

On regression representations of stochastic processes

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We construct a.s. nonlinear regression representations of general stochastic processes $(X_n)_{n \in \mathbb{N}}$. As a consequence we obtain in particular special regression representations of Markov chains and of certain m -dependent sequences. For m -dependent sequences we obtain a constructive method to check, whether these sequences have a monotone $(m+1)$ -block factor representation.

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representation as function of i.i.d. sequences * generalized two-block factor * m -dependence * Markov regression * Markov chain

1. Markov regression and standard representation

Let $X = (X_n)_{n \in \mathbb{N}}$ be a stochastic, real-valued process. The aim of this section is to construct two types of a.s. regression representations of X by an i.i.d. sequence (U_n) . One representation is of the form $X_n = f_n(X_1, \dots, X_{n-1}, U_n)$ a.s.; we call this representation ‘*Markov regression*’ (on X). A second representation is of the form $X_n = f_n(U_1, \dots, U_n)$ a.s.; we call this regression representation ‘*standard representation*’ (on U). These constructions are the counterpart for autoregressive representations in time series analysis. Here we obtain a nonlinear representation of X_n of the past and of innovations U_n (which are independent and not only orthogonal).

We need a technical proposition about quantile transformations to construct standard representations. We write λ for the Lebesgue measure and $F_{-}(t) := \lim_{s \uparrow t} F(s)$.

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Proposition 1 (Ferguson, [3, Lemma 1, p. 216]). *Let X be a real random variable with distribution function F and let U be independent of X , $R(0, 1)$ -distributed ($R(0, 1)$ is the uniform distribution over the interval $(0, 1)$). Define for $\alpha \in (0, 1)$,*

$$\tilde{F}(x, \alpha) := P[X < x] + \alpha P[X = x]. \quad (1)$$

Then

$$\tilde{F}(X, U) \stackrel{d}{=} R(0, 1) \quad (\stackrel{d}{=} \text{ is equality in distribution}), \quad (2)$$

$$F^{-1}(U) \stackrel{d}{=} X \quad (F^{-1}(t) := \inf\{s: F(s) \geq t\}) \quad (3)$$

and

$$X = F^{-1}(\tilde{F}(X, U)) \quad \text{a.s.} \quad (4)$$

Since a proof of this result seems to be not easy accessible in the literature, we provide a proof of this well-known result.

Proof. Let $D \subset \mathbb{R}$ denote the set of discontinuities of F , then

$$\begin{aligned} & P[\tilde{F}(X, U) \in A] \\ &= P[\tilde{F}(X, U) \in A, X \in D] \\ &\quad + P[\tilde{F}(X, U) \in A, X \in D^c] P[\tilde{F}(X, U) \in A, X \in D] \\ &= \sum_{x \in D} P[\tilde{F}(x, U) \in A] P[X = x] \\ &= \sum_{x \in D} P[F_-(x) + U(F(x) - F_-(x)) \in A] (F(x) - F_-(x)) \\ &= \sum_{x \in D} \frac{\lambda(A \cap (F_-(x), F(x)])}{F(x) - F_-(x)} (F(x) - F_-(x)) \\ &= \sum_{x \in D} \lambda(A \cap (F_-(x), F(x)]) = \lambda(A \cap \bar{D}), \end{aligned}$$

where $\bar{D} := \bigcup_{x \in D} (F_-(x), F(x)]$. Further (D^c is the complement of D)

$$P[\tilde{F}(X, U) \in A, X \in D^c] = P[F(X) \in A, X \in D^c] = \lambda(A \cap \bar{D}^c).$$

In the proof we used that U and $\{X = x\}$ are independent for all $x \in D$. We conclude that

$$P[\tilde{F}(X, U) \in A] = \lambda(A \cap \bar{D}) + \lambda(A \cap \bar{D}^c) = \lambda(A)$$

and this proves (2).

From the definition of the pseudo-inverse follows

$$P(F^{-1}(U) \leq t) = P(U \leq F(t)) = F(t) = P(X \leq t)$$

which proves (3) and

$$\{F^{-1}(\tilde{F}(X, U)) \leq t\} = \{\tilde{F}(X, U) \leq F(t)\} = \{X \leq t\} \quad \text{a.s.}$$

which proves (4). \square

The a.s. representation in (4) has some useful applications in stochastic ordering [11]. If F is continuous, then $\tilde{F}(X, U) = F(X)$.

We next consider the multivariate generalization of Proposition 1. Let $X = (X_1, \dots, X_n)$ be a random vector in \mathbb{R}^n and let $F_1, F_{2|1}, \dots, F_{n|1, \dots, n-1}$ denote the first marginal distribution function respectively the conditional distribution function of X_k given X_1, \dots, X_{k-1} . Let V_1, \dots, V_n be i.i.d. $R(0, 1)$ -distributed random variables and define the multivariate quantile transform

$$\begin{aligned} Y_1 &:= F_1^{-1}(V_1), \\ Y_k &:= F_{k|1, \dots, k-1}^{-1}(V_k | Y_1, \dots, Y_{k-1}), \quad 2 \leq k \leq n. \end{aligned} \quad (5)$$

For this transformation see [8, 9, 10] and [11]. Note that $Y = (Y_1, \dots, Y_n)$ is of the form $f(V)$ with $V = (V_1, \dots, V_n)$, where the i th component $f_i(V) = f_i(V_1, \dots, V_i)$.

Proposition 2. (a) $X \stackrel{d}{=} Y$.

(b) There exists an i.i.d. $R(0, 1)$ -sequence $U = (U_i)_{1 \leq i \leq n}$ such that

$$X = f(U) \quad \text{a.s., as defined above.} \quad (6)$$

Proof. (a) The proof of (a) in the case $n = 2$ is as follows:

$$\begin{aligned} P(Y_1 \leq a, Y_2 \leq b) &= P(Y_1 \leq a, V_2 \leq F_{2|1}(b | Y_1)) \\ &= \int_{-\infty}^a P(V_2 \leq F_{2|1}(b | t)) dF_1(t) \\ &= \int_{-\infty}^a F_{2|1}(b | t) dF_1(t) \\ &= P(X_2 \leq b, X_1 \leq a). \end{aligned}$$

The general case follows by induction.

