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Relative stability in strictly stationary random sequences

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Abstract

Relative stability results for weakly dependent and strongly mixing strictly stationary sequences are established. As a consequence, some infinite memory models, including ARCH(1) processes, are relatively stable.

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1. Introduction and result

Let $\{\xi_k\}_{k\in\mathbb{Z}}$ be a strictly stationary random sequence defined on a probability space $(\Omega, \mathcal{F}, P)$, taking values on the real line $\mathbb{R}$. Write c_n for a sequence of real numbers and $S_n = \sum_{k=1}^n \xi_k$. We shall denote the indicator set of A by I_A, the almost sure convergence by $\to_{\text{a.s.}}$, the convergence in probability by $\to_P$, the weak convergence by $\to_w$, $a_n \sim b_n$ means $a_n/b_n \to 1$ as $n \to \infty$ and $a_n = o(b_n)$, $a_n = O(b_n)$ means $a_n/b_n \to 0$ as $n \to \infty$ and $\overline{\lim}_{n\to\infty} a_n/b_n < \infty$, respectively.

We call $\{\xi_k\}_{k\in\mathbb{N}}$ *relatively stable* if $c_n^{-1}S_n \to_P 1$. Relative stability for Bernoulli trials was established in [1]. By the Birkhoff Ergodic Theorem (cf. [6, Chapter 2]), if $E[\xi_0] = 1$ and $\{\xi_k\}_{k\in\mathbb{Z}}$ is ergodic, then $\{\xi_k\}_{k\in\mathbb{N}}$ is almost surely relatively stable with $c_n = n$.

In this paper, we focus on the more subtle case when the first moment does not exist. It is well-known (cf. [25, pp. 312–316]) that relative stability of non-negative strongly mixing (to

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be defined) strictly stationary sequences is linked to the notion of slow variation in the limit. Namely, it has been proved (cf. [42, Theorem 1]) that $c_n^{-1} S_n \to_P 1$ if and only if

$$\lim_{n\to\infty} E[c_n^{-1} S_n \wedge x_n] = 1, \quad x_n = o(r_n), r_n \to \infty \text{ as } n \to \infty,$$

where r_n is a real sequence. This condition is not easy to use directly. However under some technical assumption it can be replaced by the *uniform integrability* of normalized sums of truncated ξ_k (as it has been shown in [42]). In particular for uniformly strong mixing sequence (to be defined) the latter can be verified via Proposition 2. Thus we get ([42, Theorem 3]): $\{\xi_k\}_{k\in\mathbb{N}}$ is relatively stable if and only if

$$\frac{E[\xi_0 I_{[\xi_0\le\lambda x]}]}{E[\xi_0 I_{[\xi_0\le x]}]} \to 1 \quad \text{as } x \to \infty, \text{ for all (some) } \lambda > 0$$

(i.e. $E[\xi_0 I_{[\xi_0\le x]}]$ is a *slowly varying* function). For independent and identically distributed (i.i.d.) random variables this is the famous Khinchine–Feller stability result established in [28] (see also [19, p. 236]).

The aim of this paper is to establish criteria for relative stability for a wide class of nonnegative strictly stationary sequences with the first moment *slowly divergent* (i.e. $E[\xi_0 I_{[\xi_0\le x]}]$ is a slowly varying function and $E[\xi_0] = \infty$). In order to state the main result the following notation is required $U_q(x) = E[|\xi_0|^q I_{[|\xi_0|\le x]}]$,

$$b_n^2 = \sup\{x > 0; nU_2(x) \ge x^2\}, \qquad a_n = a_n(\delta) = \left(\frac{b_n^\delta U_2(b_n)}{U_{2+\delta}(b_n)}\right)^{\frac{2}{\delta}}, \tag{1.1}$$

where $x, q, \delta > 0$. For a strictly stationary random sequence $\{X_k\}_{k\in\mathbb{Z}}$ and $j, m, q_n \in \mathbb{N}$ define

$$\mathcal{B}_t(\{X_k\}, b_n, q_n, m) = |\mathrm{Cov}[\exp\{itb_n^{-1} S_{mq_n}\}, \exp\{itb_n^{-1}(S_{q_n(2+m)} - S_{q_n(1+m)})\}]|,$$

$S_k = \sum_{i=1}^k X_i$. The advantage of this dependence condition is that there is no uniformity both in indices and classes of functions or sets. This should benefit in calculations for particular strictly stationary stochastic models.

Let $\{r_k\}_{k\in\mathbb{Z}}$ be an i.i.d. sequence of Rademacher random variables, i.e. such that $P[r_k = 1] = P[r_k = -1] = 1/2$, independent of ξ_k (sharing the same probability space). Our first result provides *very general criterion* for relative stability of nonnegative strictly stationary sequences.

Theorem 1. *Suppose $\{\xi_k\}_{k\in\mathbb{Z}}$ is a strictly stationary sequence such that $U_2(x)$ is a slowly varying function, $E[\xi_0^2] = \infty$, and for a fixed $\delta > 0$*

$$\lim_{n\to\infty} \frac{n}{a_n} \mathcal{B}_t(\{r_k\xi_k\}, b_n, \lfloor a_n \rfloor, m) = 0, \quad t \in \mathbb{R}, m \in \mathbb{N}. \tag{1.2}$$

Then $\{\xi_k^2\}_{k\in\mathbb{N}}$ is relatively stable with normalizing $c_n = b_n^2$.

Since this paper deals with distributions for which $U_2(\infty) = \infty$, where $U_2(x)$ varies slowly, usually a_n also varies slowly (see examples). Therefore in order to satisfy condition (1.2) the term $\frac{n}{a_n}$ has to be compensated by an appropriate decrease of $\mathcal{B}_t$. One route to attain the latter goes through the weak dependence introduced by Doukhan and Louhichi (cf. [14,11]). Set

$$\mathrm{Lip}(h) = \sup_{(x_1,\ldots,x_n)\neq(y_1,\ldots,y_n)\in\mathbb{R}^n} \frac{|h(x_1,\ldots,x_n) - h(y_1,\ldots,y_n)|}{|x_1 - y_1| + \cdots + |x_n - y_n|}.$$

A random sequence X_k is (ε, Ψ)-weakly dependent if there exist a function $\Psi : \mathbb{N}^2 \times \mathbb{R}^2 \to (0, \infty)$ and a sequence $\varepsilon(q) = \varepsilon_q(\{X_k\}) \to 0$ as $q \to \infty$, satisfying

$$|\mathrm{Cov}(g_1(X_{\nu_1}, \dots, X_{\nu_l}), g_2(X_{\nu_{l+1}}, \dots, X_{\nu_m}))| \le \Psi(l, m, \mathrm{Lip}(g_1), \mathrm{Lip}(g_2)) \cdot \varepsilon_q(\{X_k\}),$$

where $1 \le \nu_1 < \cdots < \nu_l < \nu_l + q = \nu_{l+1} < \nu_{l+2} < \cdots < \nu_m, l, \nu_l \in \mathbb{N}$ and functions $g_1 : \mathbb{R}^l \to \mathbb{R}, g_2 : \mathbb{R}^{m-l} \to \mathbb{R}$ are bounded by 1. If we set $\Psi(u, v, f, g) = u\mathrm{Lip}(g)$ or $\Psi(u, v, f, g) = u\mathrm{Lip}(f) \cdot v\mathrm{Lip}(g)$, then we deal with θ or κ-weak dependence, respectively.

An alternative route is strong mixing conditions (cf. [6]). For $n \in \mathbb{N}$ set

$$\alpha(n) = \alpha_n(\{\xi_k\}) = \sup\{|P(B \cap A) - P(A)P(B)|;\ A \in \mathcal{F}_{-\infty}^0, B \in \mathcal{F}_n^\infty\},$$
$$\rho(n) = \rho_n(\{\xi_k\}) = \sup\{|\mathrm{Corr}(F, G)|;\ F \in \mathcal{L}^2_{\mathrm{real}}(\mathcal{F}_{-\infty}^0), G \in \mathcal{L}^2_{\mathrm{real}}(\mathcal{F}_n^\infty)\},$$
$$\varphi(n) = \varphi_n(\{\xi_k\}) = \sup\{|P(B \mid A) - P(B)|;\ A \in \mathcal{F}_{-\infty}^0, B \in \mathcal{F}_n^\infty\},$$

where $\mathcal{F}_k^m = \sigma(\{\xi_i; k \le i \le m\})$. It is well known that $2\alpha(n) \le \varphi(n)$ and $\rho(n) \le 2\sqrt{\varphi(n)}$ (cf. [6, Proposition 3.11 on p. 76]). We say that $\{\xi_k\}_{k\in\mathbb{Z}}$ is strongly mixing if $\alpha(n) = o(1)$, φ-mixing or uniformly (strong) mixing if $\varphi(n) = o(1)$, and ρ-mixing if $\rho(n) = o(1)$.

