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On a characterization of optimal predictors for nonstationary ARMA processes

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This note contains a characterization of predictors for nonstationary ARMA processes. Moreover, we give the weak law of large numbers for those processes.

nonstationary ARMA process * linear prediction * weak law of large numbers

1. Introduction

Let the complex-valued ARMA process $\{y_k,\ k\in\mathbb{Z}\}$ be given by the equation

$$A(q^{-1})y_k = C(q^{-1})e_k \tag{1}$$

where $\{e_k,\ k\in\mathbb{Z}\}$ is the orthogonal-valued process such that $Ee_k=0$, $0<m\leq E|e_k|^2\leq M$, $k\in\mathbb{Z}$, and

$$A(q^{-1}) = 1+a_1q^{-1}+a_2q^{-2}+\cdots+a_nq^{-n},$$

$$C(q^{-1}) = 1+c_1q^{-1}+c_2q^{-2}+\cdots+c_nq^{-n},\quad n\in\mathbb{N},$$

with q^{-1} being the backward shift operator i.e. $q^{-1}y_k=y_{k-1}$.

A process $\{y_k,\ k\in\mathbb{Z}\}$ given by (1) is said to be regular if the all roots of the polynomials $A(q^{-1})$ and $C(q^{-1})$ belong strictly inside the unit circle. In this case there exists the unique solution of (1) such that the linear spaces spanned by the processes $\{y_k,\ k\leq l\}$ and $\{e_k,\ k\leq l\}$, $l\in\mathbb{Z}$, coincide [7]. Moreover, it has been proved there that the linear h-step predictor $\{\hat{y}_{k+h|k},\ k\in\mathbb{Z}\}$ has the similar characterization as in the stationary case.

A more general model

$$\tilde{A}_k(q^{-1})y_k = \tilde{C}_k(q^{-1})e_k,\quad k\in\mathbb{Z}, \tag{2}$$

with time dependent coefficients

$$\tilde{A}_k(q^{-1}) = 1+a_1(k)q^{-1}+\cdots+a_n(k)q^{-n},$$

$$\tilde{C}_k(q^{-1}) = 1+c_1(k)q^{-1}+\cdots+c_n(k)q^{-n},$$

$a_j(k)$, $c_j(k) \in \mathbb{C}$, $j = 1, \ldots, n$, $k \in \mathbb{Z}$, and an orthogonal, stationary, or UBLS process $\{e_k, k \in \mathbb{Z}\}$ i.e. a process such that (cf. [12])

$$E\left|\sum_{j=1}^{m} \alpha_j e_{k_j+h}\right|^2 \leq ME\left|\sum_{j=1}^{m} \alpha_j e_{k_j}\right|^2$$

for some constant $M > 0$ and every $\alpha_j \in \mathbb{C}$, $j = 1, 2, \ldots, n$, $k_1, \ldots, k_m$, $h \in \mathbb{Z}$, was discussed in several papers (cf. [2, 3, 9, 10]). To compare our results with those contained in the above papers we need to recall their main states.

Let $G(t, s)$ and $H(t, s)$ be Green functions associated with the

$$\tilde{A}(q^{-1})y_k = 0, \qquad \tilde{C}(q^{-1})e_k = 0,$$

respectively.

Suppose that $E|e_k|^2 = \sigma^2 = \text{const}$. If

$$\sum_{s=-\infty}^{k} |G(k, s)| < \infty, \quad k \in \mathbb{Z}, \tag{3}$$

and if there exists a constant M such that

$$\sum_{j=0}^{q} |c_j(k)| < M, \quad k \in \mathbb{Z}, \tag{4}$$

then the MA(∞) process

$$y_k = e_k + \sum_{r=1}^{\infty}\left[\sum_{j=1}^{\min(r,n)} [G(k, k+j-r)c_j(k+j-r) - G(k, k-r)]\right] e_{k-r} \tag{5}$$

is a second-order, purely nondeterministic, mean zero process which is a solution of (2) [2, Theorem 1 and Corollary; 9, Theorem 2.1].

