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Almost sure estimates for the concentration neighborhood of Sinai's walk

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Abstract

We consider Sinai's random walk in random environment. We prove that infinitely often (i.o.) the size of the concentration neighborhood of this random walk is bounded almost surely. We also get that i.o. the maximal distance between two favorite sites is bounded almost surely.

MSC: 60G50; 60J55

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1. Introduction and results

In this paper we are interested in Sinai's walk, i.e. a one-dimensional random walk in a random environment with three conditions on the random environment: two necessary hypotheses for getting a recurrent process (see [1]) which is not a simple random walk and a hypothesis of regularity which allows us to have a good control on the fluctuations of the random environment. The asymptotic behavior of such a walk was discovered by Sinai [2]: this walk is sub-diffusive and at an instant n it is localized in the neighborhood of a well defined point of the lattice. The almost sure behavior of this walk, originally studied by Deheuvels and Révész [3], has been investigated by Hu and Shi [4]. We denote Sinai's walk as $(X_n, n \in \mathbb{N})$; let us define the local

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time $\mathcal{L}$ at k ($k \in \mathbb{Z}$) within the interval of time $[1, T]$ ($T \in \mathbb{N}^*$) of $(X_n, n \in \mathbb{N})$:

$$\mathcal{L}(k, T) \equiv \sum_{i=1}^{T} \mathbb{I}_{\{X_i = k\}}. \tag{1.1}$$

$\mathbb{I}$ is the indicator function; notice that k and T can be deterministic or random variables. Let $V \subset \mathbb{Z}$; we define

$$\mathcal{L}(V, T) \equiv \sum_{j \in V} \mathcal{L}(j, T) = \sum_{i=1}^{T} \sum_{j \in V} \mathbb{I}_{\{X_i = j\}}. \tag{1.2}$$

Now, let us introduce the following random variables:

$$\mathcal{L}^*(n) = \max_{k \in \mathbb{Z}} (\mathcal{L}(k, n)), \qquad \mathbb{F}_n = \left\{k \in \mathbb{Z}, \mathcal{L}(k, n) = \mathcal{L}^*(n)\right\}, \tag{1.3}$$

and for all $0 \leq \beta < 1$ define

$$Y_{n,\beta} = \inf_{x \in \mathbb{Z}} \min \left\{ k \geq 0 : \sum_{i=x-k}^{x+k} \mathcal{L}(i, n) \geq \beta n \right\}. \tag{1.4}$$

$\mathcal{L}^*(n)$ is the maximum of the local times (for a given instant n), $\mathbb{F}_n$ is the set of all the favorite sites and $Y_{n,\beta}$ is the size of the interval where the walk spends more than a proportion β of its time. The first almost sure results on the local time are given by Révész [5]; he notices and shows in a special case that $\mathcal{L}^*$ can be very big (see also [6]). Shi [7] proves the result in the general case. As regards $\mathbb{F}_n$, in Hu and Shi [8] it is proven that the maximal favorite site is almost surely transient and that it has the same almost sure behavior as the walk itself (see also [9]). Until now, the random variable $Y_{n,\beta}$ has not been studied a lot for Sinai's walk. In Andreoletti [10] it is proven that, for $\beta = 1/2$, in probability, this random variable is very small compared to the typical fluctuations of Sinai's walk. Here we are interested in the almost sure behavior of $Y_{n,\beta}$. For all β we prove that the "$\liminf_n$" of this random variable is almost surely (a.s.) bounded. We will see that the result we give for $Y_{n,\beta}$ implies the result of Révész about $\mathcal{L}^*$ and has interesting consequences for the favorite sites.

A second step in the study of $Y_{n,\beta}$ would be to study the "$\limsup_n$" for this random variable. We notice that if $\liminf_n \mathcal{L}^*(n)\phi(n)/n = \text{cte} > 0$ a.s. then $\limsup_n Y_{n,\beta}/\phi(n) = \text{cte} \in]0 + \infty]$ a.s., but is $\phi(n)$ a good asymptotic for the "$\limsup$" of $Y_{n,\beta}$? Notice that Dembo et al. [11] show that $\liminf \mathcal{L}^*(n) \log\log\log n/n = \text{cte} \in]0 + \infty[$ a.s. We also point out that, in a recent work, Shi and Zindy [12] get $\limsup Y_{n,\beta}/\log\log\log n = \text{cte} \in]0 + \infty]$ a.s. (see formula 7.1 of the aforementioned work).

1.1. Definition of Sinai's walk

Let $\alpha = (\alpha_i, i \in \mathbb{Z})$ be a sequence of i.i.d. random variables taking values in $(0, 1)$ defined on the probability space $(\Omega_1, \mathcal{F}_1, Q)$; this sequence will be called a *random environment*. A random walk in a random environment (R.W.R.E.) $(X_n, n \in \mathbb{N})$ is a sequence of random variables taking values in $\mathbb{Z}$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

- for every fixed environment α, $(X_n, n \in \mathbb{N})$ is a Markov chain with the following transition probabilities, for all $n \geq 1$ and $i \in \mathbb{Z}$:

$$\mathbb{P}^{\alpha}\left[X_n = i+1 | X_{n-1} = i\right] = \alpha_i,$$
$$\mathbb{P}^{\alpha}\left[X_n = i-1 | X_{n-1} = i\right] = 1-\alpha_i. \tag{1.5}$$

We denote as $(\Omega_2, \mathcal{F}_2, \mathbb{P}^{\alpha})$ the probability space associated with this Markov chain.

- $\Omega = \Omega_1 \times \Omega_2$, $\forall A_1 \in \mathcal{F}_1$ and $\forall A_2 \in \mathcal{F}_2$, $\mathbb{P}[A_1 \times A_2] = \int_{A_1} Q(\mathrm{d}w_1) \int_{A_2} \mathbb{P}^{\alpha(w_1)}(\mathrm{d}w_2)$.

The probability measure $\mathbb{P}^{\alpha}[.|X_0 = a]$ will be denoted as $\mathbb{P}^{\alpha}_a[.]$, the expectation associated with $\mathbb{P}^{\alpha}_a$ as $\mathbb{E}^{\alpha}_a$, and the expectation associated with Q as $\mathbb{E}_Q$.

Now we introduce the hypothesis that we will use throughout this work. The two following hypotheses are the necessary hypotheses:

$$\mathbb{E}_Q\left[\log \frac{1-\alpha_0}{\alpha_0}\right] = 0, \tag{1.6}$$

$$\mathrm{Var}_Q\left[\log \frac{1-\alpha_0}{\alpha_0}\right] \equiv \sigma^2 > 0. \tag{1.7}$$

Solomon [1] shows that under (1.6), for Q-almost all environments the random walk $(X_n, n \in \mathbb{N})$ is recurrent and (1.7) implies that it is not reduced to the simple random walk. In addition to (1.6) and (1.7) we will consider the following hypothesis of regularity: there exists $0 < \eta_0 < 1/2$ such that

$$\sup\{x, Q[\alpha_0 \geq x] = 1\} \geq \eta_0 \quad \text{and} \quad \sup\{x, Q[\alpha_0 \leq 1-x] = 1\} \geq \eta_0. \tag{1.8}$$

We call the random walk in a random environment previously defined with the three hypotheses (1.6)–(1.8) *Sinai's random walk.*

1.2. Main results

First, let us give the properties that we already know for $Y_{n,\beta}$. By definition, for all β and n, $Y_{n,\beta}$ is positive and $Y_{n,0} = 0$; moreover it is non-decreasing in β because the larger β is the larger k has to be, to get $\sum_{i=x-k}^{x+k} \mathcal{L}(i,n) \geq \beta n$. At this point, we would like to recall the following result of Shi [7] concerning the maximum of the local time $\mathcal{L}^*$ and give a consequence for $\liminf_n Y_{n,\beta}$:

Theorem 1.1. *Assume (1.6)–(1.8) hold; there exists $c_1 > 0$ such that*

$$\mathbb{P}\left[\limsup_n \frac{\mathcal{L}^*(n)}{n} \geq c_1\right] = 1. \tag{1.9}$$

Notice that in Gantert and Shi [13] the following more accurate result is proven: there exists $c_2 > 0$ such that

$$\mathbb{P}\left[\limsup_n \frac{\mathcal{L}^*(n)}{n} = c_2\right] = 1. \tag{1.10}$$

We can get the following corollary of (1.10) just by inspection.

