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The first hitting time of the integers by symmetric Lévy processes

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Abstract

For one-dimensional Brownian motion, the exit time from an interval has finite exponential moments and its probability density is expanded in exponential terms. In this note we establish its counterpart for certain symmetric Lévy processes. Applying the theory of Pick functions, we study properties of the Laplace transform of the first hitting time of the integer lattice as a meromorphic function in detail. Its density is expanded in exponential terms and the poles and the zeros of a Pick function play a crucial role.

Intermediate results concerning finite exponential moments are also obtained for a class of nonsymmetric Lévy processes.

MSC: 60G51; 60J45; 60G52

Keywords: Lévy process; Probabilistic potential theory; Pick function; Fractional linear transformations

1. Introduction

Let $B(t)$ be a standard Brownian motion starting at x. We denote the probability and the expectation by P_x^{BM} and E_x^{BM}, respectively. If we set $\Psi^{BM}(\xi) = (1/2)\xi^2$ for $\xi \in \mathbb{R}$ we have $E_x^{BM}\left[\exp(i\xi B(t))\right] = \exp\left(i\xi x - \Psi^{BM}(\xi)t\right)$. We fix $L > 0$, denote by $L\mathbb{Z}$ the lattice set $\{Lm | m \in \mathbb{Z}\}$, and by $T_{L\mathbb{Z}}^{BM}$ its first hitting time by $B(t)$: $\inf\{t > 0 | B(t) \in L\mathbb{Z}\}$. Let $\mathbb{R}/L\mathbb{Z}$ be the quotient space of $\mathbb{R}$ with the equivalence relation that $x \sim y \leftrightarrow x - y \in L\mathbb{Z}$, i.e., $\mathbb{R}/L\mathbb{Z}$ is the circle with the length L. We denote by $\tilde{B}(t)$ the projection of $B(t)$ on $\mathbb{R}/L\mathbb{Z}$ and by $p_{\mathbb{R}/L\mathbb{Z}}^{BM}(t, x, y)$ its probability density.

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We introduce a meromorphic function

$$R^{BM}(z) = \sum_{n\in\mathbb{Z}} \frac{-1}{z + \Psi^{BM}(2n\pi/L)} = \frac{-L}{\sqrt{2z}} \coth(\sqrt{2z}L/2)$$

and a sequence $\rho_k = \Psi^{BM}(2k\pi/L)$ for $k \in \mathbb{N}\cup\{0\}$ so that the poles of $R^{BM}(z)$ are $(-\rho_k)_{k\in\mathbb{N}\cup\{0\}}$. Let $\zeta_k = (1/2)(2k-1)^2\pi^2/L^2$ for $k \in \mathbb{N}$. Then the zeros of $R^{BM}(z)$ are $(-\zeta_k)_{k\in\mathbb{N}}$. The sequence $(\rho_k)_{k\in\mathbb{N}\cup\{0\}}$ appears in the expansion

$$p^{BM}_{\mathbb{R}/L\mathbb{Z}}(t,x,y) = \sum_{k\in\mathbb{Z}} \frac{1}{L} \exp\left(i2k\pi(y-x)/L - \rho_{|k|}t\right), \tag{1}$$

where $t > 0$ and $x, y \in \mathbb{R}/L\mathbb{Z}$. The sequence $(\zeta_k)_{k\in\mathbb{N}}$ appears in the expansion

$$P^{BM}_x[T^{BM}_{L\mathbb{Z}} \in dt]/dt = \sum_{k=1}^{\infty} \frac{2(2k-1)\pi}{L^2} \sin\left(\frac{(2k-1)\pi x}{L}\right) \exp(-\zeta_k t), \tag{2}$$

where $t > 0$ and $0 < x < L$. The equality $\zeta_k = \Psi^{BM}((2k-1)\pi/L)$ turns out to be merely a coincidence in view of our extension in Theorem 3.1 to the Lévy process case.

Let $X(t)$ be a Lévy process starting from $x \in \mathbb{R}$. The probability and the expectation of $X(t)$ are denoted by P_x and E_x, respectively. We set $T(L\mathbb{Z}) = \inf\{t > 0 | X(t) \in L\mathbb{Z}\}$ and call it the first hitting time of the integer lattice by $X(t)$. We denote by $\tilde{X}(t)$ the projection of $X(t)$ on $\mathbb{R}/L\mathbb{Z}$.

Let $\Psi(\xi)$ be the characteristic exponent of the Lévy process $X(t)$ such that $E_x[e^{i\xi X(t)}] = e^{i\xi x - \Psi(\xi)t}$. We will assume the conditions (3), (4) and (12) that are sufficient for existence of transition density $p(t,x,y)$ for $X(t)$. We denote that for $\tilde{X}(t)$ by $p_{\mathbb{R}/L\mathbb{Z}}(t,x,y)$.

In [3] the author studies $T(L\mathbb{Z})$ in the case $X(t) = X_\alpha(t)$, a symmetric α-stable Lévy process with $1 < \alpha \le 2$, and proves that its Laplace transform $q \mapsto E_x[e^{-qT(L\mathbb{Z})}]$ can be extended to a meromorphic function and is holomorphic on a neighborhood of the origin. Finiteness of some exponential moments of $T(L\mathbb{Z})$ follows from this but the abscissa of convergence is not specified. The density of $T(L\mathbb{Z})$ is only shown to exist and be square-integrable.

In the present paper we extend to a wider class of Lévy processes and strengthen the result to obtain an expansion of the density of $T(L\mathbb{Z})$. The crucial steps in the proof are an application of the theory of Pick functions and an upper bound of meromorphic functions based on a property of fractional linear transformations.

More precisely, we study an instance of Pick function defined by $R(z) = \sum_{n\in\mathbb{Z}}(-1)/(z + \Psi(2n\pi/L))$. We will show, in the proof of Theorem 3.1(a), that the poles and the zeros of $R(z)$ lie on the nonpositive real axis and are interlacing, where two sequences are said interlacing if one member of one of them lies between each pair of neighboring terms of the other. We redefine $(\rho_k)_{k\in\mathbb{N}\cup\{0\}}$ and $(\zeta_k)_{k\in\mathbb{N}}$ as two increasing and interlacing sequences of nonnegative real numbers such that $(-\rho_k)_{k\in\mathbb{N}\cup\{0\}}$ are the poles of $R(z)$ and $(-\zeta_k)_{k\in\mathbb{N}}$ the zeros. These sequences will appear in (19):

$$p_{\mathbb{R}/L\mathbb{Z}}(t,x,y) = \sum_{k=0}^{\infty} a_k(x,y)\exp(-\rho_k t)$$

and in (14):

