(x)| \le ct\,\mathscr{G}^0_{-k}(t,x) = \Phi^{-1}(t)^{-k}\left(\frac{1}{t\,\Phi^{-1}(t)^d} \wedge \theta(|x|)\right), \quad k = 0, 1 \tag{3.2}$$

for any $0 < t \le T$ and $x \in \mathbb{R}^d$. The constant c depends only on $k, d, T, a, a_1, \alpha_1, b, \beta$ and C_0.

With the above result, we can obtain the second gradient estimate for the isotropic unimodal Lévy process whose jumping kernel satisfies (1.4) and (1.5).

Proposition 3.2. *Suppose that ϕ is a nondecreasing function with $\phi(0) = 0$ satisfying (1.6) and (1.7), and that Lévy measure $J(|y|)dy$ satisfies (1.4) and (1.5) with $0 < \beta \le 1$. Then, its corresponding transition density $x \mapsto p(t,x)$ is in $C_b^\infty(\mathbb{R}^d)$ and satisfies gradient estimates*

$$|\nabla_x^k p(t,x)| \le ct\,\mathscr{G}^0_{-k}(t,x) = \Phi^{-1}(t)^{-k}\left(\frac{1}{t\,\Phi^{-1}(t)^d} \wedge \theta(|x|)\right), \quad k = 0, 1, 2 \tag{3.3}$$

for any $0 < t \le T$ and $x \in \mathbb{R}^d$. The constant c depends only on $k, d, T, a, a_1, \alpha_1, b, \beta$ and C_0.

In the next subsection, we prove Propositions 3.1 and 3.2.

3.2. Proof of Propositions 3.1 and 3.2

In this subsection, we will combine some results in [14,11,12] to prove Proposition 3.1. Recall that we have assumed that $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ is non-increasing differentiable function satisfying $\int_{\mathbb{R}^d} (1 \wedge |y|^2)\nu(|y|)dy < \infty$. In this subsection, instead of the function Φ, we mainly use

$$\varphi(r) := \begin{cases} \dfrac{r^2}{\int_0^r s^{d+1}\nu(s)ds}, & 0 < r \le 1, \\ \varphi(1)r^2, & r > 1, \end{cases} \tag{3.4}$$

Note that the integral $\int_0^r s^{d+1}\nu(s)ds$ above is finite because of our assumption $\int_{\mathbb{R}^d} (1 \wedge |y|^2)\nu(|y|)dy < \infty$.

To prove Propositions 3.1 and 3.2 at once, we need to consider the following conditions on Lévy measure $\nu(|y|)dy$ which is slightly more general than (3.1). We assume that there exist constants $a > 0$, $0 < \beta \le 1$ and $\ell \ge 0$ such that

$$\nu(r) \le ar^{-\ell}\exp(-br^\beta), \quad r > 1. \tag{3.5}$$

Also, we assume that there exist $a_3 > 0$ and $\alpha_3 \in (0, 2]$ such that

$$a_3\Big(\frac{R}{r}\Big)^{\alpha_3} \le \frac{\varphi(R)}{\varphi(r)}, \quad 0 < r \le R < \infty. \tag{3.6}$$

For instance, when X is an isotropic Lévy process in Proposition 3.1 we have $\frac{s}{a\phi(s)} \le \nu(s)s^{d+1} \le \frac{as}{\phi(s)}$, which implies $\varphi(r) \asymp \Phi(r)$. Using this and Lemma (2.1) we obtain (3.6) with $\alpha_3 = \alpha_1$. Thus, the conditions in Proposition 3.1 imply (3.5) and (3.6).

Under (3.6), we have $\varphi(r) \le cr^{\alpha_3}$ for $r \le 1$ so that

$$c^{-1}r^{-\alpha_3} \le \int_0^r \frac{s^{d+1}}{r^2}\nu(s)ds \le \int_0^r s^{d-1}\nu(s)ds \le \int_0^1 s^{d-1}\nu(s)ds, \quad r \le 1.$$

Thus, letting $r \downarrow 0$ we obtain $\int_0^1 s^{d-1}\nu(s)ds = \infty$. Now we record the counterpart of Lemma (2.1). Following the proof of Lemma (2.1), we obtain

$$\frac{\varphi(R)}{\varphi(r)} \le \Big(\frac{R}{r}\Big)^2, \quad 0 < r \le R. \tag{3.7}$$

In addition, since ν is non-increasing, we have

$$\varphi(r)^{-1} = r^{-2}\int_0^r s^{d+1}\nu(s)ds \ge r^{-2}\int_0^r s^{d+1}\nu(r)dr = \frac{r^d\nu(r)}{d+2}, \quad r < 1. \tag{3.8}$$

In this subsection except the proofs of Propositions 3.1 and 3.2 we will always assume that ν satisfies (3.5) and (3.6). Let X be the Lévy process with Lévy measure $\nu(|y|)dy$, and $\xi \mapsto \psi(|\xi|)$ be the characteristic exponent of X. First note that $\nu(\mathbb{R}^d) = \int_{\mathbb{R}^d}\nu(|y|)dy = \infty$ because $\int_0^1 s^{d-1}\nu(s)ds = \infty$. Also, since X is isotropic, characteristic exponent of X is also isotropic function. Define $\Psi(r) := \sup_{|y|\le r}\psi(|y|)$ and let $\mathcal{P}(r) := \int_{\mathbb{R}^d}\big(1 \wedge \frac{|y|^2}{r^2}\big)\nu(|y|)dy$ be the Pruitt function for X. By [1, Lemma 1 and Proposition 2], we have that for $r > 0$,

$$\frac{2}{\pi^2 d}\mathcal{P}(r^{-1}) \le \psi(r) \le \Psi(r) \le \pi^2\psi(r) \le 2\pi^2\mathcal{P}(r^{-1}), \quad r > 0. \tag{3.9}$$

Using (3.9), we can prove the following lemma.

Lemma 3.3. *Assume that $\nu(|y|)dy$ satisfies (3.5) and (3.6). Then, $\Psi(r)$ is comparable to $\varphi(r^{-1})^{-1}$, i.e., there exists a constant $c > 0$ such that*

$$c^{-1}\varphi(r^{-1})^{-1} \le \Psi(r) \le c\varphi(r^{-1})^{-1}, \quad r > 0. \tag{3.10}$$

Proof. We claim that

$$\mathcal{P}(r) \asymp \varphi(r)^{-1} \quad \text{for} \quad r > 0. \tag{3.11}$$

First assume $r \le 1$ and observe that

$$\begin{aligned}\mathcal{P}(r) &= \int_{\mathbb{R}^d}\Big(1 \wedge \frac{|z|^2}{r^2}\Big)\nu(z)dz \\ &= c(d)\left(r^{-2}\int_0^r s^{d+1}\nu(s)ds + \int_r^1 s^{d-1}\nu(s)ds + \int_1^\infty s^{d-1}\nu(s)ds\right) \\ &=: c(d)\big(I_1 + I_2 + I_3\big).\end{aligned}$$

By the definition of φ we have $I_1 = \varphi(r)^{-1}$. To estimate I_2, let us define $k := \lfloor \frac{\log r}{\log 2} \rfloor$, the largest integer smaller than or equal to $\frac{\log r}{\log 2}$. Then we have

$$0 \le I_2 \le \sum_{i=0}^{k} \int_{2^i r}^{2^{i+1} r} s^{d-1} \nu(s) ds =: \sum_{i=0}^{k} I_{2i}.$$

Using (3.6), we have

$$\begin{aligned} I_{2i} &\le (2^i r)^{-2} \int_{2^i r}^{2^{i+1} r} s^{d+1} \nu(s) ds \le (2^i r)^{-2} \int_0^{2^{i+1} r} s^{d+1} \nu(s) ds \\ &= 4\varphi(2^{i+1} r)^{-1} \le a_3 2^{2-\alpha_3 (i+1)} \varphi(r)^{-1}. \end{aligned}$$

Thus,

$$0 \le I_2 \le \sum_{i=0}^{k} I_{2i} \le \frac{2^{2-\alpha_3}}{\varphi(r)} \sum_{i=0}^{k} 2^{-\alpha_3 i} \le \frac{c_1}{\varphi(r)}. \tag{3.12}$$

Also, using (3.5) and (3.6) we obtain

$$0 \le I_3 \le a \int_1^\infty s^{d-\ell-1} \exp(-bs^\beta) ds = c_2 \le \frac{c_2 \varphi(1)}{a_3 \varphi(r)},$$

where we used $a_3 \le a_3 \left(\frac{1}{r}\right)^{\alpha_3} \le \frac{\varphi(1)}{\varphi(r)}$ for the last inequality. Combining estimates of I_1, I_2 and I_3 we have proved the claim (3.11) for $r \le 1$.

Now assume $r > 1$. Then we have

$$\begin{aligned} \mathcal{P}(r) &= \int_{\mathbb{R}^d} \left(1 \wedge \frac{|z|^2}{r^2}\right) \nu(z) dz \\ &= c(d) \left(r^{-2} \int_0^1 s^{d+1} \nu(s) ds + \int_1^\infty \left(1 \wedge \frac{s^2}{r^2}\right) s^{d-1} \nu(s) ds \right) \\ &:= c(d) (\varphi(r)^{-1} + I_4). \end{aligned}$$

Also, using (3.5) we have

$$0 \le I_4 \le \int_1^\infty \frac{s^2}{r^2} s^{d-1} \nu(s) ds \le a r^{-2} \int_1^\infty s^{d-\ell+1} \exp(-bs^\beta) ds \le c_3 r^{-2}.$$

Using $\varphi(r) = \varphi(1) r^2$ for $r \ge 1$ we obtain that $\mathcal{P}(r) \asymp r^{-2} \asymp \varphi(r)^{-1}$ for $r > 1$, which implies (3.11) for $r > 1$. Therefore, (3.11) holds for any $r > 0$. Combining (3.11) and (3.9) we conclude the lemma. □

Using (3.10), (3.6) and (3.7) we obtain the following weak scaling condition for Ψ: there exists a constant $c > 0$ such that

$$c^{-1} \left(\frac{R}{r}\right)^{\alpha_3} \le \frac{\Psi(R)}{\Psi(r)} \le c \left(\frac{R}{r}\right)^2, \quad 0 < r \le R < \infty. \tag{3.13}$$

Let $p_t(x)$ be a transition density of X. Since X is isotropic, $x \mapsto p_t(x)$ is also isotropic function for any $t > 0$. By an abuse of notation we also denote the radial part of the heat kernel $p_t(x)$ of X as $p_t(r)$, $r > 0$.

To obtain gradient estimate for $p_t(x)$, we first follow the proof of [14, Proposition 3.1] to construct a $(d+2)$-dimensional Lévy process Y whose heat kernel estimate implies gradient estimate of X. The construction of this $(d+2)$-dimensional process Y in [14] is highly motivated by [8, Theorem 1.1].

Lemma 3.4. *Assume that isotropic unimodal Lévy measure ν satisfies (3.5) and (3.6). Then there exists an isotropic Lévy process Y in $\mathbb{R}^{d+2}$ such that its characteristic exponent is $\xi \mapsto \psi(|\xi|)$, $\xi \in \mathbb{R}^{d+2}$. Let $\nu_1(|x|)$ and $q_t(|x|)$ be the jumping kernel and heat kernel of Y, respectively. Then for any $r > 0$,*

$$q_t(r) = -\frac{1}{2\pi r}\frac{d}{dr}p_t(r) \tag{3.14}$$

and

$$\nu_1(r) = -\frac{1}{2\pi r}\nu'(r). \tag{3.15}$$

Proof. The existence of Y and (3.14) are immediately followed by [14, Proposition 3.1]. Note that using (3.9) and (3.13) we have

$$\lim_{\rho\to\infty}\frac{\psi(\rho)}{\log\rho} \geq \lim_{\rho\to\infty}\frac{\Psi(\rho)}{\pi^2\log\rho} \geq \lim_{\rho\to\infty}\frac{c_1\rho^{\alpha_3}}{\log\rho} = \infty,$$

which is one of the conditions in [14, Proposition 3.1]. For (3.15), we just need to follow the corresponding part in the proof of [14, Theorem 1.5]. Here we provide a brief sketch for the proof for reader's convenience; As in the proof of [14, Theorem 1.5], without using the assumption that $-\nu'(r)/r$ is non-increasing, one can show that there exists an isotropic Lévy process $X^{(d+2)}$ in $\mathbb{R}^{d+2}$ with jumping kernel $\nu_1(dy)$ and that the characteristic exponent of $X^{(d+2)}$ is $\psi(r)$. Thus, $X^{(d+2)}$ and Y are identical in law, which concludes the proof. To show this, only [14, (8) and (9)] are used, which follow directly from the fact that ν is isotropic, unimodal measure satisfying $\int_{\mathbb{R}^d}\left(|y|^2\wedge 1\right)\nu(dy) < \infty$. □

We emphasize here that we do not impose the condition (1.5) on ν. Thus the function $r \to \nu_1(r)$ in the above lemma may not be non-increasing.

Now we are going to establish heat kernel estimates for the process Y obtained in Lemma 3.4, which will imply heat kernel estimate and gradient estimate of X as a consequence of (3.14). To do this, we will check conditions **(E), (D), (P)** and **(C)** (when $\beta < 1$) in [12] for the process X and Y, and apply [12, Theorem 4] and [11, Theorem 1].

First, we verify the condition **(E)** in [12]. Recall $\Psi(r) = \sup_{|y|\leq r}\psi(|y|)$.

Lemma 3.5. *Assume that isotropic unimodal Lévy measure ν satisfies (3.5) and (3.6). Then for any $n, m \in \mathbb{N}$, there exists a constant $c = c(n,m) > 0$ such that*

$$\int_{\mathbb{R}^n} e^{-t\psi(|z|)}|z|^m dz \leq c\,\Psi^{-1}(t^{-1})^{n+m}, \quad t > 0.$$

Proof. By (3.9) and (3.13) we have that for $0 < t$,

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-t\psi(|z|)}|z|^m dz &\leq c_1\int_0^{\Psi^{-1}(t^{-1})} r^{n+m-1}dr + c_1\int_{\Psi^{-1}(t^{-1})}^{\infty} e^{-\pi^{-2}t\,\Psi(r)}r^{n+m-1}dr\\ &\leq c_2\,\Psi^{-1}(t^{-1})^{n+m} + c_1\int_{\Psi^{-1}(t^{-1})}^{\infty} e^{-c_3 t\,\Psi(\Psi^{-1}(t^{-1}))(r/\Psi^{-1}(t^{-1}))^{\alpha_3}}\\ &\quad\times r^{n+m-1}dr\\ &= \left(c_2 + c_1\int_1^\infty e^{-c_3 s^{\alpha_1}}s^{n+m-1}dr\right)\Psi^{-1}(t^{-1})^{n+m}\\ &= c_4\,\Psi^{-1}(t^{-1})^{n+m},\end{aligned}$$

where we have used the change of variables with $s = \frac{r}{\Psi^{-1}(t^{-1})}$ in the last line. □

Note that Lemma 3.5 for $(n, m) = (d, 1)$ and $(n, m) = (d + 2, 1)$ implies the condition **(E)** in [12] for the process X and Y, respectively.

