

# On convergence determining and separating classes of functions

Douglas Blount<sup>a</sup>, Michael A. Kouritzin<sup>b,\*</sup>

<sup>a</sup> *Department of Mathematics and Statistics, Arizona State University, Tempe, AZ 85287-1804, USA*

<sup>b</sup> *Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, T6G 2G1, Canada*

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## Abstract

Herein, we generalize and extend some standard results on the separation and convergence of probability measures. We use homeomorphism-based methods and work on incomplete metric spaces, Skorokhod spaces, Lusin spaces or general topological spaces. Our contributions are twofold: we dramatically simplify the proofs of several basic results in weak convergence theory and, concurrently, extend these results to apply more immediately in a number of settings, including on Lusin spaces.

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## 1. Introduction

Suppose that  $\{P_n\}$  and  $P$  are Borel probability measures on a topological space  $E$ . Then, it is often desirable to deduce that  $P_n$  converges weakly to  $P$  by merely showing  $\int_E f dP_n$  converges to  $\int_E f dP$  for each  $f$  in a small subset of the bounded real-valued functions on  $E$ . For example, to show convergence of martingale problem solutions, one may wish to use functions selected from a common domain of the operators associated with the martingale problems. To specify minimal conditions on such a subset of functions, Ethier and Kurtz [5] defined a subset of bounded, continuous functions that *strongly separate points*. Furthermore, when  $E$  is Polish they showed that any subset  $\mathcal{M}$  of the continuous, bounded functions that strongly separates points and is also an algebra is *convergence determining* on the set of Borel probability measures (see Theorem 3.4.5(b) of [5]). This means that  $\int_E f dP_n \rightarrow \int_E f dP$  for all  $f \in \mathcal{M}$  implies

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\* Corresponding author. Tel.: +1 780 492 2704; fax: +1 780 492 6826.

E-mail addresses: [blount@math.asu.edu](mailto:blount@math.asu.edu) (D. Blount), [mkouritz@math.ualberta.ca](mailto:mkouritz@math.ualberta.ca) (M.A. Kouritzin).

weak convergence of the Borel probability measures  $P_n \Rightarrow P$ . In the case that  $E$  is merely a separable metric space, they also showed that the uniformly continuous functions with bounded support is still *convergence determining*, as are the continuous functions with compact support provided  $E$  is also locally compact (see Proposition 3.4.4 of [5]). Such convergence determining results can also be used to establish important convergence results for Skorokhod spaces, like Theorem 12.6 of Billingsley [3] and Corollary 3.9.2 of Ethier and Kurtz [5]. However, even the basic convergence determining result, Theorem 3.4.5(b) of [5], is not general enough to handle such things as Lusin spaces or nuclear space duals, neither of which are Polish yet are critically important for probability theory.

Another essential problem is the separation of probability measures: one wants conditions on a class of functions  $\mathcal{M}$  such that  $\int_E f dQ = \int_E f dP$  for all  $f \in \mathcal{M}$  implies that  $Q = P$ . This can be important, for example, to show that a martingale problem is well posed or to identify a limit point of a relatively compact family of probability measures. In this regard, Ethier and Kurtz [5] define such separating classes of functions  $\mathcal{M}$  in terms of the *separating points* property. They show (see Theorem 3.4.5(a) of [5]) that the separation of probability measures holds on Polish spaces if  $\mathcal{M}$  separates points and is also an algebra of continuous, bounded functions. Moreover, in Theorem 2.1.4 of [7], Kallianpur and Xiong establish separation of Radon probability measures on completely regular topological spaces using all the continuous, bounded functions as the separating class  $\mathcal{M}$ . Neither of these results handle such cases as general probabilities on Lusin spaces.

The purpose of this note is to extend and generalize all the results mentioned in the previous two paragraphs. Motivated in part by the work of Bhatt and Karandikar [1], we use homeomorphism methods to capture the notion of strongly separating points and to transfer convergence determining problems from a metric space, or more generally a topological space, onto a pre-compact subset of  $\mathbb{R}^{\mathcal{M}}$ , where we have additional structure.