$$P_x[T(L\mathbb{Z}) \in dt]/dt = \sum_{k=1}^{\infty} b_k(x)\exp(-\zeta_k t),$$

which is our main result. The coefficients $a_k(x, y)$ and $b_k(x)$ are specified in (19) and (14). Fractional linear transformations appear in the following context. In Theorem 2.1 we prove that the analytic continuation of $0 < q \mapsto E_x[e^{-qT(L\mathbb{Z})}]$ is given by $R(q; x)/R(q)$, where $R(z; x) = \sum_{n\in\mathbb{Z}}(-\exp(i2n\pi x/L))/(z + \Psi(2n\pi/L))$. Since $(-\rho_k)_{k\in\mathbb{N}\cup\{0\}}$ are poles of order 1 of both $R(z; x)$ and $R(z)$, they are removable singularities of $R(z; x)/R(z)$. The poles of $R(z; x)/R(z)$ are $(-\zeta_k)_{k\in\mathbb{N}}$ and are all of order 1. In Lemma 3.5 we derive an upper bound of $|R(z; x)/R(z)|$ on the lines $\{-\rho_k + iy | y \in \mathbb{R}\}$ using the fact that $(-1)/(z + \Psi(2n\pi/L))$ lies on the image of $t \in \mathbb{R} \mapsto (-1)/(z + t)$, which is the circle with the center $i/(2\Im z)$ and the radius $1/(2|\Im z|)$. Finally by a contour integral of $R(z; x)/R(z)$ involving the formula (17) we obtain in Theorem 3.1 an expansion of $E_x[e^{-qT(L\mathbb{Z})}; T(L\mathbb{Z}) > h]$, that is connected with the density by differentiation.

This paper is organized as follows. In Section 2 we prove Theorem 2.1 where we assume conditions (3) and (4) and state that $E_x[e^{-qT(L\mathbb{Z})}] = R(q; x)/R(q)$, which implies some exponential moments of $T(L\mathbb{Z})$ are finite for certain nonsymmetric Lévy process $X(t)$. A stable Lévy process (with the index greater than 1) plus a drift is an instance and we give explicit calculations for the case $X(t) = B(t) + \mu t$, a Brownian motion with drift. The method of proof are the probabilistic potential theory, the Poisson summation formula, and the fact that analytic continuation over a neighborhood of the origin of the Laplace transform of a probability measure implies finiteness of some exponential moments.

In Section 3 we focus on symmetric Lévy processes and add the condition (12) that enables the argument in the proof of Lemma 3.5. Theorem 3.1 is the main theorem and it contains the expansion (14). In Remark 3.6 after the proof of Theorem 3.1 we suggest that our method also applies to the nonsymmetric cases. The short subSection 3.1 is devoted to (19). Its method of proof is the Poisson summation formula.

2. Finiteness of exponential moments in the nonsymmetric case

Let $\Psi(\xi)$ be the characteristic exponent of the Lévy process $X(t)$ such that $E_x[e^{i\xi X(t)}] = e^{i\xi x - \Psi(\xi)t}$. Fix $L > 0$ and for some $q > 0$ we assume

$$\int_{\mathbb{R}} \left| \frac{1}{q + \Psi(\xi)} \right| d\xi < \infty, \tag{3}$$

$$\sum_{n\in\mathbb{Z}} \left| \frac{1}{q + \Psi(2n\pi/L)} \right| < \infty. \tag{4}$$

It is known that if (3) and (4) hold for some $q > 0$ then they hold for all $q > 0$. Let $u^q(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ix\xi}(q + \Psi(\xi))^{-1} d\xi$ which is called the q-potential. By (3), $u^q(x)$ is bounded and continuous and it holds $\Psi(\xi) = 0$ if and only if $\xi = 0$.

For instance, let $1 < \alpha \le 2$, $C > 0$, $\beta \in [-1, 1]$, $\mu \in \mathbb{R}$, and $\Psi(\xi) = C|\xi|^\alpha (1 - i\beta \operatorname{sgn}(\xi) \tan(\pi\alpha/2)) - i\mu\xi$. This exponent corresponds to a (nonsymmetric) stable Lévy process plus a drift μt and satisfies (3) and (4). In Example 2.2 after the proof of Theorem 2.1 we will study the case $\Psi(\xi) = (1/2)\xi^2 - i\mu\xi$, namely the Brownian motion with drift.

Before we state the theorem we introduce notations for some meromorphic functions.

Definition 2.1. We set $R(z; x) = \sum_{n\in\mathbb{Z}}(-\exp(i2n\pi x/L))/(z+\Psi(2n\pi/L))$ and $R(z) = R(z; 0)$ for $z \in \mathbb{C}$ and $x \in \mathbb{R}$.

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By (4) the map $z \mapsto R(z; x)$ is well-defined and holomorphic on the set $\mathbb{C} \backslash \{-\Psi(2n\pi/L) \,|\, n \in \mathbb{Z}\}$. Each pole $z = -\Psi(2n\pi/L)$ of $R(z)$ is of order 1 and is possibly also a pole of $R(z; x)$. Hence it is a removable singularity of $z \mapsto R(z; x)/R(z)$ such that

$$\lim_{z \to -\Psi(2n\pi/L)} \frac{R(z; x)}{R(z)} = \frac{\sum_{k:\, \Psi(2k\pi/L) = \Psi(2n\pi/L)} \exp(i2k\pi x/L)}{\#\{k \in \mathbb{Z} \,|\, \Psi(2k\pi/L) = \Psi(2n\pi/L)\}}. \tag{5}$$

The following theorem immediately implies that some exponential moments of $T(L\mathbb{Z})$ are finite.

Theorem 2.1. *Assume (3) and (4). Let $X(t)$ be started at $x \in \mathbb{R}$ and let $T(L\mathbb{Z}) = \inf\{t > 0 \,|\, X(t) \in L\mathbb{Z}\}$ where $L\mathbb{Z} = \{Lm \,|\, m \in \mathbb{Z}\}$. Then there exists an $r > 0$, that may depend only on Ψ and L, such that the both sides of $E_x[e^{-qT(L\mathbb{Z})}] = R(q; x)/R(q)$ are finite and the equality holds for any $q \in \mathbb{C}$ with $\Re q > -r$.*

Proof. We first review some facts from the potential theory (see e.g. Bertoin [1], Chapter II). Let $q > 0$ and $T(y) = \inf\{t > 0 \,|\, X(t) = y\}$ for $y \in \mathbb{R}$. It follows from Corollary II.20, Theorem II.19 in [1], and (3) that a single point is regular for itself for $X(t)$ and the q-resolvent has the density such that $U^q(x, dy) = u^q(y - x)dy$, where u^q is bounded, continuous, positive, and satisfies

$$\frac{1}{q + \Psi(\xi)} = \int_{-\infty}^{\infty} e^{ix\xi} u^q(x)dx, \qquad u^q(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\xi} \frac{1}{q + \Psi(\xi)} d\xi \tag{6}$$

and $E_x[e^{-qT(y)}] = u^q(y - x)/u^q(0)$ for all x and y. The capacitary measure $\mu^q_{L\mathbb{Z}}$ is defined by $\mu^q_{L\mathbb{Z}}(A) = q \int_{-\infty}^{\infty} E_x[e^{-qT(L\mathbb{Z})}; X(T(L\mathbb{Z})) \in A]dx$ for any Borel set $A \subset \mathbb{R}$ and is supported by $L\mathbb{Z}$. Since the set $L\mathbb{Z}$ is translation invariant, $\mu^q_{L\mathbb{Z}}$ assigns the same mass $\mu^q_{L\mathbb{Z}}(\{0\})$ for each point of $L\mathbb{Z}$. By Theorem II.7 in [1] we have $\hat{E}_x[e^{-qT(L\mathbb{Z})}]dx = \mu^q_{L\mathbb{Z}} U^q(dx)$, where $\hat{E}_x$ is concerned with the dual process $\hat{X}(t)$ defined by $\hat{X}(t) - \hat{X}(0) = -(X(t) - X(0))$. By exploiting this identity for the dual and by taking the density, we have

$$E_x[e^{-qT(L\mathbb{Z})}] = \mu^q_{L\mathbb{Z}}(\{0\}) \sum_{m \in \mathbb{Z}} u^q(Lm - x) \tag{7}$$

for a.e. x. We denote the right hand side of (7) by $f(x)$.