In view of the above definitions (1.2) holds if one of these is satisfied

$$\frac{n}{b_n}\theta_{\lfloor a_n \rfloor}(\{\xi_k\}) \to 0, \qquad \frac{na_n}{b_n^2}\kappa_{\lfloor a_n \rfloor}(\{\xi_k\}) \to 0, \qquad \frac{n}{a_n}\alpha_{\lfloor a_n \rfloor}(\{\xi_k\}) \to 0, \quad \text{as } n \to \infty.$$

There exist models that do not satisfy the strong mixing condition (cf. [6, Example 2.15, p. 58] and [15]). Nevertheless such sequences are exponentially fast θ-weakly dependent (cf. [15, Theorem 1]) and therefore Theorem 1 applies if one can verify (1.2). This can be done (see Example 1) for ARCH(1) processes which are interesting from the point of view of modeling the financial time series that exhibit time-varying volatility [17]. Moreover, the non-exponential rate in (1.2) can be attained for some strongly mixing sequences (see Example 3).

Our second result is a dependent analog of *Raikov's principle*. It allows us to establish new relative stability results.

Theorem 2. *Suppose $\{\xi_k\}_{k\in\mathbb{Z}}$ with $E[\xi_0^2 I_{[|\xi_0|\le x]}]$ slowly varying is strongly mixing. Then the following statements are equivalent:*

(1) $\mathcal{L}(b_n^{-1}\sum_{k=1}^n r_k\xi_k) \to \mathcal{N}(0, 1)$;
(2) $\{b_n^{-2}(\sum_{k=1}^n r_k\xi_k I_{[|\xi_k|\le b_n]})^2\}_{n\in\mathbb{N}}$ *is uniformly integrable;*
(3) $\{b_n^{-2}\sum_{k=1}^n \xi_k^2 I_{[|\xi_k|\le b_n]}\}_{n\in\mathbb{N}}$ *is uniformly integrable;*
(4) $b_n^{-2}\sum_{k=1}^n \xi_k^2 \to_P 1$.

In applied probability theory we deal with strongly mixing ARMA and stochastic volatility models with regularly varying noise or solutions to stochastic recurrence equations with moments slowly divergent (cf. [13]). Thus, if we know that the structure of the solution is of the form $r_k|\xi_k|$, where $\{|\xi_k|\}_{k\in\mathbb{Z}}$ is strongly mixing, then automatically we get the *Central Limit Theorem* (CLT) via relative stability. In this context it is also worth noting that non-normal limit theorems for such strongly mixing models are obtained in [2] and therefore this work completes these results. The other advantage of Theorem 2 lies in the fact that for some dependent sequences, there are results, which allow us to establish uniform integrability for "rademacherized" random sequences. This is the case for the well-known Bradley's CLT with infinite variances, where by Theorem 2 squares are relatively stable (see Example 2). Nevertheless, there exists (cf. Example 4) a strictly stationary sequence satisfying condition (2), for which the CLT does not hold.

The paper is organized as follows. Section 2 is of independent interest and deals with uniform integrability. In Section 3 we provide some connections between CLT and relative stability together with the proof of Theorem 2. Section 4 contains proof of Theorem 1 while Section 5 examples.

2. Uniform integrability

In this section some new results on uniform integrability are presented. Let $\boldsymbol{\Psi}$ be the class of all convex functions $\Psi : [0, \infty) \to [0, \infty)$, $\Psi(0) = 0$, such that $t^{-1}\Psi(t) \to \infty$ as $t \to \infty$ and satisfying the Δ_2 condition for all t, i.e.

$$\Psi(2t) \le c\,\Psi(t), \quad t \ge 0, \text{ for some } c > 0. \tag{2.3}$$

The following is a stronger version of the de la Vallée Poussin criterion for the uniform integrability (cf. [27, Lemma 5], [33, Theorem 22]).

Lemma 1. *A class of random variables $\{Z_s\}$ is uniformly integrable if and only if $\sup_s E[\Psi(|Z_s|)] < \infty$, for some $\Psi \in \boldsymbol{\Psi}$.*

Proof of Lemma 1. Sufficiency. Let $M = \sup_s E[\Psi(|Z_s|)]$ and t_ε be such that $t^{-1}\Psi(t) > \varepsilon^{-1}M$ for $t \ge t_\varepsilon$. Thus for $t \ge t_\varepsilon$ we have

$$\{|Z_s| > t\} \subseteq \left\{|Z_s| < \Psi(|Z_s|)\frac{\varepsilon}{M}\right\}$$

and

$$\sup_s E[|Z_s| I_{[|Z_s|>t]}] \le \sup_s E[\Psi(|Z_s|) I_{[|Z_s|>t]}]\frac{\varepsilon}{M} \le \varepsilon.$$

Necessity. Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence of numbers such that $0 = u_0 < u_1 < u_2 < \cdots < u_k < \cdots$, where $2u_k \le u_{k+1}$ and

$$\sup_s E[|Z_s| I_{[|Z_s|>u_k]}] \le 2^{-2k}, \quad k \ge 1.$$

Define

$$\psi(x) = \sum_{k=0}^{\infty} 2^k I_{[u_k, u_{k+1})}(x), \qquad \Psi(t) = \int_0^t \psi(x)dx.$$

It follows from the definition of ψ that Ψ is continuous, non-decreasing and $\Psi(0) = 0$. Moreover, if $0 \le t_1 < t_2$ then

$$\begin{aligned}
\Psi\left(\frac{t_1+t_2}{2}\right) &= \int_0^{t_1} \psi(x)\,dx + \int_{t_1}^{\frac{t_1+t_2}{2}} \psi(x)\,dx \\
&= \int_0^{t_1} \psi(x)\,dx + \frac{1}{2}\int_{t_1}^{\frac{t_1+t_2}{2}} \psi(x)\,dx + \frac{1}{2}\int_{\frac{t_1+t_2}{2}}^{t_2} \psi\left(x - \frac{t_2-t_1}{2}\right) dx \\
&\le \int_0^{t_1} \psi(x)\,dx + \frac{1}{2}\int_{t_1}^{\frac{t_1+t_2}{2}} \psi(x)\,dx + \frac{1}{2}\int_{\frac{t_1+t_2}{2}}^{t_2} \psi(x)\,dx \\
&= \frac{1}{2}(\Psi(t_1) + \Psi(t_2)).
\end{aligned}$$

Hence Ψ is convex in the sense of Jensen and by the continuity it is convex in the usual way (cf. [22, pp. 71–72]). Furthermore, $\psi(t) \to \infty$ as $t \to \infty$ so $t^{-1}\Psi(t) \to \infty$ as $t \to \infty$. By the choice of u_k for $k > 0$

$$2^k I_{[u_k,u_{k+1})}(2x) \le 2^k I_{[\frac{1}{2}u_k,\frac{1}{2}u_{k+1})}(x) \le 2^k I_{[u_{k-1},u_k)}(x) + 2^k I_{[u_k,u_{k+1})}(x),$$

so for $x \ge u_1$ it yields $\psi(2x) \le 3\psi(x)$. It is easy to see that $\psi(2x) \le 3\psi(x)$ for $0 < 2x < u_1$. On the other hand if $0 < x < u_1 \le 2x < u_2$ then $\psi(2x) \le 3\psi(x)$ (from $0 < x < u_1$ and $2x \ge u_2$ it follows $x \ge u_1$). Therefore $\psi(2x) \le 3\psi(x)$, $x \ge 0$. Whence

$$\Psi(2t) = 2\int_0^t \psi(2x)dx \le 6\Psi(t), \quad t \ge 0$$

and Ψ satisfies (2.3). Consequently,

$$\begin{aligned} E[\Psi(|Z_s|)] &= \int_0^\infty \psi(x)P[|Z_s| > x]\,dx = \sum_{k=0}^\infty \int_{u_k}^{u_{k+1}} \psi(x)P[|Z_s| > x]\,dx \\ &\le \sum_{k=0}^\infty 2^k \int_{u_k}^{u_{k+1}} P[|Z_s| > x]\,dx \le \sum_{k=0}^\infty 2^k\, E[|Z_s| I_{[|Z_s|>u_k]}] \\ &\le \sum_{k=1}^\infty \frac{1}{2^k} + E|Z_s| = 1 + E|Z_s|. \end{aligned}$$

Since $\{Z_s\}$ is uniformly integrable $\sup_s E|Z_s| < \infty$. Thus we have $\sup_s E[\Psi(|Z_s|)] < \infty$, for some $\Psi \in \boldsymbol{\Psi}$. This is the desired conclusion. □

The next result is on triangular arrays of random variables $\{\xi_{nk}, 1 \le k \le k_n, n \ge 1\}$. Set $S_{nk} = \sum_{i=1}^k r_i\xi_{ni}$, $S_n = S_{nk_n}$ and $U_n^2 = \sum_{k=1}^{k_n} \xi_{nk}^2$.

