

Decomposable stationary distribution of a multidimensional SRBM

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Abstract

We focus on the stationary distribution of a multidimensional semimartingale reflecting Brownian motion (SRBM) on a nonnegative orthant. Assuming that the stationary distribution exists and is decomposable—equal to the product of two marginal distributions, we prove that these marginal distributions are the stationary distributions of some lower dimensional SRBMs, whose data can be explicitly computed through that of the original SRBM. Thus, under the decomposability condition, the stationary distribution of a high dimensional SRBM can be computed through those of lower dimensional SRBMs. Next, we derive necessary and sufficient conditions for some classes of SRBMs to satisfy the decomposability condition.

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1. Introduction

We are concerned with a d -dimensional semimartingale reflecting Brownian motion (SRBM) that lives on the nonnegative orthant $\mathbb{R}_+^d$, where $\mathbb{R}_+$ is the set of all nonnegative real numbers. The

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SRBM is specified by a $d \times d$ covariance matrix Σ , a drift vector $\mu \in \mathbb{R}^d$, and a $d \times d$ reflection matrix R . Namely, (Σ, μ, R) is the modeling primitives of the SRBM on $\mathbb{R}_+^d$. As usual, we assume that Σ is positive definite and R is completely- $\mathcal{S}$. See [Appendix A](#) for the definitions of matrix classes used in this paper. A (Σ, μ, R) -SRBM Z has the following semimartingale representation:

$$Z(t) = Z(0) + X(t) + RY(t) \in \mathbb{R}_+^d, \quad t \geq 0, \quad (1.1)$$

$$X = \{X(t), t \geq 0\} \text{ is a } (\Sigma, \mu)\text{-Brownian motion}, \quad (1.2)$$

$$Y(0) = 0, Y(\cdot) \text{ is nondecreasing}, \quad (1.3)$$

$$\int_0^\infty Z_i(t) dY_i(t) = 0 \quad \text{for } i = 1, \dots, d. \quad (1.4)$$

For the complete definition of an SRBM $Z = \{Z(t); t \geq 0\}$, we refer to Section A.1 of [8] (see [25,26] for more details).

Our focus is on the stationary distribution of the d -dimensional SRBM. Throughout this paper, we assume that the stationary distribution exists. As a consequence, the primitive data satisfies the following condition:

$$R \text{ is nonsingular, and } R^{-1}\mu < 0, \quad (1.5)$$

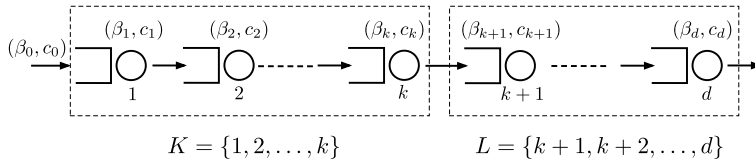
where vector inequality is interpreted entrywise. If R is either a $\mathcal{P}$ -matrix for $d = 2$ or an $\mathcal{M}$ -matrix for an arbitrary $d \geq 1$, then this condition is known to be sufficient, but generally not for $d \geq 3$ (see, e.g., [3]).

Let $J \equiv \{1, 2, \dots, d\}$. A pair (K, L) is said to be a partition of J if $K \cup L = J$ and $K \cap L = \emptyset$. We consider conditions for the stationary distribution of an SRBM in $\mathbb{R}_+^d$ to be the product of two marginal distributions associated with a partition (K, L) of the set J . Such a stationary distribution is said to be *decomposable* with respect to K and L . We have two major contributions for the decomposability of the stationary distribution.

We first characterize, in [Theorem 1](#), two marginal distributions associated with a partition (K, L) under the decomposability assumption. We prove that they are the stationary distributions of some $|K|$ - and $|L|$ -dimensional SRBMs, where $|U|$ denotes the cardinality of a set U . We also identify the data for these lower dimensional SRBMs. Thus, under the decomposability assumption, we can obtain the stationary distribution of the original SRBM by computing those of the lower dimensional ones.

However, this characterization of the marginal distributions is not sufficient for the decomposability. So, we next consider necessary and sufficient conditions for the decomposability. We obtain those conditions for several classes of SRBMs ([Theorem 2](#) and [Corollary 2](#)). These classes include diffusion limits of tandem queues and of queueing networks that have two sets of nodes with feed-forward routing between these two sets. Note that the decomposability does not mean a complete separation of such a network into two subnetworks. We illustrate a tandem queue next.

Consider a d -station generalized Jackson network in series, which is referred to as a tandem queue. In this tandem queue, the interarrival times to station 1 are assumed to be i.i.d. with mean $1/\beta_0$ and coefficient of variation (CV) c_0 . The service times at station i are assumed to be i.i.d. with mean $1/\beta_i$ and CV c_i , $i \in J$ (see [Fig. 1](#)). The diffusion limit of queue length process for this tandem queue is known to be the d -dimensional SRBM with the following reflection matrix R , the covariance matrix Σ , and the drift vector μ (see, e.g., Section 7.5 of [4]).

Fig. 1. The d -station tandem queue partitioned into two blocks.

$$R_{ij} = \begin{cases} 1, & i = j \text{ for } i = 1, 2, \dots, d, \\ -1, & j = i - 1 \text{ for } i = 2, \dots, d, \\ 0, & \text{otherwise,} \end{cases} \quad (1.6)$$

$$\Sigma_{ij} = \beta_0 \times \begin{cases} c_{i-1}^2 + c_i^2, & i = j \text{ for } i = 1, 2, \dots, d, \\ -c_{i-1}^2, & j = i - 1 \text{ for } i = 2, \dots, d, \\ -c_i^2, & j = i + 1 \text{ for } i = 1, \dots, d - 1, \\ 0, & \text{otherwise,} \end{cases} \quad (1.7)$$

$$\mu_i = \beta_{i-1} - \beta_i \quad \text{for } i = 1, \dots, d. \quad (1.8)$$

For example, when $d = 3$, R and Σ are given by

$$R = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad \Sigma = \beta_0 \begin{pmatrix} c_0^2 + c_1^2 & -c_1^2 & 0 \\ -c_1^2 & c_1^2 + c_2^2 & -c_2^2 \\ 0 & -c_2^2 & c_2^2 + c_3^2 \end{pmatrix}.$$

We assume that Σ is nonsingular and condition (1.5) is satisfied, which is equivalent to $\beta_0 < \beta_i$ for $i = 1, 2, \dots, d$. It follows from [16] that the SRBM Z has a unique stationary distribution π . For each $k \in \{1, 2, \dots, d - 1\}$, let

$$K \equiv \{1, 2, \dots, k\}, \quad L \equiv \{k + 1, \dots, d\}.$$

See Fig. 1 for an illustration of these sets. It will be shown in Corollary 2 that if

$$c_0 = c_i \quad \text{for } i = 1, \dots, k, \quad (1.9)$$

then the d -dimensional stationary distribution is decomposable with respect to K and L .

We are motivated by applications of SRBMs and our recent work [10]. Multidimensional SRBMs have been widely used for queueing applications [27] and some areas including mathematical finance [23]. For these applications, it is important to obtain the stationary distribution in a tractable form. However, this is a very hard problem even for $d = 2$ unless the d -dimensional SRBM has a product form stationary distribution. A multidimensional distribution is said to have a product form if it is the product of one dimensional marginal distributions.

It is shown in [17] that this product form holds for the stationary distribution of an SRBM Z if and only if the following skew symmetry condition

$$2\Sigma = R\text{diag}(R)^{-1}\text{diag}(\Sigma) + \text{diag}(\Sigma)\text{diag}(R)^{-1}R^T \quad (1.10)$$

is satisfied, where for a matrix A , $\text{diag}(A)$ denotes the diagonal matrix whose entries are diagonals of A , and A^T denotes the transpose of A . Furthermore, under (1.10), the one-dimensional marginal stationary distribution in the i th coordinate has the exponential distribution with mean

$1/\alpha_i$, where column vector $\alpha \equiv (\alpha_1, \dots, \alpha_d)^T$ is given by

$$\alpha = -2\text{diag}(\Sigma)^{-1}\text{diag}(R)R^{-1}\mu. \quad (1.11)$$

Thus, the stationary distribution of Z is explicitly obtained under the skew symmetry condition (1.10). However, the condition (1.10) may be too strong in some applications. In particular, (1.11) is independent of covariances Σ_{ij} for $i \neq j \in J \equiv \{1, 2, \dots, d\}$. Still, product form based approximation is often used even though its accuracy cannot be assessed when condition (1.10) is not satisfied; see, for example, [18].

This product form based approximation may be improved by the decomposability. For example, let us consider the SRBM for the tandem queue depicted in Fig. 1, and assume the decomposability condition (1.9). Then, its $|K|$ -dimensional marginal is of product-form, which can be computed easily. Its $|L|$ -dimensional marginal is the stationary distribution of an $|L|$ -dimensional SRBM. If one adopts the algorithm in [6] to compute the stationary distribution of a d -dimensional SRBM, the computational complexity is in the order of d^{3n} , where n is the degree of polynomials used in the numerical approximation; see the discussion on page 78 of [6]. Therefore, using our decomposition result, the original d -dimensional stationary distribution can be computed in the order of $|L|^{3n} + d$ operations, a huge saving as compared with d^{3n} when $|L|$ is small or moderate.