Moreover, if additionally

$$\sum_{s=-\infty}^{k} |H(k, s)| < \infty, \quad k \in \mathbb{Z}, \tag{6}$$

then the linear spaces generated by $\{e_l, l \leq k\}$ and $\{y_l, l \leq k\}$ coincides for every $k \in \mathbb{Z}$ [2, Theorem 3].

One can see that the above results can be extended to ARMA processes with UBLS noise process $\{e_k, k \in \mathbb{Z}\}$ after using the Niemi's method [7].

In [10] there are given simpler conditions than the above ones under which there exists a purely nondeterministic solution of (2). Namely, it is stated that there exists a purely nondeterministic solution (5) of (2) if $E|e_k|^2 < M < \infty$ and all zeros of the polynomials $\tilde{A}(z^{-1})$ and $\tilde{C}(z^{-1})$ lie in the region $|z| < \lambda < 1$. We note that then the process (5) is UBLS. The above given conditions allowed to solve the prediction problem for the solution (5) of the ARMA equation (2). The least square linear

h-step predictor $\hat{y}_{k+h|k}$ of y_{k+h} based on the process $\{y_l, l \leq k\}$ is as follows:

$$\hat{y}_{k+h|k} = \sum_{r=1}^{\infty} \left[\sum_{j=1}^{\min(r,n)} [G(k+h, k+h+j-r)c_j(k+h+j-r) - G(k+h, k+h-r)] \right] e_{k-r}$$

and can be recursively obtained from

$$\sum_{j=0}^{n} a_j(k+h)\hat{y}_{k+h-j|k} = \sum_{j=h}^{n} c_j(k+h)e_{k+h-j}, \quad a_0(k) = c_0(k) = 1, \qquad k \in \mathbb{Z},$$

where $e_k = y_k - \hat{y}_{k|k-1}$ [10, Theorem 3.1; 9, Theorems 3.1 and 3.2].

Necessary and sufficient conditions for invertibility of ARMA$(0, q)$ model can be found in [3, Theorem 3.1].

We give here simple sufficient conditions for the existence of a purely nondeterministic solution of (2), where we do not assume stationarity or UBLS property of the processes $\{\tilde{A}(q^{-1})y_k, k \in \mathbb{Z}\}$, $\{\tilde{C}(q^{-1})e_k, k \in \mathbb{Z}\}$ and $\{e_k, k \in \mathbb{Z}\}$. Moreover, we have a simple form of that solution and we show that it is unique in the class of purely nondeterministic processes. Now we give sufficient conditions for the equivalence of the linear spaces spanned by the processes $\{e_l, l \leq k\}$ and $\{y_l, l \leq k\}$, $k \in \mathbb{Z}$, and we solve the prediction problem. A simple characterization of the 1-step linear predictor being an ARMA process useful in applications (cf. [5]) is also obtained. Then we get a recursive representation of the h-step predictor. This predictor is also a linear combination (MA(∞)) of the past observations with coefficients easily computable. A comparison of the approaches of [2, 3, 9, 10] and ours permits us to give a property of the Green functions.

Here we treat only ARMA(n, n) models as every ARMA(p, q) model can be rewritten as an ARMA(n, n) one with $n = \max(p, q)$.

2. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ denote a probability space and let $H = L^2(\Omega, \mathcal{F}, P)$ be a Hilbert space of random variables with zero means and finite variances. Let $\{H_k, k \in \mathbb{Z}\}$, $H_k \subseteq H_{k+1} \subseteq H$, $k \in \mathbb{Z}$, denote a wide-sense filtration. L^2-process $\{x_k, k \in \mathbb{Z}\}$ is said to be $\{H_k, k \in \mathbb{Z}\}$ adapted iff $x_k \in H_k$, $k \in \mathbb{Z}$. By $\{H_k^x, k \in \mathbb{Z}\}$ we denote the filtration generated by the process $\{x_k, k \in \mathbb{Z}\}$.