Proposition 1. *$\{S_n^2\}_{n\in\mathbb{N}}$ is uniformly integrable if and only if $\{U_n^2\}_{n\in\mathbb{N}}$ is uniformly integrable.*

Proof of Proposition 1. Assume $\{S_n^2\}$ is uniformly integrable. Thus by Lemma 1 we see that for some $\Psi \in \boldsymbol{\Psi}$ $\sup_n E[\Psi(S_n^2)] < \infty$. Let $\Phi(t) = \Psi(t^2)$. Since Ψ is convex and non-decreasing, for $0 \le t_1 < t_2$,

$$\begin{aligned} \Phi\left(\frac{t_1+t_2}{2}\right) &= \Psi\left(\left(\frac{t_1+t_2}{2}\right)^2\right) \le \Psi\left(\frac{t_1^2+t_2^2}{2}\right) \\ &\le \frac{1}{2}(\Psi(t_1^2) + \Psi(t_2^2)) = \frac{1}{2}(\Phi(t_1) + \Phi(t_2)), \end{aligned}$$

that is, Φ is convex. It follows that $\Phi : [0,\infty) \to [0,\infty)$, $\Phi(0) = 0$, Φ is non-decreasing, continuous and by

$$\Phi(2t) = \frac{\Psi(4t^2)}{\Psi(2t^2)}\frac{\Psi(2t^2)}{\Psi(t^2)}\Psi(t^2) \le c^2\Phi(t), \quad t > 0,$$

it also satisfies (2.3). Therefore the Burkholder–Davis–Gundy inequality (cf. [9, Theorem 1, p. 425]) yields that for some $0 < A < B < \infty$

$$\begin{aligned} A\sup_n E\left[\Psi(U_n^2)\right] &= A\sup_n E\left[\Phi(U_n)\right] \le \sup_n E\left[\Phi(\max_{1\le k\le k_n}|S_{nk}|)\right] \\ &\le B\sup_n E[\Psi(U_n^2)]. \end{aligned} \tag{2.4}$$

Now observe that for $1 \le k \le k_n$ the conditional distributions

$$(r_1\xi_{n1}, \ldots, r_n\xi_{nk_n}) \quad \text{and} \quad (r_1\xi_{n1}, \ldots, -r_k\xi_{nk}, \ldots, -r_n\xi_{nk_n})$$

given $\xi_{n1}, \ldots, \xi_{nk_n}$ are the same (i.e. sign independent). Thus, by a variant of Lévy's inequality (cf. [23, (6.25.1), p. 473])

$$P\left[\max_{1\le k\le k_n} |S_{nk}| > t \mid \xi_{n1}, \xi_{n2}, \ldots, \xi_{nk_n}\right] \le 2P[|S_{nk_n}| > t \mid \xi_{n1}, \xi_{n2}, \ldots, \xi_{nk_n}],$$
$$t \ge 0,$$

and $E[\Phi(X)] = \int_0^\infty \phi(t)P[|X| > t]dt$ (e.g. [23, Eq. (4.2.8)]), we obtain

$$\begin{aligned}
&\sup_n E\left[\Phi(\max_{1\le k\le k_n} |S_{nk}|)\right] \\
&\quad = \sup_n E\left[E\left[\Phi\left(\max_{1\le k\le k_n} |S_{nk}|\right) \mid \xi_{n1}, \xi_{n2}, \ldots, \xi_{nk_n}\right]\right] \\
&\quad = \sup_n E\left[\int_0^\infty \phi(t)P\left[\max_{1\le k\le k_n} |S_{nk}| > t \mid \xi_{n1}, \xi_{n2}, \ldots, \xi_{nk_n}\right]dt\right] \\
&\quad \le \sup_n E\left[\int_0^\infty \phi(t)\cdot 2P[|S_n| > t \mid \xi_{n1}, \xi_{n2}, \ldots, \xi_{nk_n}]dt\right] \\
&\quad = \sup_n E\left[2E\left[\Phi(|S_n|) \mid \xi_{n1}, \xi_{n2}, \ldots, \xi_{nk_n}\right]\right] = \sup_n 2E[\Phi(|S_n|)] \\
&\quad = 2\sup_n E[\Psi(S_n^2)] < \infty,
\end{aligned} \tag{2.5}$$

where $\Phi(x) = \int_0^x \phi(t)dt$ (cf. [44, Theorem 4.141, p. 69]). Combining (2.4) with (2.5) yields

$$A\sup_n E[\Psi(U_n^2)] \le 2\sup_n E[\Psi(S_n^2)] < \infty.$$

Consequently, by Lemma 1 the sequence $\{U_n^2\}$ is uniformly integrable. For the converse statement use the upper bound in (2.4). $\square$

The next result together with Lemma 1 is very useful for obtaining the uniform integrability (see Example 2(b)) and is a *generalization* of Proposition 6.8 on p. 156 in [29] and Proposition 11 in [43]. For non-stationary random sequence $\{X_k\}_{k\in\mathbb{N}}$ set

$$Z_n = \sup_{1\le k\le n}\left|\sum_{k=1}^n X_k\right|, \qquad M_n = \max_{1\le k\le n}|X_k|,$$
$$\varphi_n = \sup_{k\in\mathbb{N}}\sup\{|P(B \mid A) - P(B)|;\, P(A) > 0,\, A \in \mathcal{F}_1^k,\, B \in \mathcal{F}_{n+k}^\infty\},$$

where $\mathcal{F}_k^m$ is the σ-field generated by $X_k, X_{k+1}, \ldots, X_m$, $m \in \mathbb{N}$.

Proposition 2. *Suppose $\Phi : [0,\infty) \to [0,\infty)$ is an increasing function such that $\Phi(0) = 0$ and condition (2.3) holds. Let $n > m \ge 1$, $\tau \in (0,1)$ and*

$$t_\tau = \inf\{t > 0;\, \varphi_m + P[Z_n > t] \le c^{-2}\tau\}.$$

If $E[\Phi(Z_n)] < \infty$, *then*

$$E[\Phi(Z_n)] \le \frac{c^2}{1-\tau}(E[\Phi(mM_n)] + \Phi(t_\tau)). \tag{2.6}$$

Proof of Proposition 2. Fix $\tau \in (0, 1)$ and take $t > t_\tau$. By Proposition 9 in [43] for any positive s, t, u,

$$P[Z_n > s + 2t + u] \le P[mM_n > u] + (\varphi_m + P[Z_n > t])P[Z_n > s], \quad n > m \ge 1.$$

Therefore,

$$\begin{aligned} E[\Phi(Z_n)] &= E\left[\Phi\left(4\frac{Z_n}{4}\right)\right] \le c^2 E\left[\Phi\left(\frac{Z_n}{4}\right)\right] \\ &= c^2\int_0^\infty P[Z_n > 4x]\Phi(dx) = c^2\left(\int_0^t + \int_t^\infty\right) P[Z_n > 4x]\Phi(dx) \\ &\le c^2\left(\Phi(t) + \int_t^\infty (P[mM_n > x] + (\varphi_m + P[Z_n > x])P[Z_n > x])\Phi(dx)\right) \\ &\le c^2\left(\Phi(t) + E[\Phi(mM_n)] + (\varphi_m + P[Z_n > t])\int_t^\infty P[Z_n > x]\Phi(dx)\right) \\ &\le c^2(\Phi(t) + E[\Phi(mM_n)]) + \tau E[\Phi(Z_n)]. \end{aligned}$$

Since the above inequality holds for arbitrary $t > t_\tau$, we have thus proved (2.6). □

3. Central limit theorems and relative stability

The relative stability is closely related to central limit theorems. In particular, Raikov [35, Theorem 1] proved that for a centered i.i.d. sequence $\{\xi_k\}_{k\in\mathbb{N}}$ the CLT holds if and only if $\{\xi_k^2\}_{k\in\mathbb{N}}$ is relatively stable (see also [21, Theorem 4, p. 143]). This was generalized to martingale differences, when normalization is made by variances and the weak invariance principle (WIP) replaces the CLT (cf. e.g. [30, Corollary 6]). A similar result for arbitrary dependent triangular arrays, originated for trigonometrical series by Salem and Zygmund [39, p. 334], has been established by McLeish [32, Theorem 2.1].

In the case of strongly mixing strictly stationary sequences $\{\xi_k\}_{k\in\mathbb{Z}}$ with symmetric $\mathcal{L}(S_n)$, the CLT holds with normalization b_n if and only if $E[b_n^{-2}S_n^2 \wedge x]$ is a slowly varying sequence in the limit (cf. [41, Theorem 1]). As it has been shown in [26], the latter can be reduced to the uniform integrability. This yields in particular that for strongly mixing centered and strictly stationary sequences $\{\xi_k\}_{k\in\mathbb{Z}}$ with $\sigma_n^2 = E[S_n^2] \to \infty$, we have $\mathcal{L}(\sigma_n^{-2}S_n) \to_w \mathcal{N}(0, 1)$, where $\mathcal{N}(0, 1)$ is the standard normal distribution, if and only if $\{\sigma_n^{-2}S_n^2\}_{n\in\mathbb{N}}$ is uniformly integrable (cf. [12, Theorem 3]). While there are many examples where $\{\sigma_n^{-2}S_n^2\}_{n\in\mathbb{N}}$ is not uniformly integrable (cf. [6]) $\{\xi_k^2\}_{k\in\mathbb{N}}$ is always relatively stable by the Birkhoff Ergodic Theorem since $E[\xi_k^2] < \infty$.