In recent years, for $d = 2$, the tail asymptotics of the stationary distribution including decay rates have been well studied (e.g., see [1,8,9]) even though condition (1.10) is not satisfied. Those decay rates may be used for better approximations of a two-dimensional stationary distribution as recently shown for a two dimensional reflecting random walk in [20]. We hope that such a two-dimensional approximation can be used to develop better approximations for the stationary distribution of a high dimensional SRBM. Thus, weaker conditions than the product form will be useful to facilitate an approximation for a higher dimensional SRBM. A decomposability condition that is checkable by modeling primitives will considerably widen the applicability of a multidimensional SRBM.

We are also inspired by geometric interpretations in [10] for the product form characterization. That work focuses on the product form, but considers characterizations that are different from the skew symmetric condition (1.10). Among them, Corollary 2 of [10] shows that, for each pair of $i \neq j \in J$, the corresponding two-dimensional, marginal distribution is equal to the stationary distribution of some two-dimensional SRBM if the d -dimensional stationary distribution has a product form, that is, condition (1.10) is satisfied. This motivates us to consider the lower dimensional SRBMs corresponding to the marginal distributions under the decomposability.

Other than the product form case, there are some two-dimensional SRBMs whose stationary distributions are obtained in closed form. For example, in [12,13], the stationary densities have finite sums of exponential functions. However, those results are only obtained for very limited cases, and their expressions for the stationary densities are quite different from decomposability. Thus, we do not have immediate benefit of using them to study decomposability. However, we may consider them as another type of approximation. Thus, it may be interesting to combine them with decomposability to get better approximations.

This paper consists of four sections. We present our results, Theorems 1 and 2 and their corollaries, in Section 2. We discuss basic facts and preliminary results in Section 3.1. Theorems 1 and 2 are proved in Sections 3.2 and 3.3, respectively. We finally remark some future work in Section 4.

We will use the following notation unless otherwise stated (see Table 1).

Table 1

A summary of basic notation.

J	$\{1, 2, \dots, d\}$, and, for $U, V \subset J$
$A^{(U,V)}$	The $ U \times V $ submatrix of a d -dimensional square matrix A whose row and column indices are taken from U and V , respectively
x^U	$ U $ -dimensional vector with $x_i^U = x_i$ for $i \in U$, where x_i^U is the i th entry of x^U
$\uparrow x^U$	d -dimensional vector x with $x_i = x_i^U$ for $i \in U$ and $x_i = 0$ for $i \in J \setminus U$
$f^U(x^U)$	$f(\uparrow x^U)$ for function f from $\mathbb{R}^d$ to $\mathbb{R}$
$\langle x, y \rangle$	$\sum_{i=1}^d x_i y_i$ for $x, y \in \mathbb{R}^d$

For example, for $i \in U$, the i th entry of $A^{(U,V)}x^V$ is $\sum_{j \in V} [A^{(U,V)}]_{ij}x_j^V$, where both T_{ij} and $[T]_{ij}$ denote the (i, j) -entry of a matrix T .

2. Main results

To state our main results, we need the following notation. Let

$$Q = R^{-1}.$$

For a non-empty set $U \subset J$ such that $Q^{(U,U)}$ is invertible, define

$$\tilde{\Sigma}(U) = (Q^{(U,U)})^{-1}(Q\Sigma Q^T)^{(U,U)}((Q^{(U,U)})^{-1})^T, \quad (2.1)$$

$$\tilde{\mu}(U) = (Q^{(U,U)})^{-1}(Q\mu)^U, \quad (2.2)$$

$$\tilde{R}(U) = (Q^{(U,U)})^{-1}. \quad (2.3)$$

Theorem 1. Assume that R is completely- $\mathcal{S}$ and that the d -dimensional SRBM $Z = \{Z(t), t \geq 0\}$ has a stationary distribution π . Let K and L be a non-empty partition of J . Assume that $Z^K(0)$ and $Z^L(0)$ are independent under the stationary distribution π . Then, for $U = K$ and $U = L$,

- (a) $Q^{(U,U)}$ is invertible, and its inverse matrix $\tilde{R}(U)$ is completely- $\mathcal{S}$.
- (b) The $|U|$ -dimensional $(\tilde{\Sigma}(U), \tilde{\mu}(U), \tilde{R}(U))$ -SRBM has a stationary distribution that is equal to the distribution of $Z^U(0)$ under π .

Remark 1. If R is a $\mathcal{P}$ -matrix, then R is invertible and R^{-1} is also a $\mathcal{P}$ -matrix by Lemma 6 of [10]. Therefore part (a) is immediate when R is a $\mathcal{P}$ -matrix since each $\mathcal{P}$ -matrix is completely- $\mathcal{S}$. However, even under condition (1.5), the completely- $\mathcal{S}$ property of R does not imply that R^{-1} has the same property as shown by the following example:

$$R = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \quad \text{implies} \quad R^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix},$$

where R is completely- $\mathcal{S}$, but R^{-1} is not. Thus, part (a) of the theorem is not obvious at all.

Theorem 1 will be proved in Section 3.2, and the corollary below is proved in Appendix B.

Corollary 1. Under the same assumptions of Theorem 1, for $i \in J$, assume that $Z^{(i)}(0)$ and $Z^{J \setminus \{i\}}(0)$ are independent under the stationary distribution π . Then, $Z^{(i)}(0)$ has the exponential

distribution with mean $1/\lambda_i$, where

$$\lambda_i = \Delta_i Q_{ii}, \quad (2.4)$$

$$\Delta_i = -\frac{2\langle \mu, (Q^T)^{(i)} \rangle}{\langle (Q^T)^{(i)}, \Sigma(Q^T)^{(i)} \rangle} > 0, \quad (2.5)$$

and $(Q^T)^{(i)}$ is the i th column of Q^T .

Remark 2. The parameter λ_i in (2.4) has a geometric interpretation; see (B.1) in the proof of Corollary 1. Note that λ_i uses information on covariance Σ_{ij} in general, so it may be different from α_i of (1.11), although we must have $\alpha_i = \lambda_i$ if the stationary distribution has a product form. This fact can quickly be checked by using Lemma 2 in [10], which characterizes the skew symmetric condition in terms of moment generating functions (see Appendix B for details). Since the exponential distribution in Corollary 1 is obtained under the weaker condition than the product form condition (1.10) for $d \geq 3$, it is intuitively clear that one should use λ_i instead of α_i in the product form approximation of a stationary distribution. We will further discuss this issue in Section 4.

In general, (a) and (b) of Theorem 1 are necessary but not sufficient for $Z^K(0)$ and $Z^L(0)$ to be independent under the stationary distribution π . For example, if $J = \{1, 2\}$, then the marginal exponential distributions are determined by the mean $1/\lambda_i$ for $i = 1, 2$ by Corollary 1, but these marginals are not sufficient for the skew symmetric condition, which is equivalent to that $Z^{(1)}(0)$ and $Z^{(2)}(0)$ are independent. This is because a condition weaker than the decomposability condition is used in the proof of Theorem 1 and therefore of Corollary 1. This fact will be detailed in Section 4. Thus, it would be interesting to seek additional conditions which imply the decomposability. However, to identify these extra conditions is generally a hard problem, so we consider a relatively simple situation. For this, we consider SRBMs arising from queueing networks that have two sets of stations with feed-forward routing between these two sets.

Theorem 2. Assume that R is completely- S and that the d -dimensional SRBM $Z = \{Z(t), t \geq 0\}$ has a stationary distribution π . Let K and L be a non-empty partition of J , and assume that

$$R^{(K,L)} = 0. \quad (2.6)$$

If $Z^K(0)$ and $Z^L(0)$ are independent under π and if $Z^K(0)$ is of product form under π , then

$$\begin{aligned} 2\Sigma^{(K,K)} &= R^{(K,K)} \text{diag}(R^{(K,K)})^{-1} \text{diag}(\Sigma^{(K,K)}) \\ &\quad + \text{diag}(\Sigma^{(K,K)}) \text{diag}(R^{(K,K)})^{-1} (R^{(K,K)})^T, \end{aligned} \quad (2.7)$$

$$2\Sigma^{(L,K)} = R^{(L,K)} \text{diag}(\Sigma^{(K,K)}) \text{diag}(R^{(K,K)})^{-1}. \quad (2.8)$$

Conversely, if Σ and R satisfy (2.7) and (2.8), and if the $|L|$ -dimensional $(\Sigma^{(L,L)}, \tilde{\mu}(L), R^{(L,L)})$ -SRBM has a stationary distribution, then $Z^K(0)$ and $Z^L(0)$ are independent under π and $Z^K(0)$ is of product form under π .

