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Shift-coupling

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Shift-coupling means pasting together the paths of two processes modulo a random shift. This concept can be related to the invariant σ-field in a similar way as ordinary coupling is related to the tail σ-field. We give an expository account of this relationship, implicit in work of Berbee and Greven. In developing these relations we introduce the concept of coupling with respect to a sub-σ-field.

coupling * Markov chain * tail σ-field * invariant σ-field * harmonic function

1. Introduction

The word 'coupling' can be used in a wide sense to mean any argument which studies stochastic processes by constructing versions of the processes on a common probability space. In a narrower sense, 'the coupling method' has been used to mean the technique of proving asymptotics of particular Markov processes by constructing versions with different initial distributions whose sample paths ultimately coincide. Such a construction is called a 'successful coupling'. Here are two fundamental results in this area. See Section 2 for definitions relevant to this paper.

Theorem 1. *Let P be a Markov kernel on a Polish space E. Let $X=(X_n;\, 0\leq n<\infty)$ and $X'=(X'_n;\, 0\leq n<\infty)$ be Markov chains with transition kernel P and initial distributions μ and μ'. The following are equivalent.*

(i) *X has trivial tail σ-field, for each μ.*
(ii) *X is mixing, for each μ.*
(iii) *There exists a successful coupling of X and X', for each pair μ, μ'.*
(iv) *$\|\mu P^n-\mu' P^n\|\to 0$ as $n\to\infty$, for each pair μ, μ'.*
(v) *$\|\boldsymbol{P}(\theta_n X\in\cdot)-\boldsymbol{P}(\theta_n X'\in\cdot)\|\to 0$ as $n\to\infty$, for each pair μ, μ'.*
(vi) *All bounded space-time harmonic functions are constant.* $\square$

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Theorem 1 is a qualitative result: each property either holds or does not hold. Here is a related quantitative result.

Theorem 2. *Let $Z=(Z_n\colon 0\leq n<\infty)$ and $Z'=(Z'_n\colon 0\leq n<\infty)$ be arbitrary stochastic processes taking values in a Polish space E.*

(i) *If T is the coupling epoch in any coupling of (Z, Z'), then*

$$\|\boldsymbol{P}(\theta_n Z\in\cdot)-\boldsymbol{P}(\theta_n Z'\in\cdot)\|\leq 2\boldsymbol{P}(T>n),\quad 0\leq n<\infty.$$

(ii) *There exists a coupling whose coupling epoch T satisfies*

$$\|\boldsymbol{P}(\theta_n Z\in\cdot)-\boldsymbol{P}(\theta_n Z'\in\cdot)\|=2\boldsymbol{P}(T>n),\quad 0\leq n<\infty.\qquad\square \tag{1}$$

A coupling satisfying (1) is called a *maximal coupling.* Assertion (i) is the easy *coupling inequality*: see (8) for a variation. Note that in Theorem 2 the processes are not required to be Markov. In one sense this is spurious generality, since for general Z the process $(\theta_n Z\colon 0\leq n<\infty)$ is Markov.

The circle of ideas surrounding Theorems 1 and 2 was well known to experts by the end of the 1970s. The recent book of Lindvall (1992) gives a detailed account of these results (his Theorem 21.12 and equation (14.1)) and their history and applications, which we shall not repeat here, except for occasional remarks. The purpose of this paper is to give an exposition of a parallel set of ideas, which are essentially known but seem not *well*-known even to experts, and which are not covered in Lindvall (1992). These concern the analogous results when the tail σ-field is replaced by the invariant σ-field $\mathcal{I}$ and the notion of 'successful coupling' is replaced by 'successful shift-coupling,' in which a random time-shift is allowed (see Section 2 for precise definition). It turns out there is the following simple analog of the highlights of Theorem 1, but no entirely satisfying analog of Theorem 2 is known.

Theorem 3. *Under the assumptions of Theorem* 1, *the following are equivalent.*

(i) *X has trivial invariant σ-field, for each μ.*

(iii) *There exists a successful shift-coupling of X and X', for each pair μ, μ'.*

(vi) *All bounded harmonic functions are constant.*

As discussed below, Theorem 3 is essentially known, but hard to find explicitly stated in the literature, apart from the equivalence of (i) and (vi) which is routine (see, e.g., Lindvall, 1992, Theorem 21.8). To our knowledge, shift-coupling of general processes was first discussed by Berbee (1979, Theorems 4.3.3 and 4.4.9). Berbee shows that in the setting of Theorem 2, the existence of a successful shift-coupling is equivalent to the following analog of (v) of Theorem 1,

$$\left\| n^{-1}\sum_{i=0}^{n-1}\boldsymbol{P}(\theta_n Z\in\cdot)-n^{-1}\sum_{i=0}^{n-1}\boldsymbol{P}(\theta_n Z'\in\cdot)\right\|\to 0\quad\text{as } n\to\infty. \tag{2}$$

Berbee was primarily interested in studying mixing conditions for non-Markovian processes, and did not point out the connection with invariant σ-fields in the Markov case, or seek to establish any analog of maximal coupling.

In the Markov case, constructing a successful shift-coupling is equivalent to constructing randomized stopping times T and T' such that

$$T<\infty, \qquad T'<\infty, \qquad X'_{T'}=X_T \quad \text{a.s.} \tag{3}$$

This question was studied in detail by Greven (1987a), who obtained the following result.

