

The evolution of a spatial stochastic network

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Abstract

The asymptotic behavior of a stochastic network represented by a birth and death processes of particles on a compact state space is analyzed. In the births, particles are created at rate λ_+ and their location is independent of the current configuration. Deaths are due to negative particles arriving at rate λ_- . The death of a particle occurs when a negative particle arrives in its neighborhood and kills it. Several killing schemes are considered. The arriving locations of positive and negative particles are assumed to have the same distribution. By using a combination of monotonicity properties and invariance relations it is shown that the configurations of particles converge in distribution for several models. The problems of uniqueness of invariant measures and of the existence of accumulation points for the limiting configurations are also investigated. It is shown for several natural models that if $\lambda_+ < \lambda_-$ then the asymptotic configuration has a finite number of points with probability 1. Examples with $\lambda_+ < \lambda_-$ and an infinite number of particles in the limit are also presented.

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1. Introduction

In this paper one studies the asymptotic behavior of configurations of points in a compact state space H . Particles of two types, “+” and “−”, arrive on H according to some arrival process.

— A “+” particle stays at its arriving site x , adding therefore a new point at the location x to the current configuration.

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— A “−” particle arriving at x kills a point of the configuration if there is one in a specified neighborhood of x . In any case a “−” particle disappears.

These natural models of birth and death processes of particles occur in several domains.

(1) *Queueing systems.*

When there are only a finite number of possible locations x_1, \dots, x_p for the particles, this model is equivalent to p single-server queues where $+$ are jobs arriving in one of the queues of the system and the rate of arrival of $-$'s at location x_k , $1 \leq k \leq p$, is simply the rate of service of the k th queue.

(2) *Stochastic networks.*

In the context of a wireless network, the “+” particles represent the requests for transmission at some location. A “−” particle is the capacity of service available in the neighborhood of a point at some moment. The assumption that the $+$ closest to a $-$ is transmitted is an approximation of the fact that, for energy dissipation reasons, this $+$ transmits with the highest rate. See [7] for related mathematical models and [24].

(3) *Biological networks.*

Growth models of protein networks can be represented by points in a three-dimensional cube, the birth of points corresponding to the local extension of filaments. See [1,14].

(4) *Matching problems in theoretical computer science.*

These are for multi-dimensional on-line bin packing problems. In this setting, a $+$ is an item A which is alone in its bin; once there is an item whose size matches “well” with that of A , both of them are stored in the bin and A is therefore removed. See [3,20,9] for example.

(5) *Statistics.*

Ferrari et al. [6] presents a method of simulation of the invariant distribution of reversible processes on configurations of points in \mathbb{R}^d with constant birth rate and whose death rate for a particle is proportional to its multiplicity.

In the following H is a compact metric space and $\mathcal{M}(H)$ denotes the space of non-negative Radon measures on H carried by points. See [22,5] for the main definitions and results on $\mathcal{M}(H)$. When the state M_0 of the initial configuration has n points z_1, z_2, \dots, z_n , it will be described as an element of $\mathcal{M}(H)$, a point process on H , i.e. $M_0 = \delta_{z_1} + \delta_{z_2} + \dots + \delta_{z_n}$, where δ_x is the Dirac mass at $x \in H$.

Starting from M_0 , if the next particle arrives at the location X_1 and its class is given by $I_1 \in \{+, -\}$, then the next state of the configuration is M_1 defined by

$$M_1 = M_0 + \mathbb{1}_{\{I_1=+\}}\delta_{X_1} - \mathbb{1}_{\{I_1=-\}}\delta_{t_1(X_1, M_0)}, \quad (1)$$

where $t_1(X_1, M_0)$ is the (possible) location of the point of M_0 which is removed. It may happen that no point is removed; the Dirac mass at $t_1(X_1, M_0)$ is understood as the 0 measure on H in this case. Several definitions for $t_1(X_1, M_0)$ are now presented depending on the model considered. The function $(x, y) \mapsto d(x, y)$ denotes the distance of the metric space H .

The cases of local interaction

In this setting a “−” particle arriving at $x \in H$ can only kill a point located in a ball $B(x, 1)$ of radius 1 around x . If $B(x, 1)$ does not contain a point of the configuration, the “−” disappears.

(1) *Local Greedy policy (LG).* The point $t_1(x, M)$ is the point of M which is the closest to x and at distance less than 1: i.e. a $y \in H$ such that

$$d(x, y) = \inf\{d(x, z) : z \in H, M(\{z\}) \neq 0 \text{ and } d(x, z) < 1\}.$$

By convention, if there is not such a point the Dirac measure $\delta_{t_1(x, M)}$ is defined as the 0 measure. In the case where there are several points achieving the above infimum, $t_1(x, M)$ is chosen at random among the corresponding locations.

- (2) *Local Random policy (LR)*. The point $t_1(x, M)$ is chosen at random in the subset $\{z \in H : M(\{z\}) \neq 0 \text{ and } d(x, z) < 1\}$.
- (3) *Local One-sided policy (LO)*. It is assumed that H is a subset of \mathbb{R}^d for some $d \geq 1$. The point $t_1(x, M)$ is $y \in H$ such that

$$d(x, y) = \inf\{d(x, z) : z \in H, M(\{z\}) \neq 0, d(x, z) < 1 \text{ and } z \geq x\},$$

where the inequality $z \geq x$ is understood coordinate by coordinate. These policies occur in the context of matching problems. See [9].

Global interaction

In this case there is no constraint of locality to kill a point.

- (1) *Global Greedy policy (GG)*. The point $t_1(x, M)$ is $y \in H$ such that

$$d(x, y) = \inf\{d(x, z) : z \in H, M(\{z\}) \neq 0\}.$$

- (2) *Global One-sided policy (GO)*. The point $t_1(x, M)$ is $y \in H \subset \mathbb{R}^d$ such that

$$d(x, y) = \inf\{d(x, z) : z \in H, M(\{z\}) \neq 0 \text{ and } z \geq x\},$$

Related spatial processes

When the processes of arrival of positive and negative particles are independent Poisson processes, the system can also be described as a continuous time Markov process. The associated infinitesimal generator Ω can be expressed as follows: For a convenient functional F on the space $\mathcal{M}(H)$,

$$\begin{aligned} \Omega(F) = & \lambda_+ \int_H (F(\eta + \delta_x) - F(\eta)) \mu(dx) \\ & + \lambda_- \int_H (F(\eta - \delta_y) - F(\eta)) \delta(x, y, \eta) \eta(dy) \mu(dx), \end{aligned}$$

where μ is the distribution of X_1 . For example, for the *LG* policy, the death rate δ is defined as

$$\delta(x, y, \eta) = \mathbb{1}_{\{d(x, y) < 1 \wedge d(x, \eta - \delta_y)\}},$$

where $a \wedge b = \min(a, b)$ and, for $y \in H$ and $\mu \in \mathcal{M}(H)$,

$$d(y, \mu) \stackrel{\text{def.}}{=} d(y, \{z \in H : \mu(\{z\}) \neq 0\}).$$

This is the point of view of Garcia and Kurtz [8] where Markov processes with general birth rates and constant death rates are introduced for the evolution of point processes on a *non-compact* state space. See [19] for a survey. The main problem analyzed in this case is the construction of a Markov process with values in the space of point processes: due to the interaction and the non-compact state space, it is not possible to order the jump instants so that the existence result is not straightforward. Additionally Penrose [19] presents some limit results (the law of large numbers and the central limit theorem) but from a spatial point of view: the

asymptotic behavior *at some fixed time* of some additive functional of the point process in a ball whose diameter goes to infinity.

In this paper, one does not use explicitly the characterization of these processes by their infinitesimal generator. The state space being compact, the existence results are straightforward. Limit results are investigated not with respect to a spatial component but with respect to *long time behavior*: when does the configuration converge in law? Since the state space is not of finite dimension, the classical tools using a Markovian approach, like Lyapunov functions (see Chapters 8 and 9 of [21] for example), seem to be more difficult to use. The dynamic which is investigated in this paper is specific but as will be seen it already leads to some non-trivial problems: in some cases, at equilibrium, the associated stochastic process will not live in the space of Radon measures for example. See Section 6.

The stability property

The rates of arrival of “+” and “−” particles are denoted respectively by λ_+ and λ_- and the particles are represented by a sequence (I_n, X_n) where, for $n \geq 1$, $I_n \in \{+, -\}$ is the type of the n th particle and $X_n \in H$ is its location. When there are more “−” particles than “+” particles, i.e. $\lambda_+ < \lambda_-$, it is likely that the distribution of the configuration should converge to a random point process having a finite number of points with probability 1. This property will be referred to as the *stability property* of the system.

For the GG policy, this property holds: with some independence assumptions, the total number of points evolves as a reflected random walk on integers with the negative drift $\lambda_+ - \lambda_-$. In this case, the geometry plays a minor role in the dynamics.