Remark 3. After an appropriate reordering of the coordinates, the condition (2.6) can be written as

$$R = \begin{pmatrix} R^{(K,K)} & 0 \\ R^{(L,K)} & R^{(L,L)} \end{pmatrix}.$$

In this case, the covariance matrix Σ and the drift vector μ are partitioned as

$$\Sigma = \begin{pmatrix} \Sigma^{(K,K)} & (\Sigma^{(L,K)})^\top \\ \Sigma^{(L,K)} & \Sigma^{(L,L)} \end{pmatrix}, \quad \mu = (\mu^K, \mu^L)^\top.$$

Note that (2.7) is the skew symmetric condition for $(\Sigma^{(K,K)}, \mu^K, R^{(K,K)})$ -SRBM. A geometric interpretation for this condition is given in Theorem 1 and its corollaries in [10]. One wonders if a similar interpretation can be found for (2.8); so far we have not been able to obtain one.

This theorem is proved in Section 3.3. The next corollary is for an SRBM arising from the d station tandem queue, which was discussed in Section 1 (see Fig. 1). We omit its proof because it is an immediate consequence of Theorem 2.

Corollary 2. Assume that the (Σ, μ, R) -SRBM has a stationary distribution π , where the reflection matrix R , the covariance matrix Σ , and the drift vector μ are given by (1.6)–(1.8), respectively. For each positive integer $k \leq d-1$, set $K = \{1, \dots, k\}$ and $L = J \setminus K$. Then $Z^K(0)$ and $Z^L(0)$ are independent under π if $c_0 = c_1 = \dots = c_k$. Furthermore, for $k = 1$, $c_0 = c_1$ is also necessary for this decomposability.

3. Proofs of main results

We will prove Theorems 1 and 2. For this, we first discuss about equations to characterize the stationary distribution and some basic facts obtained from the decomposability.

3.1. The stationary distribution

Assume the SRBM has a stationary distribution. The stationary distribution must be unique [6]. Our first tool is the basic adjoint relationship (BAR) that characterizes the stationary distribution. For this, we first introduce the boundary measures for a distribution π on $(\mathbb{R}_+^d, \mathcal{B}(\mathbb{R}_+^d))$, where $\mathcal{B}(\mathbb{R}_+^d)$ is the Borel σ -field on $\mathbb{R}_+^d$. They are defined as

$$v_i(B) = \mathbb{E}_\pi \left[\int_0^1 1\{Z(t) \in B\} dY_i(t) \right], \quad B \in \mathcal{B}(\mathbb{R}_+^d), \quad i \in J.$$

Our BAR is in terms of moment generating functions, which are defined as

$$\begin{aligned} \varphi(\theta) &= \mathbb{E}_\pi [e^{\langle \theta, Z(0) \rangle}], \\ \varphi_i(\theta) &= \mathbb{E}_\pi \left[\int_0^1 e^{\langle \theta, Z(t) \rangle} dY_i(t) \right], \quad i \in J, \end{aligned}$$

where $\mathbb{E}_\pi$ is the expectation operator when $Z(0)$ has the distribution π .

Because for each $i \in J$, $Y_i(t)$ increases only when $Z_i(t) = 0$, one has $\varphi_i(\theta)$ that depends on $\theta^{J \setminus \{i\}}$ only. Therefore,

$$\varphi_i(\theta) = \varphi_i(\uparrow \theta^{J \setminus \{i\}}).$$

Note that $Y_i(t)$ and $Y_j(t)$ may simultaneously increase for $j \neq i$ but it occurs with probability 0 by Lemma 4.5 of [11]. So, we do not need to care about the other θ_j 's in the variable θ of $\varphi_i(\theta)$.

For a (Σ, μ, R) -SRBM, its data can be alternatively described in terms of d -dimensional polynomials, which are defined as

$$\begin{aligned}\gamma(\theta) &= -\frac{1}{2}\langle\theta, \Sigma\theta\rangle - \langle\mu, \theta\rangle, \quad \theta \in \mathbb{R}^d, \\ \gamma_i(\theta) &= \langle R^{(i)}, \theta\rangle, \quad \theta \in \mathbb{R}^d, \quad i \in J,\end{aligned}$$

where $R^{(i)}$ is the i th column of the reflection matrix R . Obviously, those polynomials uniquely determine the primitive data, Σ, μ and R . Thus, we can use those polynomials to discuss everything about the SRBM instead of the primitive data themselves.

The following lemma is critical in our analysis. Eq. (3.1) below is the moment generating function version of the standard basic adjoint relationship. We still refer to it as BAR.

Lemma 1. (a) Assume π is the stationary distribution of a (Σ, μ, R) -SRBM. For $\theta \in \mathbb{R}^d$, $\varphi(\theta) < \infty$ implies $\varphi_i(\theta) < \infty$ for $i \in J$. Furthermore,

$$\gamma(\theta)\varphi(\theta) = \sum_{i=1}^d \gamma_i(\theta)\varphi_i(\theta) \quad (3.1)$$

holds for $\theta \in \mathbb{R}^d$ such that $\varphi(\theta) < \infty$. (b) Assume that π is a probability measure on $\mathbb{R}_+^d$ and that ν_i is a positive finite measure whose support is contained in $\{x \in \mathbb{R}_+^d : x_i = 0\}$ for $i \in J$. Let φ and φ_i be the moment generating functions of π and ν_i , respectively. If $\varphi, \varphi_1, \dots, \varphi_d$ satisfy (3.1) for each $\theta \in \mathbb{R}^d$ with $\theta \leq 0$, then π is the stationary distribution and ν_i is the corresponding boundary measure on $\{x \in \mathbb{R}_+^d : x_i = 0\}$.

For (a), the fact that (3.1) holds for $\theta \leq 0$ is a special case of the standard basic adjoint relationship (BAR); see, e.g., equation (7) in [6]. When some components of θ are allowed to be positive in (3.1), readers are referred to the proof of Lemma 4.1(a) in [8] to see how to rigorously derive the relationship. For (b), we refer to Theorem 1.2 of [7] (see also [19] for a more general class of reflecting processes). In [7], BAR (3.1) is given using differential operators; see, for example, again equation (7) in [6]. Under the condition of our lemma, that BAR (7) is satisfied for all functions f of the form

$$f(x) = e^{(\theta, x)} \quad x \in \mathbb{R}_+^d, \quad \text{for each } \theta \in \mathbb{R}^d \text{ with } \theta \leq 0. \quad (3.2)$$

By the analytic extension, BAR (7) in [6] continues to hold for functions f when θ is replaced by $(z_1, \dots, z_d)$ where each z_i is a complex variable with $\Re z_i \leq 0$. This means that BAR is satisfied for finite Fourier transforms. Since a continuous function with a compact support is uniformly approximated by a sequence of finite Fourier transforms, one can argue that BAR (7) in [6] holds for all bounded functions whose first- and second-order derivatives are continuous and bounded. A complete proof for the equivalence of (3.1) to BAR (7) in [6] can be found in the Appendix of the arXiv version of [10].

In the rest of this paper, whenever we write $\varphi(\theta)$, we implicitly assume it is finite.

Let K and L be a non-empty partition of J . In this paper, we consider conditions for $Z^K(0)$ to be independent of $Z^L(0)$ under the stationary distribution π . The independence is equivalent to

$$\varphi(\theta) = \varphi^K(\theta^K)\varphi^L(\theta^L), \quad (3.3)$$

where, for $U \subset J$,

$$\varphi^U(\theta^U) = \varphi(\uparrow\theta^U).$$

The next lemma shows how the boundary measure is decomposed under (3.3).

Lemma 2. Let K and L be a non-empty partition of J . Assume that $Z^K(0)$ and $Z^L(0)$ are independent under the stationary distribution π . Assume $\varphi(\theta) < \infty$. Then

$$\varphi_j(\theta) = \varphi_j^K(\theta^K)\varphi_j^L(\theta^L), \quad j \in K, \quad (3.4)$$

where, for $U \subset J$,

$$\varphi_j^U(\theta^U) = \varphi_j(\uparrow\theta^U).$$

Proof. We first prove that for $i \neq j$,

$$\lim_{\theta_j \downarrow -\infty} \varphi_i(\theta) = 0. \quad (3.5)$$

By the monotone convergence theorem, we have

$$\lim_{\theta_j \downarrow -\infty} \varphi_i(\theta) = \mathbb{E}_\pi \left[\int_0^1 e^{\langle \theta^{J \setminus \{i,j\}}, Z^{J \setminus \{i,j\}}(t) \rangle} 1(Z_j(t) = 0) dY_i(t) \right].$$

By (1.4), $Y_i(t) = \int_0^t 1(Z_i(s) = 0) dY_i(s)$ for all $t \geq 0$. Thus,

$$\begin{aligned} & \int_0^1 e^{\langle \theta^{J \setminus \{i,j\}}, Z^{J \setminus \{i,j\}}(t) \rangle} 1(Z_j(t) = 0) dY_i(t) \\ &= \int_0^1 e^{\langle \theta^{J \setminus \{i,j\}}, Z^{J \setminus \{i,j\}}(t) \rangle} 1(Z_j(t) = 0, Z_i(t) = 0) dY_i(t), \end{aligned}$$

which equals zero almost surely by Lemma 4.5 of [11] (see also Theorem 1 of [22]). Therefore, we have proved (3.5). Hence, for each $j \in J$ and $\theta \leq 0$, dividing both sides of (3.1) by θ_j and letting $\theta_j \downarrow -\infty$, we have

$$\begin{aligned} - \lim_{\theta_j \downarrow -\infty} \left(\frac{1}{2} \Sigma_{jj} \theta_j + \mu_j \right) \varphi(\theta) &= R_{jj} \varphi_j(\theta) + \sum_{i \neq j} R_{ji} \lim_{\theta_j \downarrow -\infty} \varphi_i(\theta) \\ &= R_{jj} \varphi_j(\theta). \end{aligned} \quad (3.6)$$