Theorem 4. *Under the assumptions of Theorem* 1, *for each pair* μ, μ' *there exist randomized stopping times* T, T' *such that*

(i) $$\eta(\cdot)\equiv \sum_{n=0}^{\infty} \boldsymbol{P}(X_n\in\cdot, T>n) \quad \textit{and} \quad \eta'(\cdot)\equiv \sum_{n=0}^{\infty} \boldsymbol{P}(X'_n\in\cdot, T'>n)$$

are mutually singular,

(ii) $$\boldsymbol{P}(T=\infty)+\boldsymbol{P}(T'=\infty)=\lim_{n\to\infty} n^{-1}\left\| \sum_{i=0}^{n-1} (\mu-\mu')P^i \right\|$$
$$=\sup\left\{\int h \,\mathrm{d}(\mu-\mu')\text{: } h \textit{ harmonic}, |h|\leq 1\right\}. \qquad \square$$

This immediately gives the implication (vi)$\to$(iii) in Theorem 3. Since the rest of Theorem 3 is routine, we refer to it as 'essentially known'.

Theorem 4 is close to being an analog of the maximal coupling result, Theorem 2. Greven (1987a) showed that for transient chains the measures η and η' are unique. But in the recurrent case it is clear (see Section 7) that uniqueness cannot hold. Greven (1987a) gives some results for the recurrent case, and Greven (1987b) introduces a notion of 'short couplings' in the null-recurrent setting.

Greven's proof of Theorem 4 used constructions ('flooding schemes') similar to those used in proofs of Theorem 2. The purpose of this paper is to show that some results about shift-coupling can be deduced from Theorems 1 and 2 about coupling. In Section 3 we show that Theorem 3 can be deduced directly from Theorem 1. This section is intended for the non-expert reader of Lindvall (1992). The remainder of the paper is aimed at specialists. In Section 6 we prove that Theorem 2 leads to an abstract result on maximal shift-couplings (Theorem 15), weaker than Theorem 4. As pointed out in Section 5, the structure of this argument is closely related to one argument using Theorem 2 to prove the existence of a successful coupling in Theorem 1. This approach is similar in spirit to that of Berbee (1979), who also used maximal couplings to show that (2) was equivalent to existence of successful shift-couplings.

Terminology note. The phrase 'shift-coupling' is our invention, but otherwise we mostly follow Lindvall (1992).

2. Definitions

Let $Z=(Z_0, Z_1, \ldots)$ and $Z'=(Z'_0, Z'_1, \ldots)$ be discrete time stochastic processes on a Polish state space $(E, \mathscr{E})$. For $x=(x_0, x_1, \ldots)\in E^\infty$ define the shift-maps θ_n, $0\le n<\infty$, by $\theta_n x=(x_n, x_{n+1}, \ldots)$; define θ_∞ by $\theta_\infty x=(\Delta, \Delta, \ldots)$ where Δ is a fixed state not in E.

A pair of processes $\hat{Z}$ and $\hat{Z}'$ is a *coupling* of Z and Z' if $\hat{Z}$ and $\hat{Z}'$ have the same distributions as Z and Z', respectively. A random time T in $\{0, 1, \ldots, \infty\}$ is a *coupling epoch* if

$$\theta_T\hat{Z}=\theta_T\hat{Z}'.$$

A coupling with coupling epoch T is *successful* if $\boldsymbol{P}(T<\infty)=1$. Say that Z and Z' *admit coupling* if there exists a successful coupling of Z and Z'.

Call a coupling $\hat{Z}$ and $\hat{Z}'$ together with a pair of random times T and T' a *shift-coupling* if

$$\theta_T\hat{Z}=\theta_{T'}\hat{Z}'.$$

Call T and T' *shift-coupling epochs.* Note that $\{T=\infty\}=\{T'=\infty\}$. A shift-coupling is *successful* if $\boldsymbol{P}(T<\infty)=1$, and Z and Z' *admit shift-coupling* if there exists a successful shift-coupling of Z and Z'.

Put

$$\mathscr{T}_n=\theta_n^{-1}\mathscr{E}^\infty=\text{the post-}n\ \sigma\text{-field},$$

$$\mathscr{T}=\bigcap_{n=0}^{\infty}\mathscr{T}_n=\text{the tail } \sigma\text{-field},$$

$$\mathscr{I}=\{B\in\mathscr{E}^\infty\colon \theta_1^{-1}B=B\}=\text{the invariant } \sigma\text{-field}.$$

For a sub-σ-field $\mathscr{A}$ of $\mathscr{E}^\infty$ let $\boldsymbol{P}(Z\in\cdot)_{\mathscr{A}}$ denote the restriction of $\boldsymbol{P}(Z\in\cdot)$ to $\mathscr{A}$. Say Z is *$\mathscr{A}$-trivial* if $\boldsymbol{P}(Z\in\cdot)_{\mathscr{A}}$ is a 0–1-measure.

For a bounded signed measure ν write $\|\nu\|=\sup_A \nu(A)-\inf_A \nu(A)=(\text{mass of } \nu^+)+(\text{mass of } \nu^-)$ for the total variation norm.

3. First proof of Theorem 3

Suppose (iii) holds. Let $h: E\to\mathbb{R}$ be bounded harmonic. Then $h(X_n)$ is a bounded martingale, for any initial distribution. Consider the shift-coupling of (iii), and apply the a.s. convergence theorem for martingales:

$$\int h\,\mathrm{d}\mu'=Eh(X'_0)=E\lim_n h(X'_n)=E\lim_m h(X_m)=\int h\,\mathrm{d}\mu,$$

because the limits are a.s. equal by existence of a successful shift-coupling. This equality holds for all μ' and μ, and hence h is constant.