(b) Since $f(V) \stackrel{d}{=} X$ we obtain from Proposition 1 in [7] the existence of a measure preserving transformation $\varphi: (\Omega, \Sigma) \rightarrow (\Omega, \Sigma)$ such that

$$X = f(U) \quad \text{a.s.} \quad (7)$$

where $U_i = V_i \circ \varphi$, $1 \leq i \leq n$, are again i.i.d. $R(0, 1)$ -distributed random variables. \square

Skorohod (1976) proved for random variables X, Y with values in Borel spaces and a given $R(0, 1)$ -distributed random variable V independent of X, Y the existence of a random variable U and measurable functions f, g such that

$$\begin{aligned} X &= f(Y, U) \quad \text{a.s.,} \\ U &= g(X, Y, V) \quad \text{is independent of } Y. \end{aligned} \quad (8)$$

The following theorem extends this result to stochastic processes. Furthermore, in the case of real stochastic processes we obtain an explicit representation.

Let $X = (X_1, X_2, \dots)$ be a real valued stochastic process and let $V = (V_1, V_2, \dots)$ be an i.i.d. sequence of $R(0, 1)$ -distributed random variables, V independent of X . Define

$$\begin{aligned} U_1 &:= \tilde{F}_1(X_1, V_1) \quad (\tilde{F}_1 \text{ as in (1), } F_1 \text{ distribution function of } X_1), \\ Z_1 &:= F_1^{-1}(U_1), \end{aligned} \quad (9)$$

and let for $k \geq 2$,

$$\begin{aligned} \tilde{F}_{k|1, \dots, k-1}(x, v | z_1, \dots, z_{k-1}) &:= P(X_k < x | Z_1 = z_1, \dots, Z_{k-1} = z_{k-1}) \\ &\quad + vP(X_k = x | Z_1 = z_1, \dots, Z_{k-1} = z_{k-1}), \\ U_k &:= \tilde{F}_{k|1, \dots, k-1}(X_k, V_k | Z_1, \dots, Z_{k-1}), \\ Z_k &:= F_{k|1, \dots, k-1}^{-1}(U_k | Z_1, \dots, Z_{k-1}), \end{aligned} \quad (10)$$

where $F_{k|1, \dots, k-1}$ is the conditional distribution function of X_k given X_1, \dots, X_{k-1} .

Theorem 3. Let $Z = (Z_1, Z_2, \dots)$ then:

- (a) $Z = X$ a.s.
- (b) $U = (U_1, U_2, \dots)$ is an i.i.d. $R(0, 1)$ -distributed random sequence.
- (c) U_k and (X_1, \dots, X_{k-1}) are independent.

We call the representation $X_1 = f_1(U_1)$, $X_k = f_k(X_1, \dots, X_{k-1}, U_k)$ in (9), (10), *Markov-regression representation* of X .

Proof. The equality $Z_1 = X_1$ follows from (4). We continue by induction on k . Assume that $(Z_1, \dots, Z_k) = (X_1, \dots, X_k)$ a.s. Since $P^{(U_{k+1} | Z_1 = z_1, \dots, Z_k = z_k)}$ is $R(0, 1)$ -distributed for all z_1, \dots, z_k we have that U_{k+1} and $(Z_1, \dots, Z_k) = (X_1, \dots, X_k)$ a.s. are independent.

From

$$\begin{aligned} \{Z_{k+1} \leq t\} &= \{F_{k+1|1, \dots, k}^{-1}(U_{k+1} | Z_1, \dots, Z_k) \leq t\} \\ &= \{U_{k+1} \leq F_{k+1|1, \dots, k}(t | Z_1, \dots, Z_k)\} \\ &= \{\tilde{F}_{k+1|1, \dots, k}(X_{k+1}, V_{k+1} | Z_1, \dots, Z_k) \leq F_{k+1|1, \dots, k}(t | Z_1, \dots, Z_k)\} \\ &= \{X_{k+1} \leq t\} \quad \text{a.s.} \end{aligned}$$

we conclude that $X_{k+1} = Z_{k+1}$ a.s. Because U_{k+1} and (X_1, \dots, X_k) are independent, we have that U_{k+1} and U_1, \dots, U_k (functions of $X_1, \dots, X_k, V_1, \dots, V_k$) are independent. \square

The existence of a Markov regression representation for processes with values in Borel spaces is immediate from Theorem 3 (but is nonconstructive).

In the case that $(X_n)_{n \in \mathbb{N}}$ is an m -Markov chain (for some $m \in \mathbb{N}$), i.e. the conditional distribution of X_{n+m+1} given the past $\{X_1, \dots, X_{m+n}\}$ only depends on $\{X_{n+1}, \dots, X_{n+m}\}$ the Markov regression representation in Theorem 3 specializes to:

Corollary 4. *Let $X = (X_n)$ be an m -Markov chain. Then there exists an i.i.d. sequence $U = (U_1, U_2, \dots)$ of $R(0, 1)$ -distributed random variables and a sequence of measurable functions (f_n) such that*

$$\begin{aligned} X_n &= f_n(X_{n-m}, \dots, X_{n-1}, U_n) \quad \text{a.s.} \quad (n \geq m+1), \\ U_n &\text{ independent of } (X_1, \dots, X_{n-1}). \quad \square \end{aligned} \quad (11)$$

For the case of a Markov chain ($m = 1$), see [6, p. 155]. By Theorem 3 the method of pathwise constructions of stochastic models is equivalent to constructions in distribution. One can characterize further distributional properties as in Corollary 4. E.g. if (X_n) is a Markov chain and a martingale, then X_n has a representation $X_n = f_n(X_{n-1}, U_n)$ with $\int_0^1 f_n(x, u) du = x$ for all x .

The following alternative construction of a standardization sequence $U = (U_1, U_2, \dots)$ of $X = (X_1, X_2, \dots)$ will be of interest in connection with m -dependent sequences. This i.i.d. sequence U is a.s. equal to the sequence U in Theorem 3. We will explain this in Remark 16.