Corollary 1.2. *Assume (1.6)–(1.8) hold and let $0 < \beta \leq c_2$; then*

$$\mathbb{P}\left[\liminf_n Y_{n,\beta} = 1\right] = 1. \tag{1.11}$$

We notice that the main weakness of this corollary is that we have no information about the constant c_2. Moreover we would like to know what the behavior of $\liminf_n Y_{n,\beta}$ is when β is close to 1. In this paper we prove the following theorem showing that for all β, $\liminf_n Y_{n,\beta}$ is almost surely bounded:

Theorem 1.3. *Assume (1.6)–(1.8) hold; there exists $c_3 > 0$ such that for all $0 \leq \beta < 1$*

$$\mathbb{P}\left[\liminf_n Y_{n,\beta} \leq c_3(1-\beta)^{-2}\right] = 1. \tag{1.12}$$

We notice that the size of the interval where the local time is near to 1, i.e. when β gets close to 1, grows at most like $1/(1-\beta)^2$. The strength of our result compared to Corollary 1.2 is that it works for all fixed β and especially for the interesting case when β is close to 1. We will discuss the possible improvements of our result in Section 4.

Using a similar method we also get the following result concerning the maximal distance between two favorite sites.

Theorem 1.4. *Assume (1.6)–(1.8) hold; there exists $c_4 > 0$ such that*

$$\mathbb{P}\left[\liminf_n \max_{(x,y)\in\mathbb{F}_n^2} |x-y| \leq c_4\right] = 1. \tag{1.13}$$

In words, (1.13) says that $\mathbb{P}$-almost surely, infinitely often, the maximal distance between two favorite sites is bounded.

1.3. About the proof of the results

We have used a similar method to Andreoletti [10], and also an extension for Sinai's walk of Proposition 3.1 of Gantert and Shi [13]. We will give the details of the proof in such a way that the reader will understand the $(1-\beta)^{-2}$ dependence occurring in Theorem 1.3. However, less important details of proof, already present in Andreoletti [10], have not been repeated here.

This paper is organized as follows. In Section 2 we give the proof of Theorem 1.3, in Section 3 we prove Theorem 1.4, and finally in Section 4 we make remarks and point out open problems. In the Appendix we give the needed estimates for the environment and the proofs of the most important ones.

2. Proof of Theorem 1.3

We begin with the following elementary remark: By definition we have

$$\liminf_n Y_{n,\beta} \leq c_3(1-\beta)^{-2} \Longleftrightarrow \bigcap_N \bigcup_{n\geq N} \left\{Y_{n,\beta} \leq c_3(1-\beta)^{-2}\right\}. \tag{2.1}$$

We define $\tilde{c}_3(\beta) \equiv c_3(1-\beta)^{-2}$ and for all $x \in \mathbb{Z}$ the set $\theta_\beta(x) = [x - \tilde{c}_3(\beta), x + \tilde{c}_3(\beta)]$; notice that $\tilde{c}_3(\beta)$ is not necessarily an integer but for simplicity we will disregard that. We have

$$\left\{\max_x \mathcal{L}\left(\theta_\beta(x), n\right) \geq \beta n\right\} \subseteq \left\{Y_{n,\beta} \leq \tilde{c}_3(\beta)\right\}, \tag{2.2}$$

so we get that

$$\mathbb{P}\left[\liminf_n Y_{n,\beta} \leq \tilde{c}_3(\beta)\right] \geq \mathbb{P}\left[\bigcap_N \bigcup_{n \geq N} \left\{\max_x \mathcal{L}\left(\theta_\beta(x), n\right) \geq \beta n\right\}\right]$$
$$\geq \mathbb{P}\left[\limsup_n Z_{n,\beta} > \beta\right], \tag{2.3}$$

where

$$Z_{n,\beta} = \frac{\max_x \mathcal{L}\left(\theta_\beta(x), n\right)}{n}.$$

Now assume that the following two propositions are true:

Proposition 2.1. *For all* $0 \leq \beta < 1$

$$\mathbb{P}\left[\limsup_n Z_{n,\beta} = const \in [0, \infty]\right] = 1. \tag{2.4}$$

Proposition 2.2. *For all* $0 \leq \beta < 1$ *and* n

$$\mathbb{P}\left[Z_{n,\beta} > \beta\right] \geq \frac{1}{4}. \tag{2.5}$$

From Proposition 2.2, we easily get that

$$\mathbb{P}\left[\limsup_n Z_{n,\beta} > \beta\right] \geq \frac{1}{4}, \tag{2.6}$$

and now using (2.4) together with (2.6), we get that

$$\mathbb{P}\left[\limsup_n Z_{n,\beta} > \beta\right] = 1, \tag{2.7}$$

because a random variable which has both the property of being almost surely a constant and a strictly positive probability of being larger than another constant β is necessarily almost surely larger than this constant β. Theorem 1.3 follows from (2.3) and (2.7).

Our goal now is to prove Propositions 2.1 and 2.2. Notice that Proposition 2.1 is a simple extension for Sinai's walk of Proposition 3.1 of Gantert and Shi [13]. One can find the details of the proof in the referenced paper; this is how it is applicable in our case:

2.1. Proof of Proposition 2.1

Define $f(\alpha, (X_m)) = \limsup_n \frac{\max_x \mathcal{L}(\theta_\beta(x), n)}{n}$; following the method of Gantert and Shi [13] it is enough to prove the two following facts: *Fact* 1: for Q-a.a. α, $f(\alpha, (X_m))$ is constant for $\mathbb{P}^\alpha$-a.a. realizations of (X_n, n); and *Fact* 2: $f(\alpha) \equiv f(\alpha, (X_m))$ is a constant for Q-a.a. α. Let $x \in \mathbb{Z}$; define

$$T_x = \begin{cases} \inf\{k \in \mathbb{N}^*, X_k = x\} \\ +\infty, \quad \text{if such a } k \text{ does not exist.} \end{cases} \tag{2.8}$$

The key point for the proof of these two facts is that for all $x \in \mathbb{Z}$ ($T_x < +\infty$ $\mathbb{P}^\alpha$-a.s. for Q-a.a. α) because Sinai's walk is recurrent for Q-a.a. environments. So we can apply the three steps of the proof of Gantert and Shi [13] (pp. 168–169): the first two provide Fact 1, the third one Fact 2. Notice that, here, we need a result for $\max_x \mathcal{L}\left(\theta_\beta(x), n\right)$, where $\theta_\beta(x)$ is a bounded interval, whereas in Gantert and Shi [13] $\max_x \mathcal{L}(x, n)$ is studied; this difference does not change the computations. ■