As noted by Anantharam and Foss (see [7]), the situation is quite different in the case of local interaction. The stability property is quite challenging to prove, even for the one-dimensional circle $\mathbb{T}_1(T)$ of length $T > 0$ for example. Mathematically, the stability property is formally defined as follows: There exists a probability distribution Q on the set $\mathcal{M}(H)$ of finite Radon measures on H such that

- (a) *Invariance*: if $M_0 \stackrel{\text{dist.}}{=} Q$, then $M_1 \stackrel{\text{dist.}}{=} Q$.
- (b) *Convergence*: if $M_0 \in \mathcal{M}(H)$ and (M_n) is the sequence of consecutive configurations, then (M_n) converges in distribution to Q .

It is quite natural to consider a Markovian approach to investigate the stability property. The sequence (M_n) can be seen as a Markov chain on $\mathcal{M}(H)$. In fact, as will be seen, the space $\mathcal{M}(H)$ will prove to be too small for investigating these questions properly; a larger space of measures has to be defined. One may try to prove the Harris ergodicity property of (M_n) which would give directly the properties (a) and (b). See [18] for example. In our case, due to the dynamics of the process and the complexity of the space $\mathcal{M}(H)$, a Markovian approach does not seem to work for a general state space H . Furthermore, as will be shown, symmetry properties play an important role in these questions. It does not seem that they can be really taken into account with a Markovian approach to tackle the general case. By using an interesting but specific Lyapunov function, Leskelä and Unger [10] proposed recently an alternative proof of the stability property in the case of the one-dimensional torus.

For the existence of an invariant distribution, the approach used in this paper consists in replacing the problem of finding a distribution Q which is invariant by Eq. (1) by the problem of establishing the existence of a random variable M on $\mathcal{M}(H)$ such that

$$M \circ \theta = M + \mathbb{1}_{\{I_1=+\}} \delta_{X_1} - \mathbb{1}_{\{I_1=-\}} \delta_{I_1(X_1, M_0)}, \quad (2)$$

holds almost surely, where θ is a shift operator on a convenient probability space. This method goes back to Loynes [13] studying the stability of a reflected random walk associated with a stationary sequence.

Loynes' method (1962), which can be seen as a backward coupling, has been used to study stochastic recursions in \mathbb{R}_+^d associated with several queueing systems. See [17] and the references therein. Robert [20] used it to study a bin packing algorithm related to the GO policy defined above. Propp and Wilson (1996) used this method (under the name “coupling from the past”) in the context of the Ising model. See [11] for a discussion on backward couplings. As always with backward couplings, a monotonicity property is the main ingredient for proving the existence of a random measure M solution of Eq. (2). It turns out that the existence result holds in a quite general framework. See [2] for a general presentation of the analysis of stochastic recursions.

In a second step, *invariance relations and symmetry properties* provide key arguments for proving the main results of the paper, i.e. that such an M is unique and that the convergence property (b) holds. These invariance relations have an important impact as will be seen, since the solutions of Eq. (2) have a finite mass with probability 1 when they hold. On the other hand, we exhibit examples for which these symmetry relations fail and the solution M has an infinite mass with probability 1.

Outline of the paper

The paper is mainly devoted to policies with local interactions when the proportion p_+ of “+”, $p_+ = \lambda_+ / (\lambda_+ + \lambda_-)$, is less than 1/2. Section 2 introduces the main definitions and notation required, in particular, to deal with point processes which may have an infinite number of points. Theorem 1 of Section 3 shows that, under general conditions, there always exists a random point process with possible accumulation points such that the invariance relation (a) holds for all policies listed above. In this general setting, a stronger result is shown for the local random policy: the invariant point process has a finite number of points with probability 1, i.e. the stability property holds. Section 4 considers a homogeneous case when H is a compact metric group, like the d -dimensional torus or the d -dimensional sphere. It is proved that there exists a unique random point process M satisfying Relation (2) which has a finite mass with probability 1 and such that convergence property (b) holds. This proves in particular the stability property for homogeneous state spaces for all policies with local interaction. Section 5 studies a simple non-homogeneous setting $H = [0, T]$ for the LG policy. It is proved that the stability property also holds in this case. Section 6 considers one-sided policies on $H = [0, T_1] \times \dots \times [0, T_d]$; it is shown that Properties (a) and (b) also hold in this case but with a limiting point process having an infinite number of points with probability 1.

2. Evolution equations for point processes

The main notation and definitions concerning point processes are first introduced. See [22] for the general definitions and results on Radon measures, [5] for an introduction to random point processes and [16] on stationary point processes.

2.1. Point processes

It is assumed throughout the paper that H is a compact metric space (think of a bounded closed subset of \mathbb{R}^d for example). A point process M on H is a non-negative Borel measure on

H carried by points, i.e., such that for any Borel subset A of H one has $M(A) \in \mathbb{N} \cup \{+\infty\}$. Define $\mathcal{M}^*(H)$ as the set of all point processes. If $M \in \mathcal{M}^*(H)$, $S(M)$ denotes the set of its accumulation points,

$$S(M) \stackrel{\text{def.}}{=} \{y \in H : \forall \varepsilon > 0, M(B(y, \varepsilon)) = +\infty\}.$$

Note that $S(M)$ is in particular a closed set. The space $\mathcal{M}(H)$ is the subset of $\mathcal{M}^*(H)$ of Radon non-negative measures with finite mass, i.e., the set of elements $M \in \mathcal{M}^*(H)$ such that $M(H) < +\infty$. As will be seen, for some policies the state space $\mathcal{M}(H)$ is not always appropriate for studying the asymptotic behavior of configurations of points in H .

If $f : H \rightarrow \mathbb{R}$ is some Borel function,

$$\langle f, M \rangle \stackrel{\text{def.}}{=} \int_H f(x) M(dx),$$

in particular $\langle \mathbb{1}_A, M \rangle = M(A)$ if A is a Borel subset of A . A sequence of point processes (M_n) in $\mathcal{M}(H)$ will be said to converge to $M \in \mathcal{M}^*(H)$ if the sequence $\langle f, M_n \rangle$ converges in distribution to $\langle f, M \rangle$, for any continuous function f with compact support in $H - S(M)$.

The ordering of point processes is defined as follows.

Definition 1. If M and $P \in \mathcal{M}^*(H)$, one writes $M \ll P$ if the relation $M(A) \leq P(A)$ holds for any Borel subset A of H .

If $M \ll P$, the elements in the support of M are in the support of P .

Extension of the definition of the functional $t_1(\cdot, \cdot)$ on $H \times \mathcal{M}^(H)$*

For $x \in H$, the variable $t_1(x, M)$ has been defined in Section 1 for $M \in \mathcal{M}(H)$, i.e. when the point process has only a finite number of points. Since the space $\mathcal{M}(H)$ is not closed for the topology of weak convergence, one has to define it when there are accumulation points. Furthermore this will allow us:

- (1) To have a limiting evolution equation for the possible limiting points of the sequence (M_n) of the successive states of the configuration.
- (2) To properly define the problem of uniqueness of the invariant distribution for Eq. (1).

The definition of $t_1(\cdot, M)$ is extended to an arbitrary element M of $\mathcal{M}^*(H)$. For that, one denotes by ∂ a cemetery state for which δ_∂ is the null measure. The variable $t_1(x, M)$ is defined as ∂ when $M(B(x, 1)) = 0$ (no point to kill) and in any of the following situations.

— LG policy:

- (1) there exists an accumulation point $a \in S(M)$ such that $d(x, a) < 1$, $M(\{a\}) = 0$ and $M(B(x, \varepsilon)) = 0$ for all $\varepsilon \leq d(x, a)$;
- (2) there exists $0 < r < 1$ such that $M(B(x, \varepsilon)) = 0$ for all $\varepsilon \leq r$ and the set $\{y \in H : d(x, y) = r, M(\{y\}) \neq 0\}$ is infinite.

— LR policy: when $M(B(x, 1)) = +\infty$.

— LO policy: like LG policy, on replacing balls $B(x, \varepsilon)$, $x \in H$, $\varepsilon > 0$ by $B(x, \varepsilon) \cap \{y : y \geq x\}$.

This definition gives the following proposition.

Proposition 1. For the LG, LO, LR policies, if (M_n) is a non-decreasing sequence, for the order \ll , of point processes of $\mathcal{M}(H)$, such that $M_{n+1}(H) \leq M_n(H) + 1$, if $M \in \mathcal{M}^*(H)$ is its limit, then the convergence in distribution

$$\lim_{n \rightarrow +\infty} \delta_{t_1(x, M_n)} = \delta_{t_1(x, M)}$$

holds in $\mathcal{M}(H - S(M))$.

Proof. One considers only the LG policy; the arguments are similar for the other policies.

If $0 < M(B(x, \varepsilon)) < +\infty$ for some $\varepsilon > 0$, then the sequence $(t_1(x, M_n))$ is constant after some finite rank, so the convergence holds trivially. Different cases have to be considered.