As observed before, the equivalence of (i) and (vi) is routine. We will now prove that (i) implies (iii), by using the implication (i) implies (iii) of Theorem 1.

Define a new kernel by

$$R = \tfrac{1}{2}P + \tfrac{1}{2}I, \tag{4}$$

where I is the identity kernel. Each step of an R-chain may be marked in the obvious way as a P-step or an I-step, and deleting the I-steps from an R-chain yields a P-chain. Now suppose we are given a successful coupling of R-chains with initial distributions μ and μ'. It is clear that, by deleting the I-steps of each chain, we get a successful shift-coupling of P-chains with initial distributions μ and μ'. Thus by the implication (i) implies (iii) of Theorem 1, it is enough to prove:

Lemma 5. *Let X and Z be Markov chains with initial distribution μ and transition kernels P and R related by* (4). *If X has trivial invariant σ-field, then Z has trivial tail σ-field.*

Proof. Let (ξ_i) be independent with $\boldsymbol{P}(\xi_i = 1) = \boldsymbol{P}(\xi_i = 0) = \frac{1}{2}$. Let $J(n) = \sum_{i=1}^{n} \xi_i$. For fixed k it is clear (by recurrence of simple symmetric random walk) that we can successfully couple versions of $(J(n), n \geq 0)$ and $(k + J(n), n \geq 0)$. For $x \in E^{\infty}$ write $x \circ J = (x_{J(n)})_{n \geq 0}$. Fix x, and fix some A in the tail σ-field on E^{∞}. Using the coupling above,

$$\boldsymbol{P}(x \circ J \in A) = \boldsymbol{P}((\theta_k(x)) \circ J \in A).$$

But the event $\{x \circ J \in A\}$ is in the exchangeable σ-field of (ξ_i), so has probability 0 or 1 by the Hewitt–Savage law. Taking the union over k,

$$\boldsymbol{P}(x \circ J \in A) = \boldsymbol{P}((x, J) \in B) = 0 \text{ or } 1, \tag{5}$$

where

$$B = \{(x, j)\colon \theta_k(x) \circ j \in A \text{ for some } k\},$$

and j denotes a sequence $(j(n))_{n \geq 0}$. Now let X be the Markov chain, independent of J. For deterministic j the set $\{x\colon (x, j) \in B\}$ is invariant, and so by hypothesis

$$\boldsymbol{P}((X, j) \in B) = 0 \text{ or } 1. \tag{6}$$

Using Fubini's theorem, (5) and (6) imply

$$\boldsymbol{P}((X, J) \in B) = 0 \text{ or } 1.$$

But now $Z = X \circ J$ is a chain with transition kernel R. For the tail event A,

$$\begin{aligned} \boldsymbol{P}(Z \in A) &= \boldsymbol{P}(X \circ J \in A) \\ &= \boldsymbol{P}((X, J) \in B) \quad \text{(by (5), conditioning on } X = x) \\ &= 0 \text{ or } 1, \end{aligned}$$

and the lemma is established. $\square$

Remark. Here is an area of potential application of Theorem 3. Let Q be a non-negative (but not necessarily stochastic) matrix on E, which for simplicity let us assume to be countable. Let Π be the set of probability distributions π on E which are Q-invariant, that is $\pi(t)=\sum_s \pi(s)Q(s,t)$ for all t. Suppose Π is non-empty. For $\pi \in \Pi$ we can define a Markov transition matrix on $\{s: \pi(s)>0\}$ by

$$P_\pi(s,t)=\pi(t)Q(t,s)/\pi(s).$$

Standard theory says

(i) each $\pi \in \Pi$ can be represented as an integral mixture over the *extreme* elements of Π;

(ii) π is extreme in Π iff P_π has no non-constant bounded harmonic functions and iff the invariant σ-field is trivial for π.

In this setting, Theorem 3 gives a probabilistic characterization of the extreme π's as those which allow shift-coupling. For an application of this characterization in the context of random trees see Aldous (1991, Section 5).

4. Maximality with respect to a sub-σ-field

One of several ways to prove existence of successful couplings in Theorem 1 is by using the existence of a maximal coupling (Theorem 2). We now set out an abstract approach to this argument. At first sight this approach may seem complicated, but the point is that a parallel argument will be given for shift-couplings in the subsequent sections.

Call C a *coupling event* for Z and Z' if there is a coupling $\hat{Z}$ and $\hat{Z}'$ such that $\hat{Z}=\hat{Z}'$ on C. Clearly, if T is a coupling epoch and $0\leq n<\infty$ then $\{T\leq n\}$ is a coupling event for $\theta_n Z$ and $\theta_n Z'$. On the other hand, $\{T\leq n\}$ is not necessarily a coupling event for Z and Z'. To remedy this we introduce a new coupling concept.