On one hand, we prove the continuity of the left hand side of (7) as a function of x.

By $E_x[e^{-qT(y)}] = u^q(y - x)/u^q(0)$ and the continuity of $u^q(x)$, it holds $\lim_{|x-y| \to 0} E_x[e^{-qT(y)}] = 1$. Let $m \in \mathbb{Z}$. Since Lm is regular for $L\mathbb{Z}$ we have $E_{Lm}[e^{-qT(L\mathbb{Z})}] = 1$. As $x \to Lm$, we have

$$1 \geq E_x[e^{-qT(L\mathbb{Z})}] \geq E_x[e^{-qT(Lm)}] \to 1.$$

Hence $x \mapsto E_x[e^{-qT(L\mathbb{Z})}]$ is continuous at $Lm \in L\mathbb{Z}$.

Let $z \notin L\mathbb{Z}$. For any x it holds $E_x[e^{-qT(L\mathbb{Z})}] \geq E_x[e^{-qT(z)}] E_z[e^{-qT(L\mathbb{Z})}]$ and $E_z[e^{-qT(L\mathbb{Z})}] \geq E_z[e^{-qT(x)}] E_x[e^{-qT(L\mathbb{Z})}]$ by the strong Markov property. Making $x \to z$ we obtain

$$\liminf_{x \to z} E_x[e^{-qT(L\mathbb{Z})}] \geq E_z[e^{-qT(L\mathbb{Z})}] \geq \limsup_{x \to z} E_x[e^{-qT(L\mathbb{Z})}],$$

which means continuity at z. Now the proof of continuity on $\mathbb{R}$ is completed.

On the other hand, we prove that $(-\mu^q_{L\mathbb{Z}}(\{0\})/L) R(q; x)$ is a continuous version of the right hand side of (7) as a function of x. Since $u^q(x)$ is positive and $\int_{-\infty}^{\infty} u^q(x)dx = 1/q$, the function $f(x)$ defined after (7) is integrable on $[0, L]$. By (4) the Fourier coefficient $(c_k)_{k \in \mathbb{Z}}$ defined by

$$c_k = \frac{1}{L} \int_0^L \exp(-i2k\pi x/L) f(x)dx$$

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satisfies $\sum_{k\in\mathbb{Z}}|c_k| < \infty$ since

$$c_k = \frac{\mu^q_{L\mathbb{Z}}(\{0\})}{L}\int_{-\infty}^{\infty}\exp(-i2k\pi x/L)u^q(-x)dx = \frac{\mu^q_{L\mathbb{Z}}(\{0\})}{L(q+\Psi(2k\pi/L))}.$$

By Corollary 1.8 in Chapter VII of [7] we have $f(x) = \sum_{k\in\mathbb{Z}}c_k\exp(i2k\pi x/L)$ for a.e. $x \in [0, L]$ and the right hand side is a continuous function. This equality holds for a.e. $x \in \mathbb{R}$ by periodicity. Since $E_x[e^{-qT(L\mathbb{Z})}] = f(x)$ a.e. we have

$$E_x[e^{-qT(L\mathbb{Z})}] = \sum_{k\in\mathbb{Z}} c_k\exp(i2k\pi x/L) = (-\mu^q_{L\mathbb{Z}}(\{0\})/L)R(q; x), \tag{8}$$

where the second equality follows from Definition 2.1. Since the quantities in (8) are continuous in x the two equalities hold for $x \in \mathbb{R}$. Setting $x = 0$ we have $1 = (-\mu^q_{L\mathbb{Z}}(\{0\})/L)R(q)$. The ratio between respective sides of this equality and (8) yields

$$E_x[e^{-qT(L\mathbb{Z})}] = \frac{R(q; x)}{R(q)} \tag{9}$$

for $q > 0$.

In the final part of proof we check that $q \mapsto E_x[e^{-qT(L\mathbb{Z})}]$ admits analytic continuation over a neighborhood of the origin. Note that

$$\frac{R(q; x)}{R(q)} = \frac{1+\sum_{n\in\mathbb{Z},n\neq 0}\exp(i2n\pi x/L)q/(q+\Psi(2n\pi/L))}{1+\sum_{n\in\mathbb{Z},n\neq 0} q/(q+\Psi(2n\pi/L))} \tag{10}$$

for $q > 0$.

Set $f_n(z) = z/(z + \Psi(2n\pi/L))$ for $z \in \mathbb{C}$ and $n \in \mathbb{Z}\setminus\{0\}$. Each of them is a meromorphic function having only one simple pole at $z = -\Psi(2n\pi/L)$. Set also $f_0(z) = 1$. By (4), we have $\Psi(2n\pi/L) \neq 0$ for all $n \neq 0$ and $\lim_{n\to\infty}|\Psi(2n\pi/L)| = \infty$. Hence $\rho = \frac{1}{3}\inf_{n\in\mathbb{Z}\setminus\{0\}} |\Psi(2n\pi/L)| > 0$.

For $n \in \mathbb{Z}\setminus\{0\}$ and $z \in \mathbb{C}$ with $|z| < \rho$, we have the uniform bound

$$\begin{aligned}|f_n(z)| &\leq \frac{\rho}{|\Psi(2n\pi/L)|-\rho}\\ &\leq \frac{\rho}{\frac{1}{3}|\Psi(2n\pi/L)| + \frac{2}{3}\inf_{n\in\mathbb{Z}\setminus\{0\}}|\Psi(2n\pi/L)|-\rho}\\ &= \frac{\rho}{\frac{1}{3}|\Psi(2n\pi/L)|+\rho}.\end{aligned}$$

Since the rightmost hand is a summable sequence, the sequence of holomorphic functions $\sum_{|n|<N} f_n(z)$ converges uniformly on $\{|z| < \rho\}$. Hence $g(z; 0) = \sum_{n\in\mathbb{Z}} f_n(z)$ is a holomorphic function on $\{|z| < \rho\}$ such that $g(0; 0) = 1$ and hence $g(z; 0) \neq 0$ on $\{|z| < r\}$ with some $r \in (0, \rho)$. It is clear that r may depend only on Ψ and L.

If we fix $x \in \mathbb{R}$ and set $g(z; x) = \sum_{n\in\mathbb{Z}} e^{i2n\pi x/L} f_n(z)$ then we can verify that $z \mapsto g(z; x)$ is holomorphic on $\{|z| < \rho\}$ by $|e^{i2n\pi x/L}| = 1$ and the argument in the last paragraph.