2.2. *Proof of Proposition 2.2*

To prove this proposition we use a quite similar method to Andreoletti [10]; first let us recall the following decomposition of the measure $\mathbb{P}$. Let $\mathcal{C}_n \in \sigma(X_i, i \leq n)$ and $G_n \subset \Omega_1$; we have

$$\mathbb{P}[\mathcal{C}_n] \equiv \int_{\Omega_1} Q(\mathrm{d}\omega) \int_{\mathcal{C}_n} \mathrm{d}\mathbb{P}^{\alpha(\omega)} \tag{2.9}$$

$$\geq \int_{G_n} Q(\mathrm{d}\omega) \int_{\mathcal{C}_n} \mathrm{d}\mathbb{P}^{\alpha(\omega)}. \tag{2.10}$$

So assume that for all $\omega \in G_n$ and n, $\int_{\mathcal{C}_n} \mathrm{d}\mathbb{P}^{\alpha(\omega)} \equiv d_1(\omega, n) > \text{const} > 0$ and assume that $Q[G_n] \equiv d_2(n) > \text{const}' > 0$; we get that for all n

$$\mathbb{P}[\mathcal{C}_n] \geq d_2(n) \times \min_{w \in G_n} (d_1(w, n)) > \text{const}'' > 0. \tag{2.11}$$

So choosing $\mathcal{C}_n = \{\max_x \mathcal{L}\left(\theta_\beta(x), n\right) \geq \beta n\}$, we have to extract from Ω_1 a subset G_n sufficiently small to get that $\min_{w \in G_n} (d_1(w, n)) > \text{const}' > 0$ (Proposition 2.11) but sufficiently large to have $d_2(n) > \text{const} > 0$ (Proposition 2.10). The largest part of the proof is for constructing such a G_n (Section 2.2.1 and Appendix B).

2.2.1. *Construction of G_n (arguments for the random environment)*

For completeness we begin with some basic notions originally introduced by Sinai [2].

The random potential and the valleys

Let

$$\epsilon_i \equiv \log \frac{1 - \alpha_i}{\alpha_i}, \quad i \in \mathbb{Z}, \tag{2.12}$$

and define:

Definition 2.3. The random potential $(S_m, m \in \mathbb{Z})$ associated with the random environment α is defined in the following way:

$$S_k = \begin{cases} \sum_{1 \leq i \leq k} \epsilon_i, & \text{if } k > 0, \\ -\sum_{k+1 \leq i \leq 0} \epsilon_i, & \text{if } k < 0, \end{cases}$$

$$S_0 = 0.$$

Definition 2.4. We will say that the triplet $\{M', m, M''\}$ is a *valley* if

$$S_{M'} = \max_{M' \leq t \leq m} S_t, \tag{2.13}$$

$$S_{M''} = \max_{m \leq t \leq M''} S_t, \tag{2.14}$$

$$S_m = \min_{M' \leq t \leq M''} S_t. \tag{2.15}$$

If m is not unique we choose the one with the smallest absolute value.

Definition 2.5. We will call the following quantity the *depth of the valley* $\{M', m, M''\}$ and we will denote it as $d([M', M''])$:

$$\min(S_{M'} - S_m, S_{M''} - S_m). \tag{2.16}$$

Now we define the operation of *refinement*.

Definition 2.6. Let $\{M', m, M''\}$ be a valley and let M_1 and m_1 be such that $m \leq M_1 < m_1 \leq M''$ and

$$S_{M_1} - S_{m_1} = \max_{m \leq t' \leq t'' \leq M''} (S_{t'} - S_{t''}). \tag{2.17}$$

We say that the couple (m_1, M_1) is obtained by a *right refinement* of $\{M', m, M''\}$. If the couple (m_1, M_1) is not unique, we will take the one such that m_1 and M_1 have the smallest absolute value. In a similar way we define the *left refinement* operation.

We define $\log_2 = \log\log$; throughout this section we will suppose that n is large enough such that $\log_2 n$ is positive.

Definition 2.7. Let $n > 3$ and $\Gamma_n \equiv \log n + 12 \log_2 n$; we say that a valley $\{M', m, M''\}$ contains 0 and is of depth larger than Γ_n if and only if

1. $0 \in [M', M'']$,
2. $d\left([M', M'']\right) \geq \Gamma_n$,
3. if $m < 0$, $S_{M''} - \max_{m \leq t \leq 0} (S_t) \geq 12 \log_2 n$,
   if $m > 0$, $S_{M'} - \max_{0 \leq t \leq m} (S_t) \geq 12 \log_2 n$.

The basic valley $\{M_n{}', m_n, M_n\}$

We recall the notion of a *basic valley* introduced by Sinai and denoted here as $\{M_n{}', m_n, M_n\}$. The definition that we give is inspired by the work of Kesten [14]. First let $\{M', m_n, M''\}$ be the smallest *valley that contains* 0 *and is of depth larger than* Γ_n. Here ‘smallest’ means that if we construct, with the operation of refinement, other valleys in $\{M', m_n, M''\}$, such valleys will not satisfy one of the properties of Definition 2.7. $M_n{}'$ and M_n are defined from m_n in the following way: if $m_n > 0$

$$M_n{}' = \sup\left\{l \in \mathbb{Z}_-, l < m_n, S_l - S_{m_n} \geq \Gamma_n, S_l - \max_{0 \leq k \leq m_n} S_k \geq 12 \log_2 n\right\}, \tag{2.18}$$

$$M_n = \inf\left\{l \in \mathbb{Z}_+, l > m_n, S_l - S_{m_n} \geq \Gamma_n\right\} \tag{2.19}$$

while if $m_n < 0$

$$M_n{}' = \sup\left\{l \in \mathbb{Z}_-, l < m_n, S_l - S_{m_n} \geq \Gamma_n\right\}, \tag{2.20}$$

$$M_n = \inf\left\{l \in \mathbb{Z}_+, l > m_n, S_l - S_{m_n} \geq \Gamma_n, S_l - \max_{m_n \leq k \leq 0} S_k \geq 12 \log_2 n\right\} \tag{2.21}$$

and if $m_n = 0$

$$M_n{}' = \sup\left\{l \in \mathbb{Z}_-, l < 0, S_l - S_{m_n} \geq \Gamma_n\right\}, \tag{2.22}$$

$$M_n = \inf\left\{l \in \mathbb{Z}_+, l > 0, S_l - S_{m_n} \geq \Gamma_n\right\}. \tag{2.23}$$

$\{M_n{}', m_n, M_n\}$ exists with a Q probability as close to 1 as we need. In fact it is not difficult to prove the following lemma.

Lemma 2.8. *Assume (1.6)–(1.8) hold; for all n we have*

$$Q\left[\{M_n{}', m_n, M_n\} \neq \varnothing\right] = 1 - o(1). \tag{2.24}$$

We denote as $o(1)$ *a positive decreasing function of* n *such that* $\lim_{n\to\infty} o(1) = 0$.

