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Divergent sums over excursions

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Abstract

Criteria for the almost sure divergence or convergence of sums of functions of excursions away from a recurrent point in the state space of a Markov process are proved. These are applied to the excursions from 0 of reflecting diffusions; in particular, reflecting Brownian motion, to derive some interesting sample path properties for these processes.

Keywords: Excursion formula; Local time; Diffusion; Sample path properties

1. Statement of main results

Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ be a canonical standard Markov process and 0 a recurrent point of the state space which is regular for itself, but is neither a trap nor a holding point. Introduce the following additional objects:

- σ the hitting time of X at 0: $\inf\{t: t > 0, X(t) = 0\}$
- L the local time for X at 0 normalized so that $Ee^{-\sigma} = E\int_0^\infty e^{-S}\,dL_S \quad (E \equiv E^0)$
- β the right continuous inverse of L: $\beta_t = \inf\{s: L_S > t\}$
- β^- Left continuous inverse of L: $\beta_t^- = \lim_{r \uparrow t} \beta_r$, $t > 0$, $\beta_0^- = 0$
- G the random set of strictly positive s at which $s = \beta_t^-$ for some t with $\beta_t^- < \beta_t$. Or, more simply, the left-hand end points of the excursion intervals of X away from 0.
- τ a truncation map taking Ω to itself: $\tau X = X \circ \tau = X'$ where $X'(\omega)(t) = X_t(\omega)$ for $t < \sigma(\omega), X'(\omega)(t) = 0$ for $t \geq \sigma(\omega)$. Note that $\sigma \circ \tau = \sigma$.
- $\hat{p}$ the excursion measure: $\hat{p}(A)$ is the expected number of excursions of X which have the property A during any interval of time that the local time increases by a unit amount. The reader is referred to Blumenthal's (1992) book, in particular Chapter III, Section 3, for a detailed study of the excursions of a Markov process.

Let f be a bounded nonnegative function on $[0, \infty) \times \Omega$ which has the property that $f(\cdot, \tau(\cdot))$ is $\mathscr{B}([0, \infty)) \times \mathscr{F}$ measurable where $\mathscr{F}$ is the $\hat{p}$-completion of the σ-algebra

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generated by the coordinate functions. Put[1]

$$\Sigma_1(f;a,b) = \sum_{s\in G,\,\beta(a)<s\le\beta(b)} f(L_s, \tau\circ\theta_s),^1$$

$$\Sigma_2(f;a,b) = \sum_{s\in G,\,a<s\le b} f(s, \tau\circ\theta_s).$$

Theorem 1. (i) $\Sigma_1(f;0,a) = \infty$ *a.s. if and only if* $E\{\Sigma_1(f;0,a)\} = \infty$. (ii) *Assume that* $s \to f(s,\omega)$ *is decreasing for every* ω *and that*

$$\Sigma_2(f;\gamma,a) \text{ is finite a.s. for every } \gamma, \quad 0<\gamma\le a. \tag{1.1}$$

Then $\Sigma_2(f;0,a) = \infty$ *a.s. if and only if* $E\{\Sigma_2(f;0,a)\} = \infty$.

Excursion formulas relate the expectations of sums such as those which occur in Theorem 1 to integrals involving $\hat{p}$. In particular, for $0 \le a < b < \infty$,

$$E\{\Sigma_1(f;a,b)\} = \int_a^b \hat{p}(f_t)\,\mathrm{d}t, \tag{1.2}$$

$$E\{\Sigma_2(f;a,b)\} = E\left\{\int_a^b \hat{p}(f_s)\,\mathrm{d}L_s\right\}. \tag{1.3}$$

For a proof of (1.3) in the case that f does not depend on s, see Blumenthal (1992, III.3); for the general case, see Maisonneuve (1975); and for (1.2), see Section 2.

For some diffusions on $[0,\infty)$, in particular reflecting Brownian motion, and excursions away from the origin, it is possible to calculate the excursion measure of some events quite explicitly. In conjunction with Theorem 1 and the excursion formulas, these calculations lead to some interesting zero–one type laws for the small time behavior of the sample paths. Below are several such results including the two discovered by Knight using different methods. No doubt the reader can think of other possibilities.