Since (9) and (10) imply $E_x[e^{-qT(L\mathbb{Z})}] = g(q; x)/g(q; 0)$ for all $q \in (0, r)$ and $g(z; x)/g(z; 0)$ is holomorphic on $\{|z| < r\}$ we have $E_x[e^{-qT(L\mathbb{Z})}] = g(q; x)/g(q; 0) < \infty$ for all q with $\Re q > -r$ by the method of proof of Theorem 2 in Lukacs–Szasz [5]. □

Example 2.2 (*Brownian Motion with Drift*)**.** Let $B(t)$ be a standard Brownian motion. Set $q > 0$, $0 < x < L$, $\mu \in \mathbb{R}$ and $X(t) = x + B(t) + \mu t$ so that $\Psi(\xi) = \frac{1}{2}\xi^2 - i\mu\xi$.

By the optional stopping theorem we can deduce that

$$E_x[e^{-qT(L\mathbb{Z})}] = \frac{e^{\mu(L-x)}\sinh\sqrt{2q+\mu^2}x + e^{-\mu x}\sinh\sqrt{2q+\mu^2}(L-x)}{\sinh\sqrt{2q+\mu^2}L}, \tag{11}$$

which is equivalent to the formula 2.3.0.1 in Borodin–Salminen [2, p. 233].

The quantities appearing in Theorem 2.1 are computed as follows. First we invert the Fourier transform $\mathcal{F}u^q(\xi) = 1/(q + \frac{1}{2}\xi^2 - i\mu\xi)$ of the q-potential to obtain

$$u^q(x) = \frac{1}{\sqrt{2q+\mu^2}}\exp\left(\mu x - \sqrt{2q+\mu^2}|x|\right).$$

If we set $x = 0$ in (7) we have $E_0[e^{-qT(L\mathbb{Z})}] = 1$ and

$$\begin{aligned}
\frac{1}{\mu^q_{L\mathbb{Z}}(\{0\})} &= \sum_{m\in\mathbb{Z}} u^q(Lm) \\
&= \frac{1}{\sqrt{2q+\mu^2}}\sum_{m\in\mathbb{Z}}\exp\left(\mu Lm - \sqrt{2q+\mu^2}L|m|\right) \\
&= \frac{1}{\sqrt{2q+\mu^2}}\left(\frac{1}{1-e^{-(\sqrt{2q+\mu^2}-\mu)L}} + \frac{e^{-(\sqrt{2q+\mu^2}+\mu)L}}{1-e^{-(\sqrt{2q+\mu^2}+\mu)L}}\right) \\
&= \frac{\sinh\sqrt{2q+\mu^2}L}{2\sqrt{2q+\mu^2}\sinh((\sqrt{2q+\mu^2}-\mu)\frac{L}{2})\sinh((\sqrt{2q+\mu^2}+\mu)\frac{L}{2})}.
\end{aligned}$$

By (11) and (8) we have

$$R(q;x) = \frac{-L\left(e^{\mu(L-x)}\sinh\sqrt{2q+\mu^2}x + e^{-\mu x}\sinh\sqrt{2q+\mu^2}(L-x)\right)}{2\sqrt{2q+\mu^2}\sinh((\sqrt{2q+\mu^2}-\mu)\frac{L}{2})\sinh((\sqrt{2q+\mu^2}+\mu)\frac{L}{2})}.$$

Since both sides of this equality are meromorphic functions of $q \in \mathbb{C}$ and coincide with each other on the positive real axis, they have common poles and coincide with each other outside the poles. Here are several consequences. On one hand, $q = -2\pi^2n^2/L^2 + i\mu 2n\pi/L$ where $n \in \mathbb{Z}$ are the poles of $R(q;x)$ and $R(q)$ but they are removable singularities of $R(q;x)/R(q)$. On the other hand, $q = -\mu^2/2$ is a removable singularity of $R(q;x)$ and $R(q)$. The poles of $R(q;x)/R(q)$ are $q = -\mu^2/2 - \pi^2k^2/(2L^2)$ where $k \in \mathbb{N}$. Hence the abscissa of convergence of $E_x[e^{-qT(L\mathbb{Z})}]$ is $q = -\mu^2/2 - \pi^2/(2L^2)$.

The density of $T(L\mathbb{Z})$ is obtained in Remark 3.6 after the proof of Theorem 3.1.

3. Expansion of density in the symmetric case

In Section 3 we assume (3), (4), the symmetry of $X(t)$, and the following. Note first that the symmetry implies $\Psi(-\xi) = \Psi(\xi)$ and that $\Psi(\xi)$ is real and nonnegative. For $n \in \mathbb{Z}$ we define $I_n \subset \mathbb{R}$ as the open interval $(\Psi(2(n-1)\pi/L), \Psi(2n\pi/L))$ if $\Psi(2(n-1)\pi/L) < \Psi(2n\pi/L)$, as $(\Psi(2n\pi/L), \Psi(2(n-1)\pi/L))$ if $\Psi(2n\pi/L) < \Psi(2(n-1)\pi/L)$, and as $\emptyset$ if $\Psi(2(n-1)\pi/L) = \Psi(2n\pi/L)$. We assume

$$\sup_{t\in[0,\infty)} \#\{n \in \mathbb{Z} | t \in I_n\} \le M_0 \text{ for some } M_0 > 0. \tag{12}$$

For instance, if $\Psi(\xi)$ is strictly increasing for $0 < \xi < \infty$, (12) holds with $M_0 = 2$. Other examples are symmetric stable Lévy processes plus jumps of length $\pm L/2$ with exponential

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waiting times: $\Psi(\xi) = C|\xi|^\alpha + a(1-\cos(L\xi/2))$ where $1 < \alpha < 2$, $C > 0$, and $a > 0$. If the parameters satisfy $2a > (2^\alpha - 1)C(2\pi/L)^\alpha$ we have $0 < \Psi(\pm 4\pi/L) < \Psi(\pm 2\pi/L)$ and

$$I_{-1} = I_2 = (\Psi(4\pi/L), \Psi(2\pi/L)) \subsetneq I_0 = I_1 = (0, \Psi(2\pi/L)).$$

Since $\Psi(\xi)$ is strictly increasing for all sufficiently large ξ, the condition (12) holds and any $t \in I_2$ satisfies $\#\{n \in \mathbb{Z} | t \in I_n\} \supset \{-1, 0, 1, 2\}$, which implies $4 \le M_0 < \infty$.

Before we state Theorem 3.1 we introduce the notation for the poles and the zeros of $R(z)$ that is defined in Definition 2.1.

Definition 3.1. Let $(\rho_k)_{k\in\mathbb{N}\cup\{0\}}$ be the strictly increasing sequence of real numbers such that $\{\Psi(2n\pi/L) | n \in \mathbb{Z}\} = \{\rho_k | k \in \mathbb{N}\cup\{0\}\}$ and let μ_k be the number of multiplicity of appearances $\#\{n \in \mathbb{Z} | \rho_k = \Psi(2n\pi/L)\}$ of the value ρ_k in the list $(\Psi(2n\pi/L); n \in \mathbb{Z})$.