For $0 < \beta \le 1$ and $\ell \ge 0$, we define non-increasing functions f and $\tilde f$ by

$$f(r) := \begin{cases} \dfrac{\varphi(1)}{r^{d+1}\varphi(r)}, & r \le 1, \\ r^{-\ell-1}\exp(-br^{\beta}), & r > 1 \end{cases} \quad \text{and}$$

$$\tilde f(r) := \begin{cases} \dfrac{\varphi(1)}{r^{d}\varphi(r)}, & r \le 1, \\ r^{-\ell}\exp(-br^{\beta}), & r > 1 \end{cases} \tag{3.16}$$

The functions f and $\tilde f$ above are non-increasing since for any $0 < r \le R \le 1$,

$$\frac{1}{r^d\varphi(r)} = r^{-1}\int_0^r \left(\frac{s}{r}\right)^{d+1}\nu(s)ds = \int_0^1 t^{d+1}\nu(rt)dt \ge \int_0^1 t^{d+1}\nu(Rt)dt = \frac{1}{R^d\varphi(R)}.$$

Here we used that ν is nonincreasing. Note that by (3.5) and (3.8),

$$\frac{\nu(r)}{r} \le cf(r) \quad \text{and} \quad \nu(r) \le c\tilde f(r) \qquad \text{for} \quad r > 0 \tag{3.17}$$

In the next lemma we verify the condition **(D)** in [12] for both X and Y. In fact, we are going to verify **(D)** for X with the above $\tilde f$ and $\gamma = d$, while we use f and $\gamma = d + 1$ to verify **(D)** for Y. Let $B_d(x, r) := \{y \in \mathbb{R}^d : |x - y| < r\}$ and recall that $\mathrm{diam}(A) = \sup\{|x - y| : x, y \in A\}$ and $\nu_1(r) = -\frac{1}{2\pi r}\nu'(r)$.

Lemma 3.6. *Assume that ν satisfies (3.5) and (3.6). Then both $\nu(\mathbb{R}^d)$ and $\nu_1(\mathbb{R}^{d+2}) = \int_{\mathbb{R}^{d+2}} \nu_1(|x|)dx$ are infinite, and there exists $c > 0$ such that*

$$\nu(A) \le c\tilde f(\delta(A))[\mathrm{diam}(A)]^d, \qquad A \in \mathcal{B}(\mathbb{R}^d). \tag{3.18}$$

and

$$\nu_1(A) = \int_A \nu_1(|x|)dx \le cf(\delta(A))[\mathrm{diam}(A)]^{d+1}, \qquad A \in \mathcal{B}(\mathbb{R}^{d+2}). \tag{3.19}$$

for some $c > 0$, where $\delta(A) := \inf\{|y| : y \in A\}$.

Proof. We have already showed that $\nu(B_d(0, 1)) = \nu(\mathbb{R}^d) = \infty$. For any $A \in \mathcal{B}(\mathbb{R}^d)$, using (3.17) we have

$$\nu(A) = \int_A \nu(|y|)dy \le \nu(\delta(A))[\mathrm{diam}(A)]^d \le c\tilde f(\delta(A))[\mathrm{diam}(A)]^d.$$

This concludes (3.18).

Using $\nu'(r) \le 0$, (3.1), the integration by parts and the fact $\nu(B_d(0, 1)) = \infty$ we have

$$\begin{aligned}\nu_1(\mathbb{R}^{d+2}) &\ge \int_{B_{d+2}(0,1)} \nu_1(|y|)dy = c(d)\int_0^1 r^{d+1}\nu_1(r)dr = c_1 \liminf_{\varepsilon\downarrow 0}\int_\varepsilon^1 -r^d\nu'(r)dr \\ &= c_1 \liminf_{\varepsilon\downarrow 0}\left(-[r^d\nu(r)]_\varepsilon^1 + d\int_\varepsilon^1 r^{d-1}\nu(r)dr\right) \\ &= c_1 \liminf_{\varepsilon\downarrow 0}\left(\varepsilon^d\nu(\varepsilon) + d\int_\varepsilon^1 r^{d-1}\nu(r)dr - \nu(1)\right) \\ &\ge c_2\nu(B_d(0, 1)) - c_1\nu(1) = \infty.\end{aligned}$$

Now it remains to prove (3.19). First observe that using the integration by parts, we have that for any $0 < r < R$,

$$\begin{aligned}\int_r^R s^{d+1}\nu_1(s)ds &= -\frac{1}{2\pi}\int_r^R s^d\nu'(s)ds = \frac{1}{2\pi}\left(-[s^d\nu(s)]_r^R + d\int_r^R s^{d-1}\nu(s)ds\right)\\ &\le \frac{1}{2\pi}\left(r^d\nu(r)+\nu(r)\,d\int_r^R s^{d-1}ds\right) = \frac{1}{2\pi}\nu(r)R^d\end{aligned} \tag{3.20}$$

where we used that ν is non-increasing. Now denote $r := \delta(A)$ and $l :=$diam(A).

When $l \ge r/2$, using $A \subset \{y \in \mathbb{R}^{d+2} : r \le |y| \le r+l\}$ we obtain

$$\begin{aligned}\nu_1(A) &\le \nu_1(\{y : r \le |y| \le r+l\}) = c(d)\int_r^{r+l} s^{d+1}\nu_1(s)ds\\ &\le \frac{c(d)}{2\pi}\nu(r)(r+l)^d \le c_1\frac{\nu(r)}{r}l^{d+l} \le c_3 f(r)l^{d+1},\end{aligned}$$

where we used (3.20) and (3.17) for the last line.

When $l < r/2$, choose a point $y_0 \in \bar{A}$ with $|y_0| = r$. Since $A \subset B_{d+2}(y_0, l) \setminus B_{d+2}(0, r)$, there exists $c_4 = c_4(d) > 0$ such that

$$\int_{|y|=s} \mathbf{1}_A(y)\sigma(dy) \le c_4 l^{d+1}$$

for any $s \in [r, r+l]$. Thus, by (3.20) and (3.17) we have

$$\begin{aligned}\nu_1(A) &\le \nu_1(B(y_0, l) \setminus B(0, r)) \le c_5\int_r^{r+l} l^{d+1}\nu_1(s)ds \le c_5\frac{l^{d+1}}{r^{d+1}}\int_r^{r+l} s^{d+1}\nu_1(s)ds\\ &\le \frac{c_5}{2\pi}\frac{l^{d+1}}{r^{d+1}}(r+l)^d\nu(r) \le c_6 l^{d+1}\frac{\nu(r)}{r} \le c_7 f(r)l^{d+1},\end{aligned}$$

which proves (3.19). □

Recall $\Psi(r) = \sup_{|y|\le r}\psi(|y|)$.

Lemma 3.7. *Assume that ν satisfies (3.5) and (3.6). For every $\kappa < 1$, there exists $c = c(\kappa) > 0$ such that*

$$\int_{\{y\in\mathbb{R}^d:|y|>r\}} \exp\big(b\kappa|y|^\beta\big)\nu(dy) \le c\,\Psi\Big(\frac{1}{r}\Big), \quad r > 0 \tag{3.21}$$

and

$$\int_{\{y\in\mathbb{R}^{d+2}:|y|>r\}} \exp\big(b\kappa|y|^\beta\big)\nu_1(dy) \le c\,\Psi\Big(\frac{1}{r}\Big), \quad r > 0 \tag{3.22}$$

Proof. Since (3.21) can be derived directly from the estimate of I_2 below, we only prove (3.22) here. Using the integration by parts, we have

$$\begin{aligned}\int_{|y|>r} \exp(b\kappa|y|^\beta)\nu_1(dy) &= c(d)\int_r^\infty \exp(b\kappa t^\beta)t^d(-\nu'(t))dt\\ &= c(d)\left(\left[\exp(b\kappa t^\beta)t^d(-\nu(t))\right]_r^\infty + \int_r^\infty (\exp(b\kappa t^\beta)t^d)'\nu(t)dt\right).\\ &:= c(d)\big(I_1 + I_2\big).\end{aligned}$$

For I_1, by (3.17) $\lim_{t\to\infty} e^{b\kappa t^\beta} t^d \nu(t) \le \lim_{t\to\infty} a e^{-b(1-\kappa)t^\beta} t^{d-\ell} = 0$, so $I_1 = e^{b\kappa r^\beta} r^d \nu(r) \le c_1 \varphi(r)^{-1}$. Now let us estimate I_2. First we observe that

$$\frac{d}{dt}\left(\exp(b\kappa t^\beta) t^d\right) \le c_2 \begin{cases} t^{d-1}, & t \le 1 \\ \exp(b\kappa t^\beta) t^{d+\beta-1}, & t > 1. \end{cases}$$

Thus, for $r \ge 1$ we have

$$\int_r^\infty (\exp(b\kappa t^\beta)t^d)'\nu(t)dt \le c_2 \int_r^\infty \exp(-b(1-\kappa)t^\beta) t^{d-\ell+\beta-1} dt \le c_3 r^{-2} = \frac{c_3\varphi(1)}{\varphi(r)}.$$

For $r < 1$, using above estimate, (3.12) and (3.6) we get

$$\begin{aligned}\int_r^\infty (\exp(b\kappa t^\beta)t^d)'\nu(t)dt &= \left(\int_r^1 + \int_1^\infty\right)(\exp(b\kappa t^\beta)t^d)'\nu(t)dt \\ &\le c_2\left(\int_r^1 t^{d-1}\nu(t)dt + \int_1^\infty \exp(-b(1-\kappa)t^\beta)t^{d-\ell+\beta-1}dt\right) \\ &\le \frac{c_4}{\varphi(r)} + c_3 \le \frac{c_5}{\varphi(r)}.\end{aligned}$$

Combining above two inequalities and (3.10), we obtain $I_1 + I_2 \le c_6 \Psi(\frac{1}{r})$. Therefore, we have proved the lemma. □

Using Lemma 3.7, we verify the condition **(P)** in [12] for both X and Y. We continue to use the non-increasing functions f and $\tilde{f}$ defined in (3.16).

Lemma 3.8. *Assume that isotropic unimodal Lévy measure ν satisfies (3.5) and (3.6). Then, there exists $c > 0$ such that*

$$\int_{\{y\in\mathbb{R}^d:|y|>r\}} \tilde{f}\left(s \vee |y| - \frac{|y|}{2}\right)\nu(dy) \le c\tilde{f}(s)\Psi(\frac{1}{r}), \quad r, s > 0 \tag{3.23}$$

and

$$\int_{\{y\in\mathbb{R}^{d+2}:|y|>r\}} f\left(s \vee |y| - \frac{|y|}{2}\right)\nu_1(dy) \le cf(s)\Psi(\frac{1}{r}), \quad r, s > 0 \tag{3.24}$$

Proof. We only prove (3.24) here, since (3.23) can be verified similarly. We claim that for any $0 < \beta \le 1$, there exists $c_1 > 0$ such that for any $s, t > 0$,

$$f(s \vee t - \frac{t}{2}) \le c_1 f(s) \exp(b\kappa t^\beta) \tag{3.25}$$

where $\kappa = \frac{1}{2}(2^{-\beta} + 1)$. First we define

$$f_1(r) := \begin{cases} \dfrac{\varphi(1)}{r^{d+1}\varphi(r)}, & r \le 2 \\ r^{-\ell-1}\exp(-br^\beta), & r > 2. \end{cases}$$

Then, since $f(r) = f_1(r)$ for $r \in (0, 1] \cup (2, \infty)$ we have

$$c_2^{-1} f(r) \le f_1(r) \le c_2 f(r), \quad r > 0. \tag{3.26}$$

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Now assume $s \vee t > 2$. Then, using $1 \vee \frac{s}{2} \leq s \vee t - \frac{t}{2}$ and triangular inequality,

$$f(s \vee t - \frac{t}{2}) = (s \vee t - \frac{t}{2})^{-\ell-1} \exp\big(-b(s \vee t - \frac{t}{2})^\beta\big) \leq (1 \vee \frac{s}{2})^{-\ell-1} \exp(-b(s - \frac{t}{2})^\beta)$$
$$\leq (1 \vee \frac{s}{2})^{-\ell-1} \exp(-bs^\beta) \exp(b(\frac{t}{2})^\beta) \leq c_3 f(s) \exp(b(\frac{t}{2})^\beta).$$

Here in the last inequality we used $\ell \geq 0$ and $\exp(-bs^\beta) \leq c_3 f(s)$ for $0 < s \leq 2$. When $s \leq 2$ and $t \leq 2$, using (3.26), (3.6) and (3.7) with $s \vee t - \frac{t}{2} \geq \frac{s}{2}$ we obtain

$$f(s \vee t - \frac{t}{2}) \leq c_2 f_1(s \vee t - \frac{t}{2}) = \frac{c_2 \varphi(1)}{(s \vee t - \frac{t}{2})^{d+1} \varphi(s \vee t - \frac{t}{2})}$$
$$\leq \frac{c_4 \varphi(1)}{s^{d+1} \varphi(s)} \leq c_4 f_1(s) \leq c_5 f(s).$$

Here we used $\frac{\varphi(s)}{\varphi(s \vee t - \frac{t}{2})} = \frac{\varphi(s)}{\varphi(s/2)} \frac{\varphi(s/2)}{\varphi(s \vee t - \frac{t}{2})} \leq 4a_3^{-1}$ which follows from (3.6) and (3.7). Thus, we conclude (3.25). Combining (3.25) and Lemma 3.7, we have proved the lemma. □

Now we obtain a priori heat kernel estimates for the process X and Y. To state the results, we need to define *generalized inverse of* φ by $\varphi^{-1}(t) := \inf\{s > 0 : \varphi(s) \geq t\}$. Using (3.6) and [1, Remark 4], we obtain

$$c^{-1}\Big(\frac{R}{r}\Big)^{1/2} \leq \frac{\varphi^{-1}(R)}{\varphi^{-1}(r)} \leq c\Big(\frac{R}{r}\Big)^{1/\alpha_3} \tag{3.27}$$

and

$$a_3^{-1}\varphi(\varphi^{-1}(r)) \leq r \leq a_3 \varphi(\varphi^{-1}(r)), \tag{3.28}$$

which are counterparts of (2.8). First we apply [11, Theorem 3] to obtain the regularity of the transition density $p_t(x)$ of X.

Proposition 3.9. *Let X be an isotropic unimodal Lévy process in $\mathbb{R}^d$ with jumping kernel $\nu(|y|)dy$ satisfying (3.5) and (3.6) with $0 < \beta \leq 1$. Then $x \to p_t(x) \in C_b^\infty(\mathbb{R}^d)$ and for any $k \in \mathbb{N}_0$ there exists $c_k > 0$ such that*

$$|\nabla_x^k p_t(x)| \leq c_k \varphi^{-1}(t)^{-k} \left(\varphi^{-1}(t)^{-d} \wedge \frac{t}{|x|^d \varphi(|x|)}\right) \tag{3.29}$$

for any $t > 0$ and $x \in \mathbb{R}^d$.