Now let on $(\Omega, \mathcal{F}, P)$ together with the filtration $\{H_k, k \in \mathbb{Z}\}$ be defined adapted process $\{e_k, k \in \mathbb{Z}\}$ such that $e_k \in H_k$, $e_k \perp H_{k-1}$, $E|e_k|^2 = \sigma_k^2$, $k \in \mathbb{Z}$.

Consider the nonstationary ARMA-equation

$$A_k(q^{-1})y_k = C_k(q^{-1})e_k, \quad k \in \mathbb{Z}, \tag{7}$$

where

$$A_k(q^{-1}) = 1 + a_1(k)q^{-1} + \cdots + a_n(k)q^{-n},$$
$$C_k(q^{-1}) = 1 + c_1(k)q^{-1} + \cdots + c_n(k)q^{-n},$$

and $a_j(k)$, $c_j(k)\in\mathbb{C}$, $j=1,\dots,n$, $k\in\mathbb{Z}$, are such that there exist the limits $\lim_{k\to-\infty} a_j(k):=a_j$, $\lim_{k\to-\infty} c_j(k):=c_j$, $j=1,\dots,n$.

Define the following polynomials:

$$A(z^{-1})=1+a_1z^{-1}+\cdots+a_nz^{-n},$$

$$C(z^{-1})=1+c_1z^{-1}+\cdots+c_nz^{-n},$$

and let $r=r(A)+\varepsilon$, $\varepsilon>0$, $r(A)=\max\{|z_j|, A(z_j^{-1})=0, 1\le j\le n\}$, $s=s(C)+\varepsilon'$, $\varepsilon'>0$, $s(C)=\max\{|z_j|, C(z_j^{-1})=0, 1\le j\le n\}$.

The model (7) is said to be AR-regular iff

$$\delta_k^2=\sum_{j=1}^{\infty} r^{2j}\sigma_{k-j}^2<\infty \tag{8}$$

and MA-regular iff

$$\sum_{j=1}^{\infty} s^{2j}\delta_{k-j}^2<\infty \tag{9}$$

for sufficiently small $k\in\mathbb{Z}$. The model (7) which is AR- and MA-regular we call regular.

We see that the above regularity conditions are satisfied under the classical restrictions of [7] for the ARMA-models considered by Niemi [7].

3. A nonstationary ARMA model

The following theorem extends results by Niemi [7, Theorems 2.1 and 2.2] to a larger class of ARMA processes.

Theorem 1. *Suppose that ARMA-model* (7) *is AR-regular. Then there exists the unique $\{H_k, k\in\mathbb{Z}\}$-adapted, purely nondeterministic L^2-solution $\{y_k, k\in\mathbb{Z}\}$ of* (7). *Moreover, if the model* (7) *is regular then $H_k^e=H_k^y$, $k\in\mathbb{Z}$.*

Proof. Following the results of [1, Chapter 9, (9.3.8)] we can write the model (7) in the form

$$y_k=\boldsymbol{B}\boldsymbol{x}_k+e_k,\quad \boldsymbol{x}_{k+1}=\boldsymbol{A}(k)\boldsymbol{x}_k+\boldsymbol{K}(k)e_k,\qquad k\in\mathbb{Z}, \tag{10}$$

where

$$\boldsymbol{x}_k=[x_k^{(1)},\dots,x_k^{(n)}]',$$

$$x_k^{(n-j)}=\sum_{i=j+1}^{n}[c_i(k+j)e_{k-i+j}-a_i(k+j)y_{k-i+j}],\quad j=0,1,\dots,n-1, \tag{11}$$

$$A(k)=\begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n(k+n) \\ 1 & 0 & \cdots & 0 & -a_{n-1}(k+n-1) \\ 0 & 1 & \cdots & 0 & -a_{n-2}(k+n-2) \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & -a_1(k+1) \end{bmatrix},$$

$$K(k)=\begin{bmatrix} c_n(k+n)-a_n(k+n) \\ \vdots \\ c_1(k+1)-a_1(k+1) \end{bmatrix}, \qquad B=[0 \ \cdots \ 0 \ 1].$$