These results and results from the previous section have some interesting consequences. In order to present them let $\{Y_k\}_{k\in\mathbb{Z}}$ be a strictly stationary sequence and $\{b_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers. Denote $Y_{nk} = Y_k I_{[|Y_k|\le b_n]}$, $T_{nm} = \sum_{k=1}^m (Y_{nk} - E[Y_{nk}])$, $\tau_{nm}^2 = E[T_{nm}^2]$, $T_n = T_{nn}$, $\tau_n^2 = E[T_n^2]$. The following result is proved in [26].

Proposition 3. *Suppose* $\{Y_k\}_{k\in\mathbb{Z}}$ *is a strongly mixing strictly stationary sequence such that*

$$\tau_n^{-1}\sum_{k=1}^n Y_k I_{[|Y_k|>b_n]} \to_P 0,$$

for some $b_n \to \infty$ such that $\tau_n \to \infty$. Then the necessary and sufficient condition for

$$\mathcal{L}\left(\tau_n^{-1}\left(\sum_{k=1}^{n} Y_k - nE[Y_1 I_{[|Y_1|\le b_n]}]\right)\right) \to_w \mathcal{N}(0,1)$$

is the uniform integrability of $\{\tau_n^{-2}T_n^2\}_{n\in\mathbb{N}}$.

Let $\{V_k\}_{k\in\mathbb{Z}}$ be a non-negative strictly stationary sequence and c_n be a sequence of positive numbers. Denote $Z_{nj} = \sum_{k=1}^{j} V_k I_{[V_k\le c_n]}$, $\vartheta_n = E[Z_n] = E[Z_{nn}]$. The next result is Theorem 2 in [42].

Proposition 4. *Suppose $\{V_k\}_{k\in\mathbb{Z}}$ is a non-negative strongly mixing strictly stationary sequence such that $\vartheta_n^{-1}\sum_{k=1}^{n} V_k I_{[V_k>c_n]} \longrightarrow_P 0$ for some $c_n \to \infty$. Then*

$$\vartheta_n^{-1}\sum_{k=1}^{n} V_k \longrightarrow_P 1 \quad \text{as } n\to\infty$$

if and only if $\{\vartheta_n^{-1}Z_n\}_{n\in\mathbb{N}}$ is uniformly integrable.

The first consequence is Raikov's property for strongly mixing strictly stationary sequences stated in Theorem 2. To see this note that by Theorem 6.6 on p. 199 in [6] the sequence $\{r_k\xi_k\}_{k\in\mathbb{Z}}$ is strongly mixing. Therefore it suffices to apply Propositions 1, 3 and 4.

For other consequences observe that by the proofs of Theorem 3 in [26] and Theorem 2 in [42] Propositions 3 and 4 are true under Condition B:

$$B(\upsilon_n) = B_\theta(\{X_k\},\upsilon_n) = \max_{1\le k+l\le n} |\mathrm{Cov}[\exp\{i\theta\upsilon_n^{-1}S_l\}, \exp\{i\theta\upsilon_n^{-1}S_k\}]| = o(1),$$

$S_k = \sum_{i=1}^{k} X_i$, for $\upsilon_n \to \infty$ and any $\theta\in\mathbb{R}$, $k,l\in\mathbb{N}$, which is less restrictive than "strong mixing" conditions (cf. [25, Proposition 5.2]). In particular the ergodicity in the Birkhoff almost sure relative stability can be surprisingly replaced by the *weaker* Condition B (cf. [7, Problem 16, p. 120]).

Proposition 5. *Suppose $\{V_k\}_{k\in\mathbb{Z}}$ is a non-negative strictly stationary sequence such that $E[V_1] = 1$. Then $n^{-1}\sum_{k=1}^{n} V_k \to_{\text{a.s.}} 1$ if and only if Condition B with $\upsilon_n = n$ and $X_k = V_k$ holds.*

Proof of Proposition 5. The "only if" statement is contained in Proposition 3.1, [25]. For the "if" statement set $c_n = n$ and note that $n^{-1}\sum_{k=1}^{n} V_k I_{[V_k>c_n]} \to_P 0$. Further, by the Jensen inequality for any convex function Ψ

$$E\left[\Psi\left(n^{-1}\sum_{k=1}^{n} V_k I_{[V_k\le c_n]}\right)\right] \le E[\Psi(V_1 I_{[V_1\le c_n]})].$$

Whence by Lemma 1 conditions of Proposition 4 are met and therefore $\{V_k\}_{k\in\mathbb{N}}$ is relatively stable with normalizing n. On the other hand by Theorem 6.21 on p. 113 in [7] $n^{-1}\sum_{k=1}^{n} V_k$ converges a.s. Thus the limit has to be 1. □

Finally, by Proposition 5 and martingale CLT we get that the ergodicity in the Billingsley–Ibragimov CLT for strictly stationary martingale differences $\{X_k\}_{k\in Z}$ (cf. [3, Theorem 23.1, p. 206]) can be replaced by the *weaker* condition $B_\theta(\{X_k^2\}, n) = o(1)$, $\theta\in\mathbb{R}$.

4. Proof of Theorem 1

For the proof of Theorem 1 we need two lemmas. The first one is well-known and follows from Theorem 8.1.3 on p. 332 in [4].

Lemma 2. *Suppose $\delta > 0$. Then $U_2(x)$ varies slowly if and only if*

$$\frac{U_{2+\delta}(x)}{x^\delta U_2(x)} \to 0 \quad \text{as } x \to \infty. \tag{4.7}$$

If $x^2 P[|\xi_0| > x]$ varies slowly then it is possible to describe the ratio in (4.7).

Lemma 3. *Suppose $U_2(\infty) = \infty, L(x) = x^2 P[|\xi_0| > x] > 0$, for $x \geq A$, has continuous derivative such that $xL'(x) = o(L(x))$. Then $L(x), U_2(x)$ vary slowly and*

$$\frac{x^\delta U_2(x)}{U_{2+\delta}(x)} \sim \frac{\delta}{L(x)} \int_A^x \frac{L(u)}{u} du, \quad \delta > 0. \tag{4.8}$$

Proof of Lemma 3. By the remark on p. 7 in [40] (the end of Section 1.2) L varies slowly since L is positive and has continuous derivative such that $xL'(x) = o(L(x))$. Thus applying the well known formula

$$x^p P[|\xi_0| > x] + E[|\xi_0|^p I_{[|\xi_0| \leq x]}] = p \int_0^x y^{p-1} P[|\xi_0| > y] dy, \quad p > 0, \tag{4.9}$$

(with $p = 2$) by Theorem 2 on p. 283 in [19] we obtain that $U_2(x) \sim 2 \int_A^x \frac{L(u)}{u} du$ and the slow variation of $U_2(x)$. On the other hand by (4.9) with $p = 2 + \delta$ and l'Hôpital's rule (recall $U_2(\infty) = \infty$) and $xL'(x) = o(L(x))$ we get

$$\begin{aligned}
\frac{x^\delta L(x)}{U_{2+\delta}(x)} &= \frac{x^\delta L(x)}{-x^\delta L(x) + (2+\delta) \int_A^x u^{\delta-1} L(u) du} \\
&\sim \frac{\delta x^{\delta-1} L(x) + x^\delta L'(x)}{-\delta x^{\delta-1} L(x) - x^\delta L'(x) + (2+\delta) x^{\delta-1} L(x)} \\
&\sim \frac{\delta + o(1)}{-\delta + o(1) + (2+\delta)} \sim \frac{\delta}{2}.
\end{aligned}$$

Hence

$$\frac{x^\delta U_2(x) L(x)}{U_{2+\delta}(x)} \sim \frac{\delta}{2} U_2(x) \sim \delta \int_A^x \frac{L(u)}{u} du.$$

This proves Lemma 3. □

Proof of Theorem 1. Assume $U_2(x)$ is a slowly varying function and $U_2(\infty) = \infty$. It is well known that $\{b_n\}_{n \in \mathbb{N}}$ satisfies the asymptotic equation $b_n^2 \sim nU_2(b_n)$ and b_n^2 is a slowly varying sequence with index 1 (cf. [19, Chapter IX, Section 8]). Moreover, by Theorem 2 on p. 283 in [19] we have $x^2 P[|\xi_0| > x] = o(U_2(x))$, therefore by (4.9) with $p = 2$ we get $U_2(b_n) \sim E[\xi_0^2 \wedge b_n]$.