Proof. Define $h(t) := \frac{1}{\Psi^{-1}(t^{-1})}$ as in [11]. Note that by (3.10) and (3.28) we have

$$h(t) \asymp \varphi^{-1}(t), \qquad t > 0. \tag{3.30}$$

Applying [11, Theorem 3] for the process X, $p_t(x) \in C_b^\infty(\mathbb{R}^d)$ and for any $k \in \mathbb{N}$, $\gamma \in [1, d]$ and $n > \gamma$ we have constants $c_{k,n}$ satisfying

$$|\nabla_x^k p_t(x)| \leq c_{k,n}(h(t))^{-d-k} \min\left\{1, \frac{t[h(t)]^\gamma}{|x|^\gamma \varphi(|x|)} e^{-b(|x|/4)^\beta} + \left(1 + \frac{|x|}{h(t)}\right)^{-n}\right\}.$$

Note that we already verified [11, (8)] at Lemma 3.5. Thus, using $h(t) \asymp \varphi^{-1}(t)$ we obtain

$$|\nabla_x^k p_t(x)| \leq \tilde{c}_{k,n} \varphi^{-1}(t)^{-d-k}$$

Also, taking $\gamma = d$, $n = d+2$ and using $h(t) \asymp \varphi^{-1}(t)$ we get

$$
\begin{aligned}
|\nabla_x^k p_t(x)| &\le c_{k,n} \left(h(t)^{-k} \frac{t}{|x|^d \varphi(|x|)} e^{-b(|x|/4)^\beta} + h(t)^{-k} |x|^d \left(1 + \frac{|x|}{h(t)}\right)^{-2} \right) \\
&\le c\varphi^{-1}(t)^{-k} \left(\frac{t}{|x|^d \varphi(|x|)} + \left(\frac{\varphi^{-1}(t)}{|x|} \wedge 1\right)^2 \frac{1}{|x|^d} \right) \\
&\le \tilde{c}_{k,n} \varphi^{-1}(t)^{-k} \frac{t}{|x|^d \varphi(|x|)}.
\end{aligned}
$$

The last inequality is straightforward when $|x| < \varphi^{-1}(t)$ and it follows from (3.27) and (3.28) when $|x| \ge \varphi^{-1}(t)$. Therefore, we conclude that

$$
|\nabla_x^k p_t(x)| \le c_k \varphi^{-1}(t)^{-k} \left(\varphi^{-1}(t)^{-d} \wedge \frac{t}{|x|^d \varphi(|x|)} \right). \quad \square
$$

Note that the gradient estimates in Proposition 3.9 is same as the ones in [13, Proposition 3.2] except that the gradient estimates in [13, Proposition 3.2] is for $t \le T$ (see Remark 2.3).

Combining above estimates with Lemmas 3.5, 3.6 and 3.8, we can apply [11, Theorem 1] for the process X and Y. Here is the result.

Lemma 3.10. *Assume that ν satisfies (3.5) and (3.6) and $\beta = 1$. Then for any $T \ge 1$, there exists a constant $c > 0$ such that*

$$
p_t(x) \le ct \exp(-\frac{b}{4}|x|) \quad \text{and} \quad q_t(x) \le ct\varphi^{-1}(t)^{-1} \exp(-\frac{b}{4}|x|) \tag{3.31}
$$

for any $0 < t \le T$ and $|x| > \varphi^{-1}(T)$.

Proof. Define $h(t) := \frac{1}{\Psi^{-1}(t^{-1})}$ as in [11] and denote $q_t(|x|) = q_t(x)$. Applying Lemmas 3.5, 3.6 and 3.8 to [11, Theorem 1] for the process Y in Lemma 3.4, we have that for any $t, r > 0$,

$$
\begin{aligned}
q_t(r) &\le c_1 h(t)^{-1} \left(h(t)^{-d-1} \wedge \left[tf(r/4) + h(t)^{-d-1} \exp\left(-c_2 \frac{r}{h(t)} \log(1 + \frac{r}{h(t)})\right) \right] \right) \\
&\le c_3 \varphi^{-1}(t)^{-1} \left(\varphi^{-1}(t)^{-d-1} \wedge \left[tf(r/4) + \varphi^{-1}(t)^{-d-1} \exp\Big(- c_4 \frac{r}{\varphi^{-1}(t)} \right.\right. \\
&\qquad \left.\left. \times \ \log(1 + c_5 \frac{r}{\varphi^{-1}(t)})\Big) \right] \right).
\end{aligned}
$$

First observe that using $f(\frac{r}{4}) = (\frac{r}{4})^{-\ell-1} \exp(-\frac{b}{4}r)$ for $r > 4$ we obtain

$$
t\varphi^{-1}(t)^{-1} f(\frac{r}{4}) \le c_6 t r^{-\ell-1} \exp(-\frac{b}{4}r) \le c_7 t \exp(-\frac{b}{4}r), \quad r > 4. \tag{3.32}
$$

Let $c(T) > 4$ be a constant which is large enough to satisfy

$$
\frac{c_4}{2\varphi^{-1}(T)} \log(1 + c_5 \frac{c(T)}{\varphi^{-1}(T)}) \ge \frac{b}{4}, \quad r > 1.
$$

Then using (3.27) in the second inequality, for any $0 < t \le T$ and $r > c(T)$ we have

$$
\begin{aligned}
&\varphi^{-1}(t)^{-d-1}\exp\big(-c_4\frac{r}{\varphi^{-1}(t)}\log(1+c_5\frac{r}{\varphi^{-1}(t)})\big)\\
&\le \varphi^{-1}(t)^{-d-1}\exp\big(-\frac{c_4 r}{2\varphi^{-1}(t)}\log(1+c_5\frac{c(T)}{\varphi^{-1}(T)})\big)\exp\big(-\frac{c_4 r}{2\varphi^{-1}(t)}\log(1+c_5\frac{c(T)}{\varphi^{-1}(T)})\big)\\
&\le \varphi^{-1}(t)^{-d-1}\exp\big(-c_8\frac{r}{\varphi^{-1}(t)}\big)\exp\big(-\frac{c_4 r}{2\varphi^{-1}(T)}\log(1+c_5\frac{c(T)}{\varphi^{-1}(T)})\big)\\
&\le \varphi^{-1}(t)^{-d-1}\exp(-c_8\frac{r}{\varphi^{-1}(t)}-\frac{b}{4}r) \le \frac{c_9 t}{r^{d+1}\varphi(r)}\exp(-\frac{b}{4}r) \le c_9 t\exp(-\frac{b}{4}r)
\end{aligned}
$$

where $c_9 = \sup_{s\ge 1} s^{d+1}\exp(-c_8 s) < \infty$. Thus,

$$
q_t(r) \le c_{10} t\varphi^{-1}(t)^{-1}\exp(-\frac{b}{4}r), \qquad r > c(T) = \varphi(\varphi^{-1}(c(T))).
$$

Meanwhile, by (3.14) and (3.29) we have

$$
q_t(r) = \frac{1}{2\pi r}\left|\frac{r}{dr}p_t(r)\right| \le \frac{c_{11} t\varphi^{-1}(t)^{-1}}{r^{d+1}\varphi(r)} \le c_{12} t\varphi^{-1}(t)^{-1}\exp(-\frac{b}{4}r+\frac{bc(T)}{4})
$$

for $\varphi^{-1}(T) < r \le c(T)$. Therefore, combining above two estimates we conclude the estimate on q in (3.31).

Note that, applying Lemmas 3.5, 3.6 and 3.8 to [11, Theorem 1] for the process X and using $h(t) \asymp \varphi^{-1}(t)$ we have

$$
\begin{aligned}
p_t(r) \le c_{10}\Bigg(\varphi^{-1}(t)^{-d} \wedge \Bigg[t\tilde{f}(r/4) + \varphi^{-1}(t)^{-d}\exp(-c_{11}\frac{r}{\varphi^{-1}(t)})\\
\times\ \log(1+c_{12}\frac{r}{\varphi^{-1}(t)})\Bigg]\Bigg). \qquad (3.33)
\end{aligned}
$$

for any $t, r > 0$. Using (3.33), the estimate on p in (3.31) can be verified similarly. □

Now we check condition **(C)** in [12] with $r_0 = 1$, $t_p = \infty$ and $\gamma = d$ for X ($\gamma = d+1$ for Y, respectively). We need additional condition $0 < \beta < 1$ to verify the condition **(C)**.

Lemma 3.11. *Assume ν satisfies (3.5) and (3.6) with $0 < \beta < 1$. Then, there exists constant $c > 0$ such that for every $|x| \ge 2$ and $r \in (0, 1]$,*

$$
\tilde{f}(r) \le c r^{-d}\Psi(\frac{1}{r}), \qquad \int_{\{y\in\mathbb{R}^d:|x-y|\ge 1,|y|>r\}} \tilde{f}(|x-y|)\nu(dy) \le c\Psi(\frac{1}{r})\tilde{f}(|x|), \qquad (3.34)
$$

and

$$
f(r) \le c r^{-d-1}\Psi(\frac{1}{r}), \qquad \int_{\{y\in\mathbb{R}^{d+2}:|x-y|\ge 1,|y|>r\}} f(|x-y|)\nu_1(dy) \le c\Psi(\frac{1}{r})f(|x|). \quad (3.35)
$$

Proof. The first inequalities in (3.34) and (3.35) immediately follow from (3.10) and (3.16).

Let us show the second inequality in (3.35). When $|x-y| \ge \frac{|x|}{2}$, using (2.13) and triangular inequality, we have $|x|^\beta \le |x-y|^\beta + (2^\beta - 1)|y|^\beta$. Thus, using this inequality and Lemma 3.7

we obtain

$$\begin{aligned}\int_{|x-y|\geq \frac{|x|}{2}, |y|>r} f(|x-y|)\nu_1(dy) &= \int_{|x-y|\geq \frac{|x|}{2}, |y|>r} |x-y|^{-\ell-1}\exp(-b|x-y|^{\beta})\nu_1(dy)\\ &\leq \big(\frac{|x|}{2}\big)^{-\ell-1}\int_{|y|>r}\exp(-b|x|^{\beta})\exp(b(2^{\beta}-1)|y|^{\beta})\nu_1(dy)\\ &= f(|x|)\int_{|y|>r}\exp(b(2^{\beta}-1)|y|^{\beta})\nu_1(dy) \leq c_1 f(|x|)\Psi(\frac{1}{r}).\end{aligned}$$

So, it suffices to show that there exists a constant $c_2 > 0$ such that for every $|x| \geq 2$,

$$\int_{1\leq|x-y|\leq\frac{|x|}{2}} f(|x-y|)\nu_1(dy) \leq c_2 f(|x|). \tag{3.36}$$

To show this, we will divide the set $D := \{y : 1 \leq |x-y| \leq \frac{|x|}{2}\}$ into cubes with diameter 1. Let $x = (x_1, \ldots, x_{d+2})$. For $(a_1, \ldots, a_{d+2}) \in \mathbb{Z}^{d+2}$, we define $a := (\sqrt{d+2})^{-1}(a_1, \ldots, a_{d+2})$, and let

$$C_a := \prod_{i=1}^{d+2}[x_i + \frac{2a_i-1}{2\sqrt{d+2}}, x_i + \frac{2a_i+1}{2\sqrt{d+2}})$$

be a cube with length $(\sqrt{d+2})^{-1}$. Since $\mathrm{diam}(C_a) = 1$ and $x + a$ is the center of cube C_a, for any $|a| \leq \frac{|x|+1}{2}$ we have $c_5 > 0$ independent of a such that

$$\begin{aligned}\nu_1(C_a \cap D) &\leq c_3 f(\delta(C_a\cap D)) \leq c_3 f\left(\big(|x|-|a|-\frac{1}{2}\big)\vee\frac{|x|}{2}\right)\\ &\leq c_4\big(|x|-|a|\big)^{-\ell-1}\exp\big(-b|\,|x|-|a|\,|^{\beta}\big) \leq c_5|x|^{-\ell-1}\exp\Big(-b\big(|x|-|a|\big)^{\beta}\Big)\end{aligned}$$

where we used Lemma 3.6 for the first inequality and triangular inequality for the second line. Thus, using $|a| - \frac{1}{2} \leq |x-y|$ on C_a and

$$D \subset \bigcup_{1\leq|a|\leq\frac{|x|+1}{2}} C_a,$$

we obtain

$$\begin{aligned}\int_{1\leq|x-y|\leq\frac{|x|}{2}} f(|x-y|)\nu_1(dy) &\leq \sum_{1\leq|a|\leq\frac{|x|+1}{2}}\int_{C_a\cap D}|x-y|^{-\ell-1}\exp(-b|x-y|^{\beta})\nu_1(dy)\\ &\leq \sum_{1\leq|a|\leq\frac{|x|+1}{2}}(|a|-\frac{1}{2})^{-\ell-1}\exp(-b(|a|-\frac{1}{2})^{\beta})\nu_1(C_a\cap D)\\ &\leq c_6|x|^{-\ell-1}\sum_{1\leq|a|\leq\frac{|x|}{2}+1}|a|^{-\ell-1}\exp(-b|a|^{\beta})\\ &\quad\times\exp(-b(|x|-|a|)^{\beta}).\end{aligned}$$

Since $|a| \le \frac{|x|+1}{2}$, by (2.13) we have $|a|^\beta + (|x|-|a|)^\beta + 1 \ge |a|^\beta + (|x|+1-|a|)^\beta \ge |x|^\beta + (2-2^\beta)|a|^\beta$. Thus,

$$|x|^{-\ell-1} \sum_{1\le|a|\le\frac{|x|+1}{2}} \exp(-b|a|^\beta)\exp(-b(|x|-|a|)^\beta)$$
$$\le c_7|x|^{-\ell-1}\exp(-b|x|^\beta) \sum_{a\in\mathbb{Z}^d\setminus\{0\}} |a|^{-\ell-1}\exp(-b(2-2^\beta)|a|^\beta) \le c_8 f(|x|).$$

Combining above inequalities and using (3.10), we arrive (3.36). Therefore, we conclude that the second inequality in (3.35) holds.

The second inequality in (3.34) can be verified similarly so skip the proof. □

Now we have that conditions **(E), (D)** and **(C)** in [12] hold for the process Y when ν satisfies (3.5) and (3.6) with $0 < \beta < 1$. Thus, we can apply [12, Theorem 4] for both X and Y.

Lemma 3.12. *Let $T \ge 1$ and assume that ν satisfies (3.5) and (3.6) with $0 < \beta < 1$. Then, there exists a constant $c > 0$ such that*

$$p_t(r) \le ctr^{-\ell}\exp(-br^\beta) \tag{3.37}$$

and

$$\left|\frac{d}{dr}p_t(r)\right| \le ct\varphi^{-1}(t)^{-1}r^{-\ell}\exp(-br^\beta) \tag{3.38}$$

for any $0 < t \le T$ and $r \ge 4$.

Proof. Applying [12, Theorem 4] for Y and (3.10) we have that for $0 < t \le t_p = T$ and $r \ge 4r_0 = 4$,

$$q_t(r) \le c_1t\varphi^{-1}(t)^{-1}f(r) = c_1t\varphi^{-1}(t)^{-1}r^{-\ell-1}\exp(-br^\beta).$$

Combining this with (3.14), $|\frac{d}{dr}p_t(r)| \le 2\pi r q_t(r) \le c_2t\varphi^{-1}(t)^{-1}r^{-\ell}\exp(-br^\beta)$. This concludes (3.38). (3.37) immediately follows from applying [12, Theorem 4] for X. □

For reader's convenience, we put the heat kernel estimates and gradient estimates in Proposition 3.9, and Lemmas 3.10 and 3.12 together into one proposition.