Proof. One can find the proof of this lemma in Section 5.2 of Andreoletti [10]. ■

Definition 2.9. Let $c_0 > 0$, c_0', $c_3 > 0$, $0 \leq \beta < 1$ and $\omega \in \Omega_1$; we will say that $\alpha \equiv \alpha(\omega)$ is a *good environment* if there exists n_0 such that for all $n \geq n_0$ the sequence $(\alpha_i, i \in \mathbb{Z}) = (\alpha_i(\omega), i \in \mathbb{Z})$ satisfies the properties (2.25)–(2.28):

- $\{M_n{}', m_n, M_n\} \neq \varnothing,$ (2.25)
- $M_n{}' \geq -(\sigma^{-1}\log n)^2, \qquad M_n \leq (\sigma^{-1}\log n)^2,$ (2.26)
- $\mathbb{E}^{\alpha}_{m_n}\left[\mathcal{L}(\tilde{\Theta}_\beta(m_n), T_{m_n})\right] \leq \dfrac{4c_0}{\sqrt{\tilde{c}_3(\beta)}},$ (2.27)
- $(\mathbb{E}^{\alpha}_{m_n}\left[\mathcal{L}([M_n', M_n], T_{m_n})\right])^{-1} > \dfrac{c_0'}{4}$ (2.28)

where $\tilde{\Theta}_\beta(m_n) = [M_n', M_n' + 1, \ldots, m_n - \tilde{c}_3(\beta)] \cup [m_n + \tilde{c}_3(\beta), m_n + \tilde{c}_3(\beta) + 1, \ldots, M_n]$; recall that $\tilde{c}_3(\beta) = c_3(1-\beta)^{-2}$ and the definition of T_{m_n} is given in (2.8).

Define the *set of good environments*

$$G_n \equiv G_n(c_0, c_0', c_3, \beta) = \{\omega \in \Omega_1, \alpha(\omega) \textit{ is a good environment}\}. \tag{2.29}$$

G_n depends on c_0, c_0', c_3, β and n; however, we only make explicit the dependence on n.

Proposition 2.10. *Assume (1.6)–(1.8) hold; there exists* $c_0 > 0$, $c_0' > 0$, $c_3 > 0$, *and* n_0 *such that for all* $0 \leq \beta < 1$ *and* $n > n_0$

$$Q[G_n] \geq 1/2. \tag{2.30}$$

Proof. In Andreoletti [10] it has already been proven that the first two properties are true with a probability close to 1. For the third and the fourth ones we give a method, postponed to Appendix B, showing that each of them is true with a probability larger than 3/4 and therefore that they are true together with a probability larger than 1/2. ■

2.2.2. Argument for the walk (environment fixed, $\alpha \in G_n$)

In this section we assume that n is large enough such that Proposition 2.10 is true and we assume that the random environment is fixed and belongs to G_n (denoted as $\alpha \in G_n$).

Proposition 2.11. *For all* $0 \leq \beta < 1$, *n large enough and* $\alpha \in G_n$ *we have*

$$\mathbb{P}^{\alpha}_0\left[\max_x \mathcal{L}\left(\theta_\beta(x), n\right) > \beta n\right] > 1/2. \tag{2.31}$$

Recall that $\theta_\beta(x) = [x - \tilde{c}_3(\beta), x + \tilde{c}_3(\beta)]$; $\tilde{c}_3(\beta)$ *is defined just before (2.29).*

Proof. To get this result, it is enough to prove that

$$\mathbb{P}_0^\alpha\left[\mathcal{L}\left(\theta_\beta(m_n), n\right) > \beta n\right] > 1/2, \tag{2.32}$$

where m_n is defined in the paragraph just before (2.19); in fact we will prove the following equivalent fact:

$$\mathbb{P}_0^\alpha\left[\mathcal{L}\left(\Theta_\beta(m_n), n\right) \geq (1-\beta)n\right] < 1/2, \tag{2.33}$$

where $\Theta_\beta(m_n)$ is the complementary of $\theta_\beta(m_n)$ in $\mathbb{Z}$.

First we recall the two following elementary results.

Lemma 2.12. *For all n and $\alpha \in G_n$ we have*

$$\mathbb{P}_0^\alpha\left[\bigcup_{m=0}^{n}\left\{X_m \notin \left[M_n', M_n\right]\right\}\right] = \mathrm{o}(1), \tag{2.34}$$

$$\mathbb{P}_0^\alpha\left[T_{m_n} > \frac{n}{(\log n)^4}\right] = \mathrm{o}(1), \tag{2.35}$$

and we recall that $\lim_{n\to\infty} \mathrm{o}(1) = 0$.

Proof. This is a basic result for Sinai's walk; it makes use of Properties (2.25) and (2.26). One can find the details of this proof in Andreoletti [10]: Proposition 4.7 and Lemma 4.8. ■

First we use (2.34) to reduce the set $\Theta_\beta(m_n)$ to $\tilde{\Theta}_\beta(m_n)$ defined just after (2.28); we get

$$\mathbb{P}_0^\alpha\left[\mathcal{L}\left(\Theta_\beta(m_n), n\right) \geq (1-\beta)n\right] \leq \mathbb{P}_0^\alpha\left[\mathcal{L}\left(\tilde{\Theta}_\beta(m_n), n\right) \geq (1-\beta)n\right] + o(1). \tag{2.36}$$

Now using (2.35) we get

$$\begin{aligned}&\mathbb{P}_0^\alpha\left[\mathcal{L}\left(\tilde{\Theta}_\beta(m_n), n\right) \geq (1-\beta)n\right]\\ &\quad \leq \mathbb{P}_0^\alpha\left[\mathcal{L}\left(\tilde{\Theta}_\beta(m_n), n\right) \geq (1-\beta)n,\ T_{m_n} \leq \frac{n}{(\log n)^4}\right] + o(1).\end{aligned} \tag{2.37}$$

Let us define $N_0 \equiv \left[n(\log n)^{-4}\right] + 1$ and $1 - \beta_n \equiv 1 - \beta - N_0/n$. By the Markov property and the homogeneity of the Markov chain we obtain

$$\begin{aligned}&\mathbb{P}_0^\alpha\left[\mathcal{L}\left(\tilde{\Theta}_\beta(m_n), n\right) \geq (1-\beta)n, T_{m_n} \leq \frac{n}{(\log n)^4}\right]\\ &\quad \leq \mathbb{P}_{m_n}^\alpha\left[\sum_{k=1}^{n} \mathbb{I}_{\left\{X_k \in \tilde{\Theta}_\beta(m_n)\right\}} \geq (1-\beta_n)n\right].\end{aligned} \tag{2.38}$$

Define the following return times wherein $j \geq 2$:

$$T_{m_n,j} \equiv \begin{cases}\inf\{k > T_{m_n,j-1},\ X_k = m_n\},\\ +\infty, \quad \text{if such a } k \text{ does not exist.}\end{cases}$$

$$T_{m_n,1} \equiv T_{m_n} \text{ (see (2.8))}, \qquad T_{m_n,0} = 0.$$

Since by definition $T_{m_n,n} > n$, $\{\sum_{k=1}^{n} \mathbb{I}_{\{X_k \in \tilde{\Theta}_\beta(m_n)\}} \geq (1-\beta_n)n\} \subset \{\sum_{k=1}^{T_{m_n,n}} \mathbb{I}_{\{X_k \in \tilde{\Theta}_\beta(m_n)\}} \geq (1-\beta_n)n\}$, then using the definition of the local time and the Markov inequality we get

$$\mathbb{P}^{\alpha}_{m_n}\left[\sum_{k=1}^{n} \mathbb{I}_{\left\{X_k \in \tilde{\Theta}_\beta(m_n)\right\}} \geq (1-\beta_n)n\right] \leq \mathbb{P}^{\alpha}_{m_n}\left[\sum_{k=1}^{T_{m_n},n} \mathbb{I}_{\left\{X_k \in \tilde{\Theta}_\beta(m_n)\right\}} \geq (1-\beta_n)n\right]$$
$$\leq \mathbb{E}^{\alpha}_{m_n}\left[\mathcal{L}\left(\tilde{\Theta}_\beta(m_n), T_{m_n}\right)\right](1-\beta_n)^{-1}, \tag{2.39}$$

and we have used the fact that the random variables $\mathcal{L}\left(s, T_{m_n,i+1} - T_{m_n,i}\right)$ $(0 \leq i \leq n-1)$ are i.d. Using the property (2.27), there exists c_0 such that

$$\mathbb{E}^{\alpha}_{m_n}\left[\mathcal{L}\left(\tilde{\Theta}_\beta(m_n), T_{m_n}\right)\right] \leq \frac{4c_0(1-\beta)}{(c_3)^{1/2}}. \tag{2.40}$$