— If there exists $\varepsilon_0 > 0$ such that $M(B(x, \varepsilon_0)) = 0$ and $M(B(x, \varepsilon)) = +\infty$ for any $\varepsilon > \varepsilon_0$: under this assumption, this implies that any accumulation point of the sequence $(t_1(x, M_n))$ is an accumulation point of M and consequently, in the space $\mathcal{M}(H - S(M))$,

$$\lim_{n \rightarrow +\infty} \delta_{t_1(x, M_n)} = 0 = \delta_\partial = \delta_{t_1(x, M)}.$$

— Similarly, if there exists $\varepsilon_0 > 0$ such that $M(B(x, \varepsilon)) = 0$ for any $\varepsilon < \varepsilon_0$ and $M(B(x, \varepsilon_0)) = +\infty$: this implies that after some finite rank, the sequence $(t_1(x, M_n))$ is in the set $\Delta = \{y : d(x, y) = \varepsilon_0\}$. Since the LG policy chooses the point at random on Δ , as n goes to infinity, the distribution of $(t_1(x, M_n))$ will be concentrated around the accumulation points of M in Δ , so the desired convergence will hold for the Dirac masses at the corresponding points. \square

2.2. The probabilistic model

It is assumed that the arrival times of $+$ [resp. $-$] constitute a stationary marked point process (s_n^+, X_n^+) [resp. (s_n^-, X_n^-)] on \mathbb{R} and that (X_n^+) and (X_n^-) are independent stationary sequences with the same distribution (the location of points at arrival does not depend on the type). The superposition of the two stationary point processes $(s_n^+, +, X_n^+)$ and $(s_n^-, -, X_n^-)$ yields a stationary point process (s_n, I_n, X_n) where $I_n \in \{+, -\}$ is the type of the n th particle. Note that I_n is independent of X_n . Under the Palm measure \mathbb{P} of this stationary point process the sequence $(s_{n+1} - s_n, I_n, X_n)$ is stationary, i.e. its distribution is invariant with respect to the shift θ of coordinates. See [16] or [21]. The two sequences (I_n) and (X_n) will be assumed to be independent. One writes $p_+ = \mathbb{P}(I_1 = +)$ and μ is the distribution of X_1 .

2.3. Evolution equations

The evolution of the configuration describing the system is represented as a stochastic process (N_n) with values in $\mathcal{M}(H)$. For $n \in \mathbb{N}$, N_n is the state of the configuration after the n th arrival. It is defined as follows; $N_0 \in \mathcal{M}(H)$ and the following recurrence holds, for $n \geq 1$:

$$N_n = N_{n-1} + \mathbb{1}_{\{I_n = +\}} \delta_{X_n} - \mathbb{1}_{\{I_n = -, N_{n-1}(B(X_n, 1)) \neq 0\}} \delta_{t_1(X_n, N_{n-1})}, \quad (3)$$

with, for $M \in \mathcal{M}(H)$ and $x \in H$ such that $M(B(x, 1)) \neq 0$, $t_1(x, M)$ the (possible) location of the point of M which is removed from $B(x, 1)$ when a $-$ particle arrives at x in the configuration M . See Section 1 for definitions for the LG, LR and LO policies.

2.4. Stationary evolution equations

For convenience, the framework of ergodic theory will be used; see [4] for an introduction. It can be assumed that all these random variables are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with an *automorphism*, i.e. an invertible transformation $\theta : \Omega \mapsto \Omega$ such that θ leaves the probability \mathbb{P} invariant, i.e. $\theta \circ \mathbb{P} = \mathbb{P}$. In this setting, the relation

$$(s_n - s_{n-1}, X_n, I_n, n \in \mathbb{Z})(\theta(\omega)) = (s_{n+1} - s_n, X_{n+1}, I_{n+1}, n \in \mathbb{Z})(\omega)$$

holds for any $\omega \in \Omega$. The map θ is the shift for these stationary sequences. In particular for $n \in \mathbb{Z}$, $Z_n = Z_1 \circ \theta^n$ for $Z \in \{X, I\}$, where θ^n is the n th iterate of the mapping θ . One denotes by \mathcal{F}_0 the σ -field generated by the random variables $I_1 \circ \theta^n, X_1 \circ \theta^n, n \leq -1$. It is assumed throughout the paper that the dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is *ergodic*: any event $A \in \mathcal{F}$ invariant by θ , that is $\theta(A) = A$, has either probability 0 or 1.

Additionally, a family $(U_F, F \text{ finite subset of } H)$ of independent random variables on finite sets is assumed to be defined to handle the case when the point to be removed has to be chosen at random among several points. If F is a finite set, U_F is a uniformly distributed random variable in F . The formal formulation is skipped.

In this setting, a fixed point equation for random point processes is introduced; a solution $N \in \mathcal{M}^*(H)$ is such that the relation

$$N \circ \theta = N + \mathbb{1}_{\{I_1=+\}} \delta_{X_1} - \mathbb{1}_{\{I_1=-, N(B(X_1,1)) \neq 0\}} \delta_{t_1(X_1, N)} \quad (4)$$

holds almost surely. The distribution of such an N provides an invariant distribution of the Markov chain (N_n) defined by Eq. (3). Eq. (4) is the analogue, for point processes, of the formulation used by Loynes [13] to analyze Lindley's equation

$$W_n = \max(W_{n-1} + Z_{n-1}, 0), \quad n \geq 1.$$

It is reduced in this case to the problem of the existence of a finite random variable W satisfying the relation

$$W \circ \theta = \max(W + Z_1, 0), \quad (5)$$

almost surely. See [21]. The representation in the framework of ergodic theory, i.e. with the shift θ , is due to Neveu [17]. This formulation goes back to the nice paper by Ryll-Nardzewski [23] for general stationary point processes.

The invariant distribution of the continuous time process

An invariant distribution Q on $\mathcal{M}^*(H)$ of the Markov chain (N_n) defined by Eq. (3) gives the equilibrium at the instants of arrival of particles. An invariant distribution \tilde{Q} on $\mathcal{M}^*(H)$ for the corresponding continuous time jump process (N_t) on $\mathcal{M}^*(H)$ can then be defined by

$$\int_{\mathcal{M}^*(H)} F(M) \tilde{Q}(dM) = (\lambda_+ + \lambda_-) \mathbb{E} \left(s_1 \int_{\mathcal{M}^*(H)} F(M) Q(dM) \right),$$

for any non-negative Borel function F on $\mathcal{M}^*(H)$. See [16].

3. The existence of an equilibrium

The following property is essential for having the existence of an equilibrium distribution for the evolution equations (3).

Lemma 1 (Monotonicity). *For the policies LG, LR and LO, if P_0 and $Q_0 \in \mathcal{M}(H)$ are such that $P_0 \ll Q_0$ then there exists a coupling between any two sequences (P_n) and (Q_n) satisfying the evolution equation (3) with the initial conditions $N_0 = P_0$ and $N_0 = Q_0$ respectively, such that the relation $P_n \ll Q_n$ holds for all $n \geq 0$.*

Proof. It is enough to prove the lemma for the first step.

If $I_1 = +$ then $P_1 = P_0 + \delta_{X_1}$ and $Q_1 = Q_0 + \delta_{X_1}$, so the relation $P_1 \ll Q_1$ holds.

Otherwise $I_1 = -$; since $P_0(B(X_1, 1)) \leq Q_0(B(X_1, 1))$, the only interesting case is when $Q_0(B(X_1, 1)) \neq 0$.

- $P_0(B(X_1, 1)) = 0$. A point of Q_0 not in the support of P_0 is suppressed so that $P_1 = P_0 \ll Q_1$.
- $P_0(B(X_1, 1)) \neq 0$. The three policies are treated separately.
 - LR policy. Let U a uniformly distributed random variable on the set of points of Q_0 within $B(X_1, 1)$. If $P_0(\{U\}) \neq 0$, the point U is removed both for P_0 and Q_0 . Otherwise, $P_0(\{U\}) = 0$, U is removed from Q_0 and a random point of P_0 within $B(X_1, 1)$ is removed. In any case, the relation $P_1 \ll Q_1$ holds.
 - LG policy. If the point of Q_0 with minimal distance from X_1 belongs also to P_0 then it is removed for both point processes. Otherwise another point of P_0 is removed, and hence $P_1 \ll Q_1$ holds. Note that if one has to choose at random among points at the same (minimal) distance from X_1 , one proceeds as for the LR policy.
 - LO policy. The same argument as for the LG policy.

The lemma is proved. \square

A solution to the stationary evolution equation

The asymptotic behavior of the sequence (N_n) defined by Eq. (3) is analyzed in the following.