Let $V = (V_1, V_2, \dots)$ be an i.i.d. $R(0, 1)$ -distributed sequence independent of $X = (X_1, X_2, \dots)$. Let G_1 be the distribution function of X_1 and define

$$\begin{aligned} U_1 &:= \tilde{G}_1(X_1, V_1), \\ U_k &:= \tilde{G}_{k|1, \dots, k-1}(X_k, V_k | U_1, \dots, U_{k-1}) \quad (k \geq 2), \end{aligned} \quad (12)$$

where $G_{k|1, \dots, k-1}$ is the conditional distribution function of X_k given (U_1, \dots, U_{k-1}) . The functions \tilde{G} are associated to G as in the proof of Theorem 3. Similarly to the proof of Theorem 3 we obtain:

Theorem 5. (a) (U_k) is an i.i.d. $R(0, 1)$ -sequence.

$$\begin{aligned} \text{(b)} \quad X_1 &= G_1^{-1}(U_1), \\ X_k &= G_{k|1, \dots, k-1}^{-1}(U_k | U_1, \dots, U_{k-1}). \quad \square \end{aligned} \quad (13)$$

We call the representation in (13) the *standard representation* of X .

If for some $m \in \mathbb{N}$,

$$G_{k+m+1|1, \dots, k+m}(t_{k+m+1} | t_1, \dots, t_{k+m}) = g_{k+m+1}(t_{k+1}, \dots, t_{k+m+1}), \quad (14)$$

i.e. the conditional distribution of X_{k+m+1} given U_1, \dots, U_{k+m} depends only on U_{k+1}, \dots, U_{k+m} , we say that X has *m -Markov regression* on U .

Corollary 6. *If X has m -Markov regression on U , then X is a generalized $(m+1)$ -block factor, i.e. (X_n) has the representation*

$$X_n = f_n(U_{n-m}, \dots, U_{n-1}, U_n) \quad \text{a.s.}, \quad n \geq m+1. \quad \square \quad (15)$$

An interesting problem in probability theory is to find simple sufficient conditions for the existence of an $(m+1)$ -block factor representation as in (15) (cf. [13]).

2. Markov chains and m -dependence

A process (X_n) is called m -dependent ($m \in \mathbb{N}$) if $(X_n)_{n < t}$ and $(X_n)_{n \geq t+m}$ are independent for all $t \in \mathbb{N}$. It is trivial that a *generalized* $(m+1)$ -block factor $(X_n) = (f_n(U_n, U_{n+1}, \dots, U_{n+m}))$ a.s. of an i.i.d. sequence (U_n) is m -dependent.

For quite a time it was conjectured that every stationary m -dependent process has a representation as $(m+1)$ -block factor $(f(U_n, \dots, U_{n+m}))$ (here f_n is independent of n !). In [2] a two-parameter family of counterexamples is given of stationary one-dependent processes, assuming only two values, which do not have a two-block factor representation $(f(U_n, U_{n+1}))$ of an i.i.d. sequence (U_n) . It was shown in [4] that certain extremal 0-1 valued one-dependent stationary processes have a two-block factor representation while in [1] it was shown that a stationary one-dependent Markov chain with not more than four states has a two-block factor representation. There is a counterexample for five states.

In addition to the results on Markov chains in [1] it is proved that one-dependent renewal processes are two-block factors. It will be shown next that a symmetry condition implies that one-dependent Markov chains are already independent.

Proposition 7. *Let (X_n) be a stationary, one-dependent 0-1 valued Markov chain. Then (X_n) is an i.i.d. sequence.*

Proof. We use the short notation

$$[a_1 \cdots a_n] := P[X_1 = a_1, \dots, X_n = a_n].$$

From $[0] = [00] + [01] = [00] + [10]$ follows that $[01] = [10]$. In our formulas we use the convention $0/0 = 0$. By the stationarity, the one-dependence and the Markov property we have

$$[a_k]^2 = \sum_i [a_k a_i a_k] = \sum_i \frac{[a_k a_i a_k]}{[a_k a_i]} [a_k a_i] = \sum_i \frac{[a_i a_k]}{[a_i]} [a_k a_i] = \sum_i \frac{[a_k a_i]^2}{[a_i]}.$$

Thus we obtain

$$\begin{aligned} 0 &\leq \sum_i \left\{ \frac{[a_k a_i]}{\sqrt{[a_i]}} - [a_k] \sqrt{[a_i]} \right\}^2 \\ &= \sum_i \left\{ \frac{[a_k a_i]^2}{[a_i]} - 2[a_k][a_k a_i] + [a_k]^2 [a_i] \right\} \\ &= [a_k]^2 - 2[a_k]^2 + [a_k]^2 = 0. \end{aligned}$$

This implies $[a_k a_i]/\sqrt{[a_i]} = [a_k]\sqrt{[a_i]}$ for all $a_i a_k$ which is equivalent to $[a_k a_i] = [a_k][a_i]$. Combined with the Markov property this implies independence. \square

Remark 8. From the proof it follows that the statement of the proposition also holds for one-dependent Markov chains with countable state space under the condition

$$[a_1 a_2] = [a_2 a_1] \quad \text{for all } a_1, a_2.$$

For any two-valued stationary one-dependent process we have a much stronger reversibility property:

Proposition 9. *Let $(X_n)_n$ be a stationary one-dependent 0-1 valued process. Then*

$$[a_1 \cdots a_n] = [a_n \cdots a_1] \quad \text{for all } n \text{ and all } a_1, \dots, a_n \in \{0, 1\}.$$

Proof. For $n = 2$ the statement follows from $[0] = [00] + [01] = [00] + [10]$, hence $[01] = [10]$ as in the proof of Proposition 7. We use induction on n . We write

$$[1^m] := [\underbrace{1 \cdots 1}_{m \text{ times}}].$$

Assume that the statement holds for n , then for $n+1$ we denote the number of zeroes in $w = a_1 \cdots a_n a_{n+1}$ by $n_0(w)$. We continue by induction on $n_0(w)$.

If $n_0(w) = 0$ then the statement is trivial. Assume that the statement holds for $n_0 \leq k$.