Let

$$y_k^{(N)}=\sum_{j=1}^{N} BA(k,k-j+1)K(k-j)e_{k-j}+e_k, \tag{12}$$

where $A(k+1,l)=A(k)A(k,l)$, $A(l,l)=I$, $l\in\mathbb{Z}$, is the transition matrix of difference equation (10). We see that $\{y_k^{(N)}, N\in\mathbb{N}\}$ L^2-converges as $N\to\infty$ iff the series

$$\sum_{j=1}^{\infty} |BA(k,k-j+1)K(k-j)|^2\sigma_{k-j}^2 \tag{13}$$

converges, and then $y_k^*=(L^2)\lim_{N\to\infty} y_k^{(N)}$, with

$$\|y_k^*\|^2=\sum_{j=1}^{\infty} |BA(k,k-j+1)K(k-j)|^2\sigma_{k-j}^2+\sigma_k^2$$

satisfies the equation (10) and at the same time (7). We note that then the series (12) L^2-converges also for all $l\geq k$, by the ARMA equation (7), or its equivalent form (10). Thus it is suffice to show that (13) holds for sufficiently small $k\in\mathbb{Z}$, i.e. for all $k\leq k_0$ for an arbitrary chosen $k_0\in\mathbb{Z}$.

It can be seen that the convergence of (13) follows from the AR-regularity condition. Indeed, taking into account that $\lim_{k\to-\infty}(a_j(k), c_j(k)):=(a_j, c_j)$, we see that for any given $\varepsilon_1>\varepsilon$, we can choose a matrix norm $\|\ \|$ such that $\|A(k)\|^2\leq \rho(A)+\varepsilon_1$ for every $k\leq k_{\varepsilon_1}$ (cf. [4, p. 15]), where $\rho(A)$ is a spectral radius of the matrix $A:=\lim_{k\to-\infty}A(k)$. It is known that the characteristic polynomial $\det(\lambda I-A(k))=\lambda^n A_k(\lambda^{-1})$ [6, p. 104]. Hence, $\rho(A)=\max\{|\lambda_j|: \det(\lambda_j I-A)=0, 1\leq j\leq n\}=\max\{|z_j|: A(z_j^{-1})=0, 1\leq j\leq n\}=r(A)$. Moreover, the stability assumption implies that for every $\varepsilon_2>0$ there exists k_{ε_2} such that $\|K(k-j)\|\leq K+\varepsilon_2$ for every $k\leq k_{\varepsilon_2}$. Therefore for $k\leq\min\{k_{\varepsilon_1}, k_{\varepsilon_2}\}$, by (8), we have

$$\begin{aligned}&\sum_{j=1}^{\infty} |BA(k,k-j+1)K(k-j)|^2\sigma_{k-j}^2 \\ &\quad\leq \|B\|^2\sum_{j=1}^{\infty}\|A(k-1)\|^2\cdots\|A(k-j)\|^2\|K(k-j)\|^2\sigma_{k-j}^2 \\ &\quad\leq \|B\|^2\sum_{j=1}^{\infty} r^{2j}(K+\varepsilon_2)^2\sigma_{k-j}^2 \\ &\quad\leq (K+\varepsilon_2)^2\|B\|^2\delta_k^2<\infty.\end{aligned}$$

Obviously, $\{y_k^*, k\in\mathbb{Z}\}$ is $\{H_k, k\in\mathbb{Z}\}$-adapted and purely nondeterministic process (i.e. $\bigcap_{k\in\mathbb{Z}} H_k^{y^*}=\{0\}$) as

$$y_k^* \overset{L^2}{=} \sum_{j=1}^{\infty} \boldsymbol{BA}(k, k-j+1)\boldsymbol{K}(k-j)e_{k-j}+e_k \quad \text{(cf. [7])}. \tag{14}$$

We now prove that $\{y_k^*, k\in\mathbb{Z}\}$ is the unique purely nondeterministic, $\{H_k, k\in\mathbb{Z}\}$-adapted solution of (7). By (10) we have the following orthogonal decomposition

$$y_k=\boldsymbol{BA}(k, m)\boldsymbol{x}_m+\sum_{j=1}^{k-m} \boldsymbol{BA}(k, k-j+1)\boldsymbol{K}(k-j)e_{k-j}+e_k$$

for every $m<k$, $m\in\mathbb{Z}$. Letting $m\to-\infty$ we get the Wold decomposition of the solution of (10) with y_k^* as the purely nondeterministic part.