By its definition, the sequence $(-\rho_k)_{k\in\mathbb{N}\cup\{0\}}$ exhausts the poles of $R(z)$. Since $\lim_{n\to\pm\infty}\Psi(2n\pi/L) = +\infty$ by (4), $(\rho_k)_{k\in\mathbb{N}\cup\{0\}}$ is well-defined and it holds $\rho_k \to +\infty$ as $k \to +\infty$. Since $\Psi(\xi) = 0 \Leftrightarrow \xi = 0$ and $\Psi(\xi) > 0 \Leftrightarrow \xi \ne 0$, we have $\rho_0 = 0$ and $\mu_0 = 1$. Hence it holds

$$R(z) = \sum_{n\in\mathbb{Z}} \frac{-1}{z + \Psi(2n\pi/L)} = \frac{-1}{z} + \sum_{k=1}^{\infty} \frac{-\mu_k}{z+\rho_k}. \tag{13}$$

Note also that $\mu_k \ge 2$ for any $k \in \mathbb{N}$ since $\rho_k = \Psi(2n\pi/L) > 0$ implies $\rho_k = \Psi(-2n\pi/L)$. If $\Psi(\xi)$ is strictly increasing for $0 < \xi < \infty$ we have $\rho_k = \Psi(2k\pi/L)$ and $\mu_k = 2$ for $k \in \mathbb{N}$.

Inspecting each term of $R(z)$ we easily conclude the following: $R(z)$ is strictly increasing on each interval without poles; $R(z) < 0$ if $z > 0$; for each $k \in \mathbb{N}\cup\{0\}$ it holds $\lim_{t\to\pm 0} R(-\rho_k + t) = \mp\infty$ since the term $-\mu_k/(z+\rho_k)$ is dominant.

Hence $R(z)$ has a unique simple zero on each interval $(-\rho_{k+1}, -\rho_k)$.

Definition 3.2. We define $\zeta_k > 0$ by $R(-\zeta_k) = 0$ and $\zeta_k \in (\rho_{k-1}, \rho_k)$ for each $k \in \mathbb{N}$.

Note that $\lim_{k\to\infty}\zeta_k = \infty$ and the zeros and the poles of $R(z)$ on $\mathbb{R}$ are aligned as $\cdots < -\zeta_3 < -\rho_2 < -\zeta_2 < -\rho_1 < -\zeta_1 < 0 = \rho_0$. Levin [4] calls this alignment "interlacing". As we will see in the proof Theorem 3.1(a), $R(z)$ is an instance of Pick function that is studied in Theorem 1 in [4, p. 220].

Theorem 3.1. *Assume $X(t)$ is a symmetric Lévy process that satisfies (3), (4), and (12). Let the sequence $(\zeta_k)_{k\in\mathbb{N}}$ be defined in Definition 3.2. Fix an $x \in \mathbb{R}\setminus L\mathbb{Z}$.*

(a) *The sequence $(-\zeta_k)_{k\in\mathbb{N}}$ exhausts the zeros of $R(z)$. The abscissa of convergence of $E_x[e^{-qT(L\mathbb{Z})}]$ is $q = -\zeta_1$. For any $q \in \mathbb{C}$ with $\Re q > -\zeta_1$, $E_x[e^{-qT(L\mathbb{Z})}] = R(q;x)/R(q)$.*

(b) *For any $q \in \mathbb{C}$ with $\Re q > 0$ and $h > 0$ it holds*

$$E_x[e^{-qT(L\mathbb{Z})}; T(L\mathbb{Z}) > h] = \sum_{k=1}^{\infty} \frac{R(-\zeta_k; x)}{R'(-\zeta_k)} \frac{1}{q+\zeta_k} \exp(-h(q+\zeta_k)),$$

where the convergence holds uniformly and absolutely for $h \in (\varepsilon, \infty)$ for any $\varepsilon > 0$.

(c) *The law of $T(L\mathbb{Z})$ is absolutely continuous. Its density $p_{L\mathbb{Z}}(t;x)$ satisfies*

$$p_{L\mathbb{Z}}(t;x) = \sum_{k=1}^{\infty} \frac{R(-\zeta_k;x)}{R'(-\zeta_k)} \exp(-\zeta_k t), \tag{14}$$

and possesses derivatives in t of every order, which are obtained by term-by-term differentiation of the series in (14). Moreover, the convergence is uniform and absolute for $t \in (\varepsilon, \infty)$ for any $\varepsilon > 0$.

Example 3.2 (*Symmetric α-stable Lévy Process*). Set $\Psi(\xi) = C|\xi|^\alpha$ with $1 < \alpha \le 2$ and $C > 0$. Then the conditions (3), (4), and (12) hold with $M_0 = 2$.

We have $\rho_0 = 0$, $\rho_k = C(2k\pi/L)^\alpha$, $\mu_0 = 1$, $\mu_k = 2$ for $k \in \mathbb{N}$, $R(z) = (-1)/z + \sum_{k=1}^\infty (-2)/(z+\rho_k)$, and $R(z;x) = (-1)/z + \sum_{k=1}^\infty (-2)\cos(2k\pi x/L)/(z+\rho_k)$.

With some additional effort we can prove that

$$\lim_{k\to\infty} \frac{\zeta_k - \rho_{k-1}}{\rho_k - \rho_{k-1}} = t_*$$

where $t_* \in (0,1)$ is the solution of

$$\sum_{m=0}^\infty \frac{2(1-2t_*)}{\alpha(m+t_*)(m+1-t_*)} + \frac{2\pi}{\alpha}\cot\frac{\pi}{\alpha} = 0.$$

In the case $1 < \alpha < 2$ we have $t_* \neq 1/2$ and hence $\zeta_k \neq \Psi((2k-1)\pi/L)$ for all large k since $\lim_{k\to\infty}(\Psi((2k-1)\pi/L) - \rho_{k-1})/(\rho_k - \rho_{k-1}) = 1/2$.

Remark 3.3. If we make $h \to +0$ in Theorem 3.1(b) the left hand side tends to $E_x[e^{-qT(L\mathbb{Z})}]$. But this fact does not imply the convergence, as $n \to \infty$, of $\sum_{k=1}^n R(-\zeta_k;x)R'(-\zeta_k)^{-1}(q+\zeta_k)^{-1}$ with $q = 0$ or $\Re q > 0$. In the case $\Psi(\xi) = C|\xi|^\alpha$ (Example 3.2) we can prove the convergence by elementary and tedious estimates.

Example 3.4 (*Standard Brownian Motion*). Set $\Psi(\xi) = (1/2)|\xi|^2$. Then the conditions (3), (4), and (12) hold with $M_0 = 2$.