If $n_0(w) = k+1 > 0$ then $w = 1^m 0 v$ for some $m \geq 0$. Then

$$\begin{aligned} [a_1 \cdots a_n a_{n+1}] &= [1^m 0 v] = [1^m][v] - [1^{m+1}v] \\ &= [1^m][a_{m+2} \cdots a_{n+1}] - [1^{m+1}a_{m+2} \cdots a_{n+1}] \\ &= [a_{n+1} \cdots a_{m+2}][1^m] - [a_{n+1} \cdots a_{m+2}1^{m+1}] \\ &= [a_{n+1} \cdots a_{m+2}01^m] = [a_{n+1}a_n \cdots a_1] \end{aligned}$$

which proves the proposition. \square

The statement of Proposition 9 does not hold for one-dependent processes that assume three or more values. If the condition $[a_1 a_2] = [a_2 a_1]$ for all a_1, a_2 does not hold, then the statement of Proposition 7 is no longer valid as the following example shows.

Example 1. Let $(U_n)_n$ be a Bernoulli sequence with $P[U_n = 1] = p = 1 - P[U_n = 0]$ for some $p \in (0, 1)$. Define the two-block factor $(X_n)_n$ by

$$X_n = 2U_n + U_{n+1}.$$

It is easily checked that $(X_n)_n$ is a one-dependent Markov chain with state space $\{0, 1, 2, 3\}$ and transition matrix

$$\begin{pmatrix} 1-p & p & 0 & 0 \\ 0 & 0 & 1-p & p \\ 1-p & p & 0 & 0 \\ 0 & 0 & 1-p & p \end{pmatrix}$$

and apparently $(X_n)_n$ is not an i.i.d. sequence.

Under a symmetry condition we prove a general version of Proposition 7.

Proposition 10. *Let $(X_n)_n$ be a stationary, one-dependent real Markov chain and assume that*

$$P^{(X_1, X_2)} = P^{(X_2, X_1)}. \quad (16)$$

Then $(X_n)_n$ is an i.i.d. sequence.

Proof. Let $f: \mathbb{R} \rightarrow (0, 1)$ be one to one measurable, then $Y_n := f(X_n)$ also is a one-dependent Markov chain and $(\mathbb{E} Y_1)^2 = \mathbb{E}(Y_1 Y_3) = \mathbb{E}[\mathbb{E}(Y_1 Y_3 | Y_2)] = \mathbb{E}[\mathbb{E}(Y_1 | Y_2) \mathbb{E}(Y_3 | Y_2)]$. Since $\mathbb{E}[Y_1 | Y_2] = g(Y_2)$ for a measurable g we can continue by using the stationarity and (16), $\mathbb{E}(g(Y_2) \mathbb{E}[Y_3 | Y_2]) = \mathbb{E}g(Y_2) Y_3 = \mathbb{E}g(Y_1) Y_2 = \mathbb{E}g(Y_2) Y_1 = \mathbb{E}(g(Y_2) \mathbb{E}[Y_1 | Y_2]) = \mathbb{E}Z^2$, where $Z = \mathbb{E}[Y_1 | Y_2]$. Therefore $\mathbb{E}Z = \mathbb{E}Y_1$ and $(\mathbb{E}Z)^2 = \mathbb{E}(Z^2)$ imply that $Z = \mathbb{E}Z$ a.s., i.e.

$$\mathbb{E}(f(X_1) | f(X_2) = t) = \mathbb{E}f(X_1) \quad [P^{f(X_2)} \text{ a.s.}]$$

equivalently

$$\mathbb{E}(f(X_1) | X_2 = f^{-1}(t)) = \mathbb{E}f(X_1) \quad [P^{X_2} \text{ a.s.}].$$

Since this holds for all f we obtain independence. \square

We leave it as an exercise to the reader to prove that the assumption $P^{(X_1, X_2)} = P^{(X_2, X_1)}$ is equivalent to reversibility of the Markov chain, i.e. $P^{(X_1, \dots, X_n)} = P^{(X_n, \dots, X_1)}$ for all n . Of course X_n could take also values in a Borel space. By a modification of the constructions in section one we next show that one-dependent Markov chains have a three-block factor representation.

Theorem 11. *Let $(X_n)_n$ be a real Markov chain. Then there exists an $R(0, 1)$ -sequence $(U_n)_n$ and a sequence of functions g_n such that U_n is independent of $X_1, \dots, X_{n-1}, X_{n+1}, \dots$ and*

$$X_n = g_n(U_n, X_{n-1}, X_{n+1}).$$

If $(X_n)_n$ is additionally one-dependent, then there exists an independent sequence $(Y_n)_n$ and a sequence of functions $(f_n)_n$ such that X_n is a three-block factor of $(Y_n)_n$,

$$X_n = f_n(Y_{n-2}, Y_{n-1}, Y_n).$$

Proof. Let F_1 be the distribution function of X_1 and let $F_{n|n-1, n+1}$ ($n \geq 2$) be the conditional distribution function of X_n given X_{n-1}, X_{n+1} . Define $U_1 := \tilde{F}_1(X_1, V_1)$ and ($n \geq 2$) $U_n := \tilde{F}_{n|n-1, n+1}((X_n, V_n) | X_{n-1}, X_{n+1})$, where (V_n) is an i.i.d. $R(0, 1)$ -sequence independent of $(X_n)_n$.

Because $(U_n | X_{n-1} = x_{n-1}, X_{n+1} = x_{n+1})$ is $R(0, 1)$ -distributed for every x_{n-1}, x_{n+1} , the Markov property implies that U_n is independent of $(X_1, \dots, X_{n-1}, X_{n+1}, \dots)$. Analogously to Theorem 3 we have

$$X_n = F_{n|n-1, n+1}^{-1}(U_n | X_{n-1}, X_{n+1}) := g_n(U_n, X_{n-1}, X_{n+1}).$$

Define $Y_1 := X_1$, $Y_n := (X_{2n-1}, U_{2n-2})$ ($n \geq 2$) and we obtain

$$X_1 = f_1(Y_1), \quad X_2 = f_2(Y_1, Y_2), \quad X_3 = f_3(Y_2),$$

$$X_{2n} = f_{2n}(Y_n, Y_{n+1}), \quad X_{2n+1} = f_{2n+1}(Y_{n+1}), \quad n \geq 1.$$

If (X_n) is one-dependent, then $(Y_n)_n$ is an independent sequence. We can make a decent three-block factor out of this sequence by taking some i.i.d. $R(0, 1)$ -sequence $(T_N)_N$ that is independent of X , Y and U . Define the process $(Z_N)_N$ by

$$Z_{2N+1} := T_{N+1}, \quad N \geq 0,$$

$$Z_{2N} := Y_N, \quad N \geq 1.$$

It is trivial that

$$X_N = h_N(Z_N, Z_{N+1}, Z_{N+2})$$

for measurable functions h_N . \square

Remark 12. From the last proof follows that every one-dependent Markov sequence of length 3 is a two-block factor of an i.i.d. sequence.