The uniqueness of the Wold decomposition implies that there exists a unique solution of (10) in the class of purely nondeterministic processes.

Assume now that the model (7) is regular. Then by (14) we have $H_k^y\subseteq H_k^e$, $k\in\mathbb{Z}$. Using (10) we get

$$e_k=-\boldsymbol{Bx}_k+y_k,\quad \boldsymbol{x}_{k+1}=\bar{\boldsymbol{A}}(k)\boldsymbol{x}_k+\boldsymbol{K}(k)y_k,\quad \bar{\boldsymbol{A}}(k)=\boldsymbol{A}(k)-\boldsymbol{K}(k)\boldsymbol{B}. \tag{15}$$

Moreover, we see that equality

$$e_k=-\boldsymbol{B}\bar{\boldsymbol{A}}(k, m)\boldsymbol{x}_m-\sum_{j=1}^{k-m} \boldsymbol{B}\bar{\boldsymbol{A}}(k, k-j+1)\boldsymbol{K}(k-j)y_{k-j}+y_k,\quad m<k,\ k\in\mathbb{Z},$$

implies that $H_k^e\subseteq H_m^x\cup H_k^y\subseteq H_m^e\cup H_k^y$ for every $m<k$.

To prove that $H_k^e\subseteq H_k^y$ suppose that the model (7) is MA-regular, i.e. (9) holds true. Since the characteristic polynomial of the matrix $\bar{\boldsymbol{A}}(k)$ is equal to $\lambda^n C_k(\lambda^{-1})$ (cf. [5]) then by the similar considerations concerning the process $\{y_k, k\in\mathbb{Z}\}$ it can be shown that the series

$$\sum_{j=0}^{\infty} \boldsymbol{B}\bar{\boldsymbol{A}}(k, k-j+1)\boldsymbol{K}(k-j)y_{k-j}$$

L^2-converges if (9) holds true. Letting now $m\to-\infty$ we obtain $H_k^e\subseteq H_k^y$, since $\{e_k, k\in\mathbb{Z}\}$ is purely nondeterministic process as a white noise. This completes the proof of the theorem. □

Remark. *Let $G(k, l)$, $l\leqslant k$, k, $l\in\mathbb{Z}$, denote the Green functions associated with the homogeneous difference equation $\tilde{A}_k(q^{-1})y_k=0$ satisfying the AR-regularity condition. Then for every $c_k(j)\in\mathbb{C}$, $k\in\mathbb{Z}$, $j=1, 2, \ldots, n$, and $r\in\mathbb{N}$,*

$$\sum_{j=1}^{\min(r,n)} [G(k, k+j-r)c_j(k+j-r)-G(k, k-r)]=\boldsymbol{BA}(k, k-r+1)\boldsymbol{K}(k-r).$$

Proof. Consider the ARMA(n, n) equation (2) satisfying the AR-regularity conditions with independent values stationary process $\{e_k, k \in \mathbb{Z}\}$. Then, by Theorem 1 $\{y_k^*, k \in \mathbb{Z}\}$ given by (14) is the unique solution of (7) in the class of purely nondeterministic processes while $\{y_k^{\cdot\cdot}, k \in \mathbb{Z}\}$ given by (4) is other purely nondeterministic solution of (7). Therefore the processes $\{y_k^*, k \in \mathbb{Z}\}$ and $\{y_k^{\cdot\cdot}, k \in \mathbb{Z}\}$ coincide and by the Projection Lemma [1],

$$\sum_{j=1}^{\min(r,n)} [G(k, k+j-r)c_j(k+j-r) - G(k, k-r)]$$

$$= E[y_k^{\cdot\cdot}\bar{e}_{k-r}](E|e_k|^2)^{-1}$$

$$= (E[y_k^*\bar{e}_{k-r}](E|e_k|^2)^{-1}$$

$$= \boldsymbol{B}\boldsymbol{A}(k, k-r+1)\boldsymbol{K}(k-r)$$

as $\{y_k^*, k \in \mathbb{Z}\}$ and $\{y_k^{\cdot\cdot}, k \in \mathbb{Z}\}$ are Gaussian processes such that $y_k^* \in H_k^e$, and $y_k^{\cdot\cdot} \in H_k^e$, $k \in \mathbb{Z}$. □