Thus for every $\epsilon > 0$

$$\overline{\lim_{n\to\infty}} P\left[\left|\sum_{k=1}^{n} r_k \xi_k I_{[|r_k\xi_k|>b_n]}\right| > \epsilon b_n\right] \le \overline{\lim_{n\to\infty}} nP[|\xi_0| > b_n]$$

$$= \overline{\lim_{n\to\infty}} \frac{b_n^2 P[|\xi_0| > b_n]}{U_2(b_n)} \frac{nU_2(b_n)}{b_n^2} = 0 \tag{4.10}$$

and analogously

$$\overline{\lim_{n\to\infty}} P\left[\left|\sum_{k=1}^{n} \xi_k^2 I_{[\xi_k^2>b_n^2]}\right| > \epsilon b_n^2\right] = 0. \tag{4.11}$$

Set $Y_k = r_k \xi_k$ and adopt the notations from the previous section for $Y_{nk} = Y_k I_{[|Y_k|\le b_n]} - E[Y_k I_{[|Y_k|\le b_n]}]$, $T_{nm} = \sum_{k=1}^{m} Y_{nk}$, $T_n = T_{nn}$, $\tau_{nm}^2 = E[T_{nm}^2]$, $\tau_n^2 = E[T_n^2]$.

Let $j, k_n, n, p_n, q_n \in \mathbb{N}$ be such that $k_n(p_n + q_n) \le n$ and $(k_n + 1)(p_n + q_n) > n$. Using the Markov–Bernstein blocking technique we partition $\{Y_{nk}\}_{1\le k\le n}$ in k_n big and small blocks of sizes p_n and q_n, respectively by setting

$$X_{nj} = \sum_{i=(j-1)(p_n+q_n)+1}^{jp_n+(j-1)q_n} Y_{ni}, \qquad X'_{nj} = \sum_{i=jp_n+(j-1)q_n+1}^{j(p_n+q_n)} Y_{ni},$$

$$X''_n = \sum_{i=k_n(p_n+q_n)+1}^{n} Y_{ni}.$$

Since $U_2(\infty) = \infty$ thus without loss of generality we can assume that $P[\xi_0 \le x] - P[\xi_0 \le -x]$ has positive continuous density ψ (replacing the sequence ξ_k by $\xi_k + \zeta_k$, where $\{\zeta_k\}_{k\in\mathbb{Z}}$ is i.i.d. with $\mathcal{L}(\zeta_1) = \mathcal{N}(0, 1)$, if necessary). Because $U_2(x)$ varies slowly thus by Lemma 2 for every $\delta > 0$

$$\lim_{n\to\infty} \frac{U_{2+\delta}(b_n)}{b_n^{\delta} U_2(b_n)} = 0, \quad \delta > 0. \tag{4.12}$$

By l'Hôpital's rule

$$\lim_{x\to\infty} \frac{U_2^{1+\frac{\delta}{2}}(x)}{U_{2+\delta}(x)} = \lim_{x\to\infty} \frac{\left(\int_0^x u^2\psi(u)du\right)^{1+\frac{\delta}{2}}}{\int_0^x u^{2+\delta}\psi(u)du} = \left(1 + \frac{\delta}{2}\right) \lim_{x\to\infty} \frac{U_2^{\frac{\delta}{2}}(x)}{x^{\delta}}.$$

So it yields

$$\lim_{n\to\infty} \frac{U_2^{1+\frac{\delta}{2}}(b_n)}{U_{2+\delta}(b_n)} = 0, \tag{4.13}$$

by the slow variation of $U_2(x)$. Similarly,

$$\lim_{x\to\infty} \frac{U_{2+\frac{\delta}{2}}^2(x)}{U_{2+\delta}(x)U_2(x)} = \lim_{x\to\infty} \frac{2U_{2+\frac{\delta}{2}}(x)x^{2+\frac{\delta}{2}}\psi(x)}{U_{2+\delta}(x)x^2\psi(x) + U_2(x)x^{2+\delta}\psi(x)} = \lim_{x\to\infty} \frac{2\frac{U_{2+\frac{\delta}{2}}(x)}{x^{\frac{\delta}{2}}U_2(x)}}{\frac{U_{2+\delta}(x)}{x^{\delta}U_2(x)} + 1}$$

and therefore by (4.12)

$$\lim_{n\to\infty} \frac{U^2_{2+\frac{\delta}{2}}(b_n)}{U_{2+\delta}(b_n)U_2(b_n)} = 0. \tag{4.14}$$

In view of (4.13) and (4.14) and the arguments used in the proof of Lemma 1 in [26] there exist $m_n \in \mathbb{N}$ such that $m_n \to \infty$ and

$$\begin{aligned} & m_n^{\frac{\delta}{2}} \frac{U_2^{1+\frac{\delta}{2}}(b_n)}{U_{2+\delta}(b_n)} \to 0, \qquad m_n^{\frac{\delta}{2}} \frac{U^2_{2+\frac{\delta}{2}}(b_n)}{U_{2+\delta}(b_n)U_2(b_n)} \to 0, \\ & \frac{n}{a_n}\mathcal{B}_t(\{r_k\xi_k\}, b_n, \lfloor a_n \rfloor, m_n) \to 0, \end{aligned} \tag{4.15}$$

as $n \to \infty$, where a_n and b_n are defined by (1.1). Put $q_n = \lfloor a_n \rfloor$ and $p_n = m_n q_n$. By (4.15)

$$k_n = \left\lfloor \frac{n}{p_n} \right\rfloor \geq \frac{1}{m_n} \left(\frac{U_{2+\delta}(b_n)}{U_2^{1+\frac{\delta}{2}}(b_n)} \right)^{\frac{2}{\delta}} - 1 \to \infty \quad \text{as } n \to \infty.$$

Moreover (cf. [24, p. 318])

$$\begin{aligned} & \overline{\lim_{n\to\infty}} \left| E\left[e^{itb_n^{-1}\sum_{j=1}^{k_n} X_{nj}} \right] - (E[e^{itb_n^{-1}X_{n1}}])^{k_n} \right| \\ & \quad \leq \overline{\lim_{n\to\infty}} \sum_{\nu=2}^{k_n} \left| E\left[e^{itb_n^{-1}\sum_{j=1}^{\nu} X_{nj}} \right] - E\left[e^{itb_n^{-1}\sum_{j=1}^{\nu-1} X_{nj}} \right] \cdot E\left[e^{itb_n^{-1}X_{n\nu}} \right] \right| \\ & \quad \leq \overline{\lim_{n\to\infty}} (k_n - 1)\mathcal{B}_t(\{r_k\xi_k\}, b_n, \lfloor a_n \rfloor, m_n) = 0. \end{aligned}$$

Thus by the proof of Theorem 17.2.1 in [24] (cf. [6, Corollary 1.12, p. 31]), Theorem 6.6 in [6] and hypothesis, we have

$$\begin{aligned} \overline{\lim_{n\to\infty}} \left| E\left[e^{itb_n^{-1}\sum_{j=1}^{k_n} X_{nj}} \right] - \left(E\left[e^{itb_n^{-1}X_{n1}} \right] \right)^{k_n} \right| & \leq \overline{\lim_{n\to\infty}} k_n \mathcal{B}_t(\{r_k\xi_k\}, b_n, \lfloor a_n \rfloor, m_n) \\ & = 0. \end{aligned}$$

Further, by Chebyshev's inequality

$$P[|X_n''| > \epsilon\tau_n] \leq \frac{\tau^2_{n-k_n(p_n+q_n)}}{\epsilon^2\tau_n^2}.$$

Since $\tau_n^2 \sim b_n^2$, we conclude that $\tau_n^{-1}X_n'' \to_P 0$. Similarly $\tau_n^{-1}\sum_{j=1}^{k_n} X'_{nj} \to_P 0$ so that $\mathcal{L}(\tau_n^{-1}T_n)$, $\mathcal{L}(\tau_n^{-1}\sum_{j=1}^{k_n} X_{nj})$ and $\mathcal{L}^{*k_n}(\tau_n^{-1}X_{n1})$ (i.e. the distribution of the sum k_n independent copies of $\tau_n^{-1}X_{n1}$) are the same in the limit. Therefore by the Normal Convergence Criterion [31, p. 295]

$$\mathcal{L}\left(\tau_n^{-1} \left(\sum_{k=1}^{n} X_{nk} \right) \right) \to_w \mathcal{N}(0, 1)$$

if and only if for every $\varepsilon > 0$

$$k_n \frac{E[X_{n1}^2 I_{[|X_{n1}|\geq\varepsilon\tau_n]}]}{\tau_n^2} \to 0 \quad \text{as } n \to \infty. \tag{4.16}$$

By Khinchine's [44, p. 124] and Minkowski's inequalities and (4.9) with $p = (2 + \frac{\delta}{2})$ we obtain

$$\begin{aligned} E|X_{n1}|^{2+\frac{\delta}{2}} &= E\left|\sum_{i=1}^{p_n} r_i \xi_i I_{[|\xi_i|<b_n]}\right|^{2+\frac{\delta}{2}} \leq BE\left[\left(\sum_{i=1}^{p_n} \xi_i^2 I_{[|\xi_i|<b_n]}\right)^{\frac{2+\frac{\delta}{2}}{2}}\right] \\ &\leq B\left(\sum_{i=1}^{p_n} E^{\frac{4}{4+\delta}}\left[|\xi_i|^{2+\frac{\delta}{2}} I_{[|\xi_i|<b_n]}\right]\right)^{1+\frac{\delta}{4}} \\ &= Bp_n^{1+\frac{\delta}{4}} U_{2+\frac{\delta}{2}}(b_n), \quad \text{for some } B > 0. \end{aligned}$$