3. Standard representation and m -dependence

In this section we want to prove a partial converse of Corollary 6, namely if (under some assumptions) (X_n) has an $(m+1)$ -block factor representation, then (X_n) has m -Markov regression on the standard representation U in (12). In this way we obtain a constructive method to check the possibility of an $(m+1)$ -block factor representation for some subclasses of m -dependent sequences. This also justifies the notion of standard representation for (12), (13) and implies that the standardization U in (12) is the right standardization for the $(m+1)$ -block factor representation problem. We shall deal explicitly with the case $m = 1$. We begin with the following example.

Example 2. Let $V = (V_n)_{n \in \mathbb{N}}$ be an i.i.d. $R(0, 1)$ -distributed sequence and define $X_1 = V_1$, $X_n = V_{n-1} + V_n$ ($n \geq 2$). Then $(X_n)_{n \in \mathbb{N}}$ has a two-block factor representation on the standardization $(V_n)_{n \in \mathbb{N}}$. We consider the standardization $(U_n)_n$ of (12). Obviously $U_1 = X_1 = V_1$. Furthermore, $\tilde{G}_{2|1}(x, v | v_1) = P[X_2 \leq x | V_1 = v_1] = P[V_2 \leq x - v_1] = x - v_1$, $v_1 \leq x \leq v_1 + 1$. So $U_2 := \tilde{G}_{2|1}(X_2, V_2 | U_1) = X_2 - V_1 = V_2$. By

induction we obtain in a similar way $U_n = V_n \quad \forall n$, i.e. our standardization (12) produces the right standardization leading to the two-block-factor representation $X_1 = V_1, X_n = V_{n-1} + V_n \quad (n \geq 2)$.

Generalizing this example, we say that $f_1(V_1), f_2(V_1, V_2), f_3(V_2, V_3), \dots$ is a *monotone two-block factor*, if $f_i, f_i(v, \cdot)$ are monotonically nondecreasing for all i, v .

Obviously the standard representation (13) has a monotonicity property as defined here; so this assumption is necessary if the two-block factor representation is identical to the standard representation.

Theorem 13. Assume that $X_1 = f_1(V_1)$ a.s., $X_k = f_k(V_{k-1}, V_k)$ a.s. has a monotone two-block factor representation and assume that all (conditional) distribution functions $G_1, G_{k|1, \dots, k-1}$ in (13) are continuous, then the standardization U in (12) is identical to V and the standard representation (13) gives the two-block factor representation.

Proof. Since $G_1 = \tilde{G}_1$ and $G_{k|1, \dots, k-1} = \tilde{G}_{k|1, \dots, k-1}$ we obtain from (12), (13),

$$U_1 = G_1(X_1),$$

where

$$G_1(x) = P(X_1 \leq x) = P(f_1(V_1) \leq x) = P(V_1 \leq g_1(x)) = (g_1 = f_1^{-1}) = g_1(x)$$

and, therefore,

$$U_1 = g_1 \circ f_1(V_1) = V_1 \quad \text{a.s.}$$

$$U_2 = G_{2|1}(X_2 | V_1),$$

where

$$\begin{aligned} G_{2|1}(x | v_1) &= P(f_2(V_1, V_2) \leq x | V_1 = v_1) \\ &= P(f_2(v_1, V_2) \leq x) = P(V_2 \leq g_2(v_1, x)) \\ &= g_2(v_1, x) \quad (g_2(v_1, \cdot) = f_2^{-1}(v_1, \cdot)). \end{aligned}$$

Therefore,

$$\begin{aligned} U_2 &= g_2(V_1, f_2(V_1, V_2)) = V_2, \\ G_{3|12}(x | v_1, v_2) &= P(f_3(V_2, V_3) \leq x | V_1 = v_1, V_2 = v_2) \\ &= P(f_3(v_2, V_3) \leq x) = g_3(v_2, x) \end{aligned}$$

implying that

$$U_3 = g_3(V_2, f_3(V_2, V_3)) = V_3 \quad \text{a.s.}$$

The general case now follows from induction. So we obtain that our standardization yields the right standardization for the two-block factor representation, which is obtained by (13), since obviously using $U = V$ a.s.

$$\begin{aligned} G_{k|1, \dots, k-1}(\cdot | U_1, \dots, U_{k-1}) &= G_{k|1, \dots, k-1}(\cdot | V_1, \dots, V_{k-1}) \\ &= G_{k|k-1}(\cdot | V_{k-1}). \quad \square \end{aligned}$$

If the conditional distribution functions $G_{k|1,\dots,k-1}$ are not continuous, it is not possible to reconstruct (V_i) from $X = (X_i)$. We next show that the standardization (12), (13) can be applied to a version \bar{X} of X .

Theorem 14. *If X has a monotone two-block factor representation $X = f(V)$ a.s., then there exists an i.i.d. $R(0, 1)$ -sequence $(\bar{U}_i) = (\bar{U}_i)$ such that the standard representation of $\bar{X} := f(\bar{U})$ reproduces \bar{U} and $\bar{X} = f(\bar{U})$.*

Proof. Let $f_1, f_k(v_{k-1}, \cdot)$ be monotonically nondecreasing for all k, v_{k-1} with $X_1 = f_1(V_1)$, $X_k = f_k(V_{k-1}, V_k)$, $k \geq 2$.