Define the sequence (\bar{N}_n) on the probability space Ω by induction as follows: $\bar{N}_0 \equiv 0$ where, with a slight abuse of notation, 0 stands for the null point process and, for $n \geq 1$, for $\omega \in \Omega$, by using the fact that θ is an automorphism of the probability space, the point process $N_n(\omega)$ is defined by

$$\begin{aligned} \bar{N}_n(\omega) = & \bar{N}_{n-1}(\theta^{-1}(\omega)) + \mathbb{1}_{\{I_1(\theta^{-1}(\omega))=+1\}} \delta_{X_1(\theta^{-1}(\omega))} \\ & - \mathbb{1}_{\{I_1(\theta^{-1}(\omega))=-1, \bar{N}_{n-1}(B(X_1, 1))(\theta^{-1}(\omega)) \neq 0\}} \delta_{t_1(X_1, \bar{N}_{n-1})(\theta^{-1}(\omega))}. \end{aligned}$$

or in a more compact form,

$$\bar{N}_n \circ \theta = \bar{N}_{n-1} + \mathbb{1}_{\{I_1=+1\}} \delta_{X_1} - \mathbb{1}_{\{I_1=-1, \bar{N}_{n-1}(B(X_1, 1)) \neq 0\}} \delta_{t_1(X_1, \bar{N}_{n-1})}. \quad (6)$$

Lemma 2. *The sequence (\bar{N}_n) is \mathcal{F}_0 -measurable and $(\bar{N}_n \circ \theta^n) = (N_n)$, where (N_n) is the sequence defined by the recurrence (3) with $N_0 = 0$. In particular, for $n \geq 1$, the point processes N_n and \bar{N}_n have the same distribution.*

Proof. This is done easily by induction. By using the above relation

$$\begin{aligned} \bar{N}_n = & \bar{N}_{n-1} \circ \theta^{-1} + \mathbb{1}_{\{I_1 \circ \theta^{-1}=+1\}} \delta_{X_1 \circ \theta^{-1}} \\ & - \mathbb{1}_{\{I_1 \circ \theta^{-1}=-1, \bar{N}_{n-1}(B(X_1, 1)) \circ \theta^{-1} \neq 0\}} \delta_{t_1(X_1, \bar{N}_{n-1}) \circ \theta^{-1}}, \end{aligned}$$

one gets that \bar{N}_n is a functional of the random variables

$$(I_1 \circ \theta^{-k}, X_1 \circ \theta^{-k}), \quad 1 \leq k \leq n,$$

and therefore \mathcal{F}_0 -measurable. By using again the above relation and replacing $(I_1 \circ \theta^{n-1}, X_1 \circ \theta^{n-1})$ by (I_n, X_n) , this gives

$$\begin{aligned} \bar{N}_n \circ \theta^n &= \bar{N}_{n-1} \circ \theta^{n-1} + \mathbb{1}_{\{I_n=+\}} \delta_{X_n} \\ &\quad - \mathbb{1}_{\{I_n=-, \bar{N}_{n-1} \circ \theta^{n-1}(B(X_n, 1)) \neq 0\}} \delta_{t_1(X_n, \bar{N}_{n-1} \circ \theta^{n-1})}. \end{aligned}$$

Hence the sequence $(\bar{N}_n \circ \theta^n)$ satisfies the same recursion (3) with the zero measure as the initial state. It has therefore the same distribution as (N_n) with $N_0 = 0$. The lemma is proved. \square

Theorem 1 (Existence of a Unique Minimal Equilibrium). *If $p_+ < 1/2$, for the LG and LO policies, there exists a unique random variable N such that the relation*

$$N \circ \theta = N + \mathbb{1}_{\{I_1=+\}} \delta_{X_1} - \mathbb{1}_{\{I_1=-, N(B(X_1, 1)) \neq 0\}} \delta_{t_1(X_1, N)}, \quad (7)$$

holds almost surely in the space $\mathcal{M}^(H)$ and which is minimal for the order \ll : if M is a random point process satisfying Relation (7), then $N \ll M$ holds almost surely. Such a random variable is \mathcal{F}_0 -measurable.*

Let $S(N)$ be the (possibly empty) set of accumulation points of N . It is important to note that Relation (7) is valid as an identity in the set $\mathcal{M}(H - S(N))$ of Radon measures on $H - S(N)$. As will be seen, $S(N)$ is in fact a deterministic set.

Proof. As before (\bar{N}_n) denotes the sequence defined by Eq. (6). The proof is done for the LG policy. The arguments for the LO policy work in much the same way by replacing the open balls $B(x, 1)$, $x \in H$, by $B(x, 1) \cap \{y \geq x\}$.

Convergence of the sequence (\bar{N}_n) .

Since clearly $\bar{N}_0 \ll \bar{N}_1$, the above lemma shows that $\bar{N}_1 \ll \bar{N}_2$ and by induction $\bar{N}_p \ll \bar{N}_{p+1}$ for any $p_+ \geq 1$. Consequently, there exists a non-negative random measure N such that for any Borel subset A of H ,

$$N(A) = \lim_{p \rightarrow +\infty} \uparrow \bar{N}_p(A)$$

holds almost surely. The random variable N is \mathcal{F}_0 -measurable as an almost sure limit of the \mathcal{F}_0 -measurable sequence (\bar{N}_n) . Relation (6) gives that for any $n \geq 1$,

$$|\bar{N}_n(A) \circ \theta - \bar{N}_{n-1}(A)| \leq 1$$

and consequently, the set $\{N(A) = +\infty\}$ is invariant by θ and, hence, of probability 1 or 0 by the ergodicity property. This argument is used repeatedly in the following.

N is a solution of Eq. (7).

One checks the equation when $X_1 = x \in H$; this is a direct consequence of Proposition 1.

If M is a point process satisfying Relation (7) then clearly $0 \ll M$, and therefore $\bar{N}_1 \circ \theta \ll M \circ \theta$ by Eqs. (7) and (6). By induction one gets that, for any $n \geq 1$, $\bar{N}_n \ll M$ and consequently $N \ll M$. The variable N is minimal for the order \ll .

N is in $\mathcal{M}^(H)$ with probability 1.*

It remains to prove that the set $S(N)$ of accumulation points of N has almost surely an empty interior. The ergodicity property shows that $S(N)$ is a *deterministic* subset of H . If $S(N)$ does not have an empty interior, there is some $x \in H$ and $\varepsilon > 0$ such that $B(x, \varepsilon) \subset S(N)$.

Eq. (6) gives the relation

$$\begin{aligned} & \bar{N}_n \circ \theta(B(x, \varepsilon)) - \bar{N}_{n-1}(B(x, \varepsilon)) \\ &= p\mathbb{P}(X_1 \in B(x, \varepsilon)) - (1-p)\mathbb{P}(\bar{N}_{n-1}(B(X_1, 1)) \neq 0, t_1(X_1, \bar{N}_{n-1}) \in B(x, \varepsilon)). \end{aligned}$$

By integrating the above relation (note that $N_n(H)$ is bounded by n), by using the invariance of θ with respect to \mathbb{P} and the monotonicity of the sequence $(\bar{N}_{n-1}(B(x, \varepsilon)))$, one gets the inequality

$$\mathbb{P}(\bar{N}_{n-1}(B(X_1, 1)) \neq 0, t_1(X_1, \bar{N}_{n-1}) \in B(x, \varepsilon)) \leq \frac{p}{1-p}\mathbb{P}(X_1 \in B(x, \varepsilon))$$

and hence

$$\begin{aligned} & \mathbb{P}(\bar{N}_{n-1}(B(X_1, 1)) \neq 0, X_1 \in B(x, \varepsilon), t_1(X_1, \bar{N}_{n-1}) \in B(x, \varepsilon)) \\ & \leq \frac{p}{1-p}\mathbb{P}(X_1 \in B(x, \varepsilon)). \end{aligned}$$

By assumption, the non-decreasing sequence of sets

$$O_n = \{\bar{N}_n(B(X_1, 1)) \neq 0, X_1 \in B(x, \varepsilon), t_1(X_1, \bar{N}_n) \in B(x, \varepsilon)\}$$

is converging to $\cup_n O_n = \{X_1 \in B(x, \varepsilon)\}$. By letting n go to infinity in the above inequality, this gives the relation

$$\mathbb{P}(X_1 \in B(x, \varepsilon)) \leq \frac{p_+}{1-p_+}\mathbb{P}(X_1 \in B(x, \varepsilon)).$$

Since $\mathbb{P}(X_1 \in B(x, \varepsilon))$ is non-zero (otherwise one could not have accumulation points in $B(x, \varepsilon)$), this yields $p_+ \geq 1/2$. This is a contradiction. The set $S(N)$ has therefore an empty interior. The theorem is proved. \square

The next result shows that a much stronger statement holds for the local random policy: the corresponding minimal variable N has almost surely a finite mass.

Theorem 2 (Stability of Local Random Policy). *If $p_+ < 1/2$, for the local random policy, there exists a unique minimal random variable N satisfying Relation (7). The point process N has a finite mass with probability 1, $\mathbb{P}(N \in \mathcal{M}(H)) = 1$.*

Proof. One has first to check that the limit N of the sequence (\bar{N}_n) is a solution of Eq. (7).

- If $0 < N(B(x, 1)) < +\infty$, the sequence $(t_1(x, \bar{N}_n))$ is constant after some finite rank and so Eq. (7) holds.
- Otherwise, if there exists some $0 < \varepsilon_0 < 1$ such that if $\varepsilon < \varepsilon_0$ then $N(B(x, \varepsilon)) = 0$ and if $\varepsilon > \varepsilon_0$ then $N(B(x, \varepsilon)) = +\infty$. Let S_1 be the accumulation points of N in $B(x, 1)$. Because of the random choice in $B(x, 1)$, the limit points of the sequence $(t_1(x, \bar{N}_n))$ are therefore all necessarily on $S_1 \subset S(N)$. The sequence of Dirac measures $\delta_{t_1(x, \bar{N}_n)}$ converges to 0 in the set $\mathcal{M}(H - S(N))$.