Collecting what we did above, we finally get the following inequality for n large enough:

$$\mathbb{P}^{\alpha}_{0}\left[\mathcal{L}\left(\Theta(n,\beta), n\right) \geq (1-\beta)n\right] \leq \frac{8c_0}{(c_3)^{1/2}}. \tag{2.41}$$

We obtain (2.33) choosing $c_3 = 256(c_0)^2$. ■

This ends the proof of Theorem 1.3.

3. Proof of Theorem 1.4

Let $\delta > 0$ be a free parameter that will be chosen later; for all n, define

$$\mathcal{A}_n = \left\{\max_{(x,y)\in\mathbb{F}_n^2} |x-y| \leq 2c_3/\delta^2\right\}, \tag{3.1}$$

$$Z'_{n,1-\delta} = \frac{\max_{x\in\mathbb{Z}} \mathcal{L}\left(\theta_{1-\delta}(x), n\right) + \mathcal{L}^*(n)}{n}, \tag{3.2}$$

$$\mathcal{B}_n = \left\{Z'_{n,1-\delta} > 1\right\}. \tag{3.3}$$

We recall that $\mathcal{L}^*(n) = \max_{x\in\mathbb{Z}} \mathcal{L}\left(x, n\right)$ and $\theta_{(1-\delta)}(x) = [x - c_3/\delta^2, x + c_3/\delta^2]$; c_3 is the positive constant defined above. We prove Theorem 1.4 in two steps:

Step 1. First we notice that $\mathcal{B}_n \subset \mathcal{A}_n$; indeed, denoting as $\bar{\theta}_{1-\delta}(x)$ the complementary of $\theta_{1-\delta}(x)$ in $\mathbb{Z}$ and using the elementary fact that $\mathcal{L}(\mathbb{Z}, n) = n$ we have (see also Fig. 1)

$$\begin{aligned}
\mathcal{B}_n &= \bigcup_{x\in\mathbb{Z}} \left\{\sum_{k\in\theta_{(1-\delta)}(x)} \mathcal{L}\left(k,n\right) + \mathcal{L}^*(n) > n\right\} \\
&= \bigcup_{x\in\mathbb{Z}} \left\{\sum_{k\in\bar{\theta}_{(1-\delta)}(x)} \mathcal{L}\left(k,n\right) < \mathcal{L}^*(n)\right\} \\
&\subset \bigcup_{x\in\mathbb{Z}} \bigcap_{(k\in\bar{\theta}_{(1-\delta)}(x))} \{k \notin \mathbb{F}_n\} \\
&\subset \mathcal{A}_n.
\end{aligned} \tag{3.4}$$

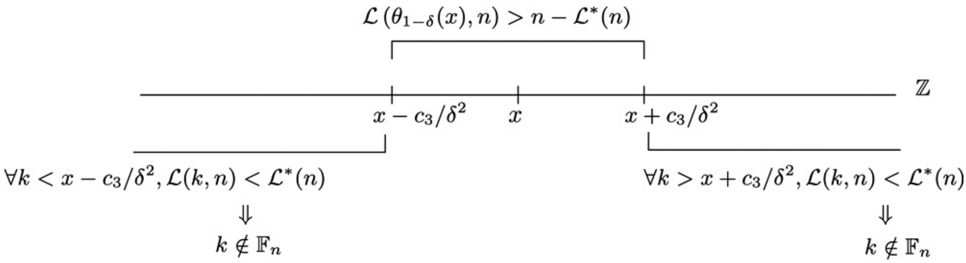

Fig. 1. $\mathcal{B}_n \subset \mathcal{A}_n$.

Now, using the definition of the "lim inf", the "lim sup" and (3.4), we get that

$$\begin{aligned}\mathbb{P}\left[\liminf_n \max_{(x,y)\in\mathbb{F}_n^2} |x-y| \le 2c_3/\delta^2\right] &\ge \mathbb{P}\left[\limsup_n \mathcal{A}_n\right]\\ &\ge \mathbb{P}\left[\limsup_n \mathcal{B}_n\right]\\ &\ge \mathbb{P}\left[\limsup_n Z'_{n,1-\delta} > 1+\delta\right]. \end{aligned} \tag{3.5}$$

Notice that the event $\{\limsup_n Z'_{n,1-\delta} > 1+\delta\}$ increases when δ decreases.

Step 2. In this second step we prove the following.

Lemma 3.1. *There exists a constant* $c_5 > 0$ *such that*

$$\mathbb{P}\left[\limsup_n Z'_{n,1-c_5} > 1 + c_5\right] = 1. \tag{3.6}$$

Proof. The method that we use here is exactly the same as the method that we have used to get (2.7), so we only recall the main ideas. With the same argument as we have used to prove Proposition 2.1 we can get that

$$\mathbb{P}\left[\limsup_n Z'_{n,1-\delta} = \text{const} \in [0, +\infty]\right] = 1, \tag{3.7}$$

so we only have to check that we can find $\delta > 0$ such that

$$\mathbb{P}\left[\limsup_n Z'_{n,1-\delta} > 1 + \delta\right] > \text{const}' > 0, \tag{3.8}$$

and to get this, it is enough to prove that we can find $\delta > 0$ such that for all n large enough

$$\mathbb{P}\left[\left(\max_{x\in\mathbb{Z}} \mathcal{L}(\theta_{1-\delta}(x), n) + \mathcal{L}^*(n)\right)/n > 1+\delta\right] > \text{const}' > 0. \tag{3.9}$$

Using Proposition 2.11, with $\beta = 1 - \delta$, for all $\alpha \in G_n$ we have

$$\mathbb{P}^\alpha\left[\left(\max_{x\in\mathbb{Z}} \mathcal{L}(\theta_{1-\delta}(x), n) + \mathcal{L}^*(n)\right)/n > 1+\delta\right] > \mathbb{P}^\alpha\left[\mathcal{L}^*(n)/n > 2\delta\right] - 1/2. \tag{3.10}$$

Note that $\mathcal{L}^*(n) \geq \mathcal{L}(m_n, n)$; moreover, the weak law of large numbers proved in Andreoletti [10] (Theorem 3.8, p. 1385) for $\mathcal{L}(m_n, n)$ implies that

$$\mathbb{P}^\alpha\left[\mathcal{L}(m_n, n)/n > (\mathbb{E}^\alpha_{m_n}[\mathcal{L}([M'_n, M_n], T_{m_n})])^{-1}(1 - o(1))\right] = 1 - o(1), \tag{3.11}$$

and we recall that $\lim_{n\to+\infty} o(1) = 0$. Now using Property (2.28) of the random environment, we know that for all $\alpha \in G_n$,

$$1/2 \geq (\mathbb{E}^\alpha_{m_n}[\mathcal{L}([M'_n, M_n], T_{m_n})])^{-1} > c'_0/4, \tag{3.12}$$

where c'_0 is a strictly positive constant; therefore, for all n large enough, (3.11) and (3.12) give

$$\mathbb{P}^\alpha\left[\mathcal{L}^*(n)/n > c'_0/16\right] \geq \mathbb{P}^\alpha\left[\mathcal{L}(m_n, n)/n > c'_0/16\right] = 1 - o(1). \tag{3.13}$$