Proposition 3.13. *Let X be an isotropic unimodal Lévy process in $\mathbb{R}^d$ with jumping kernel $\nu(|y|)dy$ satisfying (3.5) and (3.6). Then, $x \mapsto p_t(x) \in C_b^\infty(\mathbb{R}^d)$ and the following holds.*

(a) There exists a constant $c_1 > 0$ such that

$$|\nabla_x^k p_t(x)| \le c_1\varphi^{-1}(t)^{-k}\left(\varphi^{-1}(t)^{-d} \wedge \frac{t}{r^d\varphi(r)}\right), \quad t>0,\ x\in\mathbb{R}^d \quad \text{and} \quad k\in\mathbb{N}_0.$$

(b) Assume $\beta = 1$. Then for any $T \ge 1$, there exists a constant $c_2 > 0$ such that

$$|\nabla_x^k p_t(x)| \le c_2t\varphi^{-1}(t)^{-k}\exp(-\frac{b}{4}r), \quad t\in(0,T],\ |x|>\varphi^{-1}(T) \quad \text{and} \quad k=0,1.$$

(c) Assume $0 < \beta < 1$. Then for any $T \ge 1$, there exists a constant $c_3 > 0$ such that

$$|\nabla_x^k p_t(x)| \le c_3t\varphi^{-1}(t)^{-k}r^{-\ell}\exp(-br^\beta), \quad t\in(0,T],\ |x|>\varphi^{-1}(T) \quad \text{and} \quad k=0,1.$$

Proof. (a) and (b) immediately follow from Proposition 3.9 and Lemma 3.10, respectively.

(c) Observe that for any $t \in (0, T]$, $\varphi^{-1}(T) < |x| \le 4$ and $k = 0,1$,

$$|\nabla_x^k p_t(x)| \le c_1 \varphi^{-1}(t)^{-k} \frac{t}{r^d \varphi(r)} \le c_4 t \varphi^{-1}(t)^{-k} r^{-\ell} \exp(-br^\beta).$$

This and Lemma 3.12 finish the proof. □

Now we are ready to prove Propositions 3.1 and 3.2.

Proof of Proposition 3.1. Now assume that X is an isotropic Lévy process in Proposition 3.1 with Lévy measure $\nu(|y|)dy$. Recall that $\varphi(r) \asymp \Phi(r)$, and ν satisfies (3.5) with $\ell = 0$ and (3.6). Therefore, we can apply results in Proposition 3.13 with the function Φ instead of φ. Using Proposition 3.13 and (2.4), we conclude that for any $t \in (0, T]$ and $x \in \mathbb{R}^d$

$$|\nabla_x^k p_t(x)| \le c_1 t \varphi^{-1}(t)^{-k} \mathscr{G}_T(t, x) \le c_2 t \mathscr{G}^0_{-k}(t, x), \quad k = 0, 1. \quad \square$$

Proof of Proposition 3.2. (3.3) for $k = 0,1$ and that $t \mapsto p(t, x)$ is in $C_b^\infty(\mathbb{R}^d)$ immediately follow from Proposition 3.1.

Now it suffices to prove (3.3) when $k = 2$. Let X be an isotropic unimodal Lévy process with jumping kernel $J(|x|)dx$ satisfying (1.4) with $0 < \beta \le 1$ and (1.5), and let $\psi(|x|) = \psi(x)$ be a characteristic exponent of X. By Lemma 3.4, there exists an isotropic Lévy process Y in $\mathbb{R}^{d+2}$ with characteristic exponent $\psi(r)$ satisfying (3.14) and (3.15). In particular, by (1.5) and (3.15), Y is unimodal. Denote $J_1(|x|)dx$ and $q_t(|x|)$ be the Lévy density and transition density of Y respectively. Using (3.15) we have

$$2\pi \int_s^r J_1(t)dt = -\int_s^r \frac{J'(t)}{t}dt = -[\frac{J(t)}{t}]_s^r - \int_s^r \frac{J(t)}{t^2}dt = \frac{J(s)}{s} - \frac{J(r)}{r} - \int_s^r \frac{J(t)}{t^2}dt.$$

Since J_1 is non-increasing by (1.5), we obtain that for any $0 < s \le r$,

$$(r-s)J_1(r) \le \int_s^r J_1(t)dt \le \frac{J(s)}{2\pi s} \tag{3.39}$$

and

$$\begin{aligned}(r-s)J_1(s) &\ge \int_s^r J_1(r)dr \ge \frac{1}{2\pi}\left(\frac{J(s)}{s} - \frac{J(r)}{r} - \int_s^r \frac{J(s)}{t^2}dt\right) \\ &= \frac{1}{2\pi r}(J(s) - J(r)).\end{aligned} \tag{3.40}$$

We claim that there exists a constant $c > 0$ such that

$$\frac{c^{-1}}{r^{d+2}\phi(r)} \le J_1(r) \le \frac{c}{r^{d+2}\phi(r/2)}, \quad r \le 1 \quad \text{and} \quad J_1(r) \le cr^{-1}\exp(-br^\beta), \quad r > 1. \tag{3.41}$$

For $r \le 1$, letting $s = \frac{r}{2}$ in (3.39) we have $J_1(r) \le \frac{J(r/2)}{\pi r^2} \le \frac{c_1}{r^{d+2}\phi(r/2)}$ by (1.4). Also, taking $(r, s) = (Cr, r)$ with constant $C = \left(\frac{2}{a^2}\right)^{2/d} > 1$ in (3.40) we have

$$\begin{aligned}J_1(r) &\ge \frac{J(r) - J(Cr)}{2\pi C(C-1)r^2} \ge \frac{1}{2\pi C(C-1)r^2}\left(\frac{a}{r^d\phi(r)} - \frac{1}{a(Cr)^d\phi(Cr)}\right) \\ &\ge \frac{1}{C(C-1)}\left(a - \frac{1}{aC^d}\right)\frac{1}{r^{d+2}\phi(r)} = \frac{a}{2C(C-1)}\frac{1}{r^{d+2}\phi(r)},\end{aligned}$$

where we used $\phi(Cr) \ge \phi(r)$ and $a - \frac{1}{aC^d} = \frac{a}{2}$ in the second line.

When $r > 1$, letting $s = r - 1$ in (3.39) we have

$$J_1(r) \le \frac{J(r-1)}{r} \le \frac{1}{r}\exp(-b(r-1)^\beta) \le e^b \frac{1}{r}\exp(-br^\beta),$$

where we used the assumptions $r > 1$ and $0 < \beta \le 1$ for the last inequality. We have proved (3.41).

Let φ be the function (3.4) with $\nu = J_1$ and the dimension $d+2$ (instead of d). Note that (3.41) implies that φ satisfies

$$\begin{aligned} c_1\Phi(r) &\le 2c^{-1}\Phi(r/2) = \frac{c^{-1}r^2}{\int_0^r \frac{s}{\phi(s/2)}ds} \le \varphi(r) = \frac{r^2}{\int_0^r s^{d+3}J_1(s)ds} \le \frac{cr^2}{\int_0^r \frac{s}{\phi(s)}ds} \\ &= 2c\Phi(r), r < 1. \end{aligned}$$

Thus, J_1 satisfies (3.5) with $\ell = 0$ and (3.6) since Φ satisfies (2.2). Combining $\varphi(r) \asymp \Phi(r)$ and Proposition 3.13 for the process Y, we have that there is a constant $c_2 > 0$ satisfying

$$\frac{-1}{2\pi r}\frac{\partial}{\partial r}p(t,r) = q_t(r) \le c_2 t\mathscr{G}^{(d+2)}(t,r) \quad \text{and} \quad \left|\frac{d}{dr}q_t(r)\right| \le c_2 t\,\Phi^{-1}(t)^{-1}\mathscr{G}^{(d+2)}(t,r)$$

for any $0 < t \le T$ and $r > 0$. From now on, assume $t \in (0, T]$ and $x \in \mathbb{R}^d$. Also, let $r = |x|$. Combining above inequalities and (3.14) we have

$$\begin{aligned} |\nabla_x^2 p(t,x)| &= \left|\frac{\partial^2}{\partial r^2}p(t,r) + \frac{d-1}{r}\frac{\partial}{\partial r}p(t,r)\right| = 2\pi\left|\frac{d}{dr}(-rq_t(r) + (d-1)q_t(r))\right| \\ &\le 2\pi d\left(q_t(r) + r\left|\frac{d}{dr}q_t(r)\right|\right) \le c_3 t\left(1 + r\Phi^{-1}(t)^{-1}\right)\mathscr{G}^{(d+2)}(t,r) \\ &\le c_4 t\left(1 + r\Phi^{-1}(t)^{-1}\right)\mathscr{G}_T^{(d+2)}(t,r) \end{aligned} \tag{3.42}$$

where we used (2.4) for the last line. Thus, using (2.3) we obtain

$$\begin{aligned} &|\nabla_x^2 p(t,x)| \le 2c_4 t\mathscr{G}^{(d+2)}(t,r) \le 2c_4\Phi^{-1}(t)^{-d-2} = 2c_4\Phi^{-1}(t)^{-2}\mathscr{G}_T(t,r), \\ &r \le \Phi^{-1}(t). \end{aligned} \tag{3.43}$$

Also, for $\Phi^{-1}(t) < r \le \Phi^{-1}(T)$ we have

$$\begin{aligned} |\nabla_x^2 p(t,x)| &\le \frac{2c_4 tr}{\Phi^{-1}(t)}\mathscr{G}^{(d+2)}(t,r) \le \frac{2c_4 tr^2}{\Phi^{-1}(t)^2}\mathscr{G}^{(d+2)}(t,r) \\ &\le 2c_4\Phi^{-1}(t)^{-2}\frac{t}{r^d\Phi(r)} = 2c_4\Phi^{-1}(t)^{-2}\mathscr{G}_T(t,r). \end{aligned} \tag{3.44}$$

Note that above estimates are valid for any $0 < \beta \le 1$.

Now assume $0 < \beta < 1$. Let us recall that J_1 satisfies (3.5) with $\ell = 1$ and (3.6). Applying Proposition 3.13(c) for the process Y we have

$$q_t(r) \le c_5 tr^{-1}\exp(-br^\beta) \quad \text{and} \quad \left|\frac{d}{dr}q_t(r)\right| \le c_5 t\,\Phi^{-1}(t)^{-1}r^{-1}\exp(-br^\beta)$$

for $r > \Phi^{-1}(T)$. Thus, by (3.42)

$$\begin{aligned} |\nabla_x^2 p(t,x)| &\le 2\pi d\left(q_t(r) + r\left|\frac{d}{dr}q_t(r)\right|\right) \\ &\le c_6 t\left(r^{-1} + t\,\Phi^{-1}(t)^{-1}\right)\exp(-br^\beta) \le c_7 t\,\Phi^{-1}(t)^{-2}\mathscr{G}_T(t,r) \end{aligned}$$

for $r > \Phi^{-1}(T)$. Combining this with (3.43), (3.44) and (2.4) we obtain

$$|\nabla_x^2 p_t(x)| \le c_8 t\, \Phi^{-1}(t)^{-2} \mathscr{G}_T(t,x) \le c_9 \mathscr{G}^0_{-2}(t,x), \qquad 0 < t \le T, x \in \mathbb{R}^d.$$

This concludes (3.3) for $0 < \beta < 1$.

Similarly, for $\beta = 1$ using Proposition 3.13(b) for the process Y we have

$$q_t(r) \le c_{10} t \exp(-\frac{b}{4} r) \quad \text{and} \quad |\frac{d}{dr} q_t(r)| \le c_{10} t\, \Phi^{-1}(t)^{-1} \exp(-\frac{b}{4} r)$$

for $r > \Phi^{-1}(T)$. Thus,

$$\begin{aligned} |\nabla_x^2 p(t,x)| &\le 2\pi d \left(q_t(|x|) + |x| \, |\frac{d}{dr} q_t(|x|)| \right) \le c_{11} t \left(\Phi^{-1}(t)^{-1} + r \right) \exp(-\frac{b}{4} r) \\ &\le c_{12} t\, \Phi^{-1}(t)^{-2} \exp(-\frac{b}{5}|x|) \le c_{13} t\, \Phi^{-1}(t)^{-2} \mathscr{G}_T(t,r), \qquad r > \Phi^{-1}(T). \end{aligned}$$

Hence, combining this with (3.43), (3.44) and (2.4) we obtain

$$|\nabla_x^2 p_t(x)| \le c_{14} t\, \Phi^{-1}(t)^{-2} \mathscr{G}_T(t,x) \le c_{15} t \mathscr{G}^0_{-2}(t,x), \quad 0 < t \le T, x \in \mathbb{R}^d,$$

which is our desired result for $\beta = 1$. □

4. Further properties of heat kernel for isotropic Lévy process

In this section we assume that J satisfies (1.4) with $0 < \beta \le 1$ and (1.5), and that nondecreasing function ϕ satisfies (1.6) and (1.7). As in the previous section, let X be an isotropic unimodal Lévy process with jumping kernel $J(|y|)dy$ and $p(t,x)$ be the transition density of X. Also, let $\mathcal{L}$ be an infinitesimal generator of X.

Recall that δ_f is defined in (1.24). The following results are counterpart of [13, Proposition 3.3].

Proposition 4.1. *For every $T \ge 1$, there exists a constant $0 < c = c(d, T, a, a_1, \alpha_1, b, \beta, C_0)$ such that for every $t \in (0, T]$ and $x, y, z \in \mathbb{R}^d$,*

$$|p(t,x) - p(t,y)| \le c \left(\frac{|x-y|}{\Phi^{-1}(t)} \wedge 1 \right) t \left(\mathscr{G}(t,x) + \mathscr{G}(t,y) \right), \tag{4.1}$$

and

$$\left| \delta_p(t,x;z) \right| \le c \left(\frac{|z|}{\Phi^{-1}(t)} \wedge 1 \right)^2 t \left(\mathscr{G}(t, x \pm z) + \mathscr{G}(t,x) \right), \tag{4.2}$$

Proof. (a) Since (4.1) is clearly true when $\Phi^{-1}(t) \le |x-y|$ by (3.3), we assume that $\Phi^{-1}(t) \ge |x-y|$. Let $\alpha(\theta) = x + \theta(y-x)$, $\theta \in [0,1]$ be a segment from x to y. Then, for any $\theta \in [0,1]$ we have

$$|\alpha(\theta)| \ge |x| - |x - \alpha(\theta)| \ge |x| - |x-y| \ge |x| - \Phi^{-1}(t),$$

here we used $|x-y| \le \Phi^{-1}(t)$ for the last inequality. Thus, we obtain

$$\begin{aligned}|p(t,x)-p(t,y)| &= \Big|\int_0^1 \alpha'(\theta)\cdot\nabla_x p(t,\alpha(\theta))\,d\theta\Big| \le \int_0^1 |\alpha'(\theta)||\nabla_x p(t,\alpha(\theta))|\,d\theta \\ &\le c_1 \int_0^1 |\alpha'(\theta)|\,\Phi^{-1}(t)^{-1} t\mathscr{G}(t,\alpha(\theta))\,d\theta \\ &\le c_1|x-y|\Phi^{-1}(t)^{-1}t\mathscr{G}(t,|x|-\Phi^{-1}(t)) \\ &\le c_2\,|x-y|\Phi^{-1}(t)^{-1}t\mathscr{G}(t,x).\end{aligned}$$

Here we used (3.3) with $k=1$ for the second line and (2.10) for the last line. This concludes (4.1).