This, the inequality $x^2 \leq |x|^{2+\frac{\delta}{2}}$ for $|x| \geq 1$, (4.9) and (4.15) yield

$$\begin{aligned} \overline{\lim_{n\to\infty}}\, k_n \frac{E[X_{n1}^2 I_{[|X_{n1}|\geq\varepsilon\tau_n]}]}{\tau_n^2} &\leq \overline{\lim_{n\to\infty}} \frac{nE|X_{n1}|^{2+\frac{\delta}{2}}}{p_n \varepsilon^{\frac{\delta}{2}} \tau_n^{2+\frac{\delta}{2}}} \leq \overline{\lim_{n\to\infty}} \frac{Bp_n^{\frac{\delta}{4}} U_{2+\frac{\delta}{2}}(b_n)}{\varepsilon^{\frac{\delta}{2}} b_n^{\frac{\delta}{2}} U_2(b_n)} \\ &\leq \overline{\lim_{n\to\infty}} \frac{B(m_n a_n)^{\frac{\delta}{4}} U_{2+\frac{\delta}{2}}(b_n)}{\varepsilon^{\frac{\delta}{2}} b_n^{\frac{\delta}{2}} U_2(b_n)} \\ &\leq \overline{\lim_{n\to\infty}} \frac{B\varepsilon^{-\frac{\delta}{2}} m_n^{\frac{\delta}{4}} U_{2+\frac{\delta}{2}}(b_n)}{\sqrt{U_{2+\delta}(b_n) U_2(b_n)}} = 0. \end{aligned}$$

Thus (4.16) holds so that $\mathcal{L}(\tau_n^{-1} T_n)$ is asymptotically $\mathcal{N}(0, 1)$. Further by $\tau_n^2 \sim b_n^2$, $\mathcal{L}(b_n^{-1} T_n)$ is asymptotically $\mathcal{N}(0, 1)$ too. Now observe that for $1 \leq k \leq n$ the conditional distributions

$$(Y_1, \ldots, Y_n) \quad \text{and} \quad (Y_1, \ldots, -Y_k, \ldots, -Y_n)$$

given $\xi_1, \ldots, \xi_n$ are the same. Thus, by the arguments used for (2.5) we obtain

$$P\left[\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} \frac{Y_i}{b_n}\right| > t\right] \leq 2P\left[\left|\sum_{i=1}^{n} \frac{Y_i}{b_n}\right| > t\right]. \tag{4.17}$$

Since b_n^2 is 1-regularly varying for any $0 \leq s < t \leq 1$

$$b_{\lfloor nt\rfloor - \lfloor ns\rfloor} \sim b_n \cdot \sqrt{t-s} \quad \text{as } n \longrightarrow \infty,$$

by the Convergence of Types Theorem

$$\mathcal{L}\left(\frac{S_{\lfloor nt\rfloor} - S_{\lfloor ns\rfloor}}{b_n}\right) \longrightarrow_w \mathcal{N}(0, t-s) = \mathcal{W}(t) - \mathcal{W}(s),$$

where $\mathcal{W}$ is the Wiener process, $S_n = \sum_{k=1}^{n} Y_k$. By (1.2), for $m_n = o(b_n), m_n \to \infty$, the Cramér–Wold device [3, Theorem 7.7, p. 49]

$$\mathcal{L}\left(\frac{S_{\lfloor ns\rfloor - m_n}}{b_n}, \frac{S_{\lfloor nt\rfloor} - S_{\lfloor ns\rfloor}}{b_n}\right) \to_w (\mathcal{W}(s), \mathcal{W}(t) - \mathcal{W}(s)),$$

and by (1.2) the limiting random variables are independent. Further, by the choice of m_n

$$\mathcal{L}\left(\frac{S_{\lfloor ns\rfloor}}{b_n}, \frac{S_{\lfloor nt\rfloor} - S_{\lfloor ns\rfloor}}{b_n}\right) \to_w (\mathcal{W}(s), \mathcal{W}(t) - \mathcal{W}(s)),$$

and we get the convergence of finite dimensional distributions. In view of [3, Theorems 15.5 and 19.1], this yields the Weak Invariance Principle if we prove the tightness of $\{b_n^{-1} S_{[nt]}\}_{n\in\mathbb{N}}$, i.e.:

$$\lim_{\delta\downarrow 0} \overline{\lim_{n\to\infty}} P\left[\sup_{|t-s|<\delta} b_n^{-1}|S_{\lfloor nt\rfloor} - S_{\lfloor ns\rfloor}| > \epsilon\right] = 0, \tag{4.18}$$

for any $\epsilon > 0$. To this end let $s \le t$, $0 < \delta < 1$ and take partition of $(0, 1]$ at the points $k\delta$, $k \in \mathbb{N}$. Then either $k\delta \le s \le t \le (k+1)\delta$ and

$$\left\{|S_{\lfloor nt\rfloor} - S_{\lfloor ns\rfloor}| > \epsilon b_n\right\} \subseteq \left\{|S_{\lfloor nt\rfloor} - S_{\lfloor nk\delta\rfloor}| > \frac{\epsilon}{2} b_n\right\} \cup \left\{|S_{\lfloor ns\rfloor} - S_{\lfloor nk\delta\rfloor}| > \frac{\epsilon}{2} b_n\right\},$$

or $k\delta \le s \le (k+1)\delta \le t \le (k+2)\delta$ and $\{|S_{\lfloor nt\rfloor} - S_{\lfloor ns\rfloor}| > \epsilon b_n\}$ is contained in

$$\left\{|S_{\lfloor ns\rfloor} - S_{\lfloor nk\delta\rfloor}| > \frac{\epsilon}{3} b_n\right\} \cup \left\{|S_{\lfloor nt\rfloor} - S_{\lfloor n(k+1)\delta\rfloor}| > \frac{\epsilon}{3} b_n\right\}$$
$$\cup \left\{|S_{\lfloor n(k+1)\delta\rfloor} - S_{\lfloor nk\delta\rfloor}| > \frac{\epsilon}{3} b_n\right\}.$$

Consequently,

$$\begin{aligned} & P\left[\sup_{|t-s|<\delta} |S_{\lfloor nt\rfloor} - S_{\lfloor ns\rfloor}| > \epsilon b_n\right] \\ & \le P\left(\bigcup_{k=0}^{\lfloor \frac{1}{\delta}\rfloor} \left\{\sup_{\lfloor nk\delta\rfloor \le r \le \lfloor n(k+1)\delta\rfloor} |S_r - S_{\lfloor nk\delta\rfloor}| > \frac{\epsilon}{3} b_n\right\}\right) \\ & \le \frac{1}{\delta} P\left(\max_{j \le \lfloor n\delta\rfloor + 1} |S_j| > \frac{\epsilon}{3} b_n\right). \end{aligned} \tag{4.19}$$

Now, the regular variation of b_n yields that for sufficiently large n we have by (4.17)

$$\frac{1}{\delta} P\left[\max_{j\le\lfloor n\delta\rfloor+1} |S_j| > \frac{\epsilon}{3} b_n\right] \le \frac{2}{\delta} P\left[|S_{\lfloor n\delta\rfloor+1}| > \frac{\epsilon}{3} b_n\right] \to \frac{2}{\delta} P\left[|\mathcal{N}| > \frac{\epsilon}{3\sqrt{\delta}}\right],$$
as $n \to \infty$.