In the first two applications X is a standard reflecting Brownian motion on $[0,\infty)$. In Theorem 2 we get a 0–1 law for the maximums over excursion intervals compared with their lengths. In Theorem 3 we get a lower function 0–1 law for the "areas" under excursions. Throughout we suppose that h is a continuous, positive, increasing function on $[0,\infty)$, $h(0+)=0$. In Theorem 2 we also suppose that $r \to h(r)/\sqrt{r}$ is decreasing and tends to ∞ as r tends to 0. As usual i.o. is the abbreviation for "infinitely often."

Theorem 2. *Let* $M_s = \max\{X_r\colon s\le r\le s+\sigma\circ\theta_s\}$. *Then* $P\{M_s \ge h(\sigma_s)$ *i.o. as* $s\downarrow 0$ *in* $G\} = 0$ *or* 1 *according as* $\int_{0+} t^{-5/2}h(t)^2 \mathrm{e}^{-2h(t)^2/t}\,\mathrm{d}t$ *is finite or infinite.*

Example. $M_s \ge \sqrt{\frac{1}{4}\sigma_s \log 1/\sigma_s + \kappa\sigma_s \log\log 1/\sigma_s}$ i.o. $s\downarrow 0$ in G, a.s., if and only if $\kappa \le 1$.

[1] $f(s,\tau\circ\theta_S)(\omega) = f(s,\tau(\theta_S(\omega)))$ and $f(L_S,\tau\circ\theta_S)(\omega) = f(L_S(\omega),\tau(\theta_S(\omega)))$. Also $f_t(\cdot) = f(t,\cdot)$, $X_t(\cdot) = X(t,\cdot)$, etc.

Theorem 3. *Let* $A_s = \int_s^{s+\sigma\circ\theta_s} X_r \,\mathrm{d}r$. *Then* $P\{A_s \geq h(s)$ *i.o. as* $s \downarrow 0$ *in* $G\} = 0$ *or* 1 *according as* $\int_{0+} s^{-1/2} h(s)^{-1/3}\,\mathrm{d}s$ *is finite or infinite.*

Example. $A_s \geq s^{3/2}|\log s|^3[\log|\log s]^{\lambda}$ i.o. $s\downarrow 0$ in G, a.s., if and only if $\lambda \leq 3$.

For the next results, let X be a persistent nonsingular diffusion on $[0, \infty)$ on natural scale. We will suppose that 0 is both an exit and an entrance boundary point. Then 0 is regular for itself and is neither a trap nor a holding point; Itô and Mckean (1974; Chapter 3). We denote by $q(t, x, y)$ the transition density of X with respect to the speed measure. Theorems 4 and 5 are due to Knight: Theorems 0 and 2 in Knight (1973). Given the excursion theory our proofs seem to be a little shorter than his.

Theorem 4. $P\{X(t) > h(\beta^-(L_t))$ *i.o. as* $t\downarrow 0\} = 0$ *or* 1 *according as* $\int_{0+} h(t)^{-1} q(t,0,0)\,\mathrm{d}t$ *is finite or not.*

Theorem 5. *Let* $X^*(t) = \max\{X(r)\colon 0 \leq r \leq t\}$. *Then* $P\{X^*(\beta_t^-) > h(t)$ *i.o. as* $t\downarrow 0\} = 0$ *or* 1 *according as* $\int_{0+} h(t)^{-1}\,\mathrm{d}t$ *is finite or not.*[2]

2. Proof of Theorem 1

It is easy to see that type 1 sums are integrals of a convenient sort:

$$\Sigma_1(f;a,b)(\omega) = \int_{(a,b]\times U} f(t,u)\, Y_\omega(\mathrm{d}t, \mathrm{d}u),$$

where Y is the Poisson point process of excursions regarded as a random measure on $[0, \infty) \times U$, where U is the excursion pathspace; see Bluementhal (1992, Section III.2–3). The intensity measure of Y is the product of Lebesgue measure and $\hat{p}$. The conclusion of part (i) and formula (1.2) may thus be seen as a consequence of Campbell's theorem; see Kingman, (1993, p. 28).