Assume that the set $S(N)$ of accumulation points of N is not empty. It is known that it is deterministic; denote by

$$S^*(N) = \{y \in H : d(y, S(N)) < 1\},$$

the set of points at distance less than 1 from $S(N)$. Eq. (6) gives the relation

$$\begin{aligned} & \mathbb{E}[\bar{N}_{n+1}(S^*(N))] - \mathbb{E}[\bar{N}_n(S^*(N))] \\ &= p\mathbb{P}(X_1 \in S^*(N)) - (1-p)\mathbb{P}(\bar{N}_n(B(X_1, 1)) \neq 0, t_1(X_1, \bar{N}_n) \in S^*(N)) \end{aligned}$$

and therefore, by monotonicity, the inequality

$$\begin{aligned} \frac{p}{1-p} \mathbb{P}[X_1 \in S^*(N)] &\geq \mathbb{P}[\overline{N}_n(B(X_1, 1)) \neq 0, t_1(X_1, \overline{N}_n) \in S^*(N)] \\ &\geq \mathbb{P}[\overline{N}_n(B(X_1, 1)) \neq 0, X_1 \in S^*(N), t_1(X_1, \overline{N}_n) \in S^*(N)]. \end{aligned}$$

By definition of $S(N)$, the set $\{y \in H : N(\{y\}) \neq 0, d(y, S(N)) \geq 1\}$ is almost surely finite. If $x \in S^*(N)$, then almost surely $N(B(x, 1)) = +\infty$, and so, because of the random choice among the points of $\overline{N}_n(B(x, 1))$,

$$\lim_{n \rightarrow +\infty} \mathbb{P}[\overline{N}_n(B(x, 1)) \neq 0, t_1(x, \overline{N}_n) \in S^*(N)] = 1.$$

By using this relation in the above inequality, this gives

$$\frac{p_+}{1-p_+} \mathbb{P}[X_1 \in S^*(N)] \geq \mathbb{P}[X_1 \in S^*(N)],$$

and consequently $\mathbb{P}(X_1 \in S^*(N)) = 0$. If $a \in S^*(N)$, because of the dynamics of the process, there exists some $\varepsilon > 0$ such that $\mathbb{P}(X_1 \in B(a, \varepsilon)) > 0$. This is a contradiction. The set $S^*(N)$ is therefore empty. The theorem is proved. \square

Although the existence result is important in its own right, it is only a first step for studying the stability properties of these systems. Uniqueness and convergence results turn out to be much more challenging in general when studying stochastic recursions of the type (3). This is the main subject of the following sections.

4. Homogeneous state spaces

To stress the fact that only simple invariance relations are used, it is assumed in this section that H is a compact metrizable group. More specifically, for our study, the following properties of the state space are used. For the group operation, a multiplicative notation is used.

- (i) If $x, y \in H$ and $r > 0$ then $yB(x, r) = B(yx, r)$.
- (ii) There exists a unique Borel measure μ on H , the Haar probability measure, invariant by group operations $\tau_x : y \mapsto yx$ for $x \in H$.

See Loomis [12] or Weil [25] for an introduction. Simple examples of such a situation are:

- (1) For $d \geq 1$, the d -dimensional torus

$$\mathbb{T}_d(T) = \prod_{i=1}^d \mathbb{R}/T_i\mathbb{Z},$$

for $T = (T_i) \in \mathbb{R}_+^d$.

- (2) For $d \geq 1$, the d -dimensional sphere $\mathbb{S}_d(T)$,

$$\mathbb{S}_d(T) = \left\{ x = (x_i) \in \mathbb{R}^d : x_1^2 + \cdots + x_{d+1}^2 = T^2 \right\}.$$

In both cases, the normalized Lebesgue measure on H is the Haar probability distribution. Various compact groups of matrices provide additional examples of such situations. Note that a related setting was used by Mecke [15] to derive a key relation between the distribution and the Palm measure of a given stationary point process.

In the proofs, the local greedy policy is assumed. It is not difficult to see that similar arguments can be used for the local one-sided policy. Recall that a strong result, [Theorem 2](#), has already been

proved for the local random policy. Throughout the section the distribution of the locations of points is assumed to be μ which will be referred to as the *uniform measure* in the following. From now on and for the rest of the paper, (I_n) and (X_n) are assumed to be independent i.i.d. sequences of random variables.

Lemma 3. *The minimal solution N of Eq. (7) is a stationary point process on H : its distribution is invariant with respect to group operations, i.e.*

$$\int_H f(xy) N(dy) \stackrel{\text{dist.}}{=} \int_H f(y) N(dy),$$

for any $x \in H$ and any non-negative Borel function f on H .

Proof. By invariance of μ by translation, property (ii), the random variables X_1 and xX_1 have the same distribution. If one denotes as $\tau_x M$ the point process M shifted by x , i.e. $\tau_x M(\{y\}) = M(\{x^{-1}y\})$ for all $y \in H$, then property (i) implies that the sequence $(\tau_x \bar{N}_n)$ satisfies Relation (6) with the variable X_1 replaced by xX_1 . One concludes that the two sequences $(\tau_x \bar{N}_n)$ and (\bar{N}_n) have the same distribution and therefore that the same property holds for their limits. The lemma is proved. \square

Theorem 3. *For the LG and LO policies, if $p_+ < 1/2$ and the random variables (X_i) are i.i.d. with distribution μ on H , then, almost surely, there exists a unique point process N satisfying Relation (7) with finite mass, i.e. $\mathbb{P}(N \in \mathcal{M}(H)) = 1$.*

Proof. By recurrence relation (6), one gets that for any Borel subset of H ,

$$\begin{aligned} \mathbb{E}[\bar{N}_n(A)] - \mathbb{E}[\bar{N}_{n-1}(A)] &= p_+ \mathbb{P}[X_1 \in A] \\ &- (1 - p_+) \int \mathbb{P}[\bar{N}_{n-1}(B(x, 1)) \neq 0, t_1(x, \bar{N}_{n-1}) \in A] \mu(dx), \end{aligned}$$

where μ is the distribution of X_1 . This identity with $A = H$ and the monotonicity property give in particular that

$$(1 - p_+) \int \mathbb{P}(N(B(x, 1)) \neq 0) \mu(dx) \leq p_+.$$

Since the distribution of $N(B(x, 1))$ is, by the above lemma, independent of $x \in H$, one has

$$\mathbb{P}(N(B(x, 1)) \neq 0) \leq \frac{p_+}{1 - p_+} < 1.$$

The random variable $N(B(x, 1))$ has therefore a positive probability of being 0 and, in particular, of being finite. The ergodicity property implies that for all $x \in H$, $N(B(x, 1)) < +\infty$ almost surely. The point process N has almost surely a finite mass, $\mathbb{P}(N \in \mathcal{M}^*(H)) = 1$, since there is no accumulation point.

Let M be a point process satisfying Relation (7) and $\mathbb{P}(M \in \mathcal{M}(H)) = 1$. Eq. (7) gives that

$$M(H) \circ \theta - M(H) = \mathbb{1}_{\{I_1=+\}} - \mathbb{1}_{\{I_1=-, N(B(X_1, 1)) \neq 0\}},$$

and since the right hand side is clearly integrable, the expected value of the left hand is 0. See Lemma 12.2 of [21] for example. One gets the relation

$$p_+ = (1 - p_+) \mathbb{P}(M(B(X_1, 1)) \neq 0),$$

and one obtains the relation

$$\mathbb{P}(M(B(X_1, 1)) = 0) = \frac{1 - 2p_+}{1 - p_+} > 0. \quad (8)$$

The minimality property of N (cf. Theorem 1) gives that almost surely $N \ll M$ and so

$$\{M(B(X_1, 1)) = 0\} \subset \{N(B(X_1, 1)) = 0\}.$$

These two subsets having the same probability by Eq. (8). Relation (7) gives therefore that, almost surely,

$$\begin{aligned} (M(H) - N(H)) \circ \theta &= M(H) - N(H) + \mathbb{1}_{\{I_1 = -, N(B(X_1, 1)) \neq 0\}} - \mathbb{1}_{\{I_1 = -, M(B(X_1, 1)) \neq 0\}} \\ &= M(H) - N(H), \quad \text{a.s.} \end{aligned}$$

The non-negative random variable $M(H) - N(H)$ is invariant by θ and therefore is almost surely a constant C by the ergodicity property. Since $M(H) < +\infty$ almost surely, there exists some $m \geq 1$ such that $\mathbb{P}(M(H) = m) > 0$ and some finite subset $\{x_1, \dots, x_n\}$ of H such that

$$H = \bigcup_{\ell=1}^n B(x_\ell, 1).$$

On the event $\{M(H) = m\}$, it is easily checked that if, for $\ell = 1, \dots, n$, a total of $2m$ “−” points are sent in each ball $B(x_\ell, 1)$ and + does not occur, then all the m initial points will be removed. More precisely,

$$\{M(H) = m\} \bigcap_{\ell=1}^n \bigcap_{k=2m(\ell-1)}^{2m\ell-1} \{I_1 \circ \theta^k = -, X_1 \circ \theta^k \in B(x_\ell, 1)\} \subset \{M(H) \circ \theta^{2mn} = 0\}.$$