The latter convergence holds by (4.10) and $\mathcal{L}(b_n^{-1} T_n) \to_w \mathcal{N}(0, 1)$. On the other hand $\frac{2}{\delta} P[|\mathcal{N}| > \frac{\epsilon}{3\sqrt{\delta}}]$ tends to 0 as $\delta \downarrow 0$ thus by (4.19) we get (4.18). By the choice of b_n we see that

$$\lim_{n\to\infty} P\left[\sup_{0\le t\le 1} |T_{n\lfloor nt\rfloor} - S_{\lfloor nt\rfloor}| > \epsilon b_n\right] \le \lim_{n\to\infty} nP[|\xi_0| > b_n] = 0,$$

thus we infer that (4.18) holds for $\{T_{n\lfloor nt\rfloor}\}_{n\in\mathbb{N}}$, too. Therefore

$$\mathcal{L}(b_n^{-1} T_{n\lfloor nt\rfloor}) \to_w \mathcal{W}(t),$$

in $D(0, 1]$, and by Lemma 3 in [37] (see also Lemma 2 in [20] and [38]) we obtain that

$$b_n^{-2} \sum_{i=1}^{\lfloor nt\rfloor} Y_{ni}^2 \to_P t, \quad t \in (0, 1].$$

By this and (4.11) the sequence $\{\xi_k^2\}_{k\in\mathbb{N}}$ is relatively stable with $c_n = b_n^2$. This completes the proof of Theorem 1. $\square$

5. Examples

In this section we shall give some applications of Theorems 1 and 2 and discuss the rate in (1.2). For this, suppose $L(x)$ is as in Lemma 3. Then, substituting b_n in (4.8)

$$a_n = \left(\frac{b_n^\delta U_2(b_n)}{U_{2+\delta}(b_n)}\right)^{\frac{2}{\delta}} \sim \left(\frac{\delta}{L(b_n)}\int_A^{b_n}\frac{L(u)}{u}du\right)^{\frac{2}{\delta}},$$

where $b_n^2 \sim nU_2(b_n)$. In view of this and l'Hôpital's rule

for $L(x) = (\ln x)^{1+\beta}$, $\beta > -2$,

$$b_n^2 \sim 2n\frac{\left(\frac{1}{2}\ln n\right)^{2+\beta}}{2+\beta}, \qquad a_n \sim \left(\frac{\delta}{2(2+\beta)}\ln n\right)^{\frac{2}{\delta}}, \tag{5.20}$$

since $\int_A^x \frac{(\ln u)^{1+\beta}}{u}du \sim \frac{(\ln x)^{2+\beta}}{2+\beta}$ and for $L(x) = \frac{1}{\ln x}$

$$b_n^2 \sim 2n\ln\ln n, \qquad a_n \sim \left(\frac{\delta}{2}(\ln n)(\ln\ln n)\right)^{\frac{2}{\delta}}, \tag{5.21}$$

since $\int_A^x \frac{1}{u\ln u}du \sim \ln\ln x$.

The following example demonstrates the method for obtaining relative stability and the CLT for ARCH(1) processes. This method should apply to other time series models (e.g. GARCH).

Example 1. Consider a class of ARCH(1) processes: $\{\xi_k\}_{k\in\mathbb{N}\cup\{0\}}$ is recursively defined by

$$\xi_k = \zeta_k\sqrt{1+\xi_{k-1}^2}, \quad k \geq 1,$$

where $\{\zeta_k\}_{k\in\mathbb{N}}$ are i.i.d. and $E|\zeta_1|^p < 1, p \geq 1$. If ζ_1 has standard normal distribution, independent of ξ_0, where $\mathcal{L}(\xi_0^2) = \mathcal{L}(\sum_{k=1}^\infty \prod_{\nu=1}^k \zeta_\nu^2)$, then $\{\xi_k\}_{k\in\mathbb{N}\cup\{0\}}$ is exponentially θ-weakly dependent (cf. [11, Section 2.3]) hence $\theta_n \leq K\varrho^n, K > 0, \varrho < 1$. In fact ξ_k is exponentially fast strongly mixing if $\mathcal{L}(\zeta_1)$ is absolutely continuous (cf. [13, Proposition 6, p. 107], [10, Lemma A.2, p. 2077]). Furthermore, the marginal of $\{\xi_k\}_{k\in\mathbb{N}\cup\{0\}}$ satisfies

$$x^2P[|\xi_0| > x] \sim C, \quad C = (2-\ln 2-\gamma)^{-1} \approx 1.37054424-,$$

where γ is Euler's constant (cf. [16, Theorem 8.4.12, p. 467]).

For this model condition (1.2) is satisfied with $\delta = 1$ and $a_n = \frac{1}{4}\ln^2 n$. Now, by Theorem 2 on p. 283 in [19] $U_2(x)$ varies slowly and $U_2(\infty) = \infty$. Moreover, our case is related to (5.20) with $\beta = -1$ so $b_n^2 \sim Cn\ln n, a_n = \frac{1}{4}\ln^2 n$. Therefore

$$\begin{aligned}\frac{n}{b_n}\theta_{\lfloor a_n\rfloor} &\leq \frac{Kn}{\sqrt{Cn\ln n}}\varrho^{\lfloor\frac{\ln^2 n}{4}\rfloor}\\ &\leq Ke^{-\ln\varrho}e^{\frac{1}{2}(\ln n-\ln C-\ln\ln n)+(\ln\varrho)(\ln^2 n)} = Ke^{-\ln\varrho}e^{(\ln\varrho)(\ln^2 n)(1+o(1))} = o(1).\end{aligned}$$

Thus $\{\xi_k^2\}_{k\in\mathbb{N}}$ is relatively stable with $c_n = Cn\ln n$. Because the solution is of the form $r_k|\xi_k|$ consequently by Theorem 2 we get the CLT for $\{\xi_k\}_{k\in\mathbb{N}}$ with normalization $b_n = \sqrt{Cn\ln n}$.

In the previous example we applied relative stability to get the CLT. Theorem 2 allows us to obtain some new relative stability results if we know that the CLT for $\{r_k\xi_k\}_{k\in\mathbb{N}}$ holds.

Example 2. (a) Suppose $\sum_{k=1}^{\infty}\rho(2^k)<\infty$, $\rho(1)<1$ and $E[\xi_0^2 I_{[|\xi_0|\le x]}]$ is slowly varying. Thus by Theorem 6.6 on p. 199 in [6] and Theorem 1 in [5] we have $\mathcal{L}(b_n^{-1}\sum_{k=1}^n r_k\xi_k)\to_w \mathcal{N}(0,1)$. So that by Theorem 2 we get $b_n^{-2}\sum_{k=1}^n \xi_k^2 \to_P 1$ in this case.

(b) Suppose $\{\xi_k\}_{k\in\mathbb{Z}}$ is φ-mixing and $E[\xi_0^2 I_{[|\xi_0|\le x]}]$ is slowly varying. Then by inequality (3.8) on p. 298 in [34] $\{b_n^{-2}\max_{1\le k\le n}\xi_k^2 I_{[|\xi_k|\le b_n]}\}_{n\in\mathbb{N}}$ is uniformly integrable. Now, by Theorem 6.6 on p. 199 in [6], Proposition 2, Lemma 1, Lévy's and Chebyshev's inequalities $\{b_n^{-2}(\sum_{k=1}^n r_k\xi_k I_{[|\xi_k|\le b_n]})^2\}_{n\in\mathbb{N}}$ is uniformly integrable, too. Thus $\{\xi_k^2\}_{k\in\mathbb{N}}$ is relatively stable as it has been proved in [42].

(c) Suppose $E[\xi_0^2 I_{[|\xi_0|\le x]}]$ is slowly varying. The Rosenthal inequality (cf. [6, Theorem 11.23, p. 380; Theorem 6.6 on p. 199]) holds if $\lim_n \rho_n^*(\{\xi_k\})<1$ (see definition on p. 170 in vol. I, [6]), i.e.

$$E\left[b_n^{-4}\left(\sum_{k=1}^n r_k\xi_k I_{[|\xi_k|\le b_n]}\right)^4\right]\le C\left(\frac{n}{b_n^4}E\left[\xi_0^4 I_{[|\xi_0|\le b_n]}\right]+\left(\frac{n}{b_n^2}E[\xi_0^2 I_{[|\xi_0|\le b_n]}]\right)^2\right).$$

Thus by the definition of $\{b_n\}_{n\in\mathbb{N}}$ and Lemma 2 (with $\delta=2$) $\{b_n^{-2}(\sum_{k=1}^n r_k\xi_k I_{[|\xi_k|\le b_n]})^2\}_{n\in\mathbb{N}}$ is uniformly integrable. Therefore if $\{\xi_k\}_{k\in\mathbb{Z}}$ is also α-mixing then $\{\xi_k^2\}_{n\in\mathbb{N}}$ is relatively stable (cf. Theorem 11.25 on p. 387 in [6]).

In the following examples the rate in $\mathcal{B}_t$ conditions is discussed. For this we require some results which are of independent interest. They describe the asymptotic of tail probability for the sum of two independent random variables with heavy tailed component and are related to Proposition on p. 278 in [19] (see also [16, Lemma 1.3.1, p. 37]) and Problem 27 on p. 288 in [19].