Part (ii) of Theorem 1 does not have such a simple proof. Fix $\varepsilon > 0$ and define two increasing sequences of positive random variables as follows: $T_0 = 0$ and for $j \geq 1$,

$$S_j = \inf\{s\colon s \geq T_{j-1}, L(s+\varepsilon) = L(s)\}(= T_{j-1} + S_1 \circ \theta_{T(j-1)} \text{ for } j \geq 2),$$

$$T_j = \inf\{s\colon s \geq S_j, X(s) = 0\} = S_j + \sigma \circ \theta_{S(j)}.$$

In words the S_j are the successive times that X begins an excursion of length at least ε and the T_j are the successive (stopping) times that X is at 0 at the ends of these excursions. It is easily seen that for integers j and k with $j > k \geq 1$ we have

$$S_j = T_k + S_{j-k} \circ \theta_{T(k)}. \tag{2.1}$$

[2] Knight calls a function h *lower in local time* when this probability is 1.

Let us write $G(\varepsilon) = \{S_1, S_2, S_3, \ldots\}$, $V_k = I(S_k \le a) f(S_k, \tau \circ \theta_{S(k)})$, and

$$\Sigma_{2,\varepsilon} = \sum_{s \in G(\varepsilon), s \le a} f(s, \tau \circ \theta_s) = \sum_{k=1}^{\infty} V_k .$$

The sets $G(\varepsilon)$ increase to G as ε decreases to 0 so the random variables $\Sigma_{2,\varepsilon}$ increase to $\Sigma_2 = \Sigma_2(f;0,a)$ as ε decreases to 0. Furthermore, since $S_{k+1} - S_k \ge \varepsilon$, $E\{\Sigma_{2,\varepsilon}\} \le \|f\| a/\varepsilon$ is finite.

We will now show that

$$E\{(\Sigma_{2,\varepsilon})^2\} \le C \max[1, (E\{\Sigma_{2,\varepsilon}\})^2], \tag{2.2}$$

where C is a constant independent of ε. Let j and k be integers with $j > k \ge 1$. It is clear from (2.1) that the event $\{S_j \le a\}$ implies the event $\{S_{j-k} \circ \theta_{T(k)} \le a\}$. Also, since the function f is decreasing in its first variable, we have

$$\begin{aligned} f(S_j, \tau \circ \theta_{S(j)}) &= f(T_k + S_{j-k} \circ \theta_{T(k)}, \tau \circ \theta_{S(j-k)} \circ \theta_{T(k)}) \\ &\le f(S_{j-k} \circ \theta_{T(k)}, \tau \circ \theta_{S(j-k)} \circ \theta_{T(k)}), \end{aligned}$$

and consequently $V_j \le V_{j-k} \circ \theta_{T(k)}$. Applying this inequality and the strong Markov property we get

$$\begin{aligned} E(V_k V_j) \le E[V_k V_{j-k} \circ \theta_{T(k)}] &= E[V_k E\{V_{j-k} \circ \theta_{T(k)} | \mathscr{F}_{T(k)}\}] \\ &= E[V_k E^{X(T_k)}\{V_{j-k}\}] = (EV_k)(EV_{j-k}). \end{aligned}$$

(The necessary $\mathscr{F}_{T(k)}$-measurability of V_k is easily established and its proof is safely omitted.) Hence,

$$\begin{aligned} E(\Sigma_{2,\varepsilon})^2 &= E\sum_{k=1}^{\infty} (V_k)^2 + 2\sum_{k=1}^{\infty}\sum_{j=k+1}^{\infty} E(V_k V_j) \\ &\le \|f\| E(\Sigma_{2,\varepsilon}) + 2\sum_{k=1}^{\infty}\sum_{j=k+1}^{\infty} [EV_k][EV_{j-k}] \\ &= \|f\| E(\Sigma_{2,\varepsilon}) + 2[E(\Sigma_{2,\varepsilon})]^2, \end{aligned}$$

and (2.2) follows with $C = 2 + \|f\|$.