Taking $\delta = c'_0/8$ in (3.10) together with (3.13), and finally using Proposition 2.10 and (2.11), we get for n large enough

$$\mathbb{P}\left[\left(\max_{x\in\mathbb{Z}} \mathcal{L}\left(\theta_{1-c'_0/16}(x), n\right) + \mathcal{L}^*(n)\right)/n > 1 + c'_0/8\right] \geq 1/4. \tag{3.14}$$

This ends the proof of the lemma ($c_5 = c'_0/8$). ■

(3.5) and Lemma 3.1 provide the theorem ($c_4 = 128c_3/(c'_0)^2$). ■

4. Remarks and open problems

We have shown that using the work of Andreoletti [10] and Proposition 3.1 of Gantert and Shi [13] we can get, for all β, an upper bound for $\liminf_n Y_{n,\beta}$. We have also pointed out that the result for the concentration variable $Y_{n,\beta}$ implies both results for the maximum of the local time and results for the favorite sites. The weakness of our result is that we only get an upper bound for $\liminf_n Y_{n,\beta}$, and the lower bound cannot be deduced directly with our method.

Now, forgetting the hypothesis (1.6), let us use the following one originally introduced by Kesten et al. [15]:

$$-\infty < \mathbb{E}_Q\left[\log\frac{1-\alpha_0}{\alpha_0}\right] < 0, \tag{4.1}$$

and that there is $0 < \kappa < 1$ such that

$$0 < \mathbb{E}_Q\left[\left(\frac{1-\alpha_0}{\alpha_0}\right)^\kappa\right] = 1. \tag{4.2}$$

The first hypothesis implies that the random walk in a random environment that we get is almost surely transient for almost all environments and the second one that this random walk is a.s. sub-ballistic for a.a. environments. Thanks to the work of Gantert and Shi [13], we know that for a small β one can find $c_1 \equiv c_1(\beta) > 0$ such that

$$\liminf Y_{n,\beta} \leq c_1, \ \mathbb{P}\text{.a.s.} \tag{4.3}$$

A question that is maybe interesting to understand is how this β depends on κ. For example, can we find κ such that (4.3) is true for $\beta = 1/2$? We could say that Sinai's walk is concentrated uniformly for $0 < \beta < 1$ whereas the Kesten et al. walk is uniformly concentrated for $0 < \beta \leq \beta_c \equiv \beta_c(\kappa)$. What can we say about β_c?

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Appendix A. Basic results for birth and death processes

For completeness we recall some results of Chung [16] and Révész [5] on inhomogeneous discrete time birth and death processes.

Let x, a and b in $\mathbb{Z}$; assume $a < x < b$. The two following lemmata can be found in Chung [16] (pp. 73–76). The proof follows from the method of difference equations.

Lemma A.1. *Recalling (2.8), for all α we have*

$$\mathbb{P}_x^\alpha [T_a > T_b] = \frac{\sum_{i=a+1}^{x-1} \exp(S_i - S_a) + 1}{\sum_{i=a+1}^{b-1} \exp(S_i - S_a) + 1}, \tag{A.1}$$

$$\mathbb{P}_x^\alpha [T_a < T_b] = \frac{\sum_{i=x+1}^{b-1} \exp(S_i - S_b) + 1}{\sum_{i=a+1}^{b-1} \exp(S_i - S_b) + 1}. \tag{A.2}$$

We will also need the following elementary expressions for the local times (see [5], p. 279):

Lemma A.2. *For all α and $i \in \mathbb{Z}$, we have, if $x > i$,*

$$\mathbb{E}_i^\alpha [\mathcal{L}(x, T_i)] = \frac{\alpha_i \mathbb{P}_{i+1}^\alpha [T_x < T_i]}{(1-\alpha_x)\mathbb{P}_{x-1}^\alpha [T_x > T_i]}, \tag{A.3}$$

and if $x < i$,

$$\mathbb{E}_i^\alpha [\mathcal{L}(x, T_i)] = \frac{(1-\alpha_i)\mathbb{P}_{i-1}^\alpha [T_x < T_i]}{\alpha_x \mathbb{P}_{x+1}^\alpha [T_x > T_i]}. \tag{A.4}$$

Appendix B. Proof of the good properties for the environment

We recall that, by definition, the random potential $(S_m, m \in \mathbb{N})$ is a function of the random environment α even if, for simplicity, we have not emphasized this dependence. Here we give the main ideas for the proof of Proposition 2.10; more exactly we prove that property (2.27) is true with a probability larger than $3/4$. We can get the same result with the same arguments for property (2.28) so we will not give the details of the proof for this property (see also the proof of Proposition 3.12 of Andreoletti [10]). We begin with some

B.1. *Elementary results for sums of i.i.d. random variables*

We will always work on the right hand side of the origin, that means with $(S_m, m \in \mathbb{N})$; by symmetry we obtain the same results for $m \in \mathbb{Z}_-$.

We introduce the following stopping times, for $a > 0$:

$$V_a^+ \equiv V_a^+(S_j, j \in \mathbb{N}) = \begin{cases} \inf\{m \in \mathbb{N}^*, S_m \geq a\}, \\ +\infty, \quad \text{if such a } m \text{ does not exist.} \end{cases} \tag{B.1}$$

$$V_a^- \equiv V_a^-(S_j, j \in \mathbb{N}) = \begin{cases} \inf\{m \in \mathbb{N}^*, \ S_m \leq -a\}, \\ +\infty, \quad \text{if such an } m \text{ does not exist.} \end{cases} \tag{B.2}$$

The following lemma is an immediate consequence of the Wald equality (see [17]):

Lemma B.1. *Assume (1.6)–(1.8); let $a > 0$, $d > 0$. We have*

$$Q\left[V_a^- < V_d^+\right] \leq \frac{d + I_{\eta_0}}{d + a + I_{\eta_0}}, \tag{B.3}$$

$$Q\left[V_a^- > V_d^+\right] \leq \frac{a + I_{\eta_0}}{d + a + I_{\eta_0}}, \tag{B.4}$$

with $I_{\eta_0} \equiv \log((1 - \eta_0)(\eta_0)^{-1})$.

The following lemma is a basic fact for sums of i.i.d. random variables.

Lemma B.2. *Assume (1.6)–(1.8) hold; there exists $b > 0$ such that for all $r > 0$*

$$Q\left[V_0^- > r\right] \leq \frac{b}{\sqrt{r}}. \tag{B.5}$$

B.2. Proof of Proposition 2.10

It is in this part that the $(1-\beta)^{-2}$ dependence occurring in Theorem 1.3 will become clear. The main difficulty is in getting an upper bound for $\mathbb{E}_Q\left[\mathbb{E}_{m_n}^\alpha\left[\mathcal{L}(\tilde{\Theta}_\beta(m_n), T_{m_n})\right]\right]$.