## 2. Notation and background

In what follows,  $(E, \mathcal{T})$  or just  $E$  will denote a topological space,  $\mathcal{B}(E)$  or  $\mathcal{B}(\mathcal{T})$  will be the Borel sets, and  $M(E)$ ,  $B(E)$ ,  $C(E)$  and  $\overline{C}(E)$  will denote the Borel measurable, bounded measurable, continuous, and continuous bounded  $\mathbb{R}$ -valued functions on  $E$ , respectively. Our product spaces will always be given the product topology and  $|\cdot|$  will always denote Euclidean distance or absolute value.

First, we will define the strongly separating points property as Ethier and Kurtz [5, page 113] did, then we will give a little theory related to this property, and finally, we will provide a simple example of such a class of functions on a metric space or, more generally, a completely regular topological space.

**Definition 1.** Let  $(E, \mathcal{T})$  be a topological space and  $\mathcal{M} \subset M(E)$ . Then, (i)  $\mathcal{M}$  separates points (s.p.) if for  $x \neq y \in E$  there is a  $g \in \mathcal{M}$  with  $g(x) \neq g(y)$  and (ii)  $\mathcal{M}$  strongly separates points (s.s.p.) if, for every  $x \in E$  and neighborhood  $O_x$  of  $x$ , there is a finite collection  $\{g^1, \dots, g^k\} \subset \mathcal{M}$  such that

$$\inf_{y \notin O_x} \max_{1 \leq l \leq k} |g^l(y) - g^l(x)| > 0.$$

We also use s.s.p. below for *strongly separating points* and *strongly separate points*, depending upon which is the correct English usage.

Hence, if  $\mathcal{M}$  s.s.p., then for any  $x$  and neighborhood  $O_x$  there are  $\varepsilon > 0$  and  $\{g^1, \dots, g^k\} \subset \mathcal{M}$  such that

$$\left\{ y \in E : \max_{1 \leq l \leq k} |g^l(y) - g^l(x)| < \varepsilon \right\} \subset O_x.$$

Thus,  $\mathcal{M}$  s.s.p. that implies  $\mathcal{M}$  s.p. (in a Hausdorff space) and defines a topology  $\mathcal{T}^{\mathcal{M}}$  through the basis

$$\mathbb{B}^{\mathcal{M}} \doteq \left\{ \left\{ y \in E : \max_{1 \leq l \leq k} |g^l(y) - g^l(x)| < \varepsilon \right\}, \right. \\ \left. g^1, \dots, g^k \in \mathcal{M}, \varepsilon > 0, x \in E, k \in \mathbb{N} \right\} \quad (1)$$

on  $E$  that is finer than the original topology. This yields the following simple lemma.

**Lemma 1.** *Let  $(E, \mathcal{T})$  be a Hausdorff space,  $\mathcal{M} \subset M(E)$  and  $\Gamma(x) \doteq (g(x))_{g \in \mathcal{M}}$ . Then,  $\Gamma$  has a continuous inverse  $\Gamma^{-1} : \Gamma(E) \subset \mathbb{R}^{\mathcal{M}} \rightarrow E$  if and only if  $\mathcal{M}$  s.s.p. In particular,  $\Gamma$  is an imbedding of  $E$  in  $\mathbb{R}^{\mathcal{M}}$  if and only if  $\mathcal{M} \subset C(E)$  and  $\mathcal{M}$  s.s.p.*

**Proof.** If  $\mathcal{M}$  s.s.p., then  $\mathcal{M}$  s.p., so  $\Gamma^{-1}$  exists. Moreover,  $\mathcal{T} \subset \mathcal{T}^{\mathcal{M}}$ , so  $\Gamma^{-1}$  is continuous.  $\square$

Given a collection  $\mathcal{M} \subset M(E)$  that does not necessarily s.s.p., one can still define a topology  $\mathcal{T}^{\mathcal{M}}$  through the basis  $\mathbb{B}^{\mathcal{M}}$  and find that  $(E, \mathcal{T}^{\mathcal{M}})$  may differ from  $(E, \mathcal{T})$ . In particular, if  $\mathcal{M} = \{g_k\}_{k=0}^{\infty} \subset M(E)$  is countable, we define a single pseudometric

$$\rho(x, y) = \sum_{k=0}^{\infty} 2^{-k} (|g_k(x) - g_k(y)| \wedge 1), \quad (2)$$

which generates  $\mathcal{T}^{\mathcal{M}}$ . (See Dudley [4] p. 20 for the definition of a pseudometric.) If, in addition,  $\{g_k\}_{k=0}^{\infty}$  s.p., then (2) becomes a metric and  $\{g_k\}_{k=0}^{\infty}$  s.s.p. on  $(E, \mathcal{T}^{\mathcal{M}})$ .