Note that using (3.3) for $k=2$ and following the same argument as the above we can estimate $|\nabla p(t,x)-\nabla p(t,y)|$. Hence, we have a constant $c_3>0$ satisfying

$$|\nabla p(t,x)-\nabla p(t,y)| \le c_3|x-y|\Phi^{-1}(t)^{-2}t(\mathscr{G}(t,x)+\mathscr{G}(t,y)) \tag{4.3}$$

for $0<t\le T$ and $|x-y|\le\Phi^{-1}(t)$.
(b) (4.2) is clearly true when $\Phi^{-1}(t)\le 2|z|$. Now assume $\Phi^{-1}(t)\ge 2|z|$. Let $\alpha(\theta)=x+\theta z$, $\theta\in[-1,1]$ be a segment from $x-z$ to $x+z$. Then, for any $\theta\in[-1,1]$ we have $|\alpha(\theta)|\ge|x|-\Phi^{-1}(t)/2$, hence

$$\begin{aligned}|\delta_p(t,x;z)| &= |(p(t,x)-p(t,x-z))-(p(t,x+z)-p(t,x))| \\ &= \Big|\int_0^1 \alpha'(\theta)\cdot\nabla p(t,\alpha(\theta))-\alpha'(-\theta)\cdot\nabla p(t,\alpha(-\theta))d\theta\Big| \\ &= \Big|\int_0^1 z\cdot(\nabla p(t,\alpha(\theta))-\nabla p(t,\alpha(-\theta)))d\theta\Big| \\ &\le 4c_3\,|z|\Phi^{-1}(t)^{-2}\left(|z|t\mathscr{G}(t,|x|-\Phi^{-1}(t))\right) \\ &\le c_4\,\Phi^{-1}(t)^{-2}|z|^2t\mathscr{G}(t,x).\end{aligned}$$

Here we used $|\alpha(\theta)-\alpha(-\theta)|\le 2|z|\le\Phi^{-1}(t)$ and (4.3) for the first inequality, and (2.10) for the second one. □

Proposition 4.2. *For every $T\ge 1$, there exist constants $c_i=c_i(d,T,a,a_1,\alpha_1,b,\beta,C_0)>0$, $i=1,2$, such that for any $t\in(0,T]$ and $x\in\mathbb{R}^d$,*

$$\begin{aligned}\int_{\mathbb{R}^d}\left|\delta_p(t,x;z)\right|J(|z|)\,dz &\le c_1\int_{\mathbb{R}^d}\Big(\frac{|z|}{\Phi^{-1}(t)}\wedge 1\Big)^2 t\,(\mathscr{G}(t,x\pm z)+\mathscr{G}(t,x))\,J(|z|)\,dz \\ &\le c_2\mathscr{G}(t,x)\end{aligned} \tag{4.4}$$

Proof. By (4.2) we have

$$\begin{aligned}&\int_{\mathbb{R}^d}\left|\delta_p(t,x;z)\right|J(|z|)\,dz \\ &\le c_1\int_{\mathbb{R}^d}\Big(\frac{|z|}{\Phi^{-1}(t)}\wedge 1\Big)^2 t\,(\mathscr{G}(t,x\pm z)+\mathscr{G}(t,x))\,J(|z|)\,dz \\ &\le c_2\left(\int_{\mathbb{R}^d}\Big(\frac{|z|}{\Phi^{-1}(t)}\wedge 1\Big)^2 t\mathscr{G}(t,x+z)J(|z|)\,dz+t\mathscr{G}(t,x)\mathcal{P}(\Phi^{-1}(t))\right) \\ &=: c_2\,(I_1+I_2)\end{aligned} \tag{4.5}$$

Clearly, by (3.11) we have

$$I_2 \le c_3 \mathscr{G}(t,x). \tag{4.6}$$

To estimate I_1, we divide I_1 into two parts as

$$I_1 = \int_{|z|\le \Phi^{-1}(t)} \Big(\frac{|z|}{\Phi^{-1}(t)}\Big)^2 t\mathscr{G}(t,x+z)J(|z|)\,dz + \int_{|z|>\Phi^{-1}(t)} t\mathscr{G}(t,x+z)J(|z|)\,dz$$
$$=: I_{11} + I_{12}\,.$$

By using (2.10) in the first inequality below and (3.11) in the third, we have

$$I_{11} \le c_4 t\mathscr{G}(t,x)\int_{|z|\le \Phi^{-1}(t)} \Big(\frac{|z|^2}{\Phi^{-1}(t)^2}\wedge 1\Big)J(|z|)\,dz$$
$$\le c_4 t\mathscr{G}(t,x)\mathcal{P}(\Phi^{-1}(t)) \le c_5\mathscr{G}(t,x)\,.$$

For the estimates of I_{12}, we will use

$$J(|z|) \le c_6\theta(|z|) = c_6\mathscr{G}_T(t,z), \quad |z| > \Phi^{-1}(t), \tag{4.7}$$

which follows from (1.4) and (2.1). Using (2.4), (4.7) and (2.21), we arrive

$$I_{12} \le c_6 at\int_{|z|>\Phi^{-1}(t)} \mathscr{G}(t,x-z)\mathscr{G}(t,z)\,dz \le c_7\mathscr{G}(t,x).$$

Here we used (2.9) for the last inequality. The lemma follows from the estimates of I_{11}, I_{12} and I_2. □

4.1. Dependency of $p^{\mathfrak{K}}$ in terms of $\mathfrak{K}$

Recall that

$$\mathcal{L}^\kappa f(x) = \lim_{\epsilon\downarrow 0}\int_{|z|>\epsilon} (f(x+z)-f(x))\,\kappa(x,z)J(|z|)dz$$

where $J : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-increasing function satisfying (1.4) and (1.5) with strictly increasing function ϕ satisfying (1.6) and (1.7).

Let $\mathfrak{K} : \mathbb{R}^d \to (0,\infty)$ be a symmetric function satisfying

$$\kappa_0 \le \mathfrak{K}(z) \le \kappa_1 \quad \text{for all } z\in\mathbb{R}^d \tag{4.8}$$

where κ_0 and κ_1 are constants in (1.2). We denote $Z^{\mathfrak{K}}$ symmetric Lévy process whose jumping kernel is given by $\mathfrak{K}(z)J(|z|)$, $z\in\mathbb{R}^d$. Then the infinitesimal generator of $Z^{\mathfrak{K}}$ is a self-adjoint operator in $L^2(\mathbb{R}^d)$ and is of the following form:

$$\begin{aligned}\mathcal{L}^{\mathfrak{K}} f(x) &= \lim_{\epsilon\downarrow 0}\int_{|z|>\epsilon} (f(x+z)-f(x))\mathfrak{K}(z)J(|z|)dz \\ &= \frac{1}{2}\lim_{\epsilon\downarrow 0}\int_{|z|>\epsilon} (f(x+z)+f(x-z)-2f(x))\mathfrak{K}(z)J(|z|)dz.\end{aligned} \tag{4.9}$$

(1.4) implies that when $f\in C_b^2(\mathbb{R}^d)$, it is not necessary to take the principal value in the last line in (4.9). The transition density of $Z^{\mathfrak{K}}$(i.e., the heat kernel of $\mathcal{L}^{\mathfrak{K}}$) will be denoted by $p^{\mathfrak{K}}(t,x)$. In this section, we will observe further properties of $p^{\mathfrak{K}}(t,x)$.

Remark 4.3. The operator (1.1) satisfies all conditions in [13] with respect to the function $\tilde{\mathscr{G}}(t,x)$ and $\Phi(r^{-1})^{-1}$ except [13, (1.7)]: Recall from Remark 2.3 that $\tilde{\mathscr{G}}(t,x)$ is comparable to the function $\rho(t,x)$ in [13]. Moreover, by Lemma 3.3, The characteristic exponent of any symmetric Lévy process whose jumping kernel comparable to $J(|z|)$, is comparable to $\Phi(r^{-1})^{-1}$. Clearly (2.2) and [13, Remark 5.2] with (1.4) imply [13, (1.4), (1.5) and (1.9)]. Also, we obtain gradient estimates with respect to $\tilde{\mathscr{G}}(t,x)$ in Proposition 3.9, which are same as the gradient estimates in [13, Proposition 3.2]. Under these observations, one can follow the proofs of [13] using (1.4) instead of the condition [13, (1.7)] and see that [13, Theorems 1.1–1.3] hold under our setting.

Using the Remark 4.3, from the remainder this paper *we use* [13, Theorems 1.1–1.3] *without any further remark.*

Let $\widehat{\mathfrak{K}} := \mathfrak{K} - \frac{\kappa_0}{2}$. Then, $\frac{\kappa_0}{2} \le \widehat{\mathfrak{K}}(z) \le \kappa_1$. Let $p^{\widehat{\mathfrak{K}}}$ be the heat kernel of symmetric Lévy process $Z^{\widehat{\mathfrak{K}}}$ whose jumping kernel is $\widehat{\mathfrak{K}}(z)J(|z|)dz$ and $p^{\frac{\kappa_0}{2}}(t,x) = p(\frac{\kappa_0}{2}t,x)$ be the heat kernel of symmetric Lévy process $Z^{\frac{\kappa_0}{2}}$ whose jumping kernel is $\frac{\kappa_0}{2}J(|z|)dz$. Without loss of generality, we can assume that $Z^{\widehat{\mathfrak{K}}}$ and $Z^{\frac{\kappa_0}{2}}$ are independent. By [9, Theorem 1.2], there exists a constant $c = c(T) = c(d,T,a,a_1,\alpha_1,b,\beta,C_0,\kappa_0,\kappa_1) > 0$ such that

$$p^{\widehat{\mathfrak{K}}}(t,x) \le ct\mathscr{G}(t,x) \quad \text{for all} \quad 0 < t \le T,\ x \in \mathbb{R}^d \tag{4.10}$$

for every $\mathfrak{K}$ satisfying (4.8). Also, by Remark 4.3 we have [13, (3.21)]. We record this for the readers:

$$\frac{\partial p^{\mathfrak{K}}(t,x)}{\partial t} = \mathcal{L}^{\mathfrak{K}} p^{\mathfrak{K}}(t,x), \qquad \lim_{t\downarrow 0} p^{\mathfrak{K}}(t,x) = \delta_0(x). \tag{4.11}$$

Since $Z^{\widehat{\mathfrak{K}}}$ and $Z^{\frac{\kappa_0}{2}}$ are independent, $Z^{\mathfrak{K}}$ and $Z^{\widehat{\mathfrak{K}}} + Z^{\frac{\kappa_0}{2}}$ have same distributions. Thus, we have

$$\begin{aligned} p^{\mathfrak{K}}(t,x) &= \int_{\mathbb{R}^d} p^{\frac{\kappa_0}{2}}(t,x-y)p^{\widehat{\mathfrak{K}}}(t,y)dy \\ &= \int_{\mathbb{R}^d} p(\frac{\kappa_0}{2}t,x-y)p^{\widehat{\mathfrak{K}}}(t,y)dy. \end{aligned} \tag{4.12}$$

First we extend Propositions 3.2 and 4.1–4.2.

Proposition 4.4. *There exists a constant* $c = c(d,T,a,a_1,\alpha_1,b,\beta,C_0,\kappa_0,\kappa_1) > 0$ *such that for any* $t \in (0,T]$ *and* $x,y,z \in \mathbb{R}^d$,

$$|\nabla_x p^{\mathfrak{K}}(t,x)| \le ct\Phi^{-1}(t)^{-1}\mathscr{G}(t,x), \tag{4.13}$$

$$|p^{\mathfrak{K}}(t,x) - p^{\mathfrak{K}}(t,y)| \le ct(\Phi^{-1}(t)^{-1}|x-y| \wedge 1)(\mathscr{G}(t,x) + \mathscr{G}(t,y)), \tag{4.14}$$

$$|\delta_{p^{\mathfrak{K}}}(t,x;z)| \le ct((\Phi^{-1}(t)^{-1}|z|)^2 \wedge 1)(\mathscr{G}(t,x\pm z) + \mathscr{G}(t,x)), \tag{4.15}$$

and

$$\int_{\mathbb{R}^d} |\delta_{p^{\mathfrak{K}}}(t,x;z)|J(|z|)dz \le c\mathscr{G}(t,x). \tag{4.16}$$

Proof. (a) Using (4.12), (4.11), (3.3), (2.21) and (2.9) for each line, we obtain

$$
\begin{aligned}
|\nabla_x p^{\mathfrak{K}}(t,x)| &= \left| \nabla_x \int_{\mathbb{R}^d} p(\frac{\kappa_0}{2}t, x-y) p^{\widehat{\mathfrak{K}}}(t,y)dy \right| \\
&\le \left| \int_{\mathbb{R}^d} \nabla_x p(\frac{\kappa_0}{2}t, x-y) t\mathscr{G}(t,y)dy \right| \\
&\le c_1 \int_{\mathbb{R}^d} t\,\Phi^{-1}(t)^{-1} \mathscr{G}(\frac{\kappa_0}{2}t, x-y) \times t\mathscr{G}(t,y)dy \\
&\le c_2 t\,\Phi^{-1}(t)^{-1}\mathscr{G}((1+\frac{\kappa_0}{2})t, x) \\
&\le c_3 t\,\Phi^{-1}(t)^{-1}\mathscr{G}(t,x).
\end{aligned}
$$

(b) Using (4.12), (4.11), (4.1), (2.21) and (2.9) we obtain

$$
\begin{aligned}
|p^{\mathfrak{K}}(t,x) - p^{\mathfrak{K}}(t,y)| &\le \int_{\mathbb{R}^d} |p(\frac{\kappa_0}{2}, x-z) - p(\frac{\kappa_0}{2}, y-z)| p^{\widehat{\mathfrak{K}}}(t,z)dz \\
&\le c_1 \int_{\mathbb{R}^d} t(\frac{|x-y|}{\Phi^{-1}(t)} \wedge 1)(\mathscr{G}(\frac{\kappa_0}{2}t, x-z) + \mathscr{G}(\frac{\kappa_0}{2}t, y-z)) \\
&\quad \times t\mathscr{G}(t,z)dz \\
&\le c_2 t(\frac{|x-y|}{\Phi^{-1}(t)} \wedge 1)(\mathscr{G}((1+\frac{\kappa_0}{2})t, x) + \mathscr{G}((1+\frac{\kappa_0}{2})t, y)) \\
&\le c_3 t(\frac{|x-y|}{\Phi^{-1}(t)} \wedge 1)(\mathscr{G}(t,x) + \mathscr{G}(t,y)).
\end{aligned}
$$

(c) We use (4.12), (4.11), (4.2), (2.21) and (2.9) for each line to estimate $|\delta_{p^{\mathfrak{K}}}(t,x;z)|$.

$$
\begin{aligned}
|\delta_{p^{\mathfrak{K}}}(t,x;z)| &\le \int_{\mathbb{R}^d} |\delta_p(\frac{\kappa_0}{2}t, x-y; z)| p^{\widehat{\mathfrak{K}}}(t,y)dy \\
&\le c_1 \int_{\mathbb{R}^d} t((\Phi^{-1}(t)^{-1}|z|)^2 \wedge 1)(\mathscr{G}(\frac{\kappa_0}{2}t, x-y\pm z) + \mathscr{G}(\frac{\kappa_0}{2}t, x-y)) \\
&\quad \times t\mathscr{G}(t,y)dy \\
&\le c_2 t((\Phi^{-1}(t)^{-1}|z|)^2 \wedge 1)(\mathscr{G}((1+\frac{\kappa_0}{2})t, x\pm z) + \mathscr{G}((1+\frac{\kappa_0}{2})t, x)) \\
&\le c_3 t((\Phi^{-1}(t)^{-1}|z|)^2 \wedge 1)(\mathscr{G}(t, x\pm z) + \mathscr{G}(t,x)).
\end{aligned}
$$

(d) We use (4.12), Fubini's theorem, (4.11), (4.4), (2.21) and (2.9) for each line to estimate $\int_{\mathbb{R}^d} |\delta_{p^{\mathfrak{K}}}(t,x;z)| J(|z|)dz$.

$$
\begin{aligned}
\int_{\mathbb{R}^d} |\delta_{p^{\mathfrak{K}}}(t,x;z)| J(|z|)dz &\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\delta_p(\frac{\kappa_0}{2}t, x-y; z)| p^{\widehat{\mathfrak{K}}}(t,y)dy J(|z|)dz \\
&= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\delta_p(\frac{\kappa_0}{2}t, x-y; z)| J(|z|)dz \right) p^{\widehat{\mathfrak{K}}}(t,y)dy \\
&\le c_1 \int_{\mathbb{R}^d} \mathscr{G}(\frac{\kappa_0}{2}t, x-y) \times t\mathscr{G}(t,y)dy \\
&\le c_2 \mathscr{G}((1+\frac{\kappa_0}{2})t, x) \le c_3 \mathscr{G}(t,x). \qquad \Box
\end{aligned}
$$

Next, we obtain continuity of transition density with respect to $\mathfrak{K}$. This is the counterpart of [13, Theorem 3.5].