Proposition 6. *Suppose X, Y are independent random variables such that $x^p P[X>x]$ varies slowly at infinity for some $p>0$. Then for any $u\in(0,1]$*

$$P[X+Y>x;\,X>ux]\sim P[X>x]\quad \text{as } x\to\infty. \tag{5.22}$$

Proof of Proposition 6. Let $\delta\in(0,1)$. For $x>0$

$$\begin{aligned}P[X+Y>x;\,X>ux]&=P[X>ux]-P[X+Y\le x;\,X>ux]\\&\le P[X>ux]-P[X\le(1-\delta)x;\,Y<\delta x;\,X>ux]\\&=P[X>ux]-P[Y<\delta x;\,X>ux]\\&\quad+P[X>(1-\delta)x;\,Y<\delta x;\,X>ux]\\&=P[X>ux]\cdot P[Y\ge\delta x]+P[X>(1-\delta)x]\cdot P[Y<\delta x]\end{aligned}$$

if $u<(1-\delta)<1$, and

$$\begin{aligned}P[X+Y>x;\,X>ux]&\ge P[X>(1+\delta)x;\,Y>-\delta x;\,X>ux]\\&=P[X>(1+\delta)x;\,Y>-\delta x]\\&=P[X>(1+\delta)x]\cdot P[Y>-\delta x]\end{aligned}$$

if $u \le 1$. Consequently

$$\overline{\lim_{x\to\infty}} \frac{P[X+Y>x;\, X>ux]}{P[X>x]} \le \overline{\lim_{x\to\infty}} \left(\frac{P[X>ux]}{P[X>x]} \cdot P[Y \ge \delta x] + \frac{P[X>(1-\delta)x]}{P[X>x]} \cdot P[Y<\delta x] \right) = \frac{1}{(1-\delta)^p}$$

and

$$\underline{\lim}_{x\to\infty} \frac{P[X+Y>x;\, X>ux]}{P[X>x]} \ge \frac{P[X>(1+\delta)x]}{P[X>x]} \cdot P[Y>-\delta x] = \frac{1}{(1+\delta)^p}.$$

If $u = 1$ then

$$1 \ge \overline{\lim_{x\to\infty}} \frac{P[X+Y>x;\, X>x]}{P[X>x]} \ge \underline{\lim}_{x\to\infty} \frac{P[X+Y>x;\, X>x]}{P[X>x]} \ge \frac{1}{(1+\delta)^p}.$$

Letting $\delta \to 0$ yields the result. □

Corollary 1. *Suppose X, Y are independent symmetric random variables such that $x^p P[X>x]$ varies slowly at infinity and $E|Y|^{p+\epsilon} < \infty$, $p, \epsilon > 0$ or even $P[|Y|>x] = o(P[|X|>x])$. Then*

$$P[|X+Y|>x] \sim P[|X|>x] \quad \textit{as } x \to \infty.$$

Proof of Corollary 1. By the Markov inequality

$$\overline{\lim_{x\to\infty}} \frac{P[|Y|>x]}{P[|X|>x]} = \overline{\lim_{x\to\infty}} \frac{x^{p+\epsilon} P[|Y|>x]}{x^{p+\epsilon} P[|X|>x]} \le \overline{\lim_{x\to\infty}} \frac{E|Y|^{p+\epsilon}}{x^{p+\epsilon} P[|X|>x]} = 0. \tag{5.23}$$

Since for $u < (1-\delta)$, $\delta \in (0,1)$

$$\begin{aligned} P[X+Y>x;\, X\le ux] &\le P[X\le ux] - P[X<(1-\delta)x;\, Y\le \delta x;\, X\le ux] \\ &= P[X\le ux] \cdot P[Y>\delta x], \end{aligned}$$

by Proposition 6 we obtain the desired result. Note that in view of (5.23) the condition $E|Y|^{p+\epsilon} < \infty$ can be replaced by $P[|Y|>x] = o(P[|X|>x])$. □

Example 3. (a) (Doukhan) Let $\{X_k\}_{k\in\mathbb{Z}}$ be a centered stationary Gaussian process. Define

$$\xi_k = \exp\left\{\frac{1}{4}X_k^2\right\}.$$

Thus (cf. [18, Lemma 2, p. 175])

$$x^2 P[\xi_1^2 > x] \sim \frac{1}{\sqrt{2\pi \ln x}}.$$

Therefore

$$E[\xi_1^2 I_{[\xi_1\le x]}] \sim \frac{2}{\sqrt{2\pi}} \int_e^x \frac{du}{u\sqrt{\ln u}} \sim \frac{4}{\sqrt{2\pi}}\sqrt{\ln x}$$

and $E[\xi_1^2 I_{[\xi_1\le x]}]$ is a slowly varying function. By Lemma 3

$$b_n^2 = \frac{2n\sqrt{\ln n}}{\sqrt{\pi}}, \qquad a_n = \left(\frac{\delta \ln n}{\sqrt{2\pi}}\right)^{\frac{2}{\delta}}.$$

Suppose that the spectral density f of X_k is such that $f(x) \geq a > 0$. Therefore (cf. [13, Section 2.1.1]) if $|\mathrm{Cov}(X_0, X_k)| = O(e^{-c\sqrt{k}})$, $c > 0$, then by the heredity of α coefficient

$$\alpha_n(\{\xi_k\}) \leq \alpha_n(\{X_k\}) \leq \frac{1}{a} \sum_{k \geq n} |\mathrm{Cov}(X_0, X_k)| = O(e^{-2\sqrt{n}}).$$

Hence for $\delta = \frac{1}{2}$

$$\overline{\lim_{n\to\infty}} \frac{n}{a_n} \alpha_n(\lfloor a_n \rfloor) \leq \overline{\lim_{n\to\infty}} \exp\left\{-\frac{4}{\pi} \ln^2 n(1 + o(1))\right\} = 0$$

and $\{\xi_k^2\}_{k\in\mathbb{N}}$ is relatively stable with $c_n = \frac{2n\sqrt{\ln n}}{\sqrt{\pi}}$.

(b) Let $\{X_k\}_{k\in\mathbb{Z}}$ be a strictly stationary κ-weakly dependent sequence such that $\mathcal{L}(X_1)$ is symmetric (cf. [15, Proposition 4]). Define

$$\xi_k = X_k + \zeta_k,$$

where symmetric $\{\zeta_k\}_{k\in\mathbb{Z}}$ are independent of $\{X_k\}_{k\in\mathbb{Z}}$ and i.i.d. with $L(x) = x^2 P[|\zeta_1| > x]$, where L is slowly varying and $L(x) \to \infty$ as $x \to \infty$. If $x^2 P[|X_1| > x] \sim L(L(x))$ then by Corollary 1 $x^2 P[|\xi_1| > x]$ is slowly varying. Thus $E[\xi_1^2 I_{[|\xi_1| \leq x]}]$ is slowly varying too. By the estimate

$$\frac{n}{a_n} \mathcal{B}_t(\{r_k \xi_k\}, b_n, \lfloor a_n \rfloor, m) \leq \frac{nm^2 a_n t^2}{b_n^2} \kappa(\lfloor a_n \rfloor),$$

if $n\kappa(n) = O(1)$. Thus condition (1.2) is satisfied. Moreover for $\kappa(n) = O(\frac{\ln \ln n}{n})$ and $x^2 P[|\zeta_1| > x] \sim \frac{1}{\ln x}$ ((5.21) with $\delta = 4$) thus $\{\xi_k^2\}_{k\in\mathbb{N}}$ is relatively stable with $c_n = 2n \ln \ln n$.

(c) Let $\{X_k\}_{k\in\mathbb{Z}}$ be a strictly stationary θ-weakly dependent sequence such that $\mathcal{L}(X_1)$ is symmetric (cf. [11, Theorem 3.3, p. 46]). Define ξ_k as in the previous example but with $x^2 P[|\zeta_1| > x]$ slowly varying and $E\zeta_1^2 I_{[|\zeta_1| \leq x]} \to \infty$ as $x \to \infty$. If $P[|X_1| > x] = o(P[|\zeta_1| > x])$ then by Corollary 1 $E[\xi_1^2 I_{[|\xi_1| \leq x]}]$ is slowly varying. Suppose that $x^2 P[|\zeta_1| > x] \sim \ln^3 x$ ((5.20) with $\delta = \frac{1}{2}, \beta = 2$) and $\theta(n) = O(e^{-\sqrt[4]{n}})$. Now in condition (1.2) we have the upper bound $\frac{nmt}{b_n} \theta(\lfloor a_n \rfloor) = O(\frac{1}{\ln^2 n})$ thus $\{\xi_k^2\}_{k\in\mathbb{N}}$ is relatively stable with $c_n = 2^{-5} n \ln^4 n$.

In the last example we show that there are non-strongly mixing strictly stationary sequences for which Theorem 2 does not apply.

Example 4. Suppose $\{X_k\}_{k\in\mathbb{Z}}$ is a non-negative i.i.d. sequence independent of a standard normal random variable ζ and $x^2 P[X_1 > x]$ is slowly varying. Set $\xi_k = X_k \zeta$. By Proposition 3 in [8] (cf. [36], p. 88) $x^2 P[|\xi_1| > x]$ is slowly varying and therefore $E[\xi_1^2 I_{[|\xi_1| \leq x]}]$ is slowly varying, too. Now,

$$\mathcal{L}\left(b_n^{-1} \sum_{k=1}^{n} r_k X_k\right) \to_w \mathcal{N}(0, 1)$$

and therefore $\mathcal{L}(b_n^{-1} \sum_{k=1}^{n} r_k \xi_k)$ has limiting bilateral exponential distribution (cf. [19, p. 503]). On the other hand it is easy to see that $\{b_n^{-2} \sum_{k=1}^{n} \xi_k^2 I_{[|\xi_k| \leq b_n]}\}_{n\in\mathbb{N}}$ is uniformly integrable.

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