The following lemma establishes when it is possible to assume that a strongly separating collection is countable with no loss of generality.

**Lemma 2.** *If  $(E, \mathcal{T})$  has a countable basis and  $\mathcal{M} \subset C(E)$  s.s.p., then there is a countable collection  $\{g_k\}_{k=0}^{\infty} \subset \mathcal{M}$  that s.s.p. Moreover,  $\{g_k\}_{k=0}^{\infty}$  can be taken closed under either multiplication or addition if  $\mathcal{M}$  is.*

**Proof.** We have by the homeomorphism of Lemma 1 that  $\mathbb{B}^{\mathcal{M}}$ , defined in (1), forms a basis for  $\mathcal{T}$ . However, any basis contains a countable basis by Ex. 4-1.5 in [10], so we only need a countable number of  $g$ 's, which we denote with  $\{g_k\}_{k=0}^{\infty}$ .  $\square$

If  $\mathcal{M}$  does not s.s.p. or is not a subset of the continuous functions, then  $\mathcal{T}^{\mathcal{M}}$  may be coarser or finer than  $\mathcal{T}$ . However, the Borel  $\sigma$ -fields generated by  $\mathcal{T}^{\mathcal{M}}$  and  $\mathcal{T}$  may still be the same, as the following result establishes. Part (a) follows from Kuratowski's remarkable result.

**Lemma 3.** *Suppose that  $(E, \mathcal{T})$  is a topological space,  $\mathcal{M} \doteq \{g_k\}_{k=0}^{\infty} \subset M(E)$  s.p. and  $\rho$  is as defined in (2). Then, the Borel  $\sigma$ -fields of  $(E, \mathcal{T})$  and  $(E, \rho)$  are equal if either (a)  $\mathcal{T}$  is generated from a Polish space  $(E, d)$ , or (b)  $\{g_k\}_{k=0}^{\infty}$  s.s.p.*

**Proof.** (a) By Kuratowski's theorem (see Parthasarathy [11] Corollary I.3.3),  $G(A) \subset \mathbb{R}^{\infty}$  is Borel for any measurable  $A \subset E$  and  $G^{-1} : G(E) \rightarrow (E, d)$  is Borel measurable, where

$G(x) \doteq (g_0(x), g_1(x), \dots): (E, d) \rightarrow G(E)$ . On the other hand,  $G: (E, \rho) \rightarrow G(E) \subset \mathbb{R}^\infty$  has a continuous inverse. For both, the Borel sets are

$$\left\{ G^{-1}(\Gamma) : \Gamma \in \mathcal{B}(\mathbb{R}^\infty) \right\}.$$

(b) It follows by the definition of s.s.p. that  $\mathcal{T}^\mathcal{M} \supset \mathcal{T}$  so  $\mathcal{B}(E, \rho) \supset \mathcal{B}(\mathcal{T})$ . Conversely,  $\mathbb{B}^\mathcal{M} \subset \mathcal{B}(\mathcal{T})$  and  $\mathcal{T}^\mathcal{M}$  consists of countable unions of elements of  $\mathbb{B}^\mathcal{M}$  since  $(E, \rho)$  is homeomorphic to a subset of  $\mathbb{R}^\infty$  and therefore has a countable basis. Hence,  $\mathcal{B}(E, \rho) = \mathcal{B}(\mathcal{T}^\mathcal{M}) \subset \mathcal{B}(\mathcal{T})$ .  $\square$

For our results on convergence determining classes, it will be helpful to look at the s.s.p. property from another angle.