Let $\mathscr{A}$ be a sub-σ-field of $\mathscr{E}^\infty$. Call $\hat{Z}$ and $\hat{Z}'$ an *$\mathscr{A}$-coupling* of Z and Z' if

$$\boldsymbol{P}(\hat{Z}\in\cdot)_{\mathscr{A}}=\boldsymbol{P}(Z\in\cdot)_{\mathscr{A}} \quad \text{and} \quad \boldsymbol{P}(\hat{Z}'\in\cdot)_{\mathscr{A}}=\boldsymbol{P}(Z'\in\cdot)_{\mathscr{A}}.$$

Note that $\hat{Z}$ and $\hat{Z}'$ is a coupling iff it is an $\mathscr{E}^\infty$-coupling and that any coupling is an *$\mathscr{A}$-coupling* since $\mathscr{A}$ is contained in $\mathscr{E}^\infty$. Call an event C an *$\mathscr{A}$-coupling event* if

$$\{\hat{Z}\in B\}\cap C=\{\hat{Z}'\in B\}\cap C, \quad B\in\mathscr{A},$$

i.e., their restrictions to $\mathscr{A}$ are identical on C.

Lemma 6. *If T is a coupling epoch then $\{T\leq n\}$ is a $\mathscr{T}_n$-coupling event for all $n<\infty$.*

Proof. We have $\{\hat{Z}\in B\}=\{\theta_n\hat{Z}\in\theta_n B\}$ for $B\in\mathscr{T}_n$ and $\theta_n\hat{Z}=\theta_n\hat{Z}'$ on $\{T\leq n\}$ which yields

$$\begin{aligned}\{\hat{Z}\in B, T\leq n\}&=\{\theta_n\hat{Z}\in\theta_n B, T\leq n\}\\&=\{\theta_n\hat{Z}'\in\theta_n B, T\leq n\}\\&=\{\hat{Z}'\in B, T\leq n\}, \quad B\in\mathscr{T}_n. \qquad \square\end{aligned}$$

Proposition 7. *If T is a coupling epoch then $\{T<\infty\}$ is a $\mathcal{T}$-coupling event.*

Proof. Since $\mathcal{T}$ is contained in each $\mathcal{T}_n$ we have by Lemma 6 that

$$\{\hat{Z}\in B,\ T\leq n\}=\{\hat{Z}'\in B,\ T\leq n\}\quad \text{for } B\in\mathcal{T},$$

and thus sending $n\to\infty$ renders

$$\{\hat{Z}\in B,\ T<\infty\}=\{\hat{Z}'\in B,\ T<\infty\},\quad B\in\mathcal{T},$$

as required. □

Note that we cannot expect a 'coupling with coupling epoch T' to be a 'coupling with coupling event $\{T<\infty\}$' since this would imply $\|P(Z\in\cdot)-P(Z'\in\cdot)\|\leq 2P(T=\infty)$ while it is even possible that $\|P(Z\in\cdot)-P(Z'\in\cdot)\|=2$ and $P(T=\infty)=0$.

The usual coupling inequality extends easily to an *$\mathcal{A}$-coupling inequality* as follows. If C is an $\mathcal{A}$-coupling event then $P(\hat{Z}\in\cdot,C)_{\mathcal{A}}=P(\hat{Z}'\in\cdot,C)_{\mathcal{A}}$ and thus (with C^c the complement of C)

$$\|P(Z\in\cdot)_{\mathcal{A}}-P(Z'\in\cdot)_{\mathcal{A}}\|=\|P(\hat{Z}\in\cdot,C^c)_{\mathcal{A}}-P(\hat{Z}'\in\cdot,C^c)_{\mathcal{A}}\|,\tag{7}$$

which yields

$$\|P(Z\in\cdot)_{\mathcal{A}}-P(Z'\in\cdot)_{\mathcal{A}}\|\leq 2P(C^c).\tag{8}$$

Call an $\mathcal{A}$-coupling event C *maximal* if the inequality in (8) is an identity.

Lemma 8. *An $\mathcal{A}$-coupling event C is maximal iff $P(\hat{Z}\in\cdot,C^c)_{\mathcal{A}}$ and $P(\hat{Z}'\in\cdot,C^c)_{\mathcal{A}}$ are mutually singular.*

Proof. Clearly $P(\hat{Z}\in\cdot,C^c)_{\mathcal{A}}$ and $P(\hat{Z}'\in\cdot,C^c)_{\mathcal{A}}$ are mutually singular iff $\|P(\hat{Z}\in\cdot,C^c)_{\mathcal{A}}-P(\hat{Z}'\in\cdot,C^c)_{\mathcal{A}}\|=2P(C^c)$ which, due to (7), holds iff C is maximal. □

Say a coupling with coupling epoch T is *$\mathcal{T}_n$-maximal* if the $\mathcal{T}_n$-coupling event $\{T\leq n\}$ is maximal. Call the coupling *$\mathcal{T}$-maximal* if the $\mathcal{T}$-coupling event $\{T<\infty\}$ is maximal.

Lemma 9. *A coupling is maximal iff it is $\mathcal{T}_n$-maximal for all $n<\infty$.*

Proof. With $\mathcal{A}=\mathcal{T}_n$ the left-hand sides of (1) and (8) coincide and with $C=\{T\leq n\}$ so do the right-hand sides. □

Lemma 10. *A maximal coupling is $\mathcal{T}$-maximal.*

Proof. Since $\{T=\infty\}\subseteq\{T>n\}$, Lemmas 8 and 9 imply that $P(\hat{Z}\in\cdot,T=\infty)_{\mathcal{T}_n}$ and $P(\hat{Z}'\in\cdot,T=\infty)_{\mathcal{T}_n}$ are mutually singular, i.e.,

$$\exists B_n\in\mathcal{T}_n:\quad P(\hat{Z}\in B_n,\ T=\infty)=0\ \text{ and }\ P(\hat{Z}'\in B_n^c,\ T=\infty)=0.$$