Let (\bar{V}_i) be an i.i.d. $R(0, 1)$ -distributed sequence independent of (V_i) and consider the standard representation (12), with $\bar{U}_1 := \tilde{G}_1(X_1, \bar{V}_1)$, where

$$\begin{aligned}\tilde{G}_1(x, \alpha) &= P(X_1 < x) + \alpha P(X_1 = x) \\ &= P(f_1(V_1) < x) + \alpha P(f_1(V_1) = x) \\ &= P(V_1 < f_1^{-1}(x)) + \alpha P(V_1 \in f_1^{-1}\{x\}) \\ &= f_1^{-1}(x) + \alpha \lambda(f_1^{-1}\{x\}); \\ f_1^{-1}(x) &= \inf\{y: f_1(y) \geq x\}, \quad f_1^{-1}\{x\} = \{y: f_1(y) = x\}.\end{aligned}$$

Therefore

$$\bar{U}_1 = f_1^{-1} \circ f_1(V_1) + \bar{V}_1 \lambda(f_1^{-1}\{f_1(V_1)\}) = f_1^{-1}(X_1) + \bar{V}_1 \lambda(f_1^{-1}\{X_1\}). \quad (17)$$

Define $X'_1 := f_1(\bar{U}_1) = X_1$, $X'_2 := f_2(\bar{U}_1, V_2)$ then $(\bar{U}_1, V_2, V_3, \dots)$ are i.i.d., $R(0, 1)$ -distributed and

$$X^{(1)} := (X'_1, X'_2, X_3, X_4, \dots) \stackrel{d}{=} (X_1, X_2, X_3, X_4, \dots) = X.$$

In the next step consider

$$\begin{aligned}\tilde{G}_{2|1}(x, \alpha | u_1) &= P(X'_2 < x | \bar{U}_1 = u_1) + \alpha P(X'_2 = x | \bar{U}_1 = u_1) \\ &= P(f_2(u_1, V_2) < x) + \alpha P(f_2(u_1, V_2) = x) \\ &= P(V_2 < f_2^{-1}(u_1, x)) + \alpha P(V_2 \in \{f_2^{-1}(u_1, x)\}) \\ &= f_2^{-1}(u_1, x) + \alpha \lambda(\{f_2^{-1}(u_1, x)\})\end{aligned}$$

the generalized inverse is taken w.r.t. the second component. Then our standard construction gives

$$\begin{aligned}\bar{U}_2 &:= \tilde{G}_{2|1}(X'_2, \bar{V}_2) | \bar{U}_1 \\ &= f_2^{-1}(\bar{U}_1, f_2(\bar{U}_1, V_2)) + \bar{V}_2 \lambda(\{f_2^{-1}(\bar{U}_1, f_2(\bar{U}_1, V_2))\}) \\ &= f_2^{-1}(\bar{U}_1, X'_2) + \bar{V}_2 \lambda(\{f_2^{-1}(\bar{U}_1, X'_2)\}).\end{aligned} \quad (18)$$

Since (\bar{U}_1, \bar{U}_2) are functions of $(V_1, \bar{V}_1, V_2, \bar{V}_2)$ the sequence $(\bar{U}_1, \bar{U}_2, V_3, V_4, \dots)$ is i.i.d., $R(0, 1)$ -distributed. Define

$$X^{(2)} := (f_1(\bar{U}_1), f_2(\bar{U}_1, \bar{U}_2), f_3(\bar{U}_2, V_3), f_4(V_3, V_4), \dots) \quad (19)$$

then $X^{(2)} \stackrel{d}{=} X$.

We apply our standard construction to the third component $X'_3 := f_3(\bar{U}_2, V_3)$ of $X^{(2)}$ to obtain $\bar{U}_3 := \tilde{G}_{3|1,2}((X'_3, \bar{V}_3) | \bar{U}_1, \bar{U}_2)$, where

$$\begin{aligned}\tilde{G}_{3|1,2}(x, \alpha | u_1, u_2) &= P(f_3(\bar{U}_2, V_3) < x | \bar{U}_1 = u_1, \bar{U}_2 = u_2) \\ &\quad + \alpha P(f_3(\bar{U}_2, V_3) = x | (\bar{U}_1 = u_1, \bar{U}_2 = u_2)) \\ &= P(V_3 < f_3^{-1}(u_2, x)) + \alpha P(V_3 \in \{f_3^{-1}(u_2, x)\}).\end{aligned}$$

Therefore,

$$\bar{U}_3 = f_3^{-1}(\bar{U}_2, f_3(\bar{U}_2, V_3)) + \bar{V}_3 \lambda(\{f_3^{-1}(\bar{U}_2, f_3(\bar{U}_2, V_3))\}).$$

Again $(\bar{U}_1, \bar{U}_2, \bar{U}_3, V_4, V_5, V_6, \dots) \stackrel{d}{=} (V_1, V_2, V_3, \dots)$ and

$$X^{(3)} = (f_1(\bar{U}_1), f_2(\bar{U}_1, \bar{U}_2), f_3(\bar{U}_2, \bar{U}_3), f_4(\bar{U}_3, V_4), f_5(V_4, V_5), \dots) \stackrel{d}{=} X$$

and we can continue this process by induction. Thus we obtain that for a version \bar{X} of X we have the two-block-factor representation

$$\bar{X}_1 = f_1(\bar{U}_1), \quad \bar{X}_2 = f_2(\bar{U}_1, \bar{U}_2), \quad \bar{X}_3 = f_3(\bar{U}_2, \bar{U}_3), \dots, \quad (20)$$

where the (\bar{U}_i) are obtained from our modified standardization process.

Next we apply the standardization (12) to \bar{X} to obtain

$$\begin{aligned}U_1 &:= \tilde{G}_1(\bar{X}_1, \bar{V}_1) = f_1^{-1}(\bar{X}_1) + \bar{V}_1 \lambda(f_1^{-1}(\{\bar{X}_1\})) \\ &= f_1^{-1}(f_1(\bar{U}_1)) + \bar{V}_1 \lambda(f_1^{-1}\{f_1(\bar{U}_1)\}) \\ &= f_1^{-1}(X_1) + \bar{V}_1 \lambda(f_1^{-1}(X_1)) = \bar{U}_1,\end{aligned}$$

i.e. the standardization reproduces \bar{U}_1 . In the next step

$$\begin{aligned}U_2 &= \tilde{G}_{2|1}((\bar{X}_1, \bar{V}_2) | U_1) = \bar{U}_2, \\ U_3 &= \tilde{G}_{3|1,2}((\bar{X}_3, \bar{V}_3) | U_1, U_2) = \tilde{G}_{3|1,2}((\bar{X}_3, \bar{V}_3) | \bar{U}_1, \bar{U}_2) = \bar{U}_3,\end{aligned}$$

and so on. \square

So in general from the two-block factor representation $X = f(V)$ we construct by a modification of the standardization procedure a version \bar{X} of X with a two-block factor representation $\bar{X} = f(\bar{U})$. The standardization (12), applied to this representation reproduces \bar{U} i.e. $U = \bar{U}$ and (13), our standard regression representation, reproduces this two-block factor representation of \bar{X} .