**Lemma 4.** *Suppose  $(E, \mathcal{T})$  is a Hausdorff space and  $\mathcal{M} \subset M(E)$ . Then,  $\mathcal{M}$  s.s.p. if and only if, for any net  $\{x_i\}_{i \in I} \subset E$  and point  $x \in E$ , one has that  $g(x_i) \rightarrow g(x)$  for all  $g \in \mathcal{M}$  implies that  $x_i \rightarrow x$  in  $E$ .*

**Proof.** Suppose that  $\mathcal{M}$  s.s.p., and that there are  $\{x_i\}_{i \in I}$ ,  $x$  such that  $x_i \not\rightarrow x$ . Then, there exists a neighborhood  $O_x$ , an  $\varepsilon > 0$ , and finite  $\{g^1, \dots, g^k\} \subset \mathcal{M}$  such that for any  $j \in I$  we have an  $i \geq j$  satisfying  $x_i \notin O_x$  and

$$\max_{1 \leq l \leq k} |g^l(x_i) - g^l(x)| \geq \varepsilon > 0,$$

so there is some  $g \in \mathcal{M}$  such that  $g(x_i) \not\rightarrow g(x)$ . On the other hand, suppose that, for any net  $\{x_i\}_{i \in I} \subset E$  and point  $x \in E$ , one has that  $g(x_i) \rightarrow g(x)$  for all  $g \in \mathcal{M}$  implies that  $x_i \rightarrow x$  in  $E$ . Then, by Lemma 1, one has that

$$\mathcal{G} = \left\{ \left\{ y \in E : \max_{1 \leq l \leq k} |g^l(y) - g^l(x)| < \varepsilon \right\} : x \in E, \varepsilon > 0, g^1, \dots, g^k \in \mathcal{M}, k \in \mathbb{N} \right\} \quad (3)$$

is a basis for a topology finer than  $\mathcal{T}$ , so given  $x$ ,  $O_x$  there are  $\varepsilon > 0$  and  $g^1, \dots, g^k \in \mathcal{M}$  such that  $\{y \in E : \max_{1 \leq l \leq k} |g^l(y) - g^l(x)| < \varepsilon\} \subset O_x$ . Therefore,

$$\inf_{y \notin O_x} \max_{1 \leq l \leq k} |g^l(y) - g^l(x)| \geq \varepsilon$$

and  $\mathcal{M}$  s.s.p.  $\square$

This lemma provides a useful means to establish that classes of functions s.s.p. For example, if  $(E, d)$  is a metric space, then the non-negative, uniformly continuous functions

$$\{g_{y,k}(\cdot) \doteq (1 - kd(\cdot, y)) \vee 0 : y \in E, k \in \mathbb{N}\} \quad (4)$$

s.s.p. Moreover, they have bounded support in general and compact support on locally compact spaces since  $g_{y,k}(\cdot)$  is zero off the ball with center  $y$  and radius  $1/k$ .

Lusin spaces include Polish spaces and are very important in probability theory (see for example Meyer and Zheng [8]). A Hausdorff space  $E$  is Lusin if it is the image of a Polish space under a continuous bijection. The well-known fact that the continuous bijection can be taken to be a Borel isomorphism can be used to show there is a countable collection of bounded functions that s.s.p. Indeed, suppose that  $S$  is Polish and that  $J: S \rightarrow E$  is this continuous bijection. Then, a collection of continuous, bounded functions that s.s.p. on  $S$  can be found, for example by Eq. (4), and turned into a countable collection  $\{g_k\}_{k=0}^\infty$  by Lemma 2. It follows by Lemma 4 and the continuity of  $J$  that  $\{g_k \circ J^{-1}\}_{k=0}^\infty$  s.s.p. on  $E$ . In general, these composite functions will not be continuous as  $J^{-1}$  need not be.

When  $E$  is the dual of a nuclear space or just a completely regular topological space with collection of pseudometrics  $\mathcal{D}$ , the class of functions defined in (4) can be extended to an s.s.p. class on  $E$  by letting  $d$  range over  $\mathcal{D}$  as well.