Put $B = \limsup_{n\to\infty} B_n$ and note that $B \in \mathcal{T}$ and that $B^c = \liminf_{n\to\infty} B_n^c$ to obtain

$$\exists B \in \mathcal{T}: \quad \boldsymbol{P}(\hat{Z} \in B, T = \infty) = 0 \text{ and } \boldsymbol{P}(\hat{Z}' \in B^c, T = \infty) = 0,$$

i.e., $\boldsymbol{P}(\hat{Z} \in \cdot, T = \infty)_{\mathcal{T}}$ and $\boldsymbol{P}(\hat{Z}' \in \cdot, T = \infty)_{\mathcal{T}}$ are mutually singular. □

Remark. The converse of Lemma 10 is not true since if T is a maximal coupling epoch then $T+1$ is not; however $T+1$ is $\mathcal{T}$-maximal.

The arguments above are self-contained. We can now invoke Theorem 2, which says that a maximal coupling exists, and use Lemma 10 to give:

Proposition 11. *There exists a $\mathcal{T}$-maximal coupling of Z and Z'.* □

The last remark shows that Proposition 11 is weaker than Theorem 2, which asserted existence of a maximal coupling. Our main point is that an analogous result (Theorem 15) holds for shift-coupling, and that the qualitative results on existence of couplings and shift-couplings are simple consequences of this notion of maximality. We derive these consequences below for coupling, and in the next section for shift-coupling. The result below is due to Goldstein (1979).

Corollary 12. *Z and Z' admit coupling iff $\boldsymbol{P}(Z \in \cdot)_{\mathcal{T}} = \boldsymbol{P}(Z' \in \cdot)_{\mathcal{T}}$.*

Proof. If $\boldsymbol{P}(T<\infty) = 1$ then Proposition 7 and the $\mathcal{T}$-coupling inequality yield $\boldsymbol{P}(Z \in \cdot)_{\mathcal{T}} = \boldsymbol{P}(Z' \in \cdot)_{\mathcal{T}}$. Conversely, if $\boldsymbol{P}(Z \in \cdot)_{\mathcal{T}} = \boldsymbol{P}(Z' \in \cdot)_{\mathcal{T}}$ then Proposition 11 yields the existence of a successful coupling. □

This in turn easily implies the existence of successful couplings in Theorem 1, as follows. (Note it is routine that (iii) implies (i).)

Corollary 13. *Under the assumptions of Theorem* 1, *if*

(i) *X has trivial tail σ-field, for each μ then*

(iii) *there exists a successful coupling of X and X', for each pair μ, μ'.*

Proof. By (i), $\boldsymbol{P}(X \in \cdot)_{\mathcal{T}}$ and $\boldsymbol{P}(X' \in \cdot)_{\mathcal{T}}$ are both 0–1 measures, and so is any mixture of them. It follows that $\boldsymbol{P}(X \in \cdot)_{\mathcal{T}} = \boldsymbol{P}(X' \in \cdot)_{\mathcal{T}}$. Apply Corollary 12. □

5. Shift-coupling

As in the last section, let Z and Z' be arbitrary processes. The analog of Proposition 7 is:

Proposition 14. *If T is a shift-coupling epoch then $\{T<\infty\}$ is an $\mathscr{I}$-coupling event.*

Proof. With $B\in\mathscr{I}$ we have $\{\hat{Z}\in B\}=\{\theta_n\hat{Z}\in B\}$ which yields the second identity in

$$\begin{aligned}\{\hat{Z}\in B,\ T<\infty\}&=\bigcup_{n=0}^{\infty}\{\hat{Z}\in B,\ T=n\}\\&=\bigcup_{n=0}^{\infty}\{\theta_T\hat{Z}\in B,\ T=n\}\\&=\{\theta_T\hat{Z}\in B,\ T<\infty\}\\&=\{\theta_T\hat{Z}'\in B,\ T'<\infty\}\\&=\cdots=\{\hat{Z}'\in B,\ T'<\infty\}\\&=\{\hat{Z}'\in B,\ T<\infty\}.\qquad\square\end{aligned}$$

Call a shift-coupling *$\mathscr{I}$-maximal* if the $\mathscr{I}$-coupling event $\{T<\infty\}$ is maximal. Our main abstract result is the following analog of Proposition 11.

Theorem 15. *There exists an $\mathscr{I}$-maximal shift-coupling of Z and Z'.*

The proof occupies the next section. The point is that this notion of maximality for shift-coupling is enough to prove the qualitative results below. These are the analogs of Corollaries 12 and 13, and have identical proofs.

Corollary 16. *Z and Z' admit shift-coupling iff $\boldsymbol{P}(Z\in\cdot)_{\mathscr{I}}=\boldsymbol{P}(Z'\in\cdot)_{\mathscr{I}}$.* $\square$

Corollary 17. *Under the assumptions of Theorem* 1, *if*

(i) *X has trivial invariant σ-field, for each μ then*

(iii) *there exists a successful shift-coupling of X and X', for each pair μ, μ'.* $\square$

(Again, it is routine that (iii) implies (i).)

So Theorem 15 leads to a third proof of the basic result on existence of successful shift-couplings (the others being Greven's Theorem 4 and our Section 3 proof).