Let $\theta = \uparrow\theta^K$ for $j \in K$ in this equation, we have

$$- \lim_{\theta_j \downarrow -\infty} \left(\frac{1}{2} \Sigma_{jj} \theta_j + \mu_j \right) \varphi^K(\theta^K) = R_{jj} \varphi_j^K(\theta^K). \quad (3.7)$$

By the independence assumption, we can write

$$\varphi(\theta) = \varphi^K(\theta^K) \varphi^L(\theta^L).$$

Hence, multiplying $\varphi^L(\theta^L)$ to both sides of (3.7), then (3.6) yields (3.4) because R is a completely- $\mathcal{S}$ matrix and therefore $R_{jj} \neq 0$. $\square$

3.2. Proof of Theorem 1

Because the SRBM Z has a stationary distribution, (1.5) is satisfied. Thus, $Q \equiv R^{-1}$ exists. Let $V(t) = QZ(t)$ for $t \geq 0$. It follows from (1.1) that

$$V(t) = V(0) + QX(t) + Y(t), \quad t \geq 0. \quad (3.8)$$

Note that QX is still Brownian motion with drift vector $Q\mu$ and covariance matrix $Q\Sigma Q^T$. For an SRBM Z arising from open multiclass queueing networks, the process V is known as the *total*

workload process [15]. The key idea of our proof is to use (3.8) instead of (1.1), and (3.8) allows us to easily separate the entries of $Y(t)$ among coordinates in a partition K and L .

The most difficult part is in the proof of part (a) of the theorem because there is no hope to directly verify that $\tilde{R}(U)$ is completely- $\mathcal{S}$ as noted in Remark 1 immediately below the statement of the theorem. To verify it, we need information from the decomposability. We prove the first half of part (a) in the following lemma.

Lemma 3. $Q^{(K,K)}$ is invertible.

Proof. In this proof, we apply truncation arguments similarly to those in [9]. To this end, we introduce the following sequences of functions. For each positive integer n , let

$$g_n(s) = \begin{cases} \frac{1}{2}(s+n+2)^2, & -(n+2) < s \leq -(n+1), \\ 1 - \frac{1}{2}(s+n)^2, & -(n+1) < s \leq -n, \\ 1, & -n < s \leq n, \\ 1 - \frac{1}{2}(s-n)^2, & n < s \leq n+1, \\ \frac{1}{2}(n+2-s)^2, & n+1 < s \leq n+2, \\ 0, & s \leq -(n+2) \text{ or } s > n+2, \end{cases}$$

and let

$$f_n(u) = \begin{cases} \int_0^u g_n(s) ds, & u \geq 0, \\ -\int_u^0 g_n(s) ds, & u < 0. \end{cases}$$

Clearly, for each fixed n , $f_n(u)$ is bounded, twice continuously differentiable, and its derivatives $f'_n(u)$ and $f''_n(u)$ are bounded by 1 in absolute values. Furthermore, for each $u \in \mathbb{R}$, $f'_n(u) = g_n(u)$ is monotone in n , and for each $u \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} f_n(u) = u, \quad \lim_{n \rightarrow \infty} f'_n(u) = 1, \quad \lim_{n \rightarrow \infty} f''_n(u) = 0. \quad (3.9)$$

For each $i \in J$ and $t \geq 0$, we apply Itô's integration formula for $f_n([QZ(t)]_i)$ for each fixed n , then it follows from (3.8) that

$$\begin{aligned} f_n([V(t)]_i) - f_n([V(0)]_i) &= \int_0^t f'_n([V(u)]_i) d([QX(u)]_i + Y_i(u)) \\ &\quad + \frac{1}{2} \int_0^t f''_n([V(u)]_i) [Q\Sigma Q^T]_{ii} du. \end{aligned} \quad (3.10)$$

Because f_n , f'_n and f''_n are all bounded, we take the expectation $\mathbb{E}_\pi$ on both sides of (3.10) with $t = 1$, and obtain

$$\begin{aligned} 0 &= \int_{\mathbb{R}_+^d} f'_n([Qx]_i) [Q\mu]_i \pi(dx) + \int_{\mathbb{R}_+^d} f'_n([Qx]_i) v_i(dx) \\ &\quad + \frac{1}{2} [Q\Sigma Q^T]_{ii} \int_{\mathbb{R}_+^d} f''_n([Qx]_i) \pi(dx). \end{aligned}$$

Applying the dominated convergence theorem on the f_n'' term and the monotone convergence theorem on two f_n' terms, by letting $n \rightarrow \infty$, we have

$$(Q\mu)_i + v_i(\mathbb{R}_+^d) = 0 \quad i \in J. \quad (3.11)$$

We now assume that $Q^{(K,K)}$ is singular. Then, there exists a non-zero $|K|$ -dimensional row vector η such that

$$\eta Q^{(K,K)} = 0. \quad (3.12)$$

We will prove that (3.12) implies

$$\eta Q^{(K,L)} = 0. \quad (3.13)$$

Assuming (3.13), we now show that it leads to a contradiction, thus proving the lemma. To see this, it follows from (3.8) that

$$V^K(t) = V^K(0) + (QX)^K(t) + Y^K(t). \quad (3.14)$$

Since $V(t) = QZ(t)$, (3.12) and (3.13) imply that $\eta V^K(t) = 0$ for all $t \geq 0$. Similarly, (3.12) and (3.13) imply that $\eta(QX(t))^K = 0$ for all $t \geq 0$. Hence, (3.14) yields $\eta Y^K(t) = 0$ for all $t \geq 0$. Namely,

$$\sum_{i \in K} \eta_i Y_i^K(t) = 0, \quad t \geq 0.$$

Assume $\eta_j \neq 0$ for some $j \in K$. Since $Y_i^K(t) = Y_i(t)$, we have

$$\sum_{i \in K} \eta_i \int_0^t 1(Z_j(u) = 0) dY_i(u) = 0.$$

By Lemma 4.5 of [11], we have

$$\int_0^1 1(Z_j(u) = 0) dY_i(t) = 0 \quad \text{almost surely for each pair } i \neq j.$$

This yields

$$\eta_j \int_0^1 1(Z_j(u) = 0) dY_j(t) = 0 \quad \text{almost surely,}$$

which leads to a contradiction because $\eta_j \neq 0$ for some $j \in J$ and (3.11) together with (1.5) implies, for every $j \in J$,

$$\mathbb{E}_\pi \left(\int_0^1 1(Z_j(u) = 0) dY_j(t) \right) = v_j(\mathbb{R}_+^d) = -(Q\mu)_j > 0.$$

Now we prove (3.13). Note that $(QX)^K$ in (3.14) is a $|K|$ -dimensional Brownian motion with drift $Q^{(K,J)}\mu$ and covariance matrix

$$Q^{(K,J)} \Sigma (Q^T)^{(J,K)} = (Q \Sigma Q^T)^{(K,K)}.$$

We apply Itô's integration formula to $h_n(V^K(t))$, where $h_n(x) = e^{-f_n(\langle \eta, x \rangle)}$ for $x \in \mathbb{R}^{|K|}$. Then, we have

$$\begin{aligned}
h_n(V^K(t)) - h_n(V^K(0)) &= \sum_{i \in K} \int_0^t \frac{\partial h_n(x)}{\partial x_i} \Big|_{x=V^K(u)} d(QX)_i^K(u) \\
&\quad + \frac{1}{2} \sum_{i,j \in K} \int_0^t \frac{\partial^2 h_n(x)}{\partial x_i \partial x_j} \Big|_{x=V^K(u)} (Q \Sigma Q^T)_{ij}^{(K,K)} du \\
&\quad + \sum_{i \in K} \int_0^t \frac{\partial h_n(x)}{\partial x_i} \Big|_{x=V^K(u)} dY_i(u),
\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}
\frac{\partial h_n(x)}{\partial x_i} &= -\eta_i g_n(\langle \eta, x \rangle) h_n(x), \\
\frac{\partial^2 h_n(x)}{\partial x_i \partial x_j} &= \eta_i \eta_j (g_n^2(\langle \eta, x \rangle) + g_n'(\langle \eta, x \rangle)) h_n(x).
\end{aligned}$$

Setting $t = 1$ and taking expectation $\mathbb{E}_\pi$ on both sides, we have

$$\begin{aligned}
0 &= \sum_{i \in K} \mathbb{E}_\pi \left[\frac{\partial h_n(x)}{\partial x_i} \Big|_{x=V^K(0)} \right] (Q\mu)_i + \frac{1}{2} \sum_{i,j \in K} (Q \Sigma Q^T)_{ij}^{(K,K)} \mathbb{E}_\pi \left[\frac{\partial^2 h_n(x)}{\partial x_i \partial x_j} \Big|_{x=V^K(0)} \right] \\
&\quad + \sum_{i \in K} \mathbb{E}_\pi \left[\int_0^1 \frac{\partial h_n(x)}{\partial x_i} \Big|_{x=V^K(u)} dY_i(u) \right].
\end{aligned} \tag{3.16}$$