Since the variable is \mathcal{F}_0 -measurable, it is independent of the i.i.d. sequence $((I_1, X_1) \circ \theta^i, i \geq 0)$; one gets therefore that

$$0 < \mathbb{P}(M(H) \circ \theta^{2mn} = 0) = \mathbb{P}(M(H) = 0) = \mathbb{P}(M(H) = 0, N(H) = 0),$$

and one deduces that the constant $C = M(H) - N(H)$ is 0. The two point processes M and N coincide. The theorem is proved. \square

Proposition 2 (Convergence of Distributions of Configurations). *For the LG, LR, LO policies, if $p_+ < 1/2$ and P is some finite point process on H and (M_n) is the sequence of point processes defined by, $M_0 = P$ and*

$$M_n = M_{n-1} + \mathbb{1}_{\{I_n = +\}} \delta_{X_n} - \mathbb{1}_{\{I_n = -, M_{n-1}(B(X_n, 1)) \neq 0\}} \delta_{t_1(X_n, M_{n-1})}, \quad n \geq 1,$$

then (M_n) converges in distribution to N , the unique solution of Eq. (7).

Proof. Recall that the sequence (N_n) defined by Eq. (3) corresponds to the case where the initial state is empty. Let (\bar{M}_n) be the sequence of point processes satisfying Relation (6) with $\bar{M}_0 = P$. The sequence (\bar{N}_n) defined by Relation (6) is such that \bar{N}_0 is the empty state. By induction, it is easy to check that, for $n \geq 0$, $\bar{M}_n = M_n \circ \theta^{-n}$ and $\bar{N}_n = N_n \circ \theta^{-n}$, and therefore that \bar{M}_n has the same distribution as M_n .

The monotonicity property gives that $\bar{N}_n \ll \bar{M}_n$ holds and that if $m = P(H)$ is the number of initial points of P then necessarily

$$0 \leq \bar{M}_n(H) - \bar{N}_n(H) \leq m.$$

The limit N of (\bar{N}_n) having a positive probability of being 0, there are almost surely an infinite number of $\ell \geq 0$ such that $N(H) \circ \theta^\ell = 0$. The relation $\bar{N}_n(H) \leq N(H)$ implies that there are an infinite number of $\ell \geq 0$ such that

$$N_\ell(H) = \bar{N}_\ell(H) \circ \theta^\ell = 0.$$

For these indexes ℓ , $M_\ell(H) \leq m$ and, as in the proof of the above theorem, there is a positive probability (lower bounded by a quantity independent of the location of the points of M_ℓ) that all the $M_\ell(H)$ points are removed before a new “+” arrives. Hence, with probability 1, there exists some (random) index ℓ such that $M_n = N_n$. The convergence in distribution of (M_n) is therefore proved. \square

Corollary 1. *The distribution of N the solution of Eq. (7) is the only distribution on $\mathcal{M}(H)$ invariant under the equation*

$$M_1 = M_0 + \mathbb{1}_{\{I_1=+\}}\delta_{X_1} - \mathbb{1}_{\{I_1=-\}}\delta_{t_1(X_1, M_0)}.$$

The uniqueness result when $H = \mathbb{T}_1(T)$

This section is concluded with a uniqueness result for the one-dimensional torus. One denotes by $\mathcal{M}_\mu^*(\mathbb{T}_1(T))$ the subset of elements P of $\mathcal{M}^*(\mathbb{T}_1(T))$ with a set $S(P)$ of accumulation points negligible for μ , the Lebesgue measure. The following proposition generalizes the uniqueness result of Theorem 3 for the solution of Eq. (7).

Proposition 3 (A Uniqueness Property for the Torus). *For the LG policy, if M is a random variable in $\mathcal{M}_\mu^*(\mathbb{T}_1(T))$, the solution of Eq. (7), \mathcal{F}_0 -measurable and such that, for any accumulation point $a \in S(M)$,*

$$M((a, a + \varepsilon]) = +\infty \text{ and } M([a - \varepsilon, a)) = +\infty$$

holds almost surely for any $\varepsilon > 0$ sufficiently small, then M is the minimal solution N of Eq. (7). In particular the set $S(M)$ of accumulation points is empty

Proof. Assume that such a variable M exists. It is known that $S(M)$ is a deterministic set and $H - S(M)$ being an open set, it can be written as

$$H - S(M) = \bigcup_{n \geq 1} (a_n, b_n),$$

where (a_n) and (b_n) are sequences of elements of $S(M)$. Note that, because of the assumption on M near accumulation points, the variable $t_1(x, M)$ is well defined (i.e. not equal to ∂) for all $x \in (a_n, b_n)$, $n \geq 1$, as long as $M([x - 1, x + 1]) \neq 0$. The minimality of N implies that $N \ll M$. From Eq. (7), for $n \geq 1$ and ε sufficiently small,

$$\begin{aligned} & M([a_n + \varepsilon, b_n - \varepsilon]) \circ \theta - M([a_n + \varepsilon, b_n - \varepsilon]) \\ &= \mathbb{1}_{\{I_1=+, X_1 \in [a_n + \varepsilon, b_n - \varepsilon]\}} - \mathbb{1}_{\{I_1=-, M(B(X_1, 1)) \neq 0, t_1(X_1, M) \in [a_n + \varepsilon, b_n - \varepsilon]\}}. \end{aligned}$$

With the same argument as in the previous proof, one gets that the expected value of the left hand side of the above identity is 0 and consequently that

$$\begin{aligned} & \frac{p_+}{1 - p_+} \mathbb{P}(X_1 \in [a_n + \varepsilon, b_n - \varepsilon]) \\ &= \mathbb{P}(M([X_1 - 1, X_1 + 1]) \neq 0, t_1(X_1, N) \in [a_n + \varepsilon, b_n - \varepsilon]) \\ &= \mathbb{P}(M([X_1 - 1, X_1 + 1]) \neq 0, X_1 \in (a_n, b_n), t_1(X_1, M) \in [a_n + \varepsilon, b_n - \varepsilon]), \end{aligned}$$

due to the assumption on accumulation points. By letting ε go to 0 one gets the relation

$$\frac{p_+}{1 - p_+} \mathbb{P}(X_1 \in [a_n, b_n)) = \mathbb{P}(M([X_1 - 1, X_1 + 1]) \neq 0, X_1 \in (a_n, b_n)),$$

and by summing up these terms with respect to n and taking into account the fact that $\mu(S(M)) = 0$ one finally obtains the identity

$$\mathbb{P}(M([X_1 - 1, X_1 + 1]) = 0) = \frac{1 - 2p_+}{1 - p_+},$$

and the same equality also holds for N , but since $N \ll M$, this implies that, for almost every $x \in [0, T]$,

$$\begin{aligned} \mathbb{P}(M([x - 1, x + 1]) = 0) \\ = \mathbb{P}(N([x - 1, x + 1]) = 0) = \mathbb{P}(N([-1, 1]) = 0) = \frac{1 - 2p_+}{1 - p_+} > 0. \end{aligned}$$

This is in contradiction with the fact that M has accumulation points at some fixed points. The proposition is proved. \square

5. The case of the interval $[0, T]$

In this section one considers a simple space, the interval $[0, T]$, for which boundary effects occur, unlike in Section 4 where the homogeneity property rules out this feature. The LG is analyzed in this case and stability results are proved. It should be noted that the LO policy (Local One-sided), has a completely different qualitative behavior; it is analyzed in Section 6 in a more general setting. For the LR policy, Theorem 2 addresses this case.

The value of T is assumed to be greater than 1; otherwise the stability problem is trivial. The location of the points is an i.i.d. sequence (X_i) of uniform random variables on $[0, T]$. The variable N is the minimal solution of Eq. (7). As before, the ergodicity property and Theorem 1 give that the set $S(N)$ of accumulation points of N is deterministic.

Properties of possible accumulation points of N are now analyzed in four steps. The set $S(N)$ is assumed to be non-empty.