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Theorem 4.5. *There exists a constant $c = c(d, T, a, a_1, \alpha_1, b, \beta, C_0, \kappa_0, \kappa_1) > 0$ such that for any two symmetric functions $\mathfrak{K}_1$ and $\mathfrak{K}_2$ in $\mathbb{R}^d$ satisfying (4.8), any $t \in (0, T]$ and $x \in \mathbb{R}^d$, we have*

$$\left|p^{\mathfrak{K}_1}(t,x) - p^{\mathfrak{K}_2}(t,x)\right| \le c\|\mathfrak{K}_1 - \mathfrak{K}_2\|_\infty\, t\mathscr{G}(t,x), \tag{4.17}$$

$$\left|\nabla p^{\mathfrak{K}_1}(t,x) - \nabla p^{\mathfrak{K}_2}(t,x)\right| \le c\|\mathfrak{K}_1 - \mathfrak{K}_2\|_\infty \Phi^{-1}(t)^{-1} t\mathscr{G}(t,x), \tag{4.18}$$

and

$$\int_{\mathbb{R}^d} |\delta_{p^{\mathfrak{K}_1}}(t,x;z) - \delta_{p^{\mathfrak{K}_2}}(t,x;z)| J(|z|)dz \le c\|\mathfrak{K}_1 - \mathfrak{K}_2\|_\infty \mathscr{G}(t,x). \tag{4.19}$$

Proof. (a) $p^{\mathfrak{K}_1}(s,y)$ is uniformly bounded on $s \in [t/2, t]$ by (4.10) and $\lim_{s\to t} p^{\mathfrak{K}_2}(t-s, x-y) = \delta_0(x-y)$ by (4.11). Thus, we have

$$\lim_{s\uparrow t} \int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,y)p^{\mathfrak{K}_2}(t-s,x-y)\,dy = p^{\mathfrak{K}_1}(t,x).$$

By the similar way, we get

$$\lim_{s\downarrow 0} \int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,y)p^{\mathfrak{K}_2}(t-s,x-y)\,dy = p^{\mathfrak{K}_2}(t,x).$$

Hence, for $t \in (0, T]$ and $x \in \mathbb{R}^d$,

$$|p^{\mathfrak{K}_1}(t,x) - p^{\mathfrak{K}_2}(t,x)| = \left|\int_0^t \frac{d}{ds}\left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,y)p^{\mathfrak{K}_2}(t-s,x-y)\,dy\right)ds\right|.$$

Using (4.11) in the second line, the fact that $\mathcal{L}^{\mathfrak{K}_1}$ is self-adjoint in the third line and (4.9) in the fourth line, we have

$$\begin{aligned}
&\int_0^{t/2} \frac{d}{ds}\left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,y)p^{\mathfrak{K}_2}(t-s,x-y)\,dy\right)ds \\
&= \int_0^{t/2}\left(\int_{\mathbb{R}^d}\left(\mathcal{L}^{\mathfrak{K}_1}p^{\mathfrak{K}_1}(s,y)p^{\mathfrak{K}_2}(t-s,x-y) - p^{\mathfrak{K}_1}(s,y)\mathcal{L}^{\mathfrak{K}_2}\right.\right. \\
&\quad \left.\left.\times\, p^{\mathfrak{K}_2}(t-s,x-y)\right)dy\right)ds \\
&= \int_0^{t/2}\left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,y)\left(\mathcal{L}^{\mathfrak{K}_1} - \mathcal{L}^{\mathfrak{K}_2}\right)p^{\mathfrak{K}_2}(t-s,x-y)dy\right)ds \\
&= \frac{1}{2}\int_0^{t/2}\left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,y)\left(\int_{\mathbb{R}^d}\delta_{p^{\mathfrak{K}_2}}(t-s,x-y;z)(\mathfrak{K}_1(z) - \mathfrak{K}_2(z))J(|z|)dz\right)dy\right)ds.
\end{aligned}$$

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Hence, by using (4.16), (4.13) and the convolution inequality (2.21), we have

$$\begin{aligned}
&\left|\int_0^{t/2} \frac{d}{ds}\left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,y)p^{\mathfrak{K}_2}(t-s,x-y)\,dy\right)ds\right| \\
&\le \frac{1}{2}\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty \int_0^{t/2}\left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,y)\left(\int_{\mathbb{R}^d}\left|\delta_{p^{\mathfrak{K}_2}}(t-s,x-y;z)\right| J(|z|)dz\right)dy\right)ds \\
&\le c_1\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty \int_0^{t/2}\int_{\mathbb{R}^d} s\mathscr{G}(s,y)\mathscr{G}(t-s,x-y)\,dy\,ds \\
&\le c_2\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty \int_0^{t/2} s\big(s^{-1}+(t-s)^{-1}\big)\mathscr{G}(t,x)ds \le c_3\|\mathfrak{K}_1-\mathfrak{K}_2\|t\mathscr{G}(t,x),
\end{aligned}$$

for all $t\in(0,T]$ and $x\in\mathbb{R}^d$. By the similar way, we also obtain

$$\begin{aligned}
&\left|\int_{t/2}^{t} \frac{d}{ds}\left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,y)p^{\mathfrak{K}_2}(t-s,x-y)\,dy\right)ds\right| \\
&= \frac{1}{2}\left|\int_{t/2}^{t}\left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_2}(t-s,y)\left(\int_{\mathbb{R}^d}\delta_{p^{\mathfrak{K}_1}}(s,x-y;z)(\mathfrak{K}_1(z)-\mathfrak{K}_2(z))J(|z|)dz\right)dy\right)ds\right| \\
&\le c_1\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty \int_{t/2}^{t}\int_{\mathbb{R}^d}(t-s)\mathscr{G}(s,y)\mathscr{G}(t-s,x-y)\,dy\,ds \\
&\le c_3\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty t\mathscr{G}(t,x).
\end{aligned}$$

Therefore, we arrive

$$\begin{aligned}
&|p^{\mathfrak{K}_1}(t,x)-p^{\mathfrak{K}_2}(t,x)| \\
&\le \left|\int_0^{t/2} \frac{d}{ds}\left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,y)p^{\mathfrak{K}_2}(t-s,x-y)\,dy\right)ds\right| \\
&\quad + \left|\int_{t/2}^{t} \frac{d}{ds}\left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,y)p^{\mathfrak{K}_2}(t-s,x-y)\,dy\right)ds\right| \\
&\le 2c_3\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty t\mathscr{G}(t,x).
\end{aligned}$$

(b) Set $\widehat{\mathfrak{K}}_i(z) := \mathfrak{K}_i(z)-\kappa_0/2$, $i=1,2$. Using (4.12), (3.3), (4.17), (2.21) and (2.9), we have that for all $t\in(0,T]$ and $x\in\mathbb{R}^d$,

$$\begin{aligned}
\left|\nabla p^{\mathfrak{K}_1}(t,x)-\nabla p^{\mathfrak{K}_2}(t,x)\right| &= \left|\int_{\mathbb{R}^d}\nabla p\left(\frac{\kappa_0}{2}t,x-y\right)\left(p^{\widehat{\mathfrak{K}}_1}(t,y)-p^{\widehat{\mathfrak{K}}_2}(t,y)\right)dy\right| \\
&\le c_1\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty \Phi^{-1}(t)^{-1}t^2\int_{\mathbb{R}^d}\mathscr{G}(\frac{\kappa_0}{2}t,x-y)\mathscr{G}(t,y)\,dy \\
&\le c_2\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty \Phi^{-1}(t)^{-1}t^2t^{-1}\mathscr{G}((1+\frac{\kappa_0}{2})t,x) \\
&\le c_3\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty \Phi^{-1}(t)^{-1}t\mathscr{G}(t,x)\,.
\end{aligned}$$

(c) By using (4.12), (4.2), (4.17), (2.21) and (2.9) we have that for any $t \in (0, T]$ and $x, z \in \mathbb{R}^d$,

$$\begin{aligned}
|\delta_{p_{\mathfrak{K}_1}}(t, x; z) - \delta_{p_{\mathfrak{K}_2}}(t, x; z)| &= \left| \int_{\mathbb{R}^d} \delta_p(\frac{\kappa_0}{2} t, x - y; z) \left(p^{\widehat{\mathfrak{K}}_1}(t, y) - p^{\widehat{\mathfrak{K}}_2}(t, y) \right) dy \right| \\
&\leq c_1 \|\mathfrak{K}_1 - \mathfrak{K}_2\|_\infty \left(\Phi^{-1}(t)^{-1}|z| \wedge 1 \right)^2 t^2 \int_{\mathbb{R}^d} (\mathscr{G}(\frac{\kappa_0}{2} t, x - y \pm z)) + \mathscr{G}(t, x - y)\mathscr{G}(t, y) dy \\
&\leq c_2 \|\mathfrak{K}_1 - \mathfrak{K}_2\|_\infty \left(\Phi^{-1}(t)^{-1}|z| \wedge 1 \right)^2 t^2 \left(t^{-1}\mathscr{G}((1 + \frac{\kappa_0}{2})t, x \pm z) + \mathscr{G}((1 + \frac{\kappa_0}{2})t, x) \right) \\
&\leq c_3 \|\mathfrak{K}_1 - \mathfrak{K}_2\|_\infty \left(\Phi^{-1}(t)^{-1}|z| \wedge 1 \right)^2 t \left(\mathscr{G}(t, x \pm z) + \mathscr{G}(t, x) \right).
\end{aligned}$$

Integrating above inequality we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\delta_{p^{\mathfrak{K}_1}}(t, x; z) - \delta_{p^{\mathfrak{K}_2}}(t, x; z)| J(|z|) dz \\
&\leq c_3 t \|\mathfrak{K}_1 - \mathfrak{K}_2\|_\infty \int_{\mathbb{R}^d} \left(\frac{|z|}{\Phi^{-1}(t)} \wedge 1 \right)^2 \left(\mathscr{G}(t, x \pm z) + \mathscr{G}(t, x) \right) J(|z|) dz \\
&\leq c_4 \|\mathfrak{K}_1 - \mathfrak{K}_2\|_\infty \mathscr{G}(t, x),
\end{aligned}$$

where the last inequality follows from Proposition 4.2. □

Estimates in this section are almost same with [13, Section 2 and 3] except these: First of all, the function $\mathscr{G}$ is different from [13], hence our estimates are more precise than estimates in [13]. However, we do not have estimates for third derivatives in terms of $\mathscr{G}$ of the heat kernel in Proposition 3.2. Thus, we do not have the improvements on [13, (3.14) and (3.18)], which are used for the gradient estimate of the function $p^\kappa(t, x, y)$ in Theorems 1.1–1.4, for instance, [13, Theorem 1.1(2) and 1.2(4)].

5. Estimates of $p^\kappa(t, x, y)$

For the remainder of this paper, we always assume that $\kappa : \mathbb{R}^d \times \mathbb{R}^d \to (0, \infty)$ is a Borel function satisfying (1.2) and (1.3), that J satisfies (1.4)–(1.5) with the function ϕ satisfying (1.6) and (1.7).

For a fixed $y \in \mathbb{R}^d$, let $\mathfrak{K}_y(z) = \kappa(y, z)$ and let $\mathcal{L}^{\mathfrak{K}_y}$ be the freezing operator defined by

$$\mathcal{L}^{\mathfrak{K}_y} f(x) = \lim_{\varepsilon \downarrow 0} \int_{|z|>\varepsilon} \delta_f(x; z)\kappa(y, z)J(|z|)dz. \tag{5.1}$$

Let $p_y(t, x) := p^{\mathfrak{K}_y}(t, x)$ be the heat kernel of the operator $\mathcal{L}^{\mathfrak{K}_y}$. Note that $\mathfrak{K}_y$ satisfies (4.8) so that there exists a constant $c > 0$ such that

$$p_y(t, x) \leq ct\mathscr{G}(t, x) \quad \text{for all } x, y \in \mathbb{R}^d, t \in (0, T]. \tag{5.2}$$

By Remark 4.3 and [13, Theorem 1.1], we have a continuous function $p^\kappa(t, x, y)$ on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ solving (1.11) and it satisfies

$$p^\kappa(t, x, y) \leq ct\widetilde{\mathscr{G}}(t, x - y), \qquad 0 < t \leq T \text{ and } x \in \mathbb{R}^d$$

In this section, we will investigate further estimates and regularity of $p^\kappa(t, x, y)$. We first recall the construction of p^κ from [13, section 4]. For $t > 0$ and $x, y \in \mathbb{R}^d$, define

$$\begin{aligned}
q_0(t, x, y) &:= \frac{1}{2} \int_{\mathbb{R}^d} \delta_{p_y}(t, x - y; z)(\kappa(x, z) - \kappa(y, z))J(|z|)dz \\
&= \left(\mathcal{L}^{\mathfrak{K}_x} - \mathcal{L}^{\mathfrak{K}_y} \right) p_y(t, \cdot)(x - y).
\end{aligned} \tag{5.3}$$

For $n \in \mathbb{N}$, we inductively define the function $q_n(t,x,y)$ by

$$q_n(t,x,y) := \int_0^t \int_{\mathbb{R}^d} q_0(t-s,x,z) q_{n-1}(s,z,y) dz ds \tag{5.4}$$

and

$$q(t,x,y) := \sum_{n=0}^{\infty} q_n(t,x,y). \tag{5.5}$$

Finally we define

$$\phi_y(t,x) := \int_0^t \phi_y(t,x,s) ds = \int_0^t \int_{\mathbb{R}^d} p_z(t-s,x-z) q(s,z,y) dz ds \tag{5.6}$$

and

$$\begin{aligned} p^\kappa(t,x,y) := p_y(t,x-y) + \phi_y(t,x) = p_y(t,x-y) \\ + \int_0^t \int_{\mathbb{R}^d} p_z(t-s,x-z) q(s,z,y) dz ds. \end{aligned} \tag{5.7}$$

As [13, section 4], the definitions in (5.3)–(5.7) are well-defined. In other words, each integrand in (5.3)–(5.7) is integrable and series in (5.5) absolutely converge on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$.