Remark 15. Obviously a result similar to Theorem 13, 14 also holds for $(m+1)$ -block factor representations. While Theorem 13 is constructive, Theorem 14 indicates the applicability of the standard construction to a (not known) version of X .

Remark 16. The i.i.d. sequence U in Theorem 3 is a.s. equal to the i.i.d. sequence U in Theorem 5. The proof is essentially the same as the proof of Theorem 13. We leave it as an exercise to the reader. The consequence of this observation is that the Standard Representation $X_n = f_n(X_1, \dots, X_{n-1}, U_n)$ can also be obtained by iterating the Markov Regression $X_n = g_n(U_1, \dots, U_n)$; i.e. $X_2 = f_2(X_1, U_2) = f_2(f_1(U_1), U_2) = g_2(U_1, U_2)$ and $X_3 = f_3(X_1, X_2, U_3) = f_3(f_1(U_1), f_2(f_1(U_1), U_2), U_3) = g_3(U_1, U_2, U_3)$ etc.

The question now is: how restrictive is the assumption of a monotone two-block factor representation?

Example 3. (a) Let (V_i) be an i.i.d. $R(0, 1)$ -sequence and consider the two-block factor $X_1 = V_1$, $X_2 = V_1 - V_2$, $X_3 = V_2 - V_3, \dots$. We obtain a monotone two-block factor representation by defining $U_1 = V_1$, $U_i := 1 - V_i$, $i \geq 2$. Then

$$X_1 = U_1, \quad X_2 = U_1 + U_2 - 1, \quad X_3 = U_3 - U_2, \quad X_4 = U_4 - U_3, \dots, \quad (21)$$

is a monotone two-block factor representation.

(b) If $X_1 = V_1$, $X_2 = (V_1 - \frac{1}{2})V_2$, $X_3 = (V_2 - \frac{1}{2})V_3, \dots$, then define

$$U_1 = V_1, \quad U_i = \begin{cases} V_i & \text{if } V_{i-1} \geq \frac{1}{2}, \\ 1 - V_i & \text{if } V_{i-1} < \frac{1}{2}, \end{cases} \quad i \geq 2.$$

It is easy to check that (U_i) is an i.i.d. $R(0, 1)$ -sequence and we obtain the monotone representation (in distribution) \bar{X} of X ,

$$\bar{X}_1 = U_1, \quad \bar{X}_i = \begin{cases} (U_{i-1} - \frac{1}{2})U_i & \text{if } U_{i-1} \geq \frac{1}{2}, \\ (U_{i-1} - \frac{1}{2})(1 - U_i) & \text{if } U_{i-1} < \frac{1}{2}, \end{cases} \quad i \geq 2. \quad (22)$$

(c) If more generally than in (b) $X_1 = f_1(V_1)$, $X_i = f_i(V_{i-1}, V_i)$, $f_i \uparrow$ and for all v_{i-1} , $i, f_i(v_{i-1}, \cdot)$ is either monotonically nondecreasing or nonincreasing (i.e. $f_i(v_{i-1}, \cdot) \uparrow$ for $v_{i-1} \in V_i^+$ and $f_i(v_{i-1}, \cdot) \downarrow$ for $v_{i-1} \in V_i^-$) then define a sequence

$$U_1 := V_1, \quad U_i := \begin{cases} V_i & \text{if } U_{i-1} \in V_i^+, \\ 1 - V_i & \text{if } U_{i-1} \in V_i^-, \end{cases} \quad i \geq 2.$$

Then (U_i) is an i.i.d. $R(0, 1)$ -sequence and with $g_1 = f_1$,

$$g_i(v_{i-1}, v_i) = \begin{cases} f_i(v_{i-1}, v_i) & \text{if } v_{i-1} \in V_i^+, \\ f_i(v_{i-1}, 1 - v_i) & \text{if } v_{i-1} \in V_i^-, \end{cases}$$

the sequence $g_1(U_1), g_2(U_1, U_2), \dots$ has the same distribution as X . Therefore, X has a monotone two-block factor representation.

For the general question we use the following proposition.

Proposition 17. Let (V_n) be an i.i.d. $R(0, 1)$ -sequence and $X_1 = f_1(V_1)$, $X_n = f_n(V_{n-1}, V_n)$, $n \geq 2$, a generalized two-block factor. Furthermore, let (\bar{V}_n) be an i.i.d. $R(0, 1)$ -sequence independent of (V_n) . Then there exist an i.i.d. $R(0, 1)$ -sequence (U_n) , $U_n = h_n(V_{n-1}, V_n, \bar{V}_n)$ independent of (V_1, \dots, V_{n-1}) and functions (g_n) such that

$$X_1 = g_1(U_1), \quad X_n = g_n(V_{n-1}, U_n), \quad n \geq 2, \quad \text{and} \quad (23)$$

$g_1, g_n(v_{n-1}, \cdot)$ monotonically nondecreasing $\forall n, v_{n-1}$.