4. Linear prediction

Let the ARMA-model (7) be regular. We give a simple characterization of the h-step optimal (in L^2-sense) linear predictor $\hat{y}_{k+h|k}$ for the state y_{k+h} based on the process $\{y_l, l \leq k\}$. It is well known that $\hat{y}_{k+h|k} = \mathbb{P}_k^y y_{k+h}$, where $\mathbb{P}_k^y$ denotes the orthoprojector on the space H_k^y.

Theorem 2. *Suppose that the model* (7) *is regular. Then the optimal linear* 1*-step predictor* $\hat{y}_{k|k-1}$ *is given by the following ARMA-equation*

$$C_k(q^{-1})\hat{y}_{k|k-1} = G_k(q^{-1})y_k, \quad k \in \mathbb{Z}, \tag{16}$$

where $G_k(q^{-1}) = \alpha_1(k)q^{-1} + \cdots + \alpha_n(k)q^{-n}$, $\alpha_i(k) = c_i(k) - a_i(k)$, $1 \leq i \leq n$.

Proof. Since the vector $x_k \in H_{k-1}^y \cup H_{k-1}^e = H_{k-1}^y = H_{k-1}^e$ then by (5) we have

$$\hat{y}_{k|k-1} = \boldsymbol{B}\boldsymbol{x}_k = x_k^{(n)} = \sum_{i=1}^{n} [c_i(k)e_{k-i} - a_i(k)y_{k-i}]. \tag{17}$$

But from (10) we have $e_k = y_k - \hat{y}_{k|k-1}$ which with (17) proves (16) and ends the proof of the theorem. □

The MA-representation (14) of the process $\{y_k, k \in \mathbb{Z}\}$ and Theorem 2 lead to the following characterizations of predictor for nonstationary regular ARMA-process (cf. [7, 10]).

Theorem 3. *Assume that the ARMA-model* [7] *is regular. Then the h-step predictor* $\hat{y}_{k+h|k}$ *is given by the following equation*:

$$\sum_{j=0}^{n} a_j(k+h)\hat{y}_{k+h-j|k} = \sum_{j=h}^{n} c_j(k+h)e_{k+h-j}, \quad a_0(k)=c_0(k)=1, \qquad k\in\mathbb{Z},$$

where $e_k = y_k - \hat{y}_{k|k-1}$ *and* 1*-step predictor* $\hat{y}_{k|k-1}$ *satisfies* (16). *The predictor* $\hat{y}_{k+h|k}$ *can also be represented as follows*:

$$\hat{y}_{k+h|k} = \sum_{j=0}^{\infty} \boldsymbol{BA}(k+h, k+1)\bar{\boldsymbol{A}}(k+1, k-j+1)\boldsymbol{K}(k-j)y_{k-j}. \tag{18}$$

The variance of the prediction error

$$y_{k+h} - \hat{y}_{k+h|k} = e_{k+h} + \sum_{j=1}^{h-1} \boldsymbol{BA}(k+h, k+h-j+1)\boldsymbol{K}(k+h-j)e_{k+h-j}$$

equals to

$$E|y_k - \hat{y}_{k+h|k}|^2 = \sum_{j=0}^{h-1} |\boldsymbol{BA}(k+h, k+h-j+1)\boldsymbol{K}(k+h-j)|^2\sigma^2_{k+h-j} + \sigma^2_{k+h}.$$

Proof. The first statement of Theorem 3 follows from the ARMA equation (7).