Recall that $V^K(t) = Q^{(K,K)} Z^K(t) + Q^{K,L} Z^L(t)$. Let

$$M^K(t) = Q^{K,L} Z^L(t).$$

Because $\eta Q^{(K,K)} = 0$, we have $\eta V^K(t) = \eta M^K(t)$, and therefore, for $i \in K$,

$$\begin{aligned}
\mathbb{E}_\pi \left[\int_0^1 \frac{\partial h_n(x)}{\partial x_i} \Big|_{x=V^K(u)} dY_i(u) \right] &= \mathbb{E}_\pi \left[\int_0^1 \frac{\partial h_n(x)}{\partial x_i} \Big|_{x=M^K(u)} dY_i(u) \right] \\
&= v_i(\mathbb{R}_+^d) \mathbb{E}_\pi \left[\frac{\partial h_n(x)}{\partial x_i} \Big|_{x=M^K(0)} \right],
\end{aligned} \tag{3.17}$$

where in the second equality, we have used the fact that $M^K(t)$ is a function of $Z^L(t)$ and Lemma 2. It follows from (3.11), (3.16), and (3.17) that

$$\frac{1}{2} \sum_{i,j \in K} (Q \Sigma Q^T)_{ij}^{(K,K)} \mathbb{E}_\pi \left[\frac{\partial^2 h_n(x)}{\partial x_i \partial x_j} \Big|_{x=V^K(0)} \right] = 0,$$

or equivalently

$$\begin{aligned}
&\frac{1}{2} \sum_{i,j \in K} \eta_i \eta_j (Q \Sigma Q^T)_{ij}^{(K,K)} \mathbb{E}_\pi ((g_n^2(\langle \eta, V^K(0) \rangle) + g_n'(\langle \eta, V^K(0) \rangle)) h_n(V^K(0))) \\
&= 0.
\end{aligned} \tag{3.18}$$

By the construction of functions g_n and f_n , $g_n'(u) \geq 0$ except for $u \in (n, n+2)$, in which $g_n'(u) \in [-1, 0)$, and $e^{-f_n(u)}$ monotonically converges to e^{-u} as $n \rightarrow \infty$ and is bounded by 1

for $u \geq 0$. Furthermore, $g_n(u)$ and $g'_n(u)$ are bounded by 1 for all u and n . We have

$$\mathbb{E}_\pi \left[(g_n^2(\langle \eta, V^K(0) \rangle) + g'_n(\langle \eta, V^K(0) \rangle)) 1_{(\langle \eta, V^K(0) \rangle \leq 0)} h_n(V^K(0)) \right] \geq 0 \quad (3.19)$$

for each $n \geq 1$. By dominated convergence theorem,

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}_\pi \left[(g_n^2(\langle \eta, V^K(0) \rangle) + g'_n(\langle \eta, V^K(0) \rangle)) 1_{(\langle \eta, V^K(0) \rangle > 0)} h_n(V^K(0)) \right] \\ = \mathbb{E}_\pi \left[1_{(\langle \eta, V^K(0) \rangle > 0)} e^{-\langle \eta, V^K(0) \rangle} \right] > 0, \end{aligned}$$

where the strict inequality follows from the fact that the Lebesgue measure of set $\{z \in \mathbb{R}_+^d : \eta z^K > 0\}$ is positive and the fact that $\pi(A) > 0$ for every measurable set A that has positive Lebesgue measure [6]. Therefore, (3.19) can be sharpened for some large n_0 in such a way that, for every $n \geq n_0$,

$$\mathbb{E}_\pi \left[(g_n^2(\langle \eta, V^K(0) \rangle) + g'_n(\langle \eta, V^K(0) \rangle)) 1_{(\langle \eta, V^K(0) \rangle \leq 0)} h_n(V^K(0)) \right] > 0.$$

Thus, from (3.18) we arrive at

$$\frac{1}{2} \sum_{i,j \in K} \eta_i \eta_j (Q \Sigma Q^T)_{ij}^{(K,K)} = 0.$$

Namely,

$$\frac{1}{2} \eta Q^{(K,J)} \Sigma (Q^T)^{(J,K)} \eta^T = 0.$$

Since Σ is positive definite, for this to be true, we must have

$$\eta Q^{(K,J)} = 0, \quad (3.20)$$

thus proving (3.13). $\square$

We now return to the proof of Theorem 1. Because $V^K(t) = Q^{(K,K)} Z^K(t) + Q^{K,L} Z^L(t)$ and $Q^{(K,K)}$ is invertible by Lemma 3, we have

$$\begin{aligned} Z^K(t) + W^K(t) &= (Q^{(K,K)})^{-1} V^K(t) \\ &= (Q^{(K,K)})^{-1} V^K(0) + (Q^{(K,K)})^{-1} (QX)^K(t) + (Q^{(K,K)})^{-1} Y^K(t), \end{aligned} \quad (3.21)$$

where

$$W^K(t) = (Q^{(K,K)})^{-1} Q^{(K,L)} Z^L(t),$$

and the second equality follows from (3.14). Note that

$$(Q^{(K,K)})^{-1} (QX)^K(t)$$

is a $|K|$ -dimensional Brownian motion with drift vector $\tilde{\mu}(K) = (Q^{(K,K)})^{-1} (Q\mu)^K$ and covariance matrix

$$\tilde{\Sigma}(K) = (Q^{(K,K)})^{-1} (Q \Sigma Q^T)^{(K,K)} ((Q^{(K,K)})^{-1})^T.$$

We now apply Itô's integral formula to $f(Z^K(t) + W^K(t))$, where $f(x) \equiv e^{i\langle \theta^K, x \rangle}$ with $x \in \mathbb{R}^{|K|}$ for each fixed $\theta^K \in \mathbb{R}^{|K|}$ and $i = \sqrt{-1}$ is the imaginary unit of a complex number. (We really apply the Itô formula twice, one for $\cos(\langle \theta^K, x \rangle)$ and one for $\sin(\langle \theta^K, x \rangle)$.) We have

$$\begin{aligned} & f(Z^K(t) + W^K(t)) - f(Z^K(0) + W^K(0)) \\ &= -\frac{1}{2} \int_0^t \sum_{i,j \in K} \theta_i \theta_j [\tilde{\Sigma}(K)]_{ij} f(Z^K(u) + W^K(u)) du \\ & \quad + i \int_0^t \sum_{i \in K} \theta_i f(Z^K(u) + W^K(u)) d[(Q^{(K,K)})^{-1}(QX)^K]_i \\ & \quad + i \int_0^t \sum_{i \in K} \theta_i f(Z^K(u) + W^K(u)) \sum_{j \in K} [\tilde{R}(K)]_{ij} dY_j(u), \end{aligned}$$

where

$$\tilde{R}(K) = (Q^{(K,K)})^{-1}.$$

Because $|f(Z^K(u) + W^K(u))| \leq 1$, setting $t = 1$, we can take expectation $\mathbb{E}_\pi$ on both sides of this equation for $\theta \in \mathbb{R}^d$, we have

$$\begin{aligned} & -\frac{1}{2} \sum_{i,j \in K} \theta_i \theta_j [\tilde{\Sigma}(K)]_{ij} \mathbb{E}_\pi \left(e^{i\langle \theta^K, Z^K(0) \rangle + i\langle \theta^K, W^K(0) \rangle} \right) \\ & \quad + i \sum_{i \in K} \theta_i [\tilde{\mu}(K)]_i \mathbb{E}_\pi \left(e^{i\langle \theta^K, Z^K(0) \rangle + i\langle \theta^K, W^K(0) \rangle} \right) \\ & \quad + i \sum_{i,j \in K} \theta_i [\tilde{R}(K)]_{ij} \mathbb{E}_\pi \left(\int_0^1 e^{i\langle \theta^K, Z^K(u) \rangle + i\langle \theta^K, W^K(u) \rangle} dY_j(u) \right) = 0. \end{aligned} \quad (3.22)$$

We now use the assumption that $Z^K(0)$ and $Z^L(0)$ are independent under π . Under the independence assumption, one can show that (3.4) also holds when θ is replaced by $i\theta$, i.e.,

$$\varphi_j(i\theta) = \varphi_j^K(i\theta^K) \varphi^L(i\theta^L) \quad \text{for each } \theta \in \mathbb{R}^d. \quad (3.23)$$

Eq. (3.23) can be proved following the proof of Lemma 2 by using Riemann–Lebesgue lemma (e.g., see Section 7.1 of [24]) for (3.5) and the characteristic function version of (3.1) when θ is replaced by $i\theta$. For each $\theta^K \in \mathbb{R}^{|K|}$, let $\theta \in \mathbb{R}^d$ be the corresponding vector with $\theta^L = \theta^K (Q^{(K,K)})^{-1} Q^{(K,L)}$. For this $\theta \in \mathbb{R}^d$, it follows from the definition of $W^K(t)$ that (3.23) is equivalent to the following equality

$$\begin{aligned} & \mathbb{E}_\pi \left(\int_0^1 e^{i\langle \theta^K, Z^K(u) \rangle + i\langle \theta^K, W^K(u) \rangle} dY_j(u) \right) \\ &= \mathbb{E}_\pi \left(\int_0^1 e^{i\langle \theta^K, Z^K(u) \rangle} dY_j(u) \right) \mathbb{E}_\pi \left(e^{i\langle \theta^K, W^K(0) \rangle} \right). \end{aligned}$$