B.2.1. Preliminaries

By linearity we have

$$\mathbb{E}_{m_n}^\alpha\left[\mathcal{L}(\tilde{\Theta}_\beta(m_n), T_{m_n})\right] \equiv \sum_{j=m_n+\tilde{c}_3(\beta)}^{M_n} \mathbb{E}_{m_n}^\alpha\left[\mathcal{L}(j, T_{m_n})\right] + \sum_{j=M_n'}^{m_n-\tilde{c}_3(\beta)} \mathbb{E}_{m_n}^\alpha\left[\mathcal{L}(j, T_{m_n})\right] + 1, \tag{B.6}$$

and we recall that $\tilde{c}_3(\beta) = c_3(1-\beta)^{-2}$ with $c_3 > 0$ and $0 \leq \beta < 1$. Now using Lemma A.1 and hypothesis (1.8) we easily get the following lemma.

Lemma B.3. *Assume (1.8), for Q-a.a. environments, all ${M_n}' \leq k \leq M_n$, with $k \neq m_n$,*

$$\frac{\eta_0}{1-\eta_0}\frac{1}{\mathrm{e}^{S_k - S_{m_n}}} \leq \mathbb{E}_{m_n}^\alpha\left[\mathcal{L}(k, T_{m_n})\right] \leq \frac{1}{\eta_0}\frac{1}{\mathrm{e}^{S_k - S_{m_n}}}. \tag{B.7}$$

The following lemma is easy to prove:

Lemma B.4. *For Q-a.a. environments and $n > 3$*

$$\sum_{j=m_n+\tilde{c}_3(\beta)}^{M_n} \frac{1}{\mathrm{e}^{S_j-S_{m_n}}} \le \sum_{i=1}^{N_n+1} \frac{1}{\mathrm{e}^{a(i-1)}} \sum_{j=m_n+\tilde{c}_3(\beta)}^{M_n} \mathbb{I}_{S_j-S_{m_n}\in[a(i-1),ai[}, \tag{B.8}$$

$$\sum_{j=M'_n}^{m_n-\tilde{c}_3(\beta)} \frac{1}{\mathrm{e}^{S_j-S_{m_n}}} \le \sum_{i=1}^{N_n+1} \frac{1}{\mathrm{e}^{a(i-1)}} \sum_{j=M'_n}^{m_n-\tilde{c}_3(\beta)} \mathbb{I}_{S_j-S_{m_n}\in[a(i-1),ai[}, \tag{B.9}$$

where $a = \frac{I_{\eta_0}}{4}$, $N_n = [(\Gamma_n + I_{\eta_0})/a]$; recall that $\mathbb{I}$ is the indicator function.

Using (B.6), Lemmas B.3 and B.4, we have for all $n > 3$

$$\begin{aligned}\mathbb{E}_Q\left[\mathbb{E}^{\alpha}_{m_n}\left[\mathcal{L}(\tilde{\Theta}_\beta(m_n), T_{m_n})\right]\right] &\le 1 + \frac{1}{\eta_0}\sum_{i=1}^{N_n+1}\frac{1}{\mathrm{e}^{a(i-1)}}\mathbb{E}_Q \\ &\quad\times\left[\sum_{j=m_n+\tilde{c}_3(\beta)}^{M_n}\mathbb{I}_{S_j-S_{m_n}\in[a(i-1),ai[}\right] \\ &\quad+\frac{1}{\eta_0}\sum_{i=1}^{N_n+1}\frac{1}{\mathrm{e}^{a(i-1)}}\mathbb{E}_Q\left[\sum_{j=M'_n}^{m_n-\tilde{c}_3(\beta)}\mathbb{I}_{S_j-S_{m_n}\in[a(i-1),ai[}\right].\end{aligned} \tag{B.10}$$

The next step for the proof is to show that the two expectations $\mathbb{E}_Q[\ldots]$ on the right hand side of (B.10) are bounded by a constant depending only on the distribution Q times a polynomial in i times $1/\sqrt{\tilde{c}_3(\beta)}$:

Lemma B.5. *There exists a constant $c \equiv c(Q)$ such that for all n large enough,*

$$\mathbb{E}_Q\left[\sum_{j=m_n+\tilde{c}_3(\beta)}^{M_n}\mathbb{I}_{S_j-S_{m_n}\in[a(i-1),ai[}\right] \le \frac{c\times i^3}{\sqrt{\tilde{c}_3(\beta)}}, \tag{B.11}$$

$$\mathbb{E}_Q\left[\sum_{j=M'_n}^{m_n-\tilde{c}_3(\beta)}\mathbb{I}_{S_j-S_{m_n}\in[a(i-1),ai[}\right] \le \frac{c\times i^3}{\sqrt{\tilde{c}_3(\beta)}}. \tag{B.12}$$

B.2.2. Proof of Lemma B.5

Remark B.6. We give some details of the proof of Lemma B.5 mainly because it helps to understand the occurrence of the $(1-\beta)^{-2}$ in Theorem 1.3. Moreover it is based on a very nice cancellation that occurs between two $\Gamma_n \equiv \log n + \log_2 n$; see formulas (B.18) and (B.20). A similar cancellation is already present in Kesten [14].

Let us define the following stopping times, letting $i > 1$:

$$u_0 = 0,$$
$$u_1 \equiv V_0^- = \inf\{m > 0, S_m < 0\},$$
$$u_i = \inf\{m > u_{i-1}, S_m < S_{u_{i-1}}\}.$$

The following lemma gives a way to characterize the point m_n; it is inspired by the work of Kesten [14] and is just from inspection.

Lemma B.7. *Let $n > 3$ and $\gamma > 0$, recall $\Gamma_n = \log n + \gamma \log_2 n$, and assume $m_n > 0$, for all $l \in \mathbb{N}^*$; we have*

$$m_n = u_l \Rightarrow \begin{cases} \bigcap_{i=0}^{l-1} \left\{ \max_{u_i \le j \le u_{i+1}} (S_i) - S_{u_i} < \Gamma_n \right\} & \text{and} \\ \max_{u_l \le j \le u_{l+1}} (S_i) - S_{u_i} \ge \Gamma_n & \text{and} \\ M_n = V^+_{\Gamma_n, l} \end{cases} \tag{B.13}$$

where

$$V^+_{z,l} \equiv V^+_{z,l}\left(S_j, j \ge 1\right) = \inf\left(m > u_l, S_m - S_{u_l} \ge z\right). \tag{B.14}$$

A similar characterization of m_n if $m_n \le 0$ can be done (the case $m_n = 0$ is trivial). We will only prove (B.11); we get (B.12) symmetrically. Moreover we assume that $m_n > 0$. Computations are similar for the case $m_n \le 0$. Thinking of the basic definition of the expectation, we need an upper bound for the probability:

$$Q\left[\sum_{j=m_n+\tilde{c}_3(\beta)}^{M_n} \mathbb{I}_{S_j - S_{m_n} \in [a(i-1), ai[} = k\right].$$