In the next lemma, we will establish the upper bounds of p^κ.

Theorem 5.1. *For every $T \geq 1$ and $\delta_0 \in (0,\delta] \cap (0,\alpha_1/2)$, there are constants c_1 and c_2 such that for any $t \in (0,T]$ and $x,y \in \mathbb{R}^d$,*

$$|\phi_y(t,x)| \leq c_1 t \big(\mathscr{G}_0^{\delta_0} + \mathscr{G}_{\delta_0}^0\big)(t,x-y) \tag{5.8}$$

and

$$p^\kappa(t,x,y) \leq c_2 t \mathscr{G}(t,x-y). \tag{5.9}$$

The constants c_1 and c_2 depend on $d, T, a, a_1, \alpha_1, b, \beta, C_0, \delta_0, \delta, \kappa_0, \kappa_1$ and κ_2.

Proof. We first claim that for $n \in \mathbb{N}_0$,

$$|q_n(t,x,y)| \leq d_n \big(\mathscr{G}^0_{(n+1)\delta_0} + \mathscr{G}^{\delta_0}_{n\delta_0}\big)(t,x-y) \tag{5.10}$$

with

$$d_n := (16C(\delta_0,T)c_2)^{n+1} \prod_{k=1}^n B(\delta_0/2, k\delta_0/2) = (16Cc_2)^{n+1} \frac{\Gamma(\delta_0/2)^{n+1}}{\Gamma((n+1)\delta_0/2)}$$

where $C = C(\delta_0,T)$ is the constant in (2.20). Without loss of generality, we assume that $C \geq 1/16$.

For $n=0$, using (5.3), (1.2), (1.3) and (4.16) we have

$$\begin{aligned} |q_0(t,x,y)| &\leq \frac{1}{2} \int_{\mathbb{R}^d} |\delta_{p_y}(t,x-y;z)(\kappa(x,z)-\kappa(y,z))| J(|z|) dz \\ &\leq c_1 \big(|x-y|^{\delta_0} \wedge 1\big) \int_{\mathbb{R}^d} |\delta_{p_y}(t,x-y;z)| J(|z|) dz \\ &\leq c_2 \big(|x-y|^{\delta_0} \wedge 1\big) \mathscr{G}(t,x-y) = c_2 \mathscr{G}_0^{\delta_0}(t,x-y). \end{aligned}$$

Suppose that (5.10) is valid for n. Then for $t \le T$,

$$\begin{aligned}
|q_{n+1}(t,x,y)| &\le \int_0^t \int_{\mathbb{R}^d} |q_0(t-s,x,z)q_n(s,z,y)|dzds \\
&\le c_2 d_n \int_0^t \int_{\mathbb{R}^d} \mathscr{G}_0^{\delta_0}(t-s,x-z)\big(\mathscr{G}^0_{(n+1)\delta_0} + \mathscr{G}^{\delta_0}_{n\delta_0}\big)(x,z-y)dzds \\
&\le 16Cc_2 d_n B(\delta_0/2,(n+1)\delta_0/2)\big(\mathscr{G}^0_{(n+2)\delta_0} + \mathscr{G}^{\delta_0}_{(n+1)\delta_0}\big)(t,x-y) \\
&= d_{n+1}\big(\mathscr{G}^0_{(n+2)\delta} + \mathscr{G}^{\delta_0}_{(n+1)\delta_0}\big)(t,x-y)
\end{aligned}$$

here we used induction in the second line, and used (2.22) and (2.23) in the last line. For the third line, we need the following: let $\theta = \eta = 1$, $\gamma_1 = \delta_2 = 0$, $\delta_1 = \delta_0$ and $\gamma_2 = (n+1)\delta_0$ which satisfy conditions in Proposition 2.8(c) since $\delta_0 \in (0,\alpha_1/2)$. Then, by (2.22) we have

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} \mathscr{G}_0^{\delta_0}(t-s,x-z)\mathscr{G}^0_{(n+1)\delta_0}(s,z-y)dzds \\
&\le 4CB(\delta_0/2,(n+1)\delta_0/2)\big(\mathscr{G}^0_{(n+2)\delta_0} + \mathscr{G}^{\delta_0}_{(n+1)\delta_0} + \mathscr{G}^0_{(n+2)\delta_0}\big)(t,x-y) \\
&\le 8CB(\delta_0/2,(n+1)\delta_0/2)\big(\mathscr{G}^0_{(n+2)\delta_0} + \mathscr{G}^{\delta_0}_{(n+1)\delta_0}\big)(t,x-y).
\end{aligned}$$

Also, letting $\theta = \eta = 1$, $\gamma_1 = 0$, $\delta_1 = \delta_2 = \delta_0$ and $\gamma_2 = \delta_0$ which satisfy conditions in Proposition 2.8(c),

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} \mathscr{G}_0^{\delta_0}(t-s,x-z)\mathscr{G}^{\delta_0}_{n\delta_0}(x,z-y)dzds \\
&\le 4CB(\delta_0/2,(n+1)\delta_0/2)\big(\mathscr{G}^0_{(n+2)\delta_0} + \mathscr{G}^{\delta_0}_{(n+1)\delta_0} + \mathscr{G}^{\delta_0}_{(n+1)\delta_0}\big)(t,x-y) \\
&\le 8CB(\delta_0/2,(n+1)\delta_0/2)\big(\mathscr{G}^0_{(n+2)\delta_0} + \mathscr{G}^{\delta_0}_{(n+1)\delta_0}\big)(t,x-y).
\end{aligned}$$

Thus, (5.10) is valid for all $n \in \mathbb{N}_0$. Note that

$$\sum_{n=0}^\infty d_n \Phi^{-1}(T)^{\delta_0} := C_1(\delta_0,T) < \infty \tag{5.11}$$

since $\frac{d_{n+1}\Phi^{-1}(T)^{(n+1)\delta_0}}{d_n\Phi^{-1}(T)^{n\delta_0}} = 16Cc_2\Phi^{-1}(T)^{\delta_0}B(\delta_0/2,(n+1)\delta_0/2) \to 0$ as $n \to \infty$. So, by using (2.16) in the second line we obtain

$$\begin{aligned}
&\sum_{n=0}^\infty |q_n(t,x,y)| \le \sum_{n=0}^\infty d_n\big(\mathscr{G}^0_{(n+1)\delta_0} + \mathscr{G}^{\delta_0}_{n\delta_0}\big)(t,x-y) \\
&\le \sum_{n=0}^\infty d_n \Phi^{-1}(T)^{n\delta_0}\big(\mathscr{G}^0_{\delta_0} + \mathscr{G}^{\delta_0}_0\big)(t,x-y) = C_1\big(\mathscr{G}^0_{\delta_0} + \mathscr{G}^{\delta_0}_0\big)(t,x-y)
\end{aligned}$$

for $t \le T$. Therefore, for every $t \in (0,T]$ and $x,y \in \mathbb{R}^d$,

$$|q(t,x,y)| \le C_1\big(\mathscr{G}^0_{\delta_0} + \mathscr{G}^{\delta_0}_0\big)(t,x-y). \tag{5.12}$$

To obtain (5.8) and (5.9), we calculate that

$$\begin{aligned}
|\phi_y(t,x)| &\le \int_0^t \int_{\mathbb{R}^d} p_z(t-s,x-z)|q(s,z,y)|\,dz\,ds \\
&\le c_3 \int_0^t \int_{\mathbb{R}^d} (t-s)\mathscr{G}(t-s,x-z)\left(\mathscr{G}^0_{\delta_0}+\mathscr{G}^{\delta_0}_0\right)(s,z-y)\,dz\,ds \\
&\le c_4 t\left(\mathscr{G}^0_{\delta_0}+\mathscr{G}^{\delta_0}_0\right)(t,x-y) \\
&\le 2c_4\Phi^{-1}(T)^{\delta_0} t\mathscr{G}(t,x-y) = c_5 t\mathscr{G}(t,x-y), \qquad \text{for all } t\in(0,T].
\end{aligned}$$

Here we used (4.11) and (5.12) for the second line, (2.22) for the third line and (2.18) for the last line. Therefore, using (4.11) we arrive $p^\kappa(t,x,y) \le p_y(t,x-y)+|\phi_y(t,x)| \le c_6 t\mathscr{G}(t,x-y)$. □

We concludes this section with some fractional estimates on $p^\kappa(t,x,y)$.

Lemma 5.2. *For every $T \ge 1$ and $\gamma \in (0,1]\cap(0,\alpha_1)$, there exists a constant c_3 such that for any $t\in(0,T]$ and $x,x',y\in\mathbb{R}^d$,*

$$\left|p^\kappa(t,x,y)-p^\kappa(t,x',y)\right| \le c_3|x-x'|^\gamma\, t\Big(\mathscr{G}^0_{-\gamma}(t,x-y)+\mathscr{G}^0_{-\gamma}(t,x'-y)\Big).$$

The constant c_3 depends on $d, T, a, a_1, \alpha_1, b, \beta, C_0, \gamma, \delta, \kappa_0, \kappa_1$ and κ_2.

Proof. Assume that $x,x',y\in\mathbb{R}^d$ and $t\in(0,T]$. By (4.14) and the fact that $\gamma\le 1$, we have

$$\begin{aligned}
|p_z(s,x-z)-p_z(s,x'-z)| &\le c_1|x-x'|^\gamma s\Phi^{-1}(s)^{-\gamma}\left(\mathscr{G}(s,x-z)+\mathscr{G}(s,x'-z)\right) \\
&\le c_1|x-x'|^\gamma s\left(\mathscr{G}^0_{-\gamma}(s,x-z)+\mathscr{G}^0_{-\gamma}(s,x'-z)\right).
\end{aligned} \tag{5.13}$$

for any $0<s\le T$ and $z\in\mathbb{R}^d$. Thus, by (5.12), (5.13) and a change of the variables, we further have that for $\delta_0 := (\delta\wedge\alpha_1/4)\in(0,\delta]\cap(0,\alpha_1/2)$,

$$\begin{aligned}
|\phi_y(t,x)-\phi_y(t,x')| &\le \int_0^t\int_{\mathbb{R}^d} |p_z(t-s,x-z)-p_z(t-s,x'-z)|\,|q(s,z,y)|\,dz\,ds \\
&\le c_2|x-x'|^\gamma \int_0^t\int_{\mathbb{R}^d}(t-s)\Big(\mathscr{G}^0_{-\gamma}(t-s,x-z)+\mathscr{G}^0_{-\gamma}(t-s,x'-z)\Big) \\
&\quad\times\ (\mathscr{G}^{\delta_0}_0+\mathscr{G}^0_{\delta_0})(s,z-y)\,dz\,ds \\
&\le c_3|x-x'|^\gamma t\Big(\mathscr{G}^0_{-\gamma+\delta_0}(t,x-y)+\mathscr{G}^{\delta_0}_{-\gamma}(t,x-y)+\mathscr{G}^0_{-\gamma+\delta_0}(t,x'-y)+\mathscr{G}^{\delta_0}_{-\gamma}(t,x'-y)\Big) \\
&\le 2c_3\Phi^{-1}(T)^{\delta_0}|x-x'|^\gamma t\big(\mathscr{G}^0_{-\gamma}(t,x-y)+\mathscr{G}^0_{-\gamma}(t,x'-y)\big), \qquad \text{for all } t\in(0,T].
\end{aligned}$$

Since $\gamma<\alpha_1$, the penultimate line follows from (2.22) (with $\theta=0$), and the last line by (2.16) and (2.17). The lemma follows by combining above two estimates and (5.7). □

6. Proof of Theorems 1.1–1.4

In this section we prove the main theorems in Section 1. We first prove that the function $p^\kappa(t,x,y)$ defined by (5.7) satisfies all statements in Theorems 1.1–1.4, then we show that $p^\kappa(t,x,y)$ is the unique solution to (1.11) satisfying (i)–(iii) in Theorem 1.1.

Proof of Theorems 1.3 and 1.4. It follows from Remark 4.3 that we can apply the results in [13, Theorem 1.1–1.4] for operator (1.1) with the function $\widetilde{\mathscr{G}}(t,x)$. Note that the function $p^\kappa(t,x,y)$

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in [13, Theorems 1.1–1.4] is constructed by the same way as (5.7). Therefore, Theorems 1.3 and 1.4 except (1.20) immediately follow from Remarks 2.3 and 4.3, and [13, Theorem 1.1(iii), 1.2 and 1.3]. Finally (1.20) is proved in Lemma 5.2. □

Now we prove the lower bound estimates in Theorem 1.1 and Corollary 1.2 for the function $p^\kappa(t,x,y)$ in (5.7). By Theorems 1.3 and 1.4, we have that $(P_t^\kappa)_{t\ge 0}$ defined by $p^\kappa(t,x,y)$ in (5.7) with (1.22) is a Feller semigroup and there exists a Feller process $X=(X_t,\mathbb{P}_x)$ corresponding to $(P_t^\kappa)_{t\ge 0}$. Moreover, by (1.23) for $f\in C_b^{2,\varepsilon}(\mathbb{R}^d)$,

$$f(X_t)-f(x)-\int_0^t \mathcal{L}^\kappa f(X_s)\,ds \tag{6.1}$$

is a martingale with respect to the filtration $\sigma(X_s, s\le t)$. Therefore, by the same argument as that in [5, Section 4.4], we have the following Lévy system formula: for every function $f:\mathbb{R}^d\times\mathbb{R}^d\to[0,\infty)$ vanishing on the diagonal and every stopping time S,

$$\mathbb{E}_x\sum_{0<s\le S} f(X_{s-},X_s)=\mathbb{E}_x\int_0^S f(X_s,y)J_X(X_s,dy)ds\,, \tag{6.2}$$

where $J_X(x,y):=\kappa(x,y-x)J(|x-y|)$. For $A\in\mathcal{B}(\mathbb{R}^d)$ we define $\tau_A:=\inf\{t\ge 0: X_t\notin A\}$ be the exit time from A.

Using (6.1), (1.2) and (3.11), the proof of the following result is the same as the one in [13, Lemma 5.7]. We skip the proof.

Lemma 6.1. *Let $T\ge 1$. For each $\varepsilon\in(0,1)$ there exists $\lambda=\lambda(\varepsilon)>0$ such that for every $0<r\le\Phi^{-1}(T)$,*

$$\sup_{x\in\mathbb{R}^d}\mathbb{P}_x\left(\tau_{B(x,r)}\le\lambda\Phi(r)\right)\le\varepsilon\,. \tag{6.3}$$

We record that by (6.3), for any $x\in\mathbb{R}^d$ and $0<r\le\Phi^{-1}(T)$ we have

$$\mathbb{E}_x[\tau_{B(x,r)}]\ge\lambda(1/2)\Phi(r)\mathbb{P}_x(\tau_{B(x,r)}>\lambda\Phi(r))\ge\frac{\lambda}{2}\Phi(r)=c\Phi(r). \tag{6.4}$$

Now we are ready to prove the lower bound in (1.15).