For a metric space  $(E, d)$ , we define  $D_E[0, \infty)$  to be the space of all  $E$ -valued functions on  $[0, \infty)$  that are right continuous and have left-hand limits.  $D_E[0, \infty)$  is a metric space with metric

$$\tilde{d}(x, y) \triangleq \inf_{\lambda \in \Lambda} \left( \operatorname{esssup}_{t \geq 0} |\log \lambda'(t)| \vee \int_0^\infty e^{-u} \sup_{t \geq 0} (d(x(\lambda(t) \wedge u), y(t \wedge u)) \wedge 1) du \right)$$

for all  $x, y \in D_E[0, \infty)$ , where  $\Lambda$  is the collections of strictly increasing Lipschitz continuous functions from  $[0, \infty)$  onto itself. See also Section 3.5 of [5] for more details. Suppose that  $S$  is another metric space and that  $f : E \rightarrow S$  is continuous. Then, we define  $\tilde{f} : D_E[0, \infty) \rightarrow D_S[0, \infty)$  by  $\tilde{f}(x)(t) = f(x(t))$  for  $t \geq 0$ . Now, we list a basic result that will be used in what follows. It follows from Problems 3.11.22 and 3.11.23 of Ethier and Kurtz [5] or, alternatively, the proofs of Theorems 1.7 and 4.3(ii) of [6].

**Theorem 5.** *Let  $S$  be a metric space,  $\mathcal{H}^1 \subset C(S)$  s.s.p.,  $\mathcal{H}^2 \triangleq \{f + g : f, g \in \mathcal{H}^1\}$ , and  $\mathcal{H} = \mathcal{H}^1 \cup \mathcal{H}^2$ . Then,  $\hat{G} : D_S[0, \infty) \rightarrow (D_{\mathbb{R}}[0, \infty))^{\mathcal{H}}$  is an imbedding, where  $\hat{G}(x) = (\tilde{g}(x))_{g \in \mathcal{H}}$  for  $x \in D_S[0, \infty)$ .*

In what follows,  $\mathcal{P}(S)$  will denote the Borel probability measures on any topological space  $S$  and  $\beta_k$  will be the projection mapping onto the  $k$ th component of  $\mathbb{R}^\infty$ .

### 3. Weak convergence and separation results

We first consider general spaces and establish conditions on  $E$  and  $\mathcal{M}$  under which  $\{\hat{g}(P) \triangleq \int_E g dP\}_{g \in \mathcal{M}}$  s.s.p. on  $(\mathcal{P}(E), T^W)$ , where  $T^W$  is the topology of weak convergence of probability measures. One can see through Lemma 4 that our result is an extension of the probability measure convergence result in Theorem 3.4.5(b) of Ethier and Kurtz [5]. Due to the definition of weak convergence of probability measures as well as the desired use of compactness, we now work with *bounded* functions. Still, given a class  $\mathcal{M}$  of functions that s.s.p., one can create a class of positive, bounded functions that s.s.p. For example,  $\left\{ \frac{e^g}{1+e^g} : g \in \mathcal{M} \right\}$  is one such class.

**Theorem 6.** *Suppose that  $(E, T)$  is a topological space,  $\{P_n\} \cup \{P\} \subset \mathcal{P}(E)$ ,  $\mathcal{M} \subset B(E)$  s.s.p. and is closed under multiplication, and*

$$\int_E g dP_n \rightarrow \int_E g dP \quad \forall g \in \mathcal{M}.$$

(a) *If  $E$  has a countable basis and  $\mathcal{M} \subset \overline{C}(E)$ , then  $P_n \Rightarrow P$  on  $(E, T)$ . (b) *If  $\mathcal{M}$  is countable, then  $P_n \Rightarrow P$  on  $(E, \rho)$  and  $(E, T)$ , where  $\rho$  is defined in (2) with  $\{g_k\}_{k=0}^\infty = \mathcal{M}$ .**

**Remark 1.** The extensions of Theorem 6 over Theorem 3.4.5(b) in [5] are due to the facts that we do not require the space to be Polish nor even continuity of  $\{g_k\}_{k=0}^\infty$  in the case of (b). These extensions are important in the sense that they allow us to handle the case where  $E$  is Lusin and the class that s.s.p. is constructed from the (pre-image) Polish space as explained in the previous section. Our theorem also handles completely regular spaces like the tempered distributions and path spaces of the same (see for example the important works of Mitoma [9] and Chapter 2 of

Kallianpur and Xiong [7]), which are important in probability. Still, a central argument in our proof is motivated by the development of [5, Theorem 3.4.5(b)].