To prove a second part it is enough to see that (10) implies that

$$y_{k+h} = \boldsymbol{BA}(k+h, k+1)\boldsymbol{x}_{k+1} + \sum_{j=1}^{h-1} \boldsymbol{BA}(k+h, k+h-j+1)\boldsymbol{K}(k+h-j)e_{k+h-j} + e_{k+h}.$$

Taking into account that $\boldsymbol{x}_{k+1} \in H^y_k$ we have

$$\hat{y}_{k+h|k} = \boldsymbol{BA}(k+h, k+1)\boldsymbol{x}_{k+1}.$$

Now by (15) and the regularity conditions (8) and (9) we get (18).

The variance of the predictor error can be easily calculated. □

5. The weak law of large numbers for nonstationary ARMA-processes.

We investigate now some asymptotic properties of nonstationary ARMA-processes.

Theorem 4. *Let* $\{y_k, k\in\mathbb{Z}\}$ *be a nonstationary process. Suppose that* $\max\{E|y_k|^2, k\in\mathbb{Z}\} \le M$, $M>0$, *and* $(q^*)^k z \nrightarrow 0$ (*in* L^2*-sense*) *as* $k\to\infty$ *for any* $z\in H^y$, *where* q^* *is the adjoint shift operator.*

Then there exists a self-adjoint bounded linear operator $B: H^y \to H^y$ *with the bounded inverse such that*

$$N^{-1}\sum_{k=0}^{N-1} y_k \xrightarrow{L^2} (1 - I_{[y_k \to^{L^2} 0]})B\mathbb{P}(B^{-1}y_0 \,|\, \mathbb{I}_{B^{-1}qB}), \tag{19}$$

where I is the indicator function and $\mathbb{P}(\cdot\,|\mathbb{I}_{B^{-1}qB})$ denotes the orthoprojector on the invariant subspace of the operator $B^{-1}qB$.

Proof. If $y_k \to^{L^2} 0$, $k\to\infty$, then obviously $N^{-1}\sum_{k=0}^{N-1} y_k \to^{L^2} 0$, $N\to\infty$, and (19) holds.

Assume now that $y_k \not\to^{L^2} 0$, $k\to\infty$. Since the group of shift operators $\{q^k, k\in\mathbb{Z}\}$ is uniformly bounded then, by Theorem 5.4, Chapter II of [11], we see that there exists a self-adjoint bounded operator $B: H^y \to H^y$ with bounded inverse B^{-1} such that $U := B^{-1}qB$ is the unitary operator. Therefore, by Von Neumann's 'mean' ergodic theorem [8, p. 21], we have

$$N^{-1}\sum_{k=0}^{N-1} y_k = N^{-1}\sum_{k=0}^{N-1} q^k y_0 = BN^{-1}\sum_{k=0}^{N-1} U^k B^{-1} y_0 \xrightarrow{L^2} B\mathbb{P}(B^{-1}y_0|\mathbb{I}_U), \quad N\to\infty.$$

which ends the proof of (19). □

For the model (7) which is AR-regular we have the following theorem.

Theorem 5. *Suppose that the process $\{y_k, k\in\mathbb{Z}\}$ is AR-regular. Then*

$$N^{-1}\sum_{k=0}^{N-1} y_k \xrightarrow{L^2} 0, \quad N\to\infty,$$

iff

$$N^{-1}\sum_{l=-N+1}^{\infty} \sigma_{-l}^2 \left| \boldsymbol{B}\sum_{k=0}^{N-1} \boldsymbol{A}(k, -l+1)\boldsymbol{K}(-l)\right|^2 \to 0, \quad N\to\infty. \qquad \square$$

Proof. By (10) we have

$$y_k = \sum_{j=0}^{\infty} b_k(j) e_{k-j},$$

where $b_k(0) = 1$ and $b_k(j) = \boldsymbol{BA}(k, k-j+1)\boldsymbol{K}(k-j)$, $j > 1$. Hence

$$\sum_{k=0}^{N-1} y_k = \sum_{l=-N+1}^{\infty} e_{-l}\left(\sum_{k=0}^{N-1} b_k(k+l)\right)$$

which completes the proof as $\{e_k, k\in\mathbb{Z}\}$ is an orthogonal process. □

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