Also, $Z^K(0)$ and $W^K(0)$ are independent because $W^K(0)$ is a linear transform of $Z^L(0)$. Thus, combining these facts with (3.22) yields

$$\begin{aligned} & \left(-\frac{1}{2} \sum_{i,j \in K} \theta_i \theta_j [\tilde{\Sigma}(K)]_{ij} + \iota \sum_{i \in K} \theta_i [\tilde{\mu}(K)]_i \right) \mathbb{E}_\pi \left(e^{\iota \langle \theta^K, Z^K(0) \rangle} \right) \\ & + \iota \sum_{i,j \in K} \theta_i [\tilde{R}(K)]_{ij} \mathbb{E}_\pi \left(\int_0^1 e^{\iota \langle \theta^K, Z^K(u) \rangle} dY_j(u) \right) = 0. \end{aligned} \quad (3.24)$$

In what follows we will get the moment generating function version of (3.24) with $\theta^K \leq 0$. One may wonder why we do not directly consider this generating function version. This is because $\theta^L \equiv \theta^K (Q^{(K,K)})^{-1} Q^{(K,L)} \leq 0$ may not be true in (3.4) for $\theta^K \leq 0$.

Denote the left-hand side of (3.24) by $g(\iota\theta)$ as a function of $\iota\theta$ for $\theta \in \mathbb{R}^{|K|}$. We can replace $\iota\theta$ of $g(\iota\theta)$ by a complex vector $z \equiv (z_1, z_2, \dots, z_{|K|})$ such that $\Re z_i \leq 0$ for all $i = 1, \dots, K$, where $\Re z_i$ is the real part of z_i . Then, it is easy to see that $g(z_1, z_2, \dots, z_{|K|})$ is analytic in each z_i such that $\Re z_i < 0$ when z_j for $j \neq i$ is fixed satisfying $\Re z_j \leq 0$. Let $i = 1$ and fix an arbitrary $\theta \in \mathbb{R}^{|K|}$. Since $g(z_1, \iota\theta_2, \dots, \iota\theta_{|K|})$ converges to $g(\iota\theta) \equiv 0$ as z_1 with $\Re z_1 < 0$ continuously moves to $\iota\theta_1$, we must have

$$g(z_1, \iota\theta_2, \dots, \iota\theta_{|K|}) = 0 \quad \text{for } \Re z_1 \leq 0$$

by the so called boundary uniqueness theorem (e.g., see page 371 of Volume I of [21]). We then inductively replace $\iota\theta_i$ by z_i with $\Re z_i \leq 0$ for $i = 2, 3, \dots, |K|$, and we have

$$g(z_1, z_2, \dots, z_{|K|}) = 0, \quad \Re z_i \leq 0 \text{ for } i = 1, 2, \dots, |K|.$$

In particular, letting $z_i = \theta_i$ for real $\theta_i \leq 0$ for $i = 1, 2, \dots, |K|$, we have

$$\begin{aligned} & \left(\frac{1}{2} \sum_{i,j \in K} \theta_i \theta_j [\tilde{\Sigma}(K)]_{ij} + \sum_{i \in K} \theta_i [\tilde{\mu}(K)]_i \right) \varphi^K(\theta^K) + \sum_{i,j \in K} \theta_i [\tilde{R}(K)]_{ij} \varphi_j^K(\theta^K) \\ & = 0. \end{aligned} \quad (3.25)$$

We are now ready to prove the remaining part of Theorem 1, part (a), in the following lemma.

Lemma 4. Under the assumptions of Theorem 1, $\tilde{R}(K)$ and $\tilde{R}(L)$ are completely-S matrices.

Proof. For an arbitrarily fixed $\ell \in K$, let $\theta_j^K = 0$ in θ^K for $j \neq \ell$. We denote this vector θ^K as $(\uparrow\theta_\ell)^K$. For $j \neq \ell$, let $\theta_j = 0$ in (3.25) and divide the resulting formula by $\theta_\ell < 0$, then we have

$$\left(\frac{1}{2} \theta_\ell [\tilde{\Sigma}(K)]_{\ell\ell} + [\tilde{\mu}(K)]_\ell \right) \varphi^K((\uparrow\theta_\ell)^K) + \sum_{j \in K} [\tilde{R}(K)]_{\ell j} \varphi_j^K((\uparrow\theta_\ell)^K) = 0. \quad (3.26)$$

Similarly to (3.6),

$$- \lim_{\theta_\ell \downarrow -\infty} \left(\frac{1}{2} \theta_\ell [\tilde{\Sigma}(K)]_{\ell\ell} + [\tilde{\mu}(K)]_\ell \right) \varphi^K((\uparrow\theta_\ell)^K) = [\tilde{R}(K)]_{\ell\ell} \varphi_\ell^K(0^K).$$

Since the left-hand side of this formula is positive by (3.6), its right-hand side must be positive. Hence, $[\tilde{R}(K)]_{\ell\ell} > 0$. Furthermore, we can take sufficiently small $\theta_\ell < 0$ such that

$$\left(\frac{1}{2} \theta_\ell [\tilde{\Sigma}(K)]_{\ell\ell} + [\tilde{\mu}(K)]_\ell \right) \varphi^K((\uparrow\theta_\ell)^K) < 0.$$

By (3.26), we have, for this θ_ℓ ,

$$\sum_{j \in K} [\tilde{R}(K)]_{\ell j} \varphi_j^K((\uparrow \theta_\ell)^K) > 0. \quad (3.27)$$

Since $\varphi_j^K((\uparrow \theta_\ell)^K) > 0$ for all $j \in K$, $\tilde{R}(K)$ is an $\mathcal{S}$ -matrix. Let U be a subset of K such that $\ell \in U$ and $U \neq K$, then we can choose θ_ℓ such that $\varphi_j^K((\uparrow \theta_\ell)^K)$ is sufficiently small for $j \in K \setminus U$. This yields that $(\tilde{R}(K))^{(U,U)}$ is an $\mathcal{S}$ -matrix, and therefore we have proved that $\tilde{R}(K)$ is a completely- $\mathcal{S}$ matrix. $\square$

We now can see that (3.24) is nothing but the BAR for the $|K|$ -dimensional SRBM with data $(\tilde{\Sigma}(K), \tilde{\mu}(K), \tilde{R}(K))$ because $\tilde{R}(K)$ is completely- $\mathcal{S}$ by Lemma 4. By part (b) of Lemma 1, the $|K|$ -dimensional $(\tilde{\Sigma}(K), \tilde{\mu}(K), \tilde{R}(K))$ -SRBM has a stationary distribution that is equal to the distribution of $Z^K(0)$ under π . By the symmetric roles of K and L , the $|L|$ -dimensional $(\tilde{\Sigma}(L), \tilde{\mu}(L), \tilde{R}(L))$ -SRBM has a stationary distribution that is equal to the distribution of $Z^L(0)$ under π .

3.3. Proof of Theorem 2

First of all, we will prove if $Z^K(0)$ and $Z^L(0)$ are independent and $Z^K(0)$ is of product form under π , then (2.7) and (2.8) hold. According to Theorem 1, the distribution of $Z^{(i)}(0)$ under π is equal to the stationary distribution of $(\tilde{\Sigma}(\{i\}), \tilde{\mu}(\{i\}), \tilde{R}(\{i\}))$ -SRBM for $i \in K$. The distribution of $Z^L(0)$ under π is equal to the stationary distribution of $(\tilde{\Sigma}(L), \tilde{\mu}(L), \tilde{R}(L))$ -SRBM. Then by Lemma 1, for $\theta \in \mathbb{R}^d$ with $\theta \leq 0$,

$$-\left(\frac{1}{2} \frac{\Sigma_{ii}}{R_{ii}} \theta_i + ((R^{(K,K)})^{-1} \mu^K)_i\right) \varphi^{(i)}(\theta_i) = 1, \quad i \in K, \quad (3.28)$$

$$-\left(\frac{1}{2} \sum_{i,j \in L} \theta_i \theta_j [\tilde{\Sigma}(L)]_{ij} + \sum_{i \in L} \theta_i [\tilde{\mu}(L)]_i\right) \varphi^L(\theta^L) = \sum_{i,j \in L} \theta_i [\tilde{R}(L)]_{ij} \varphi_j^L(\theta^L). \quad (3.29)$$