Obviously if $\Sigma_2 = \infty$ a.s. then $E(\Sigma_2) = \infty$, so let us suppose that $E(\Sigma_2) = \infty$. Then $\lim_{\varepsilon \to 0} E(\Sigma_{2,\varepsilon}) = \infty$ by monotone convergence. From this and (2.2) it is straightforward to show (see Kochen and Stone (1964) for a typical argument) that the event

$$\left\{ \limsup_{\varepsilon \to 0} \Sigma_{2,\varepsilon}/E(\Sigma_{2,\varepsilon}) > r \right\}$$

has nonzero probability for some strictly positive number r from which fact it follows that the event $\{\Sigma_2 = \infty\}$, which contains that event, also has positive probability. But clearly, condition (1.1) of Theorem 1 demands that $\{\Sigma_2 = \infty\}$ is in $\bigcap_{\gamma > 0} \mathscr{F}_\gamma = \mathscr{F}_0$ and therefore this event must have probability 1 since $\mathscr{F}_0$ is trivial. This completes the proof.

3. Proof of Theorem 2

We are going to apply part (i) of Theorem 1. Let $f(s,\omega) = f(\omega) = I\{M \geq h(\sigma)\}(\omega)$ where $M = \max\{X_s\colon 0 < s \leq \sigma\}$. Then, since $\beta_a \downarrow 0$, as $a \downarrow 0$, a.s., we have

$$\{M_s \geq h(\sigma_s) \text{ i.o. as } s \downarrow 0 \text{ in } G\} = \{\Sigma_1(f;0,a) = \infty \text{ for some } a > 0\}.$$

From (1.2) and the fact that f does not depend on s and $\Sigma_1(f;0,a)$ increases with a, we have that $P\{\Sigma_1(f;0,a) = \infty \text{ for some } a > 0\} = 1$ if and only if $\hat{p}\{M > h(\sigma)\} = \infty$. From Williams (1979, p. 99), exercise at the bottom of the page, we find the formula (in slightly different notation):

$$\hat{p}\{M > x \mid \sigma = t\} = 2\sum_{k=1}^{\infty}[t^{-1}(2kx)^2 - 1]\mathrm{e}^{-2k^2x^2/t}.$$

Also[3] see Blumenthal (1992, p. 112),

$$\hat{p}\{\sigma > t\} = 1/\sqrt{\pi t}. \tag{3.1}$$

Whatever be the specific form of these quantities, we have for $\varepsilon > 0$,

$$\hat{p}[M > h(\sigma)] = \left(\int_0^{\varepsilon} + \int_{\varepsilon}^{\infty}\right)\hat{p}\{M > h(t) | \sigma = t\}\hat{p}\{\sigma \in \mathrm{d}t\} \equiv J_1 + J_2.$$

Since $x \to \hat{p}\{M > x \mid \sigma = t\}$ is decreasing and bounded by 1 and h is increasing and strictly positive, it follows that $J_2 \leq \hat{p}\{\sigma > \varepsilon\} = 1/\sqrt{\pi\varepsilon} < \infty$, and $\hat{p}[M > h(\sigma)]$ is finite or not according as J_1 is finite or not. From the above formulas we have

$$J_1 = \int_0^{\varepsilon}\sum_{k=1}^{\infty} v_k(t)\,\mathrm{d}t/2\sqrt{\pi},$$

where $v_k(t) = t^{-3/2}[4k^2h(t)^2/t - 1]\mathrm{e}^{-2k^2h(t)^2/t}$. The function $h(t)^2/t \uparrow \infty$ as $t \downarrow 0$, so we can find constants C_1 and ε such that for all t with $0 < t \leq \varepsilon$ and every $k \geq 2$,

$$0 < [4k^2h(t)^2/t - 1]\mathrm{e}^{-2k^2h(t)^2/t} \leq [C_1/k^2][h(t)^2/t]\mathrm{e}^{-2h(t)^2/t}.$$

Consequently, for another constant C_2, $\sum_{k=2}^{\infty} v_k(t) \leq C_2 t^{-5/2}h(t)^2\mathrm{e}^{-2h(t)^2/t} \equiv C_2 u(t)$ for $0 < t \leq \varepsilon$. Also $v_1(t) \sim 4u(t)$ as $t \downarrow 0$ as is easily checked. From these calculations it follows that for $\varepsilon > 0$ sufficiently small J_1 is equivalent to $\int_0^{\varepsilon} u(t)\,\mathrm{d}t$. This completes the proof of Theorem 2.