It holds $\rho_0 = 0$, $\rho_k = (\pi^2/2L^2)(2k)^2$, $\mu_0 = 1$, $\mu_k = 2$, $\zeta_k = (\pi^2/2L^2)(2k-1)^2$ for $k \in \mathbb{N}$, $R(z;x) = (-L/\sqrt{2z})\cosh(\sqrt{2z}(x - L/2))\sinh(\sqrt{2z}L/2)^{-1}$ if $0 < x < L$, and $R(z) = (-L/\sqrt{2z})\coth(\sqrt{2z}L/2)$. Since $R(-\zeta_k;x)/R'(-\zeta_k) = 2\pi(2k-1)L^{-2}\sin((2k-1)\pi x/L)$ the result in Theorem 3.1(c) coincides with (2). The formula 1.3.0.2 in [2, p. 172] is an alternative to (2).

Proof of Theorem 3.1. (a) If $\Im z > 0$ then each term in (13) satisfies $\Im(-1/(z+\Psi(2n\pi/L))) > 0$ and if $\Im z < 0$ then $\Im(-1/(z+\Psi(2n\pi/L))) < 0$. Being their sum, $R(z)$ satisfies the same property and $R(z) \neq 0$ if $z \in \mathbb{C}\setminus\mathbb{R}$. Hence $(-\zeta_k)_{k\in\mathbb{N}}$ exhausts the zeros of $R(z)$ and possible poles of $R(z;x)/R(z)$. We easily conclude that $E_x[e^{-qT(L\mathbb{Z})}]$ converges and equals to $R(q;x)/R(q)$ for all $q \in \mathbb{C}$ with $\Re q > -\zeta_1$ by Theorem 2.1 and the method of proof of Theorem 2 in [5]. The proof of (a) is completed.

(b) By (5) and Definition 3.1, $z = -\rho_k$ is a removable singularity of $R(z;x)/R(z)$ and it holds $\lim_{z\to-\rho_k}|R(z;x)/R(z)| \le 1$. We depend on the following lemma that provides an upper bound of $|R(z;x)/R(z)|$.

Lemma 3.5. *There exist constants M_1 and M_2 that may depend on L and $x \in (0,L)$ such that*

$$\sup_{k\in\mathbb{N}}\sup_{y\in\mathbb{R}}\left|\frac{R(-\rho_k+iy;x)}{R(-\rho_k+iy)}\right| \le M_1 < +\infty \tag{15}$$

and

$$\sup_{-|\Im z|<\Re z<0}\left|\frac{R(z;x)}{R(z)}\right| \le M_2 < +\infty. \tag{16}$$

Proof. Since the series in Definition 2.1 is absolutely convergent it holds

$$\sum_{n\in\mathbb{Z}} \frac{-\exp(i2n\pi x/L)}{z+\Psi(2(n-1)\pi/L)} = \exp(i2\pi x/L)R(z;x).$$

We have for $x \in (0, L)$

$$R(z;x) = \sum_{n\in\mathbb{Z}} \frac{\exp(i2n\pi x/L)}{\exp(i2\pi x/L)-1}\left(\frac{-1}{z+\Psi(2(n-1)\pi/L)} - \frac{-1}{z+\Psi(2n\pi/L)}\right)$$

and

$$|R(z;x)| \le \frac{1}{|\exp(i2\pi x/L)-1|}\sum_{n\in\mathbb{Z}}\left|\frac{-1}{z+\Psi(2(n-1)\pi/L)} - \frac{-1}{z+\Psi(2n\pi/L)}\right|.$$

We next prove $|R(z;x)| \le M_3/|\Im z|$ for all $z \in \mathbb{C}\setminus\mathbb{R}$ for some $M_3 > 0$ using a property of fractional linear transformations, where M_3 that may depend on L and x. The map $[0,\infty) \ni t \mapsto -1/(z+t)$ is an injection to the circle $C(z) = \{w \in \mathbb{C}|\ |w - i/(2\Im z)| = 1/(2|\Im z|)\}$. For $0 \le t_1 < t_2$ we denote the arc $\{-1/(z+t) \in C(z)| t \in [t_1, t_2]\}$ by $\mathrm{arc}(-1/(z+t_1), -1/(z+t_2))$ and denote the length of a curve C by $\mathrm{len}(C)$.

Set $A_n = \{-1/(z+t)| t \in I_n\} \subset C(z)$ where I_n is defined in the first paragraph in Section 3. Note that the two endpoints of A_n are $-1/(z+\Psi(2(n-1)\pi/L))$ and $-1/(z+\Psi(2n\pi/L))$ if $I_n \ne \emptyset$. By (12) each point $-1/(z+t) \in C(z)$ can belong to A_n at most M_0 times: $\sup_{t\in[0,\infty)} \#\{n \in \mathbb{Z}| -1/(z+t) \in A_n\} \le M_0$.

By the inequality $|-1/(z+t_1) - (-1/(z+t_2))| \le \mathrm{len}(\mathrm{arc}(-1/(z+t_1), -1/(z+t_2)))$ we have

$$\sum_{n\in\mathbb{Z}}\left|\frac{-1}{z+\Psi(2(n-1)\pi/L)} - \frac{-1}{z+\Psi(2n\pi/L)}\right|$$
$$\le \sum_{n\in\mathbb{Z}} \mathrm{len}(A_n) \le M_0 \mathrm{len} C(z) \le M_0 \frac{2\pi}{2|\Im z|}.$$

Hence we have $|R(z;x)| \le M_3/|\Im z|$ with $M_3 = \pi M_0/|1-\exp(i2\pi x/L)|$.

To prove (15) we may assume $y = \Im(-\rho_k + iy) > 0$ since $R(\bar{z}) = \overline{R(z)}$ for any $z \in \mathbb{C}$. Note that $\Im(-1/(-\rho_k+iy)) > 0$ and $\Im(-\mu_m/(-\rho_k+iy+\rho_m)) > 0$ for all $m \in \mathbb{N}$. The term corresponding to $m = k$ gives the following lower bound for any $k \in \mathbb{N}$ and $y > 0$:

$$|R(-\rho_k+iy)| \ge \Im R(-\rho_k+iy) > \Im\frac{-\mu_k}{-\rho_k+iy+\rho_k} = \frac{\mu_k}{y} \ge \frac{2}{y}.$$

Hence $|R(-\rho_k+iy;x)/R(-\rho_k+iy)| \le (M_3/y)/(2/y) = M_3/2$ and (15) holds with $M_1 = M_3/2$.

To prove (16) we may assume $\Im z > -\Re z > 0$, which implies $|z|^2 \le 2(\Im z)^2$ and $\Im(-1/z) = \Im z/|z|^2 \ge 1/(2\Im z)$. Since $\Im(-\mu_k/(z+\rho_k)) > 0$ for each $k \in \mathbb{N}$, we have $|R(z)| \ge \Im R(z) \ge \Im(-1/z) \ge 1/(2\Im z)$. Hence $|R(z;x)/R(z)| \le (M_3/\Im z)/(1/(2\Im z)) = 2M_3$ and (16) holds with $M_2 = 2M_3$. The proof of lemma is completed. □

We resume proving (b). We can verify using the residue theorem that, for any $q \in \mathbb{C}$ with $\Re q > 0$ and $h, t \in \mathbb{R}$,

$$\frac{e^{-qh}}{2}1_{\{h\}}(t) + e^{-qt}1_{(h,\infty)}(t) = \lim_{A\to\infty}\int_{-A}^{A} \frac{e^{-h(q+i\theta)}}{2\pi(q+i\theta)} e^{i\theta t} d\theta. \tag{17}$$

Moreover, the uniform bound

$$\sup_{A\in(2|q|,\infty), t\in\mathbb{R}} \left|\int_{-A}^{A} \frac{e^{-h(q+i\theta)}}{2\pi(q+i\theta)} e^{i\theta t} d\theta\right| < 2e^{-h\Re q}$$

can be derived using the line integral along the half circle $C_+(A) = \{Ae^{is} | 0 \le s \le \pi\}$ for the case $t > h$ and $C_-(A) = \left\{Ae^{is} | -\pi \le s \le 0\right\}$ for the case $t \le h$.