First we make a partition over the possible values of m_n and then we use Lemma B.7; we get

$$\begin{aligned} &Q\left[\sum_{j=m_n+\tilde{c}_3(\beta)}^{M_n} \mathbb{I}_{S_j - S_{m_n} \in [a(i-1), ai[} = k\right] \\ &\quad \equiv \sum_{l \ge 0} Q\left[\sum_{j=m_n+\tilde{c}_3(\beta)}^{M_n} \mathbb{I}_{S_j - S_{m_n} \in [a(i-1), ai[} = k,\ m_n = u_l\right] \\ &\quad \le \sum_{l \ge 0} Q\left[\mathcal{A}^+_{\Gamma_n, l}, \max_{u_l \le j \le u_{l+1}} (S_j) - S_{u_l} \ge \Gamma_n, \mathcal{A}^-_{\Gamma_n, l}\right] \end{aligned} \tag{B.15}$$

where

$$\mathcal{A}^+_{\Gamma_n, l} = \sum_{s=u_l+\tilde{c}_3(\beta)}^{V^+_{\Gamma_n, l}} \mathbb{I}_{\{S_j - S_{u_l} \in [a(i-1), ai[\}} = k,$$

$$\mathcal{A}^-_{\Gamma_n, l} = \bigcap_{r=0}^{l-1} \left\{ \max_{u_r \le j \le u_{r+1}} (S_r) - S_{u_r} < \Gamma_n \right\}, \qquad \mathcal{A}^-_0 = \Omega_1$$

for all $l \ge 0$. By the strong Markov property we have

$$Q\left[\mathcal{A}^+_{\Gamma_n, l}, \max_{u_l \le j \le u_{l+1}} (S_j) - S_{u_l} \ge \Gamma_n,\ \mathcal{A}^-_{\Gamma_n, l}\right] \le Q\left[\mathcal{A}^+_{\Gamma_n, 0}, V^-_0 > V^+_{\Gamma_n}\right] Q\left[\mathcal{A}^-_{\Gamma_n, l}\right]. \tag{B.16}$$

The strong Markov property gives also that the sequence $(\max_{u_r \le j \le u_{r+1}}(S_r) - S_{u_r} < \Gamma_n, r \ge 1)$ is i.i.d., and therefore

$$Q\left[\mathcal{A}^-_{\Gamma_n,l}\right] \le \left(Q\left[V_0^- < V^+_{\Gamma_n}\right]\right)^{l-1}. \tag{B.17}$$

We notice that $Q\left[\mathcal{A}^+_{\Gamma_n,0}, V_0^- > V^+_{\Gamma_n}\right]$ does not depend on l; therefore, using (B.15)–(B.17) we get

$$\begin{aligned} &Q\left[\sum_{j=m_n+\tilde{c}_3(\beta)}^{M_n} \mathbb{I}_{S_j - S_{m_n} \in [a(i-1), ai[} = k\right] \\ &\quad \le \left(1 + \left(Q\left[V_0^- \ge V^+_{\Gamma_n}\right]\right)^{-1}\right) Q\left[\mathcal{A}^+_{\Gamma_n,0},\ V_0^- > V^+_{\Gamma_n}\right]. \end{aligned} \tag{B.18}$$

Using the Markov property we obtain that

$$Q\left[\mathcal{A}^+_{\Gamma_n,0},\ V_0^- > V^+_{\Gamma_n}\right] \le Q\left[V_0^- > \tilde{c}_3(\beta)\right] \max_{0 \le x \le \tilde{c}_3(\beta)/I_{\eta_0}} \left\{Q_x\left[\mathcal{A}^+_{\Gamma_n,0}, V_0^- > V^+_{\Gamma_n}\right]\right\}. \tag{B.19}$$

I_{η_0} is given just after (B.4). To get an upper bound for $Q_x[\mathcal{A}^+_{\Gamma_n,0}, V_0^- > V^+_{\Gamma_n}]$, we introduce the following sequence of stopping times, letting $k > 0$:

$$\begin{aligned} &H_{ia,0} = 0, \\ &H_{ia,k} = \inf\{m > H_{ia,k-1},\ S_m \in [(i-1)a, ia[\}. \end{aligned}$$

Making a partition over the values of $H_{ia,k}$ and using the Markov property we get

$$\begin{aligned} &Q_x\left[\mathcal{A}^+_{\Gamma_n,0}, V_0^- > V^+_{\Gamma_n}\right] \\ &\quad \le \sum_{w \ge 0} \int_{(i-1)a}^{ia} Q_x\left[H_{ia,k} = w, S_w \in \mathrm{d}y, \bigcap_{s=0}^{w}\{S_s > 0\}, \bigcap_{s=w+1}^{\inf\{l > w, S_l \ge \Gamma_n - x\}} \{S_s > 0\}\right] \\ &\quad \le Q_x\left[H_{ia,k} < V_0^-\right] \max_{(i-1)a \le y \le ia} \left\{Q_y\left[V^+_{\Gamma_n - y} < V_y^-\right]\right\} \\ &\quad \equiv Q_x\left[H_{ia,k} < V_0^-\right] Q_{ia}\left[V^+_{\Gamma_n - ia} < V^-_{ia}\right]. \end{aligned} \tag{B.20}$$

To finish we need an upper bound for $Q_x[H_{ia,k} < V_0^-]$; we do not want to give details of the computations for this because it is not difficult. However, the reader can find these details in Andreoletti [18] pp. 142–145. We have for all $i > 1$

$$\begin{aligned} Q_x\left[H_{ia,k} < V_0^-\right] &\le Q_x\left[V_0^- > V^+_{(i-1)a}\right] \\ &\quad \times \left(1 - Q\left[\epsilon_0 < -\frac{I_{\eta_0}}{2}\right] Q_{(i-1)a - \frac{I_{\eta_0}}{4}}\left[V^+_{(i-1)a} \ge V_0^-\right]\right)^{k-1} \\ &\le \left(1 - Q\left[\epsilon_0 < -\frac{I_{\eta_0}}{2}\right] Q_{(i-1)a - \frac{I_{\eta_0}}{4}}\left[V^+_{(i-1)a} \ge V_0^-\right]\right)^{k-1}, \end{aligned} \tag{B.21}$$

and in the same way

$$Q_x\left[H_{a,k} < V_0^-\right] \leq \left(1 - Q\left[\epsilon_0 < -\frac{I_{\eta_0}}{4}\right]\right)^{k-1}. \tag{B.22}$$

So using (B.18)–(B.22), Lemmas B.1 and B.2, one can find a constant $c > 0$ that depends only on the distribution Q such that for all $i \geq 0$

$$\mathbb{E}_Q\left[\sum_{j=m_n+\tilde{c}_3(\beta)}^{M_n} \mathbb{I}_{S_j - S_{m_n} \in [a(i-1), ai[}\right] \equiv \sum_{k=1}^{+\infty} k Q\left[\sum_{j=m_n+\tilde{c}_3(\beta)}^{M_n} \mathbb{I}_{S_j - S_{m_n} \in [a(i-1), ai[} = k\right] \leq \frac{c \times i^3}{\sqrt{\tilde{c}_3(\beta)}},$$

which provides (B.11). ■

Using both (B.10) and Lemma B.5 we get that there exists $c_0 \equiv c_0(Q)$ such that

$$\mathbb{E}_Q\left[\mathbb{E}_{m_n}^{\alpha}\left[\mathcal{L}(\tilde{\Theta}_\beta(m_n), T_{m_n})\right]\right] \leq \frac{c_0}{\sqrt{\tilde{c}_3(\beta)}}. \tag{B.23}$$

Now, using the Markov inequality, we get

$$Q\left[\mathbb{E}_{m_n}^{\alpha}\left[\mathcal{L}(\tilde{\Theta}_\beta(m_n), T_{m_n})\right] \leq \frac{4c_0}{\sqrt{\tilde{c}_3(\beta)}}\right] \equiv 1 - Q\left[\mathbb{E}_{m_n}^{\alpha}\left[\mathcal{L}(\tilde{\Theta}_\beta(m_n), T_{m_n})\right] > \frac{4c_0}{\sqrt{\tilde{c}_3(\beta)}}\right] \geq 3/4.$$

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