Remarks on inhomogeneous chains. The proofs of Corollaries 13 and 17 are unchanged for inhomogeneous chains (note that the argument in Section 3 does not seem to extend so easily). However, a simple example shows that a small change in the 'routine' converses is required.

Example. Consider a Markov chain X on $E=\{0,1\}$ such that X_0 has an arbitrary distribution, X_1 takes the values 0 and 1 with equal probabilities and $X_n\equiv X_1$ for $n\geqslant 2$. Then $T\equiv 1$ is a coupling epoch but $\mathscr{T}$ is not trivial. Also $T'\equiv T\equiv 1$ are shift-coupling epochs while $\mathscr{I}$ is not trivial, because the event $\{X$ is absorbed in $1\}$ is invariant and has probability $\frac{1}{2}$ for all initial distributions.

The example indicates that to get the precise relationship between σ-fields and couplings in the inhomogeneous case, we want to consider couplings of processes started at arbitrary times.

Corollary 18. *Let $(P_n;\, 0 \le n < \infty)$ be a sequence of Markov kernels on a Polish space E. Let $X = (X_n;\, n_0 \le n < \infty)$ and $X' = (X'_n;\, n'_0 \le n < \infty)$ be Markov chains with transition kernels P_n and initial distributions μ and μ'. The following are equivalent.*

(i) *X has trivial invariant σ-field, for each n_0 and μ.*

(iii) *There exists a successful shift-coupling of X and X', for each n_0, μ, n'_0, μ'.* □

Replacing 'invariant' by 'tail' and 'shift-coupling' by 'coupling' gives the inhomogeneous version of Corollary 13.

6. Proof of Theorem 15

The proof of Theorem 15 given below relies on the existence of maximal couplings (Theorem 2). We shall recursively maximal-couple Z and Z' shifted relative to each other in all possible ways. Having constructed a candidate in this way we check in Lemma 19 that it is well-defined, show in Lemma 20 that it is a shift-coupling and finally in Lemma 21 establish $\mathscr{I}$-maximality.

Let $n_1, n_2, \ldots$ be an enumeration of the integers and $n_\infty = \infty$. Write $n^+ = \max\{0, n\}$ and $n^- = \max\{0, -n\}$. For $x = (x_0, x_1, \ldots) \in E^\infty$ denote

$$\Delta x = (\Delta, x_0, x_1, \ldots), \qquad \Delta_0 x = x \quad \text{and} \quad \Delta_n x = \Delta\Delta_{n-1} x, \quad 1 \le n < \infty.$$

Recursively, define triples

$$(Z(k), Z'(k), T(k)), \quad 1 \le k \le \infty,$$

which are independent as k varies, and a random variable K in the following three steps:

Step 1. Let $Z(1), Z'(1)$ and $T(1)$ be such that $\Delta_{n_1^+} Z(1)$ and $\Delta_{n_1^-} Z'(1)$ is a maximal coupling of $\Delta_{n_1^+} Z$ and $\Delta_{n_1^-} Z'$ with coupling epoch $T(1) + n_1^+$.

Step 2. For $2 \le k < \infty$, let $Z(k), Z'(k)$ and $T(k)$ be such that $\Delta_{n_k^+} Z(k)$ and $\Delta_{n_k^-} Z'(k)$ is a maximal coupling of processes with the distributions

$$\boldsymbol{P}(\Delta_{n_k^+} Z(k-1) \in \cdot \mid T(k-1) = \infty) \tag{9}$$

and

$$\boldsymbol{P}(\Delta_{n_k^-} Z'(k-1) \in \cdot \mid T(k-1) = \infty), \tag{10}$$

respectively, (if $\boldsymbol{P}(T(k-1) = \infty) = 0$ pick the conditional distributions in some arbitrary way) with coupling epoch $T(k) + n_k^+$.

Step 3. Put

$$K = \inf\{1 \le k < \infty\colon T(k) < \infty\} \quad (\text{interpreting } \inf\emptyset = \infty),$$

and let $Z(\infty)$ and $Z'(\infty)$ have distributions

$$P(Z(\infty)\in\cdot\,)=(P(Z\in\cdot\,)-P(Z(K)\in\cdot\,, K<\infty))/P(K=\infty) \tag{11}$$

and

$$P(Z'(\infty)\in\cdot\,)=(P(Z'\in\cdot\,)-P(Z(K)\in\cdot\,, K<\infty))/P(K=\infty), \tag{12}$$

respectively, (if $P(K=\infty)=0$ pick the distributions in some arbitrary way) and put $T(\infty)=\infty$.

Having thus defined $(Z(k), Z'(k), T(k))$, $1\leq k\leq\infty$, and K let

$$\hat{Z}=Z(K),\quad \hat{Z}'=Z'(K),\quad T=T(K)\quad\text{and}\quad T'=T+n_K$$

be our candidate for an $\mathcal{I}$-maximal shift-coupling.