By the definition of $\tilde{\Sigma}(L)$, $\tilde{\mu}(L)$ and $\tilde{R}(L)$, we can find they are

$$\begin{aligned} \tilde{\Sigma}(L) &= \Sigma^{(L,L)} + R^{(L,K)}(R^{(K,K)})^{-1} \Sigma^{(K,K)}((R^{(K,K)})^{-1})^T (R^{(L,K)})^T \\ &\quad - \Sigma^{(L,K)}((R^{(K,K)})^{-1})^T (R^{(L,K)})^T - R^{(L,K)}(R^{(K,K)})^{-1} (\Sigma^{(L,K)})^T, \\ \tilde{\mu}(L) &= \mu^L - R^{(L,K)}(R^{(K,K)})^{-1} \mu^K, \\ \tilde{R}(L) &= R^{(L,L)}. \end{aligned}$$

By the independence assumptions, we have, for $i \in K$,

$$\varphi(\theta) = \varphi^{(i)}(\theta_i) \varphi^{J \setminus \{i\}}(\theta^{J \setminus \{i\}}) = \varphi^K(\theta^K) \varphi^L(\theta^L).$$

By Lemma 2, we get

$$\varphi_i(\theta) = \varphi^{J \setminus \{i\}}(\theta^{J \setminus \{i\}}), \quad i \in K, \quad \varphi_j(\theta) = \varphi^K(\theta^K) \varphi_j^L(\theta^L), \quad j \in L.$$

Using the fact that $\varphi_i^{(i)}(0) = 1$, we can rewrite (3.28) and (3.29) as

$$-\left(\frac{1}{2} \frac{\Sigma_{ii}}{R_{ii}} \theta_i + ((R^{(K,K)})^{-1} \mu^K)_i\right) \varphi(\theta) = \varphi_i(\theta), \quad i \in K, \quad (3.30)$$

$$-\left(\frac{1}{2} \sum_{i,j \in L} \theta_i \theta_j [\tilde{\Sigma}(L)]_{ij} + \sum_{i \in L} \theta_i [\tilde{\mu}(L)]_i\right) \varphi(\theta) = \sum_{i,j \in L} \theta_i [\tilde{R}(L)]_{ij} \varphi_j(\theta). \quad (3.31)$$

We further modify (3.30) into

$$-\left(\frac{1}{2} \frac{\Sigma_{ii}}{R_{ii}} \theta_i + ((R^{(K,K)})^{-1} \mu^K)_i\right) \gamma_i(\theta) \varphi(\theta) = \gamma_i(\theta) \varphi_i(\theta). \quad (3.32)$$

Then adding (3.32) for $i \in K$ and (3.31), we have

$$\begin{aligned} & -\left(\sum_{i \in K} \left(\frac{1}{2} \frac{\Sigma_{ii}}{R_{ii}} \theta_i + ((R^{(K,K)})^{-1} \mu^K)_i\right) \gamma_i(\theta) \right. \\ & \left. + \frac{1}{2} \sum_{i,j \in L} \theta_i \theta_j [\tilde{\Sigma}(L)]_{ij} + \sum_{i \in L} \theta_i [\tilde{\mu}(L)]_i\right) \varphi(\theta) = \sum_{i=1}^d \gamma_i(\theta) \varphi_i(\theta). \end{aligned} \quad (3.33)$$

So according to Lemma 1, we have

$$\begin{aligned} \gamma(\theta) = & -\left(\sum_{i \in K} \left(\frac{1}{2} \frac{\Sigma_{ii}}{R_{ii}} \theta_i + ((R^{(K,K)})^{-1} \mu^K)_i\right) \gamma_i(\theta) \right. \\ & \left. + \frac{1}{2} \sum_{i,j \in L} \theta_i \theta_j [\tilde{\Sigma}(L)]_{ij} + \sum_{i \in L} \theta_i [\tilde{\mu}(L)]_i\right). \end{aligned} \quad (3.34)$$

Comparing the coefficients of $\theta_i \theta_j$ and coefficients of θ_i of both sides, we can get

$$-\frac{1}{2}(\Sigma_{ij} + \Sigma_{ji}) = -\frac{\Sigma_{ii}}{2R_{ii}} R_{ji} - \frac{\Sigma_{jj}}{2R_{jj}} R_{ij}, \quad i, j \in K, \quad (3.35)$$

$$-\frac{1}{2}(\Sigma_{ij} + \Sigma_{ji}) = -\frac{\Sigma_{ii}}{2R_{ii}} R_{ji}, \quad i \in K, \quad j \in L, \quad (3.36)$$

$$\begin{aligned} & R^{(L,K)} (R^{(K,K)})^{-1} \Sigma^{(K,K)} ((R^{(K,K)})^{-1})^T (R^{(L,K)})^T \\ & - \Sigma^{(L,K)} ((R^{(K,K)})^{-1})^T (R^{(L,K)})^T - R^{(L,K)} (R^{(K,K)})^{-1} (\Sigma^{(L,K)})^T = 0. \end{aligned} \quad (3.37)$$

Observe that (3.35) is equivalent to (2.7), and (3.36) is equivalent to (2.8). Under (2.7) and (2.8), (3.37) automatically holds. So we have proved if $Z^K(0)$ and $Z^L(0)$ are independent and $Z^K(0)$ is of product form under π , then (2.7) and (2.8) hold.

Next we will prove that if Σ and R satisfy (2.7) and (2.8), and the $|L|$ -dimensional $(\Sigma^{(L,L)}, \tilde{\mu}(L), R^{(L,L)})$ -SRBM has a stationary distribution, then $Z^K(0)$ and $Z^L(0)$ are independent and $Z^K(0)$ is of product form under π . First observe that if (2.7) and (2.8) holds, then (3.35)–(3.37) hold. Therefore, (3.34) holds.

Because the skew symmetry condition (2.7) holds, the $(\Sigma^{(K,K)}, \mu^{(K,K)}, R^{(K,K)})$ -SRBM has a product form stationary distribution [17]. Let $\tilde{\varphi}^K(\theta^K)$ and $\tilde{\varphi}_i^K(\theta^K)$ be the moment generating functions of the stationary distribution and i th boundary measure for this $(\tilde{\Sigma}(K), \tilde{\mu}(K), \tilde{R}(K))$ -SRBM. Then

$$-\left(\frac{1}{2} \frac{\Sigma_{ii}}{R_{ii}} \theta_i + ((R^{(K,K)})^{-1} \mu^K)_i\right) (\tilde{\varphi}^K)^{\{i\}}(\theta_i) = 1, \quad i \in K.$$

Because the $|L|$ -dimensional $(\Sigma^{(L,L)}, \tilde{\mu}(L), R^{(L,L)})$ -SRBM has a stationary distribution, $\tilde{\Sigma}(L) = \Sigma^{(L,L)}$ and $\tilde{R}(L) = R^{(L,L)}$, we have

$$-\left(\frac{1}{2} \sum_{i,j \in L} \theta_i \theta_j [\tilde{\Sigma}(L)]_{ij} + \sum_{i \in L} \theta_i [\tilde{\mu}(L)]_i\right) \tilde{\varphi}^L(\theta^L) = \sum_{i,j \in L} \theta_i [\tilde{R}(L)]_{ij} \tilde{\varphi}_j^L(\theta^L)$$

where $\tilde{\varphi}^L(\theta^L)$ and $\tilde{\varphi}_j^L(\theta^L)$ are the moment generating functions of the stationary distribution and j th boundary measure for $(\Sigma^{(L,L)}, \tilde{\mu}(L), R^{(L,L)})$ -SRBM.

Let $\tilde{\varphi}(\theta) = \tilde{\varphi}^K(\theta^K) \tilde{\varphi}^L(\theta^L)$, $\tilde{\varphi}_i(\theta) = \tilde{\varphi}_i^K(\theta^K) \tilde{\varphi}^L(\theta^L)$ for $i \in K$ and $\tilde{\varphi}_j(\theta) = \tilde{\varphi}^K(\theta^K) \tilde{\varphi}_j^L(\theta^L)$ for $j \in L$. Then we can see (3.30)–(3.33) hold with $\varphi(\theta)$, $\varphi_i(\theta)$ and $\varphi_j(\theta)$ replaced by $\tilde{\varphi}(\theta)$, $\tilde{\varphi}_i(\theta)$ and $\tilde{\varphi}_j(\theta)$. Furthermore, as (3.34) holds, we conclude

$$\gamma(\theta) \tilde{\varphi}(\theta) = \sum_{i=1}^d \gamma_i(\theta) \tilde{\varphi}_i(\theta).$$

By part (b) of Lemma 1, we know $\varphi(\theta) = \tilde{\varphi}(\theta)$, $\varphi^K(\theta^K) = \tilde{\varphi}^K(\theta^K)$ and $\varphi^L(\theta^L) = \tilde{\varphi}^L(\theta^L)$. So $\varphi(\theta) = \varphi^K(\theta^K) \varphi^L(\theta^L)$, that is, $Z^K(0)$ and $Z^L(0)$ are independent under π . Furthermore, the distribution of $Z^K(0)$ under π is equal to the stationary distribution of the $|K|$ -dimensional $(\Sigma^{(K,K)}, \mu^K, R^{(K,K)})$ -SRBM. By (2.7), the distribution of $Z^K(0)$ is of product form because the skew symmetry condition (1.10) in [17] is satisfied.