(a) *Accumulation points are at distance at least 1.*

Assume that there are at least two elements $a < b$ in $S(N)$ such that $b - a < 1$ and $(a, b) \subset H - S(N)$. Take some ε sufficiently small; Eq. (6) for the sequence (\bar{N}_n) gives the relation

$$\begin{aligned} \mathbb{E}(\bar{N}_{n+1}([a - \varepsilon, b + \varepsilon]) \circ \theta) - \mathbb{E}(\bar{N}_n([a - \varepsilon, b + \varepsilon])) \\ = p_+(b - a + 2\varepsilon) - \mathbb{P}(\bar{N}_n(B(X_1, 1)) \neq 0, t(X_1, \bar{N}_n) \in [a - \varepsilon, b + \varepsilon]), \end{aligned}$$

and by the monotonicity property of (\bar{N}_n) ,

$$\begin{aligned} \mathbb{P}(\bar{N}_n(B(X_1, 1)) \neq 0, X_1 \in [a + \varepsilon, b - \varepsilon], t(X_1, \bar{N}_n) \in [a - \varepsilon, b + \varepsilon]) \\ \leq \frac{p_+}{1 - p_+}(b - a + 2\varepsilon). \end{aligned}$$

Since a and b are almost surely accumulation points, the non-decreasing sequence of sets

$$\{\bar{N}_n(B(X_1, 1)) \neq 0, X_1 \in [a + \varepsilon, b - \varepsilon], t(X_1, \bar{N}_n) \in [a - \varepsilon, b + \varepsilon]\}$$

converges, as n goes to infinity, to the set $\{a + \varepsilon \leq X_1 \leq b - \varepsilon\}$. By taking the limit in the last inequality, one gets the relation

$$b - a - 2\varepsilon \leq \frac{p_+}{1 - p_+}(b - a + 2\varepsilon),$$

and by letting ε go to 0, this gives $p_+ \geq 1/2$. This is a contradiction. Consequently, if there are accumulation points for N , they are isolated points of $[0, T]$ at distance 1 at least.

(b) *Coupling.*

One denotes temporarily by N_T the point process, the solution of Eq. (7), associated with uniform random variables on $[0, T]$. Let $S < T$ and (\bar{N}_n^S) be the sequence defined by

$$\bar{N}_{n+1}^S \circ \theta = \bar{N}_n^S + \mathbb{1}_{\{I_1=+1, X_1 \leq S\}} \delta_{X_1} - \mathbb{1}_{\{I_1=-1, \bar{N}_n(B(X_1, 1)) \neq 0, X_1 \leq S\}} \delta_{t_1(X_1, \bar{N}_n^S)}.$$

Then, almost surely,

$$\lim_{n \rightarrow +\infty} \bar{N}_n^S \stackrel{\text{def.}}{=} M_S \stackrel{\text{dist.}}{=} N_S.$$

By induction, it is easily checked that $\bar{N}_n([0, S] \cap \cdot) \ll \bar{N}_n^S$ holds for all $n \geq 1$. The inequality \ll instead of equality comes from the fact that for $\bar{N}_n([0, S] \cap \cdot)$ a minus arriving in $[S, T]$ can kill a point in $[0, S]$. By letting n go to infinity, one obtains the relation

$$N([0, S] \cap \cdot) \ll M_S. \quad (9)$$

(c) *Patterns of accumulation points.*

Denote by $S_+(N)$ the subset of elements of $S(N)$ which have an infinite number of points of N on their right,

$$S_+(N) = \{x \in [0, T] : \exists \varepsilon_0 > 0, \forall \varepsilon \leq \varepsilon_0, N((x, x + \varepsilon)) = +\infty \text{ a.s.}\}.$$

Similarly $S_-(N)$ is defined for left neighborhoods.

Claim: There do not exist $a \in S_+(N)$ and $b \in S_-(N)$ such that $(a, b) \subset [0, T] - S(N)$. Assume there are such a and b . Eq. (7) for N gives the relation

$$\begin{aligned} N \circ \theta - N \\ = \mathbb{1}_{\{I_1=+, a < X_1 < b\}} \delta_{X_1} - \mathbb{1}_{\{I_1=-, N([X_1-1, X_1+1]) \neq 0, a < t_1(X_1, N) < b\}} \delta_{t_1(X_1, N)}, \end{aligned}$$

valid in the space $\mathcal{M}([a, b])$. Because of the assumption on a and b and of the definition of $t_1(\cdot, N)$, one has the identity

$$\begin{aligned} \{N([X_1 - 1, X_1 + 1]) \neq 0, a < t_1(X_1, N) < b\} \\ = \{N([X_1 - 1, X_1 + 1]) \neq 0, a < X_1 < b\} \end{aligned}$$

which gives the relation

$$\begin{aligned} N \circ \theta - N \\ = \mathbb{1}_{\{I_1=+, a < X_1 < b\}} \delta_{X_1} - \mathbb{1}_{\{I_1=-, N([X_1-1, X_1+1]) \neq 0, a < X_1 < b\}} \delta_{t_1(X_1, N)}. \end{aligned} \quad (10)$$

If a and b are identified, the above equality states that the point process N restricted to the torus $\mathbb{T}_1(b - a) = [a, b]$ satisfies Relation (7) for this state space. Proposition 3 shows that N on $[a, b]$ is a point process with finite mass. This is a contradiction.

(d) *Conclusion.*

Recall that N is the limit of the (\bar{N}_n) when the sequence (X_i) is i.i.d. uniformly distributed on $[0, T]$. Since (X_i) has the same distribution as $(T - X_i)$, one deduces that the distribution of N is invariant with respect to the mapping $x \mapsto T - x$. Accumulation points of N being at distance 1 at least by (a), $S(N)$ is a finite set, $S(N) = \{a_1, \dots, a_p\}$, for some $p_+ \geq 1$ and $0 \leq a_1 < a_2 < \dots < a_p \leq T$.

Assume that $a_1 \in S_+(N)$ holds. By the symmetry of N with respect to the mapping $x \rightarrow T - x$, one gets that $a_p = T - a_1 \in S_-(N)$. By (c) one has necessarily $a_2 \in S_+(N)$, and therefore $a_{p-1} \in S_-(N)$. By proceeding inductively, one deduces that there exists a $k < p$ such that $a_k \in S_+(N)$ and $a_{k+1} \in S_-(N)$. This is impossible according to (c).

Consequently, $a_1 \in S_-(N)$ and $a_1 > 0$. By the coupling result (9) above, with the same notation as in (b), one has

$$N_T([0, a_1] \cap \cdot) \ll N_{a_1}.$$

In particular a_1 is an accumulation point of N_{a_1} , and by the symmetry of N_{a_1} with respect to the mapping $x \rightarrow a_1 - x$, one gets that 0 is also an accumulation point of N_{a_1} . Consequently, $S(N_{a_1}) = \{b_1, \dots, b_q\}$, with $b_1 \in S_+(N_{a_1})$, which is impossible by what we have just proved. The set (N) is therefore empty.

The uniqueness statement of the following proposition has therefore been proved.

Proposition 4 (Stability Property for the LG Policy). *If $p_+ < 1/2$ and the random variable X_1 is uniformly distributed on $[0, T]$, then Eq. (7) has a unique minimal solution N such that $\mathbb{P}(N \in \mathcal{M}(H)) = 1$.*

If N_0 is an element of $\mathcal{M}([0, T])$ with finite mass, then the sequence (N_n) defined by Recursion (3) converges in distribution to N .

Proof. The proof of the convergence in distribution follows the same lines as in the proof of Proposition 2. \square

The distribution of the variable X_1 is now assumed to have a density h with respect to Lebesgue's measure.

Proposition 5 (Non-Uniform Distributions). *If $p_+ < 1/2$ and the distribution of X_1 has density h on $[0, 1]$ which is piecewise constant on a finite partition of sub-intervals of $[0, T]$, then the conclusions of Proposition 4 also hold in this case.*

Proof. The proof is sketched since most of the arguments have already been used on several occasions. By assumption there is a partition of $[0, T]$ into sub-intervals $(I_k, k \in K)$ and $(\alpha_k, k \in K)$ such that, for $x \in [0, T]$,

$$h(x) = \sum_{k \in K} \alpha_k \mathbb{1}_{I_k}(x).$$

The sequence (\bar{N}_n) defined by Recurrence (6) can be dominated by the sequence of point process (\tilde{N}_n) whose dynamic is modified as follows: a minus point falling into some sub-interval I_k does not kill a point in another sub-interval. In this way, for $n \geq 1$, one has clearly $\bar{N}_n \ll \tilde{N}_n$. Now, for $k \in K$, the point process \tilde{N}_n restricted to I_k is, up to a translation, simply the point process associated with uniformly distributed random variables on I_k when

$$\mathbb{1}_{\{X_1 \in I_k\}} + \mathbb{1}_{\{X_2 \in I_k\}} + \dots + \mathbb{1}_{\{X_n \in I_k\}}$$

points have been used. By Proposition 4, the point processes \tilde{N}_n , $n \geq 1$, are upper bounded by a point process with finite mass. This shows that N , the limit of (\bar{N}_n) , has almost surely a finite mass. \square

6. One-sided policies $[0, T]^d$

In this section, a multi-dimensional generalization of the results of Robert [20] is presented. It is assumed that $T = (T_i) \in \mathbb{R}_+^d$ with $T_i > 0$ for $1 \leq i \leq d$, H is defined as

$$H = \prod_{i=1}^d [0, T_i],$$

and that the locations of the points (X_i) are uniformly distributed in H .