Lemma 19. *The candidate is well-defined.*

Proof. We must show that the distributions in (11, 12) are well-defined, i.e., that

$$P(Z\in\cdot\,)\geq P(Z(K)\in\cdot\,, K<\infty) \tag{13}$$

and

$$P(Z'\in\cdot\,)\geq P(Z'(K)\in\cdot\,, K<\infty). \tag{14}$$

In order to establish (13, 14), note that (9) implies

$$P(Z(k)\in\cdot\,)=P(Z(k-1)\in\cdot\mid T(k-1)=\infty)$$

and thus the second identity in

$$\begin{aligned}P(Z(n)\in\cdot\,, K>n)&=P(Z(n)\in\cdot\mid T(n)=\infty)P(K>n)\\&=P(Z(n+1)\in\cdot\,)P(K>n)\\&=P(Z(n+1)\in\cdot\,, K>n),\end{aligned}$$

which yields inductively

$$\begin{aligned}P(Z\in\cdot\,)&=P(Z(1)\in\cdot\,, K=1)+P(Z(1)\in\cdot\,, K>1)\\&=\cdots=\sum_{k=1}^{n}P(Z(k)\in\cdot\,, K=k)+P(Z(n)\in\cdot\,, K>n).\end{aligned} \tag{15}$$

Thus $P(Z\in\cdot\,)\geq P(Z(K)\in\cdot\,, K\leq n)$ and sending $n\to\infty$ yields (13). Similarly (10) yields (14). Hence our candidate is well-defined. □

Lemma 20. *The candidate is a shift-coupling.*

Proof. Note that K is defined in terms of $T(k)$, $0\leq k<\infty$ and thus is independent of both $Z(\infty)$ and $Z'(\infty)$. Hence (11) yields

$$P(Z(\infty)\in\cdot\,, K=\infty)=P(Z(\infty)\in\cdot\,)-P(Z(K)\in\cdot\,, K<\infty),$$

which implies that $Z(K)$ has the same distribution as Z. Similarly (12) implies that $Z'(K)$ has the same distribution as Z'. Moreover,

$$\theta_{T(k)}Z(k) = \theta_{T(k)+n_k}Z'(k), \quad 1 \le k < \infty,$$

which yields $\theta_T \hat{Z} = \theta_{T'}\hat{Z}$. Hence our candidate is a shift-coupling. □

Lemma 21. *The candidate is $\mathscr{I}$-maximal.*

Proof. Note that $\theta_{n_k^-}Z(k)$ and $\theta_{n_k^+}Z'(k)$ with coupling epoch $T(k) - n_k^-$ is a *maximal coupling*. Due to Lemma 10 and Lemma 8, this means that $\boldsymbol{P}(Z(k) \in \theta_{-n_k^-}\cdot, T(k) = \infty)_{\mathscr{T}}$ and $\boldsymbol{P}(Z'(k) \in \theta_{-n_k^+}\cdot, T(k) = \infty)_{\mathscr{T}}$ are mutually singular; here $\theta_{-n} = \theta_n^{-1}$ for $n > 0$. Since, for $-\infty < n < \infty$, θ_n is one-to-one as a set map from $\mathscr{T}$ to $\mathscr{T}$ and since $\theta_{n^-}\theta_{-n^+} = \theta_{-n}$, this implies that

$$\boldsymbol{P}(Z(k) \in \cdot, T(k) = \infty)_{\mathscr{T}} \text{ and } \boldsymbol{P}(Z'(k) \in \theta_{-n_k}\cdot, T(k) = \infty)_{\mathscr{T}}$$
$$\text{are mutually singular.} \tag{16}$$

From $\boldsymbol{P}(\hat{Z} \in \cdot) = \boldsymbol{P}(Z \in \cdot)$ and (15) we obtain the identity in

$$\begin{aligned}\boldsymbol{P}(\hat{Z} \in \cdot, T = \infty) &\le \boldsymbol{P}(Z(K) \in \cdot, K > k)\\ &= \boldsymbol{P}(Z(k) \in \cdot, K > k)\\ &\le \boldsymbol{P}(Z(k) \in \cdot, T(k) = \infty),\end{aligned}$$

and similarly we have $\boldsymbol{P}(\hat{Z}' \in \cdot, T = \infty) \le \boldsymbol{P}(Z'(k) \in \cdot, T(k) = \infty)$. Combining this and (16) shows that $\boldsymbol{P}(\hat{Z} \in \cdot, T = \infty)_{\mathscr{T}}$ and $\boldsymbol{P}(\hat{Z}' \in \theta_{-n_k}\cdot, T = \infty)_{\mathscr{T}}$ are mutually singular. Since $n_1, n_2, \ldots$ is an enumeration of the integers this means that

$$\forall -\infty < n < \infty \quad \exists B_n \in \mathscr{T}:$$
$$\boldsymbol{P}(\hat{Z} \in B_n, T = \infty) = 0 \text{ and } \boldsymbol{P}(\hat{Z}' \in \theta_n B_n^c, T = \infty) = 0.$$

Put $B = \bigcup_{-\infty<n<\infty} B_n$ and note that $B^c \subseteq B_n^c$ for $-\infty < n < \infty$. Thus

$$\exists B \in \mathscr{T} \quad \forall -\infty < n < \infty:$$
$$\boldsymbol{P}(\hat{Z} \in B, T = \infty) = 0 \text{ and } \boldsymbol{P}(\hat{Z}' \in \theta_n B^c, T = \infty) = 0. \tag{17}$$

Putting $A = \bigcap_{-\infty<n<\infty} \theta_n B$ we have $A \in \mathscr{I}$, $A \subseteq B$ and $A^c = \bigcup_{-\infty<n<\infty} \theta_n B^c$. Thus

$$\exists A \in \mathscr{I}: \quad \boldsymbol{P}(\hat{Z} \in A, T = \infty) = 0 \text{ and } \boldsymbol{P}(\hat{Z}' \in A^c, T = \infty) = 0,$$

i.e., our candidate is $\mathscr{I}$-maximal and the proof of Theorem 15 is complete. □