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[3] Multiplying by $\sqrt{2}$ gives Williams' formula for $\hat{p}[\sigma > t]$. The reason for the discrepancy is that the normalization of local time chosen here entails that L is $\sqrt{2}$ times the Itô–McKean–Williams Brownian local time. From this it follows that the constant C_0 which appears in Section 4 has the value $\sqrt{2}$. This is also the value of C_0 in Section 5 as can be seen from the formula for ℓ in terms of a time change of Brownian local time in Itô and McKean (1992, Section 5.4)

4. Proof of Theorem 3

We apply Theorem 1(ii) and formula (1.3). Put $f(s,\omega) = I\{\int_0^\sigma X_r\,\mathrm{d}r > h(s)\}(\omega)$. Then, provided (1.1) holds,

$$\{A_s > h(s) \text{ i.o. as } s\downarrow 0,\ s \in G\} = \{\Sigma_2(f;0,a) = \infty \text{ for all } a > 0 \text{ sufficiently small}\}.$$

Let W denote the BES(3) process, let Z be the process $t \to Z_t = (1-t)W(t/(1-t))$, $0 \le t < 1$, $Z_1 = 0$, and let μ_b be the law of the scaled process $r \to \sqrt{b}Z_{r/b}$, $0 \le r \le b$. Then $\hat{p} = \int_0^\infty \mu_b \hat{p}\{\sigma \in \mathrm{d}b\} = \int_0^\infty \mu_b b^{-3/2}\,\mathrm{d}b/2\pi^{1/2}$ (i.e., $\mu_b(\cdot) = \hat{p}(\cdot \mid \sigma = b)$ in the notation of the previous section); see Blumenthal (1992, p. 42 and p. 112). Now if $q(t,x,y)$ denotes the transition density of X with respect to Lebesgue measure, then there exists a constant C_0 such that $E\{\mathrm{d}L(s)\} = C_0 q(s,0,0)\,\mathrm{d}s$ (see the next section). In our case $q(t,0,0) = (2/\pi t)^{1/2}$. Put $\xi = \int_0^1 Z_t\,\mathrm{d}t$, $\mu =$ law of Z, and $C_1 = (2/\pi)^{1/2}(1/2\pi^{1/2})C_0 = C_0/\pi\sqrt{2}$. Then

$$\begin{aligned}
E\{\Sigma_2(f;\gamma,a)\} &= E\left\{\int_\gamma^a \hat{p}\left[\int_0^\sigma X_r\,\mathrm{d}r > h(s)\right]\mathrm{d}L(s)\right\} \\
&= C_1\int_\gamma^a\int_0^\infty \mu\left\{\int_0^b \sqrt{b}Z_{r/b}\,\mathrm{d}r > h(s)\right\}b^{-3/2}\,\mathrm{d}b\,s^{-1/2}\,\mathrm{d}s \\
&= C_1\int_\gamma^a\int_0^\infty \mu\left\{\xi > b^{-3/2}h(s)\right\}b^{-3/2}\,\mathrm{d}b\,s^{-1/2}\,\mathrm{d}s \\
&= \frac{2}{3}C_1\int_\gamma^a\int_0^\infty \mu\left\{\xi > z\right\}z^{-2/3}\,\mathrm{d}z\,h(s)^{-1/3}s^{-1/2}\,\mathrm{d}s \\
&= C_2\int_\gamma^a h(s)^{-1/3}s^{-1/2}\,\mathrm{d}s,
\end{aligned}$$

where $C_2 = 2C_1E_\mu(\xi^{1/3}) < \infty$. The last displayed integral is finite since the integrand is bounded on $[\gamma,a]$ for $0 < \gamma \le a < \infty$; thus, (1.1) is satisfied since the exhibited sum there has finite expectation. Clearly, $s \to f(s,\omega)$ is nonincreasing for each ω since h is nondecreasing. Therefore, Theorem 1(ii) applies. Taking $\gamma = 0$ in the preceding computation, gives $E\{\Sigma_2(f;0,a)\} = C_2\int_0^a h(s)^{-1/3}s^{-1/2}\,\mathrm{d}s$ and the conclusion of Theorem 3 is apparent.