With a slight abuse of notation, we will write $H = [0, T]^d$ and if $x, y \in \mathbb{R}_+$, xy [resp. x/y] will stand for $(x_i y_i)$ [resp. (x_i/y_i)]. Similarly, if $x = (x_i) \in \mathbb{R}_+^d$, $\log x$ denotes $(\log x_i)$ and finally Δ is the subset defined as the lower boundary of H ,

$$\Delta \stackrel{\text{def.}}{=} \{x \in [0, T]^d : \exists i \in \{1, \dots, d\}, x_i = 0\}.$$

A “–” particle at x can only kill the closest particle of the point process in the orthant with the corner at x , i.e. in the set $(x + \mathbb{R}_+^d) \cap H$. In order to get a more precise characterization of the variable N of Theorem 1, the following notation has to be introduced. If $M \in \mathcal{M}^*(H)$, one denotes by $D(M)$ the “dead zone” of M for minus particles, i.e. the set of locations where no point of M can be killed by them,

$$D(M) = \left\{ y \in H : M\left((y + \mathbb{R}_+^d) \cap H\right) = 0 \right\}.$$

If M is the null measure, then $D(M) = H$ and if $P, Q \in \mathcal{M}^*(H)$ are such that $P \ll Q$, then $D(Q) \subset D(P)$.

In this context, the corresponding stationary evolution equation is given by

$$N \circ \theta = N + \mathbb{1}_{\{I_1=+\}} \delta_{X_1} - \mathbb{1}_{\{I_1=-, X_1 \notin D(N)\}} \delta_{t_1(X_1, N)}. \quad (11)$$

With the same arguments as in Theorem 1 for local policies, there exists a unique minimal N in the set $\mathcal{M}^*(H)$ with probability 1 which is the solution of Eq. (11). The variable N is the limit of the non-decreasing sequence (\bar{N}_n) defined by the recurrence

$$\bar{N}_{n+1} \circ \theta = \bar{N}_n + \mathbb{1}_{\{I_1=+1\}} \delta_{X_1} - \mathbb{1}_{\{I_1=-1, X_1 \notin D(\bar{N}_n)\}} \delta_{t_1(X_1, \bar{N}_n)}. \quad (12)$$

The following proposition establishes a specific property of this policy, namely that it exhibits an invariance with respect to scaling.

Proposition 6. *If $p_+ < 1/2$ and $\alpha = (\alpha_i) \in \mathbb{R}_+^d$ with $0 < \alpha_i \leq 1$ for $1 \leq i \leq d$, for the GO policy the minimal solution N of Eq. (11) satisfies the invariance relation*

$$\int_{[0, \alpha T]^d} f(x) N(dx) \stackrel{\text{dist.}}{=} \int_{[0, T]^d} f(\alpha x) N(dx) \quad (13)$$

for any continuous function on $[0, T]^d$. Furthermore N is almost surely a Radon measure on $[0, T]^d - \Delta$.

Proof. Relation (13) is a consequence of the following two simple facts:

- The variables (X_i) that are in $[0, \alpha T]$ have the same distribution as (αX_i) .
- Invariance by scaling of the dynamics:

$$\int_{[0, \alpha T]^d} f(x) \bar{N}_{v_n}(dx) \stackrel{\text{dist.}}{=} \int_{[0, T]^d} f(\alpha x) \bar{N}_n(dx),$$

where v_n is the first index k for which exactly n elements of the k first points are in $[0, \alpha T]^d$. Relation (13) follows from this identity on letting n go to infinity.

Eq. (12) gives the inequality

$$0 \leq \mathbb{E}(\bar{N}_n(H)) - \mathbb{E}(\bar{N}_{n-1}(H)) = p - (1 - p)\mathbb{P}(X_1 \notin D(\bar{N}_{n-1})).$$

On letting n go to infinity and by using the fact that the non-increasing sequence of sets $(D(\bar{N}_{n-1}))$ is converging to $D(N)$, one gets therefore that

$$\mathbb{P}(X_1 \notin D(N)) \leq \frac{p_+}{1 - p_+} < 1.$$

The set $D(N)$ is therefore non-empty with some positive probability.

With the ergodicity property, any accumulation point $a = (a_i) \in [0, T]^d$ of N is deterministic. Assume that $a_i > 0$ for all $1 \leq i \leq d$. One considers the case where all the a_i are such that $a_i < T_i$; the other situations are treated in a similar way by using one-sided neighborhoods of a . Take $\varepsilon_0 > 0$ sufficiently small that if $\varepsilon < \varepsilon_0$, then $a + \varepsilon \stackrel{\text{def.}}{=} (a_i + \varepsilon)$ and $a - \varepsilon \in [0, T]^d$. One writes $(\alpha_i) = (T/(a_i + \varepsilon))$; then with probability 1, $N([a - \varepsilon, a + \varepsilon]^d) = +\infty$ for all $0 < \varepsilon < \varepsilon_0$. Relation (13) implies therefore that T is also an accumulation point. This contradicts the fact that the set $D(N)$ is therefore non-empty with some positive probability. One concludes that if there exists an accumulation point of N , then necessarily one of its coordinates is null and therefore it belongs to Δ . This shows that $\mathbb{P}(N \in \mathcal{M}([0, T]^d - \Delta)) = 1$. \square

The following proposition shows that for the invariant distribution, configurations under the GO policy have an infinite number of points with probability 1. It will be shown that this property also holds for the local version of the policy.

Proposition 7 (Infinite Number of Points Near Δ). *Almost surely, any point of the set Δ is an accumulation point of the solution N of Eq. (11) for the GO policy. Furthermore, the point process \tilde{N} on \mathbb{R}_+^d defined by*

$$\tilde{N} = \int_{[0, T]^d} \delta_{-\log(u/T)} N(du)$$

is a stationary point process on \mathbb{R}_+^d , i.e. for $x \in \mathbb{R}_+^d$, the distribution of the variable \tilde{N} is invariant with respect to the translation to x :

$$\int_{\mathbb{R}^d} f(x + y) \tilde{N}(dy) \stackrel{\text{dist.}}{=} \int_{\mathbb{R}^d} f(y) \tilde{N}(dy),$$

for any continuous function f with compact support on \mathbb{R}_+^d .

Proof. Let $a \in \Delta$; it is assumed that, for example, only the first coordinate is 0, $a = (0, a_2, \dots, a_d)$. Let $0 < \delta \leq 1$, $\varepsilon > 0$ and define

$$A_\delta = [0, \delta T_1] \times \prod_{i=2}^d [a_i, a_i + \varepsilon].$$

By taking $\alpha = (\delta, 1, \dots, 1)$ and using Relation (13), one gets the identity

$$N(A_\delta) \stackrel{\text{dist.}}{=} N(A_1)$$

for all $0 < \delta \leq 1$. Since $\mathbb{P}(N(A_\delta) < +\infty)$ is either 0 or 1 and since clearly $\mathbb{P}(N(A_1 - A_\delta) \neq 0) > 0$, one gets that $\mathbb{P}(N(A_\delta) = +\infty) = 1$. By the above proposition, one has that $N(A_1 - A_\delta)$ is almost surely finite. One concludes that a is an accumulation point. Consequently, the same property holds when there are several coordinates which are 0.

Relation (13) gives that, for $\alpha \in [0, 1]^d$, the identity

$$\begin{aligned} & \left(N \left(\prod_{i=1}^d [\alpha_i y_i, \alpha_i x_i] \right), x, y \in [0, T]^d, y \leq x \right) \\ & \stackrel{\text{dist.}}{=} \left(N \left(\prod_{i=1}^d [y_i, x_i] \right), x, y \in [0, T]^d, y \leq x \right) \end{aligned}$$

holds. By taking $z = -\log \alpha$, this relation can be rewritten as

$$\begin{aligned} & \left(\tilde{N} \left(\prod_{i=1}^d [v_i + z_i, u_i + z_i] \right), u, v \in \mathbb{R}_+^d, v \leq u \right) \\ & \stackrel{\text{dist.}}{=} \left(\tilde{N} \left(\prod_{i=1}^d [v_i, u_i] \right), u, v \in [0, T]^d, v \leq u \right). \end{aligned}$$

The point process \tilde{N} is invariant with respect to the non-negative translations. \square

Corollary 2 (*Local One-Sided Policy on the Torus $\mathbb{T}_1(T)$*). *The minimal solution N_L of the equation*

$$N_L \circ \theta = N_L + \mathbb{1}_{\{I_1=+\}} \delta_{X_1} - \mathbb{1}_{\{I_1=-, X_1 \notin D(N_L), t_1(X_1, N_L) \in B(X_1, 1)\}} \delta_{t_1(X_1, N_L)}$$

for the LO policy is such that $\mathbb{P}(N_L \in \mathcal{M}((0, T))) = 1$ and every element of Δ is almost surely an accumulation point of N .

Proof. With the same arguments as before, it is not difficult to prove that the solution N of Eq. (11) is such that $N \ll N_L$ which gives the result for the accumulation points. The proof that N_L is a Radon measure on $(0, T)$ with probability 1 is sketched. As before, one first proves that accumulation points are at distance 1 at least. If there is an accumulation point other than 0, denote by $a > 0$ the smallest one which is not 0; by considering the evolution of the number of points on the interval $[a - 1, a + \varepsilon]$ for some $\varepsilon > 0$, it is not difficult to get a contradiction to the fact that $p_+ < 1/2$. \square

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