Remark. Let us show how Theorem 15 can be deduced from Greven's result, Theorem 4, in the non-Markovian form of Corollary 22 below. That corollary implies there exists $C \in \mathscr{E}^\infty$ such that for all $k \ge 0$ and $-\infty < n \le k$ we have

$$\boldsymbol{P}(\hat{Z} \in \theta_{-k}C, T = \infty) = 0 \text{ and } \boldsymbol{P}(\hat{Z}' \in \theta_n\theta_{-k}C^c, T = \infty) = 0.$$

Then $B = \limsup_{k\to\infty} \theta_{-k}C$ satisfies $B \in \mathscr{T}$ and $B^c = \liminf_{k\to\infty} \theta_{-k}C^c$, establishing (17). Continue from there to obtain Theorem 15.

7. Remarks on maximality

Although Greven (1987a) stated Theorem 4 for Markov chains, it can of course be applied to non-Markov processes Z by considering the Markov chain $(\theta_n Z;\ 0 \leqslant n < \infty)$, giving the following corollary.

Corollary 22. *Let $Z = (Z_n\colon 0 \leqslant n < \infty)$ and $Z' = (Z'_n\colon 0 \leqslant n < \infty)$ be arbitrary stochastic processes taking values in a Polish space E. Then there exists a shift-coupling such that*

$$\sum_{0 \leqslant n < \infty} \boldsymbol{P}(\theta_n \hat{Z} \in \cdot\,, T > n) \quad \textit{and} \quad \sum_{0 \leqslant n < \infty} \boldsymbol{P}(\theta_n \hat{Z}' \in \cdot\,, T' > n)$$

$$\textit{are mutually singular.} \qquad \square \tag{18}$$

In Section 6 we noted that the maximality property (18) implies our $\mathcal{I}$-maximality. Indeed it is a stronger property, since if shift-coupling epochs (T, T') satisfy (18) then typically $(T+1, T'+1)$ will not satisfy (18) but will still be $\mathcal{I}$-maximal. Loosely, $\mathcal{I}$-maximality is only 'maximality at infinity'.

To understand (18), note that a coupling is maximal iff $\boldsymbol{P}(\theta_n \hat{Z} \in \cdot\,, T > n)$ and $\boldsymbol{P}(\theta_n \hat{Z}' \in \cdot\,, T' > n)$ are mutually singular for all $n \geqslant 0$. So (18) may seem a natural extension of the latter property to shift-coupling. But a simple example convinces one that (18) is not strong enough to qualify as an intuitively correct notion of maximal shift-coupling. For the remainder of this section we concentrate on Markov chains, so (18) reduces to (i) of Theorem 4, that is

$$\eta(\cdot) \equiv \sum_{n=0}^{\infty} \boldsymbol{P}(X_n \in \cdot\,, T > n) \quad \text{and} \quad \eta'(\cdot) \equiv \sum_{n=0}^{\infty} \boldsymbol{P}(X'_n \in \cdot\,, T' > n)$$

$$\text{are mutually singular.} \tag{19}$$

Example. Let X be an irreducible recurrent Markov chain starting in a fixed state x. Let X' be the same Markov chain starting at some state $y \neq x$. Then $(T_y, 0)$ are shift-coupling epochs, where T_y is the first hitting time of X on y. And so are $(0, T'_x)$ and even $(T_y^{(r)}, 0)$, where $T_y^{(r)}$ is the rth hitting time of X on y. Each of these shift-couplings satisfies the 'maximality' property (19). But by considering the latter example, we see that (19) is not even strong enough to preclude the existence of stochastically smaller shift-coupling epochs.

We remarked earlier that Greven (1987a) proved a uniqueness property for (19) in the transient case, and Greven (1987b) discussed the existence of shift-couplings with nice tail behavior for T and T' in the null-recurrent case. See also Harison and Smirnov (1990). These papers give examples and discussion somewhat similar to ours here, but don't focus on the finite-state case. As illustrated in the example above, in seeking the intuitively correct notion of maximality it is the finite irreducible case which seems the hardest to understand!

A natural hope is that one could consider a quantity such as

$$\max(T, T'), \quad T + T', \quad \text{or} \quad \min(T, T'),$$

and prove that there exist shift-coupling epochs for which this quantity is stochastically smaller than the same quantity for all other shift-coupling epochs. The previous example shows that 'min' is unsatisfactory. The next example shows that with 'max' or 'sum' there may be no stochastically smallest shift-coupling.

Example. Consider the chain which cycles deterministically through 5 states $i_1 \to i_2 \to i_3 \to i_4 \to i_5 \to i_1 \cdots$. Let μ be uniform on $\{i_1, i_2\}$ and let μ' be uniform on $\{i_2, i_3\}$. Plainly there is a successful shift-coupling with epochs $(1, 0)$ and another with epochs $(2I, 0)$, where I is uniform on $\{0, 1\}$. But there is none with epochs $(S, 0)$ such that S is stochastically smaller than $\min(1, 2I)$.

On a more technical note, one might expect the construction in Section 6 to give a shift-coupling with some optimality property suggested by the construction itself. Suppose we take the enumeration in Section 6 to be $0, -1, 1, -2, 2, \ldots$. Then we might expect our shift-coupling to be maximal in the sense of minimizing T subject to minimizing the absolute shift $|T - T'|$. But the example above shows this is incorrect.

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