5. Proof of Theorems 4 and 5

It is well known that for a one-dimensional diffusion such as X there exists a jointly continuous local time functional $\ell(t,x) = \ell(t,x,\omega)$ which satisfies

$$E^x[\ell(t_1,y) - \ell(t_2,y)] = \int_{t_1}^{t_2} q(s,x,y)\,\mathrm{d}s$$

for $0 \le t_1 \le t_2$ and x,y in the state interval; see Ito and Mckean (1974, Section 5.4). From this and the well-known fact that local time at a point is unique up to a constant multiple, if follows that there exists a constant $C_0 > 0$ such that for any nonnegative Borel function g on $[0, \infty)$,

$$E\left\{\int_0^t g(s)\,\mathrm{d}L(s)\right\} = C_0 \int_0^t g(s)q(s,0,0)\,\mathrm{d}s.^3 \tag{5.1}$$

We need one additional formula: for $x > 0$,

$$\hat{p}\{M > x\}(\equiv \hat{p}\{\sigma_x < \sigma\}) = 1/x\sqrt{2}, \tag{5.2}$$

where σ_x is the hitting time of the point x. The proof of this formula follows verbatim the proof of the same formula in the case that X is Brownian motion; see Blumenthal (1992, pp. 110–111). Only the fact that Brownian motion is a natural scale diffusion is used in that proof.

As to the proof of Theorem 4, note first that for each $t > 0$ $\beta^-(L_t)$ is the largest s in G which does not exceed t. Hence, $h(\beta^-(L_t))$ is constant for t in $[s, s + \sigma \circ \theta_s)$. Let $f(s,\omega) = I\{M > h(s)\}(\omega)$, then

$$\begin{aligned}\{X_t > h(\beta^-(L_t)) \text{ i.o. as } t\downarrow 0\} &= \{M_s > h(s) \text{ i.o. as } s\downarrow 0,\ s \in G\}\\ &= \{\Sigma_2(f;0,a) = \infty\\ &\qquad \text{for all } a > 0 \text{ sufficiently small}\},\end{aligned}$$

provided (1.1) holds. Applying (1.3), (5.1), and (5.2) we get

$$E\{\Sigma_2(f;\gamma,a)\} = E\left\{\int_\gamma^a \hat{p}(f_s)\,\mathrm{d}L(s)\right\} = C_0\int_\gamma^a h(s)^{-1}q(s,0,0)\,\mathrm{d}s.$$

One now proceeds as in the last part of the proof of Theorem 3.

To prove Theorem 5 observe that, a.s.,

$$\begin{aligned}\{X^*(\beta_t^-) > h(t) \text{ i.o. as } t\downarrow 0\} &= \{X^*(s) > h(L_s) \text{ i.o. as } s\downarrow 0,\ s\in G\}\\ &= \{M_s > h(L_s) \text{ i.o. as } s\downarrow 0,\ s \in G\}\\ &= \{\Sigma_1(f;0,a) = \infty \text{ for all small } a\},\end{aligned}$$

where f is the same function used above. An application of (5.2), (1.2), and Theorem 1 finishes the proof.

Added in Proof. From Blumenthal (1992, p. 42) and L. Takács, 'Random walk processes and their applications in order statistics', Annals of App. Prob. 2 (1992), one can show that $\mu\{\xi \le x\} \sim \sqrt{6\gamma}x^{-2}\exp(-\gamma/x^2)$, as $x\downarrow 0$, where μ and ξ are as in §4, $\gamma = 2a_1^3/27$, and $-a_1$ (≈ -2.338) is the largest zero of the standard Airy function. An appropriate application of Theorem 1(i) then yields a companion to Theorem 3: If $h \in \uparrow$, and $b^3/h(b)^2 \to \infty$, $b \to 0$, then $P\{A_s \le h(\sigma_s) \text{ i.o. as } s \in G\downarrow 0\} = 0$ or 1 according as $\int_{0+} b^{3/2}h(b)^{-2}\exp\{-\gamma b^3/h(b)^2\}\,\mathrm{d}b$ is finite or not.

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