**Remark 2.** Since  $\overline{C}(E, \rho) \supset \overline{C}(E, T)$  when  $\{g_k\}_{k=0}^\infty \stackrel{\circ}{=} \mathcal{M} \subset B(E)$  s.s.p., (b) gives convergence on a possibly larger collection of functions than (a).

**Proof.** (a) By Lemma 2, we can assume the  $\mathcal{M} = \{g_k\}_{k=0}^\infty$  is countable and define the homeomorphism  $G : E \rightarrow G(E) \subset \mathbb{R}^\infty$ , where  $G(x) \doteq (g_0(x), g_1(x), \dots)$  and  $G(E)$  has the subspace topology. We set  $Q = PG^{-1}$ ,  $Q_n = P_nG^{-1}$  on  $\mathcal{B}(G(E))$  and let  $\mathcal{G}$  be the  $\sigma$ -field of subsets of  $\mathbb{R}^\infty$  of the form  $A = B \cup B'$ , where  $B \in \mathcal{B}(G(E))$  and  $B' \cap G(E) = \emptyset$ . Then, we note that  $\overline{\mathcal{B}}(\mathbb{R}^\infty) \subset \mathcal{G}$  and define  $\widehat{Q}(A) = Q(B)$ , with  $\widehat{Q}_n$  being defined similarly. Letting  $K(E) = \overline{G(E)}$  be the compact closure of  $G(E)$  in  $\mathbb{R}^\infty$ , one finds that  $\widehat{Q}$  and  $\widehat{Q}_n$  also define, by restriction, probabilities on  $\mathcal{B}(K(E)) \subset K(E) \cap \mathcal{G} \doteq \{K(E) \cap A : A \in \mathcal{G}\}$ , which are equal to  $Q$  and  $Q_n$  respectively on  $\mathcal{B}(G(E)) \subset K(E) \cap \mathcal{G}$ . Noting that  $\widehat{Q}(G(E)) = \widehat{Q}_n(G(E)) = 1$  and using our assumptions, one has that

$$\int_{K(E)} f \circ (\beta_0, \dots, \beta_k) d\widehat{Q}_n \rightarrow \int_{K(E)} f \circ (\beta_0, \dots, \beta_k) d\widehat{Q}$$

for polynomials  $f$ . Since  $K(E)$  is compact, the Stone–Weierstrass theorem gives  $\widehat{Q}_n(\beta_0, \dots, \beta_k)^{-1} \Rightarrow \widehat{Q}(\beta_0, \dots, \beta_k)^{-1}$  on  $\mathbb{R}^k$  for each  $k$ . Noting that  $\{\widehat{Q}_n\}$  are tight and applying the tightness and consistency argument and Lemma 3 found on pages 38–39 of Billingsley [2], we obtain  $\widehat{Q}_n \Rightarrow \widehat{Q}$  on  $\mathbb{R}^\infty$  and then on  $K(E)$  by the Portmanteau theorem. Any uniformly continuous function on  $G(E)$  extends continuously to  $K(E)$ , and we obtain  $Q_n = \widehat{Q}_n \Rightarrow \widehat{Q} = Q$  on  $G(E)$ .  $G^{-1}$ -continuity and the continuous mapping theorem yield  $P_n \Rightarrow P$ .

(b) By Lemma 3(b),  $\{P_n\} \cup \{P\}$  are probabilities on  $(E, \rho)$ , and  $\mathcal{M} = \{g_k\}_{k=0}^\infty \subset \overline{C}(E, \rho)$  s.s.p. on  $(E, \rho)$ . Thus, (b) follows from (a) applied to  $(E, \rho)$  and the fact that  $(E, T)$  has a coarser topology than  $(E, \rho)$ , which implies that  $\overline{C}(E, \rho) \supset \overline{C}(E, T)$ .  $\square$

Now, we can recover Proposition 3.4.4 of [5] in an elementary manner. Indeed, the following corollaries follow from Theorem 6(a) by taking  $\mathcal{M}$  to be the space of uniformly continuous functions with bounded support and the space of continuous functions with compact support, respectively. Both classes are clearly algebras that s.s.p. since they contain the functions defined in (4).