Lemma 6.2. *The function $p^\kappa(t,x,y)$ in (5.7) satisfies (1.15).*

Proof. Fix $T\ge 1$. Let $p_y(t,x)$ be the heat kernel of the freezing operator in (5.1), and $J_y(z):=\kappa(y,z)J(|z|)$ and $\psi_y(z)$ be the corresponding Lévy measure and characteristic exponent, respectively. By [11, Theorem 2], there exist constants $C_1,C_2>0$ such that

$$p_y(t,x)\ge C_1\Phi^{-1}(t)^{-d},\qquad t\in(0,T],\ y\in\mathbb{R}^d \text{ and } |x|\le C_2\Phi^{-1}(t). \tag{6.5}$$

Indeed, $J_y(z)dz$ is symmetric and infinite Lévy measure by (1.2) and that $J(|z|)dz$ is infinite Lévy measure. To check the condition [11, (3)], we need to show that there exists a constant $c>0$ such that

$$\int_{\mathbb{R}^d} e^{-t\psi_y(z)}|z|dz\le ch_y(t)^{-d-1},\quad 0<t,\ y\in\mathbb{R}^d \tag{6.6}$$

where $h_y(t) := \frac{1}{\Psi_y^{-1}(t^{-1})}$ and $\Psi_y(r) := \sup_{|z|\le r}\psi_y(z)$. Let $\mathcal{P}(r) := \int_{\mathbb{R}^d}\left(1 \wedge \frac{|z|^2}{r^2}\right)J(|z|)dz$ and $\mathcal{P}_y(r) := \int_{\mathbb{R}^d}\left(1 \wedge \frac{|z|^2}{r^2}\right)J_y(|z|)dz$. Then, by [11, (11)] we have

$$c_1\kappa_0\mathcal{P}(r^{-1}) \le c_1\mathcal{P}_y(r^{-1}) \le \Psi_y(r) \le 2\mathcal{P}_y(r^{-1}) \le 2\kappa_1\mathcal{P}(r^{-1}), \quad r > 0. \tag{6.7}$$

On the other hand, by the symmetry of J_y and [11, (10)], we have

$$\psi_y(z) \ge (1-\cos 1)\int_{|\xi|\le 1/|z|}|\xi\cdot z|^2 J_y(d\xi) \ge \kappa_0(1-\cos 1)\int_{|\xi|\le 1/|z|}|\xi\cdot z|^2 J(|\xi|)d\xi.$$

Since by a rotation

$$\int_{|\xi|\le 1/|z|}|\xi\cdot z|^2 J(|\xi|)d\xi = |z|^2\int_{|\xi|\le 1/|z|}\xi_i^2 J(|\xi|)d\xi, \quad i = 1,\dots,d,$$

we have

$$\psi_y(z) \ge d^{-1}\kappa_0(1-\cos 1)|z|^2\int_{|\xi|\le 1/|z|}|\xi|^2 J(|\xi|)d\xi.$$

Thus, when $|z| \le 1$ we have

$$\psi_y(z) \ge d^{-1}\kappa_0(1-\cos 1)|z|^2\int_{|\xi|\le 1}|\xi|^2 J(d\xi) \ge c_2|z|^2 = c_3\Phi(|z|^{-1}),$$

whereas by (1.4) we have

$$\psi_y(z) \ge d^{-1}\kappa_0(1-\cos 1)|z|^2\int_{|\xi|\le 1/|z|}|\xi|^2 J(d\xi) \ge c_4|z|^2\int_0^{1/|z|}\frac{s}{\phi(s)}ds = c_4\Phi(|z|^{-1})$$

for $|z| \ge 1$. Therefore, using (3.11) and (3.9) we obtain

$$\psi_y(z) \ge c_5\Phi(|z|^{-1}) \ge c_6\mathcal{P}(|z|) \ge (c_6/2)\psi(|z|). \tag{6.8}$$

Moreover, (3.11) and (6.7) also imply that $h_y(t) \asymp \Phi^{-1}(t) \asymp h(t) := \frac{1}{\Psi^{-1}(t^{-1})}$. From this and (6.8) we can follow the proof of Lemma 3.5 and obtain (6.6) as

$$\int_{\mathbb{R}^d} e^{-t\psi_y(z)}|z|dz \le \int_{\mathbb{R}^d} e^{-c_6t\psi(z)/2}|z|dz \le c_7h(t)^{-d-1} \le c_8h_y(t)^{-d-1},$$
$$0 < t,\ y \in \mathbb{R}^d. \tag{6.9}$$

Note that every constant above is independent of y. Therefore, letting $f(r) \equiv 0$ we obtain all conditions in [11, Theorem 2] so we have (6.5) where $C_1 > 0$ is independent of y.

The rest of the proof is almost identical to the one of [13, Theorem 1.4]. Note that there is minor gap in [13, (5.36)]. We provide the full details here including the correction of [13, (5.36)].

Choose $t_0 \in (0, T]$ small enough to satisfy $2c_1\Phi^{-1}(t_0)^{\delta_0} \le C_1/2$ where c_1 and δ_0 are constants in (5.8). Then, using (5.8) and (2.18) we have that for any $0 < t \le t_0$ and $x, y \in \mathbb{R}^d$,

$$\begin{aligned}|\phi_y(t,x)| &\le c_1t\left(\mathscr{G}_0^{\delta_0} + \mathscr{G}_{\delta_0}^0\right)(t, x-y) \le 2c_1\Phi^{-1}(t_0)^{\delta_0}t\mathscr{G}(t, x-y)\\ &\le 2c_1\Phi^{-1}(t_0)^{\delta_0}\Phi^{-1}(t)^{-d} \le \frac{C_1}{2}\Phi^{-1}(t)^{-d}.\end{aligned}$$

Thus, combining above inequality and (6.5) we obtain

$$p^\kappa(t,x,y) = p_y(t, x-y) + \phi_y(t,x) \ge p_y(t,x-y) - |\phi_y(t,x)| \ge \frac{C_1}{2}\Phi^{-1}(t)^{-d}$$

for $0 < t \le t_0$ and $x, y \in \mathbb{R}^d$ with $|x-y| \le C_2 \Phi^{-1}(t)$. By (1.19) and iterating at most $n_0 := \lfloor T/t_0 \rfloor + 1$ times, we obtain the following near-diagonal lower bound

$$p^\kappa(t,x,y) \ge C_3 \Phi^{-1}(t)^{-d} \quad \text{for all } t \in (0,T] \text{ and } |x-y| \le C_4 \Phi^{-1}(t) \tag{6.10}$$

for some constants $C_3, C_4 > 0$. Indeed, for $t \in (0,T]$ and $x, y \in \mathbb{R}^d$ with $|x-y| \le C_4 \Phi^{-1}(t)$ where $C_4 := C_2 (n_0/a_1)^{1/\alpha_1} \in (0, 2^{-4})$ is a sufficiently small constant satisfying

$$C_4 \Phi^{-1}(t) \le C_2 \Phi^{-1}(\frac{t}{n_0})$$

by (2.8). Let $n = \lfloor \frac{T|x-y|}{C_4 t_0 \Phi^{-1}(t)} \rfloor + 1$ and $z_1, \ldots, z_{n-1}$ be the points in the segment from x to y satisfying $|z_1 - x| = |z_{i+1} - z_i| = |y - z_{n-1}| = \frac{|x-y|}{n}$. Note that $n \le n_0$ since $|x-y| \le C_4 \Phi^{-1}(t)$. Then, by (1.15)

$$\begin{aligned} p^k(t,x,y) &= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} p^\kappa(t/n, x, w_1) p^\kappa(t/n, w_1, w_2) \ldots p^\kappa(t/n, w_{n-1}, y) \\ &\quad \times dw_1 \ldots dw_{n-1} \\ &\ge \int_{B(z_1, \frac{|x-y|}{3n})} \cdots \int_{B(z_{n-1}, \frac{|x-y|}{3n})} p^\kappa(t/n, x, w_1) p^\kappa(t/n, w_1, w_2) \ldots p^\kappa(t/n, w_{n-1}, y) \\ &\quad \times dw_1 \ldots dw_{n-1} \\ &\ge \omega_d^{n-1} \Big(\frac{|x-y|}{3n}\Big)^{-(n-1)d} \cdot \frac{C_1^n}{2^n} \Phi^{-1}(\frac{t}{n})^{-nd} \ge C_3 \Phi^{-1}(t)^{-d}. \end{aligned}$$

Here we used $\frac{|x-y|}{n} \ge \frac{T}{C_4 t_0} \Phi^{-1}(t)$, $n \le n_0$ and (2.8) for the last line. Therefore, we obtain (6.10). Now we assume $|x-y| > C_4 \Phi^{-1}(t)$ and let $\lambda > 0$ be the constant in Lemma 6.1 for $\varepsilon = 1/2$ and $\tau(z,c,t) := \tau_{B(z, c\Phi^{-1}(t))}$. Then for every $0 < t \le T$,

$$\sup_{z \in \mathbb{R}^d} \mathbb{P}_z\big(\tau(z, 2^{-2}C_4, t) \le \lambda t\big) \le \frac{1}{2}. \tag{6.11}$$

Let $\sigma = \inf\{s \ge 0 : X_s \in B(y, 2^{-2} C_4 \Phi^{-1}(t))\}$ be the hitting time of $B(y, 2^{-2} C_4 \Phi^{-1}(t))$. By the strong Markov property and (6.11) we have

$$\begin{aligned} \mathbb{P}_x\Big(X_{\lambda t} \in B(y, 2^{-1} C_4 \Phi^{-1}(t))\Big) &\ge \mathbb{P}_x\Big(\sigma \le \lambda t, \sup_{s \in [\sigma, \lambda t]} |X_s - X_\sigma| < 2^{-2} C_4 \Phi^{-1}(t)\Big) \\ &= \mathbb{E}_x\Big(\mathbb{P}_{X_\sigma}\Big(\sup_{s \in [0,\lambda t]} |X_s - X_0| < 2^{-2} C_4 \Phi^{-1}(t)\Big); \sigma \le \lambda t\Big) \\ &\ge \inf_{z \in B(y, \Phi^{-1}(t))} \mathbb{P}_z\big(\tau(z, 2^{-2}C_4, t) > \lambda t\big) \mathbb{P}_x\big(\sigma \le \lambda t\big) \\ &\ge \frac{1}{2} \mathbb{P}_x\big(\sigma \le \lambda t\big) \ge \frac{1}{2} \mathbb{P}_x\left(X_{\lambda t \wedge \tau(x, 2^{-3}C_4, t)} \in B(y, 2^{-2} C_4 \Phi^{-1}(t))\right). \end{aligned} \tag{6.12}$$

Since $|x-y| > C_4 \Phi^{-1}(t)$, we have

$$X_s \notin B\big(y, 2^{-2} C_4 \Phi^{-1}(t)\big)^c \subset B\big(x, 2^{-3} C_4 \Phi^{-1}(t)\big)^c, \quad s < \lambda t \wedge \tau(x, 2^{-3} C_4, t).$$

Thus,

$$\mathbf{1}_{\{X_{\lambda t \wedge \tau(x, 2^{-3}C_4, t)} \in B(y, 2^{-2} C_4 \Phi^{-1}(t))\}} = \sum_{s \le \lambda t \wedge \tau(x, 2^{-3}C_4, t)} \mathbf{1}_{\{X_s \in B(y, 2^{-2} C_4 \Phi^{-1}(t))\}}.$$

Therefore, by the Lévy system formula in (6.2) we obtain

$$
\begin{aligned}
&\mathbb{P}_x\left(X_{\lambda t\wedge\tau(x,2^{-3}C_4,t)} \in B(y, 2^{-2}C_4\Phi^{-1}(t))\right)\\
&= \mathbb{E}_x\left[\int_0^{\lambda t\wedge\tau(x,2^{-3}C_4,t)}\int_{B(y,2^{-2}C_4\Phi^{-1}(t))} J_X(X_s,u)\,du\,ds\right]\\
&\ge \mathbb{E}_x\left[\int_0^{\lambda t\wedge\tau(x,2^{-3}C_4,t)}\int_{B(y,2^{-2}C_4\Phi^{-1}(t))} \kappa_0 J(|X_s-u|)\mathbf{1}_{\{u:|X_s-u|\le|x-y|\}}\,du\,ds\right]. \qquad (6.13)
\end{aligned}
$$

Let w be the point in the segment from x to y satisfying $|w-y| = 3\cdot 2^{-4}C_4\Phi^{-1}(t)$. Since $|x-X_s| \le 2^{-3}C_4\Phi^{-1}(t)$, we have that for any $u \in B(w, 2^{-4}C_4\Phi^{-1}(t))$,

$$|X_s-u| \le |x-X_s| + |x-w| + |w-u| \le |x-y|$$

so that $B(w, 2^{-4}C_4\Phi^{-1}(t)) \subset B(y, 2^{-2}C_4\Phi^{-1}(t)) \cap \{u : |X_s-u| \le |x-y|\}$ for every $s < \lambda t\wedge\tau(x, 2^{-3}C_4, t)$. Thus,

$$
\begin{aligned}
&\mathbb{E}_x\left[\int_0^{\lambda t\wedge\tau(x,2^{-3}C_4,t)}\int_{B(y,\Phi^{-1}(t))} \kappa_0 J(|X_s-u|)\mathbf{1}_{\{u:|X_s-u|\le|x-y|\}}\,du\,ds\right]\\
&\ge \kappa_0\mathbb{E}_x\left[\int_0^{\lambda t\wedge\tau(x,2^{-3}C_4,t)}\int_{B(w,2^{-4}C_4\Phi^{-1}(t))} J(|x-y|)\,du\,ds\right] \qquad (6.14)\\
&\ge c_1\Phi^{-1}(t)^d J(|x-y|)\mathbb{E}_x[\lambda t\wedge\tau(x,2^{-3}C_4,t)]\\
&\ge c_2 t\,\Phi^{-1}(t)^d J(|x-y|),
\end{aligned}
$$

where we used (6.4) and (2.2) for the last line.

Therefore, combining (6.12)–(6.14) we arrive

$$
\begin{aligned}
p^\kappa(t,x,y) &\ge \int_{B(y,\Phi^{-1}(t))} p^\kappa(\lambda t,x,z)p^\kappa((1-\lambda)t,z,y)\,dz\\
&\ge \inf_{z\in B(y,2^{-1}C_4\Phi^{-1}(t))} p^\kappa((1-\lambda)t,z,y)\int_{B(y,2^{-1}C_4\Phi^{-1}(t))} p^\kappa(\lambda t,x,z)\,dz\\
&\ge c_3 t\,\Phi^{-1}(t)^{-d}\,\Phi^{-1}(t)^d J(|x-y|).
\end{aligned}
$$

for all $0 < t \le T$ and $x, y \in \mathbb{R}^d$ with $|x-y| > C_4\Phi^{-1}(t)$. □

Proof of Theorem 1.1. By Remarks 2.3 and 4.3, $p^\kappa(t,x,y)$ defined in (5.7) satisfies (1.11), (1.13) and (1.14). Also, (1.12) and (1.15) follow from Theorem 5.1 and Lemma 6.2, respectively. It remains to show the uniqueness part of Theorem 1.1. Recall that we observe in Remark 4.3 that [13, (1.9)] holds. Thus all results in [13, Sections 5.1 and 5.2] hold for our case. Since properties (i)–(iii) are stronger than ones in [13, Theorem 1.1], we now see that the proof of the uniqueness part of Theorem 1.1 is exactly same as the one of the uniqueness part of [13, Theorem 1.1]. □

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