Proof. Let G_1 be the distribution function of X_1 and let $G_{n|n-k, \dots, n-1}$ be the conditional distribution function of X_n given V_{n-k}, \dots, V_{n-1} . Define

$$U_1 := \tilde{G}_1(X_1, \bar{V}_1), \quad (24)$$

$$U_n := \tilde{G}_{n|1, \dots, n-1}(X_n, \bar{V}_n | V_1, \dots, V_{n-1}), \quad n \geq 2.$$

Since the conditional distribution of U_n given $V_1 = v_1, \dots, V_{n-1} = v_{n-1}$ is $R(0, 1)$ for all v_1, \dots, v_{n-1} we have that U_n is independent of (V_1, \dots, V_{n-1}) . Since $U_k = h_k(V_1, \dots, V_k, \bar{V}_1, \dots, \bar{V}_k)$, this implies that U_n is independent of U_1, \dots, U_{n-1} . From (4) we conclude that

$$X_1 = G_1^{-1}(U_1), \quad X_n = G_{n|1, \dots, n-1}^{-1}(U_n | V_1, \dots, V_{n-1}), \quad n \geq 2.$$

Actually, $\tilde{G}_{n|1, \dots, n-1} = \tilde{G}_{n|n-1}$ since

$$\begin{aligned} G_{n|1, \dots, n-1}(x | v_1, \dots, v_{n-1}) &= P(X_n \leq x | V_1 = v_1, \dots, V_{n-1} = v_{n-1}) \\ &= P(f_n(V_{n-1}, V_n) \leq x | V_1 = v_1, \dots, V_{n-1} = v_{n-1}) \\ &= P(f_n(v_{n-1}, V_n) \leq x) = G_{n|n-1}(x | v_{n-1}) \end{aligned}$$

(and similarly for $\tilde{G}_{n|1, \dots, n-1}$). So we have $X_n = G_{n|n-1}^{-1}(U_n | V_{n-1}) = g_n(V_{n-1}, U_n)$, where $g_n(v_{n-1}, \cdot)$ is nondecreasing. \square

Obviously from (23) we obtain a monotone two-block factor representation if $V_{n-1} = h(U_{n-1})$ for some function h . In general we obtain the following weakened monotone representation property.

Corollary 18. *Let (W_n) be an i.i.d. $R(0, 1)$ -sequence independent of (V_n) , (\bar{V}_n) and let $X_1 = f_1(V_1)$, $X_i = f_i(V_{i-1}, V_i)$, $i \geq 2$, be a generalized two-block factor. Then there exists an $R(0, 1)$ -sequence $\bar{U}_i = \tilde{h}_i(U_i, V_i, W_i)$ such that \bar{U}_i is independent of U_i and*

$$X_1 = g_1(U_1), \quad X_2 = g_2(U_1, \bar{U}_1, U_2), \quad X_3 = g_3(U_2, \bar{U}_2, U_3), \dots, \quad (25)$$

where $g_i, g_i(u_i, \bar{u}_i, \cdot)$ are monotonically nondecreasing.

Proof. From Proposition 17 we have a monotone representation, $X_1 = h_1(U_1)$, $X_n = h_n(V_{n-1}, U_n)$, $n \geq 2$. We apply (8) to obtain $V_i = \tilde{g}_i(U_i, \bar{U}_i)$ where $\bar{U}_i = \tilde{h}_i(U_i, V_i, W_i)$ is independent of U_i . Together we obtain (25). \square

Generally, we can not assert that (U_n, V_n) is independent of (V_1, \dots, V_{n-1}) (we only have separately the independence of U_n respectively V_n of (V_1, \dots, V_{n-1})). In the case that (U_n, V_n) is independent of (V_1, \dots, V_{n-1}) we obtain that in the representation (25) the sequence

$$U_1, \bar{U}_1, U_2, \bar{U}_2, \dots \text{ is an i.i.d. } R(0, 1)\text{-sequence.} \quad (26)$$

Example 4. Let (V_i) be an i.i.d. $R(0, 1)$ -sequence and let $X_i = (V_i - \frac{1}{2})^2$, $X_i = V_{i-1} \cdot (V_i - \frac{1}{2})^2$, $i \geq 2$ be a generalized two-block-factor. Then the construction of (25) is the following: $F_{X_1}(x) = 2\sqrt{x}$, $g_1(y) = (\frac{1}{2}y)^2$ and $U_1 = 2|V_1 - \frac{1}{2}|$. Let ε_i be random signs defined by $\varepsilon_i = +1$ if $V_i \geq \frac{1}{2}$ and $\varepsilon_i = -1$, else, and define $U_i = 2|V_i - \frac{1}{2}|$. Then $V_i = \frac{1}{2} + \frac{1}{2}\varepsilon_i U_i$ (and we can formally write ε_i as function of an $R(0, 1)$ -random variable \bar{U}_i). Obviously, (ε_i, U_i) is independent of V_1, \dots, V_{i-1} and we obtain from (25) the weakened monotone representation

$$X_1 = g_1(U_1), \quad X_2 = (\frac{1}{2} + \frac{1}{2}\varepsilon_1 U_1)g_1(U_2), \dots \quad (27)$$

Proposition 19. *There exists a generalized two-block factor which does not have a monotone two-block factor representation.*

Proof. Let (V_i) be an i.i.d. $R(0, 1)$ -sequence and let $X_1 = |V_1 - \frac{1}{2}|$, $X_i = V_{i-1} V_i$, $i \geq 2$. In order to show that (X_i) does not admit a monotone two-block factor representation we apply Theorem 13. So we calculate the standardization (U_i) from (12) and we show that the standard representation is not a two-block factor. Since $G_1(x) = P(X_1 \leq x) = 2x$, we obtain $U_1 = 2|V_1 - \frac{1}{2}|$. Furthermore,

$$G_{2|1}(x|u) = P(X_2 \leq x | U_1 = u) = \frac{1}{2} \left[\left(\frac{2x}{1+u} \wedge 1 \right) + \left(\frac{2x}{1-u} \wedge 1 \right) \right],$$

i.e.

$$U_2 = \frac{1}{2} \cdot \left[\frac{2V_1 V_2}{1+2|V_1 - \frac{1}{2}|} \wedge 1 + \frac{2V_1 V_2}{1-2|V_1 - \frac{1}{2}|} \wedge 1 \right].$$

With some calculations we obtain

$$G_{3|1,2}(x|u_1, u_2) = P(X_3 \leq x | U_1 = u_1, U_2 = u_2) \\ = \begin{cases} \frac{1}{2} \cdot \left[\frac{x}{(1-u_1)u_2} \wedge 1 + \frac{x}{(1+u_1)u_2} \wedge 1 \right] & \text{if } u_2 \leq \frac{1}{1+u_1}, \\ \frac{x}{2u_2-1} \wedge 1 & \text{if } u_2 > \frac{1}{1+u_1}. \end{cases}$$

From this we conclude that (X_i) does not have a monotone two-block factor representation. \square

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