4. Concluding remarks

There are two directions for future study. We first comment on the marginal distributions. In the proof of Theorem 1, we may only use the following fact to complete the proof. Random vectors $Z^K(0)$ and $W^K(0) \equiv (Q^{(K,K)})^{-1} Q^{(K,L)} Z^L(0)$ are “weakly independent through convolution” under the stationary distribution π , that is, for all $\theta \in \mathbb{R}^{|K|}$,

$$\mathbb{E}_\pi(e^{i\langle \theta, (Z^K(0) + W^K(0)) \rangle}) = \mathbb{E}_\pi(e^{i\langle \theta, Z^K(0) \rangle}) \mathbb{E}_\pi(e^{i\langle \theta, W^K(0) \rangle}), \quad (4.1)$$

where $i = \sqrt{-1}$ is again the imaginary unit.

In general, we can easily find an example such that random variables X and Y are not independent but weakly independent through convolution. However, this may not be the case for $Z^K(0)$ and $W^K(0)$ because they have more structure. To further discuss this issue, let us consider the tandem queue example of Section 1 for $d = 2$, $K = \{1\}$ and $L = \{2\}$. Since $Q^{(K,L)} = \{0\}$ and $Q^{(L,K)} = \{1\}$, we have $W^K(t) = 0$ and $W^L(t) = Z_1(t)$. Hence, (4.1) for L instead of K is equivalent to that $Z_1(0)$ and $Z_2(0)$ are weakly independent through convolution under π . We consider the implication of this observation below.

We first observe that (1.11) and Corollary 1 yield that

$$\alpha_1 = \lambda_1 = \frac{2(\beta_1 - \beta_0)}{\beta_0(c_0^2 + c_1^2)}, \quad \alpha_2 = \frac{2(\beta_2 - \beta_0)}{\beta_0(c_1^2 + c_2^2)}, \quad \lambda_2 = \frac{2(\beta_2 - \beta_0)}{\beta_0(c_0^2 + c_2^2)}.$$

Hence, $\alpha_2 = \lambda_2$ holds if and only if $c_0 = c_1$, which is indeed the skew symmetric condition. Thus, (4.1) implies that the stationary distribution π has exponential marginal distributions with parameters λ_1 and λ_2 . However, it is unclear if there exist parameters (β_i, c_i) , $i = 0, 1, 2$, for the tandem queue such that (4.1) is satisfied, but $\lambda_2 \neq \alpha_2$. Thus, it may be interesting to consider the following questions.

Question 1. What is a class of SRBM satisfying (4.1)? How can we characterize this class in terms of the modeling primitives? How much is it larger than the class satisfying the decomposability?

Question 2. Can the stationary distributions of $Z^K(0)$ and $Z^L(0)$ serve good approximations for the marginal distributions of the original stationary distribution when (4.1) does not hold? If not, for what class of SRBM can they provide good approximations?

Obviously, these two questions are closely related. Question 1 is hard to answer while Question 2 may be studied through numerical experiments. Furthermore, for $d = 2$, we know the tail asymptotics of the one-dimensional marginals and when their tail decay rates are identical with $\theta_i^{(i,r)}$ defined in (B.1) (see Theorem 2.2 and 2.3 of [9]). This may suggest the class of SRBM for which the product form approximation using $\theta_i^{(i,r)}$ is reasonable. Unfortunately, we do not have any explicit results yet for the tail decay rates for $d \geq 3$ except for some special cases. We hope that this tail decay rate problem will be solved sometime in the future, and Question 2 will be better answered then.

Another question is about sufficient conditions for the decomposability. We partially answered this question by Theorem 2. It seems hard to extend the arguments in the proof of this theorem to more general cases. Such an extension is a challenging open problem.

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Appendix A. Some matrix classes

We give definitions of some matrix classes in our arguments. One can find them in text books for matrices (e.g., see [2,5]). Let A be an n -dimensional square matrix. Then, A is called an $\mathcal{S}$ -matrix if there is an n -dimensional vector $v > 0$ such that $Av > 0$, where, for a vector u , $u > 0$ means each component of u is strictly positive. If all principal sub-matrices of A are $\mathcal{S}$ -matrices, then A is called a completely- $\mathcal{S}$ matrix. If all principal minors of A are positive, then A is called a $\mathcal{P}$ -matrix. Every $\mathcal{P}$ -matrix is an $\mathcal{S}$ -matrix (e.g., see Corollary 3.3.5 of [5]). A $\mathcal{P}$ -matrix is called an $\mathcal{M}$ -matrix if its off diagonal entries are all non-positive (see Definition 3.11.1 of [5] for a $\mathcal{K}$ -matrix, which is another name of an $\mathcal{M}$ -matrix used in [5, Section 3.13.24]).

Appendix B. Proof of Corollary 1 and some related results

To prove the corollary, we introduce some geometric objects:

$$E = \{\theta \in \mathbb{R}^d; \gamma(\theta) = 0\},$$

$$P^{(i)} = \cap_{k \in J \setminus \{i\}} \{\theta \in \mathbb{R}^d; \gamma_k(\theta) = 0\}, \quad i \in J.$$

The object E is an ellipse in $\mathbb{R}^d$. Since R is invertible and $\theta \in P^{(i)}$ implies that $\langle \theta, R^{(k)} \rangle = 0$ for $k \neq i$, $P^{(i)}$ must be a line going through the origin. Clearly, for each i , $P^{(i)}$ intersects the ellipse

E by at most two points, one of which is the origin. We denote its non-zero intersection by $\theta^{(i,r)}$ if it exists. Otherwise, let $\theta^{(i,r)} = 0$. The following lemma shows that the latter is impossible by giving an explicit formula for $\theta^{(i,r)}$. Recall that $(Q^T)^{(i)}$ be the i th column of Q^T . We refer to the following fact, obtained as Lemma 3 in [10].

Lemma 5. For each $i \in J$,

$$\theta^{(i,r)} = \Delta_i (Q^T)^{(i)}, \quad (\text{B.1})$$

where $\Delta_i > 0$ is defined in (2.5).

Proof of Corollary 1. It follows from [14] that the stationary distribution of a one-dimensional SRBM with drift $\mu < 0$ and variance σ^2 is exponential with mean $1/\lambda$, where

$$\lambda = -\frac{2\mu}{\sigma^2}. \quad (\text{B.2})$$

We apply Theorem 1 to $K = \{i\}$. According to the theorem, $Z^{(i)}(0)$ under π is a one-dimensional SRBM with variance $\tilde{\Sigma}(\{i\})$ and drift $\tilde{\mu}(\{i\})$. Set

$$\lambda_i = -\frac{2\tilde{\mu}(\{i\})}{\tilde{\Sigma}(\{i\})}.$$

By (B.1) and (B.2), to prove the corollary it suffices to verify that

$$\lambda_i = \theta_i^{(i,r)}. \quad (\text{B.3})$$

We first compute Q and $Q^{(\{i\},\{i\})}$ for this. By Lemma 5, we have

$$Q = (\Delta_1^{-1}\theta^{(1,r)}, \Delta_2^{-1}\theta^{(2,r)}, \dots, \Delta_d^{-1}\theta^{(d,r)})^T,$$

and therefore

$$Q^{(\{i\},\{i\})} = \Delta_i^{-1}\theta_i^{(i,r)}.$$

Hence,

$$\begin{aligned} \tilde{\Sigma}(\{i\}) &= \Delta_i^2 (\theta_i^{(i,r)})^{-2} \sum_{j,k} [Q]_{ij} \Sigma_{jk} [Q]_{ik} \\ &= \Delta_i^2 (\theta_i^{(i,r)})^{-2} \sum_{j,k} \Delta_i^{-1} \theta_j^{(i,r)} \Sigma_{jk} \Delta_i^{-1} \theta_k^{(i,r)} \\ &= (\theta_i^{(i,r)})^{-2} \langle \theta^{(i,r)}, \Sigma \theta^{(i,r)} \rangle \\ &= -2(\theta_i^{(i,r)})^{-2} \langle \theta^{(i,r)}, \mu \rangle, \end{aligned}$$

where the last equality is obtained since $\gamma(\theta^{(i,r)}) = 0$. Similarly, we have

$$\tilde{\mu}(\{i\}) = (\theta_i^{(i,r)})^{-1} \langle \theta^{(i,r)}, \mu \rangle.$$

Hence,

$$\lambda_i = -\frac{2\tilde{\mu}(\{i\})}{\tilde{\Sigma}(\{i\})} = \theta_i^{(i,r)}.$$

This completes the proof of Corollary 1. $\square$

In what follows, we give a short proof for $\alpha_i = \lambda_i$ when the skew symmetric condition holds. For this, we use Lemma 2 in [10], which characterizes the condition by the formula

$$\gamma(\theta) = \sum_{i=1}^d \frac{\Sigma_{ii}}{2R_{ii}} \gamma_i(\theta)(\alpha_i - \theta_i), \quad \theta \in \mathbb{R}^d.$$

Then, we immediately have $\alpha_i = \theta_i^{(i,r)} = \lambda_i$ from the fact that $\gamma(\theta^{(i,r)}) = 0$, $\gamma_i(\theta^{(i,r)}) \neq 0$ and $\gamma_i(\theta^{(j,r)}) = 0$ for $j \neq i$.

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