It follows from the bounded convergence theorem that

$$\frac{e^{-qh}}{2} P_x[T(L\mathbb{Z}) = h] + E_x[e^{-qT(L\mathbb{Z})}; T(L\mathbb{Z}) > h]$$
$$= \lim_{A\to\infty} \int_{-A}^{A} \frac{e^{-h(q+i\theta)}}{2\pi(q+i\theta)} E_x[e^{i\theta T(L\mathbb{Z})}]d\theta.$$

Since the limit exists, we may replace $\lim_{A\to\infty} \int_{-A}^{A}$ with $\lim_{n\to\infty} \int_{-\rho_n}^{\rho_n}$. The expectation $E_x[e^{i\theta T(L\mathbb{Z})}]$ coincides for $\theta \in \mathbb{R}$ with $R(-i\theta; x)/R(-i\theta)$, which is a meromorphic function of θ and whose poles are $\theta = -i\zeta_n$ for $n \in \mathbb{N}$.

We define the paths $C_1(n)$, $C_2(n)$, and $C_3(n)$ as the line segments from $-\rho_n$ to $-\rho_n - i\rho_n$, from $-\rho_n - i\rho_n$ to $\rho_n - i\rho_n$, and from $\rho_n - i\rho_n$ to ρ_n, respectively.

By the residue theorem we have

$$\int_{-\rho_n}^{\rho_n} \frac{e^{-h(q+i\theta)}}{2\pi(q+i\theta)} \frac{R(-i\theta; x)}{R(-i\theta)} d\theta$$
$$= -2\pi i \sum_{k=1}^{n} \mathrm{Res}\left(\frac{e^{-h(q+i\theta)}}{2\pi(q+i\theta)} \frac{R(-i\theta; x)}{R(-i\theta)}; \theta = -i\zeta_k \right)$$
$$+ \left(\int_{C_1(n)} + \int_{C_2(n)} + \int_{C_3(n)} \right) \frac{e^{-h(q+i\theta)}}{2\pi(q+i\theta)} \frac{R(-i\theta; x)}{R(-i\theta)} d\theta$$

where

$$\mathrm{Res}\left(\frac{e^{-h(q+i\theta)}}{2\pi(q+i\theta)} \frac{R(-i\theta; x)}{R(-i\theta)}; \theta = -i\zeta_k \right) = \frac{e^{-h(q+\zeta_k)}}{2\pi(q+\zeta_k)} \frac{R(-\zeta_k; x)}{(-i)R'(-\zeta_k)}.$$

Fix an $h > 0$. If $\Im\theta < 0$ and $\Re q > 0$ it holds $\Re(q + i\theta) = \Re q + |\Im\theta| \ge \Re q$, $|e^{-h(q+i\theta)}| = e^{-h\Re q - h|\Im\theta|}$ and $|q + i\theta| \ge |\Im(q + i\theta)| = |\Im q + \Re\theta| \ge |\Re\theta| - |\Im q|$ for any θ with sufficiently large $|\Re\theta|$.

By the estimate (15) and (16) we have

$$\left| \frac{e^{-h(q+i\theta)}}{2\pi(q+i\theta)} \frac{R(-i\theta; x)}{R(-i\theta)} \right| \le \frac{e^{-h\Re q - h|\Im\theta|}}{2\pi\Re q} M_1 \le \frac{e^{-h\Re q - h\rho_n}}{2\pi\Re q} M_1$$

for $\theta \in C_2(n)$ and

$$\left| \frac{e^{-h(q+i\theta)}}{2\pi(q+i\theta)} \frac{R(-i\theta; x)}{R(-i\theta)} \right| \le \frac{e^{-h\Re q - h|\Im\theta|}}{2\pi(|\Re\theta| - |\Im q|)} M_2 \le \frac{e^{-h\Re q - h|\Im\theta|}}{2\pi(\rho_n - |\Im q|)} M_2$$

for $\theta \in C_1(n) \cup C_3(n)$ with sufficiently large n.

The length of $C_2(n)$ is $2\rho_n$ and it holds

$$\left| \int_{C_2(n)} \frac{e^{-h(q+i\theta)}}{2\pi(q+i\theta)} \frac{R(-i\theta; x)}{R(-i\theta)} d\theta \right| \le \frac{e^{-h\Re q - h\rho_n}}{2\pi\Re q} M_1 \cdot 2\rho_n \to 0$$

as $n \to \infty$. We also have

$$\left| \left(\int_{C_1(n)} + \int_{C_3(n)} \right) \frac{e^{-h(q+i\theta)}}{2\pi(q+i\theta)} \frac{R(-i\theta; x)}{R(-i\theta)} d\theta \right| \le \frac{e^{-h\Re q}}{2\pi(\rho_n - |\Im q|)} M_2 \cdot (2/h) \to 0$$

as $n \to \infty$ for any fixed $h > 0$. Hence

$$\frac{e^{-qh}}{2} P_x[T(L\mathbb{Z}) = h] + E_x[e^{-qT(L\mathbb{Z})}; T(L\mathbb{Z}) > h]$$

$$= \lim_{n\to\infty} \int_{-\rho_n}^{\rho_n} \frac{e^{-h(q+i\theta)}}{2\pi(q+i\theta)} \frac{R(-i\theta; x)}{R(-i\theta)} d\theta$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{e^{-h(q+\zeta_k)}}{q+\zeta_k} \frac{R(-\zeta_k; x)}{R'(-\zeta_k)}.$$

We denote the rightmost side by $f(h)$. We have next to prove the continuity of $f(h)$ in $h > 0$, which implies $P_x[T(L\mathbb{Z}) = h] = 0$ and the statement of (b). For this we show that $f(h_1 + h_2)$ is continuous in $h_2 \in (h_3, \infty)$ for any fixed $h_1 > 0$ and $h_3 > 0$. If we set

$$a_k = \frac{e^{-h_1(q+\zeta_k)}}{q+\zeta_k} \frac{R(-\zeta_k; x)}{R'(-\zeta_k)}$$

we have

$$f(h_1 + h_2) = \sum_{k=1}^{\infty} a_k e^{-h_2(q+\zeta_k)}.$$

On one hand, since $\sum_{k=1}^{n} a_k$ converges as $n \to \infty$ the sequence (a_k) is bounded. On the other hand, by (4),

$$\int_0^\infty \sum_{k=1}^{\infty} e^{-h(\Re q+\zeta_k)} dh = \sum_{k=1}^{\infty} \frac{1}{\Re q + \zeta_k} < \sum_{k=1}^{\infty} \frac{1}{\Re q + \rho_{k-1}}$$

$$< \sum_{k=0}^{\infty} \frac{\mu_k}{\Re q + \rho_k} = \sum_{n\in\mathbb{Z}} \frac{1}{\Re q + \Psi(2n\pi/L)} < \infty$$

and hence $\sum_{k=1}^{\infty} \exp(-h(\Re q + \zeta_k)) < \infty$ for all $h > 0$. Since $\exp(-h(\Re q + \zeta_k))$ is monotonically decreasing we have

$$\sum_{k=1}^{\infty} \sup_{h_2 \in (h_3,\infty)} |a_k e^{-h_2(q+\zeta_k)}| \le (\sup_n |a_n|) \sum_{k=1}^{\infty} e^{-h_3(\Re q+\zeta_k)} < \infty,$$

which implies uniform and absolute convergence of $f(h_1 + h_2)$ for $h_2 \in (h_3, \infty)$. Now the continuity of $f(h_1 + h_2)$ follows and the proof of (b) is completed.

(c) Let $\Re q > 0$ and retain the notations $f(h)$ and a_k. Note that

$$\sum_{k=1}^{\infty} \sup_{h_2 \in (h_3,\infty)} \left| a_k \frac{d}{dh_2} e^{-h_2(q+\zeta_k)} \right|$$

$$= \sum_{k=1}^{\infty} \sup_{h_2 \in (h_3,\infty)} \left| a_k (q+\zeta_k) e^{-h_2(q+\zeta_k)} \right|$$

$$\le (\sup_n |a_n|) \left(\sup_n |q+\zeta_n| e^{-(h_3/2)(\Re q+\zeta_n)} \right) \sum_{k=1}^{\infty} e^{-(h_3/2)(\Re q+\zeta_k)}$$

$$\le (\sup_n |a_n|) \left(|q| + \sup_{t>0} t e^{-(h_3/2)t} \right) \sum_{k=1}^{\infty} e^{-(h_3/2)(\Re q+\zeta_k)}$$

$$< \infty,$$

which implies that $f(h_1+h_2)=\sum_{k=1}^{\infty}a_k\exp(-h_2(q+\zeta_k))$ is differentiable term by term in $h_2\in(h_3,\infty)$ for any fixed $h_1>0$ and $h_3>0$. Hence $f(h)=\sum_{k=1}^{\infty}\exp(-h(q+\zeta_k))(q+\zeta_k)^{-1}R(-\zeta_k;x)R'(-\zeta_k)^{-1}=E_x[e^{-qT(L\mathbb{Z})};T(L\mathbb{Z})>h]$ is differentiable term by term in $h>0$. We have

$$-e^{-qh}p_{L\mathbb{Z}}(h;x)=\frac{d}{dh}E_x[e^{-qT(L\mathbb{Z})};T(L\mathbb{Z})>h]=\sum_{k=1}^{\infty}\left(-e^{-h(q+\zeta_k)}\right)\frac{R(-\zeta_k;x)}{R'(-\zeta_k)}.$$

Repeating this argument we immediately verify differentiability of all order and the uniform and absolute convergence in the statement (c). □

Remark 3.6. The method of Theorem 3.1 also applies to certain nonsymmetric cases. Recall Example 2.2 where $X(t)$ is a Brownian motion with drift. The poles $q=-\zeta_k=-\mu^2/2-\pi^2k^2/(2L^2)$ of $R(q;x)/R(q)$ are located on the negative real axis. By an elementary argument we can prove two upper bounds for $|R(q;x)/R(q)|$, which are similar to (15) and (16) on the line $\{q\in\mathbb{C}|\Re q=-\mu^2/2-\pi^2(k-\frac{1}{2})^2/(2L^2)\}$ and on the set $\{q\in\mathbb{C}|-|\Im q|<\Re q<0\}$, respectively. By the argument in the proof of Theorem 3.1(b) and (c) we have

$$p_{L\mathbb{Z}}(t;x)=\sum_{k=1}^{\infty}\frac{\pi k}{L^2}\left((-1)^{k-1}e^{\mu(L-x)}+e^{-\mu x}\right)\sin\left(\frac{k\pi x}{L}\right)\exp(-\zeta_k t). \tag{18}$$

We can find an alternative expression for this formula in [2, p. 233]. In fact, (18) is a spectral representation of the formula 2.3.0.2 there.

It seems probable that other nonsymmetric Lévy process $X(t)$ admits similar expansion of $p_{L\mathbb{Z}}(t;x)$ but we have not made any progress in this direction.

3.1. Transition density for periodic process

Since the resolvent density $u^q(x)$ exists by the condition (3), the transition density $p(t,x,y)$ for $X(t)$ exists if $X(t)$ is symmetric (see Sato [6, Remark 41.13]). In Section 1 we define $\tilde{X}(t)$ as the projection of $X(t)$ on $\mathbb{R}/L\mathbb{Z}$ and $p_{\mathbb{R}/L\mathbb{Z}}(t,x,y)$ as the transition density for $\tilde{X}(t)$.

Lemma 3.7. *For $x,y\in\mathbb{R}$ and $t>0$ we have*

$$p_{\mathbb{R}/L\mathbb{Z}}(t,x,y)=\sum_{k=0}^{\infty}\frac{1}{L}\exp(-\rho_k t)\sum_{n:\Psi(-2n\pi/L)=\rho_k}\exp\left(i2n\pi(y-x)/L\right). \tag{19}$$

Proof. By the periodic sum we have $p_{\mathbb{R}/L\mathbb{Z}}(t,x,y)=\sum_{k\in\mathbb{Z}}p(t,x,y+kL)$. We define a_n as $a_n=\frac{1}{L}\int_0^L\exp\left(-i2n\pi y/L\right)p_{\mathbb{R}/L\mathbb{Z}}(t,x,y)dy$. Then it holds

$$\begin{aligned}a_n&=\frac{1}{L}\int_{-\infty}^{\infty}\exp\left(-i2n\pi y/L\right)p(t,x,y)dy\\&=\frac{1}{L}E_x[\exp\left(-i2n\pi X(t)/L\right)]\\&=\frac{1}{L}\exp\left(-i2n\pi x/L-\Psi(-2n\pi/L)t\right).\end{aligned}$$

Since $\Psi(\xi)$ is real-valued and nonnegative there exists a constant $M_4(t,q)>0$ such that $\exp(-\Psi(\xi)t)<M_4(t,q)/(q+\Psi(\xi))$ for any $\xi\in\mathbb{R}$. The condition (4) implies $\sum_{n\in\mathbb{Z}}|a_n|<\infty$.

By Corollary 1.8 in [7, Chapter VII] we have

$$\sum_{n\in\mathbb{Z}} a_n \exp(i2n\pi y/L) = p_{\mathbb{R}/L\mathbb{Z}}(t, x, y)$$

for almost all $y \in [0, L]$. Rearrangement of terms yields the desired formula in view of Definition 3.1. □

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