**Corollary 7.** Suppose that  $E$  is a separable metric space. Then, the space  $\mathcal{M}$  of uniformly continuous functions with bounded support is convergence determining.

**Corollary 8.** Suppose that  $E$  is a separable, locally compact metric space. Then, the space of continuous functions with compact support is convergence determining.

Next, Theorem 6(b) can be used to generalize Theorem 12.6, p. 136, of Billingsley [3] from finite to infinite intervals and to not-necessarily complete metric spaces.

**Theorem 9.** Let  $(E, d)$  be a separable metric space,  $\beta_{t_1, t_2, \dots, t_k}(x) \doteq (x(t_1), x(t_2), \dots, x(t_k))$  for all  $0 \leq t_1 < t_2 < \dots < t_k$ ,  $x \in D_E[0, \infty)$  be the projection function,  $\{P_n\}_{n=1}^\infty$ ,  $P \in \mathcal{P}(D_E[0, \infty))$ , and  $R$  be a countable, dense subset of  $[0, \infty)$ . Suppose that  $S$  is a Borel subset of  $D_E[0, \infty)$  satisfying  $P_n(S) = P(S) = 1$  as well as the property that  $x_n, x \in S$  and  $x_n(t) \rightarrow x(t)$  for all  $t \in R$  implies  $x_n \rightarrow x$  in  $D_E[0, \infty)$ . Then,  $P_n \Rightarrow P$  on  $D_E[0, \infty)$  if  $P_n \beta_{t_1, t_2, \dots, t_k}^{-1} \Rightarrow P \beta_{t_1, t_2, \dots, t_k}^{-1}$  for all  $t_1, \dots, t_k \in R$ .

**Proof.** If  $\{h_k\}_{k=0}^\infty \subset \overline{C}(E)$  s.s.p. and is closed under multiplication, then  $\mathcal{M} = \{g_k\}_{k=0}^\infty \stackrel{\circ}{=} \{(h^1 \circ \beta_{t^1}) \times \cdots \times (h^j \circ \beta_{t^j}) : h^i \in \{h_k\}_{k=0}^\infty, t^i \in R, j \in \mathbb{N}\} \subset B(D_E[0, \infty))$  s.s.p. on  $S$  (by Lemma 4) and is closed under multiplication. Moreover, by hypothesis,  $\int g dP_n \rightarrow \int g dP$  for all  $g \in \mathcal{M}$ , so by Theorem 6(b)  $P_n \Rightarrow P$  on  $(S, \tilde{d})$ . Finally,  $O \cap S$  is open in  $(S, \tilde{d})$  for any open  $O \subset (D_E[0, \infty), \tilde{d})$ , so  $\liminf_{n \rightarrow \infty} P_n(O) = \liminf_{n \rightarrow \infty} P_n(O \cap S) \geq P(O \cap S) = P(O)$  by the Portmanteau theorem and weak convergence holds.  $\square$

Next, we extend Corollary 3.9.2 of Ethier and Kurtz [5] to the separable metric space case using Theorem 6(a). In the following proof, we use the fact (see [5, Theorem 3.5.6]) that  $(D_E[0, \infty), \tilde{d})$  is separable if  $(E, d)$  is.

**Theorem 10.** Suppose that  $(E, d)$  is a separable metric space,  $\mathcal{M} \subset C(E)$  s.s.p.,  $\{X^n\}, X \in D_E[0, \infty)$ , and  $(g^1, \dots, g^k) \circ X^n \Rightarrow (g^1, \dots, g^k) \circ X$  in  $D_{\mathbb{R}^k}[0, \infty)$  for each  $k \in \mathbb{N}$ ,  $g^1, \dots, g^k \in \mathcal{M}$ . Then,  $X^n \Rightarrow X$  in  $D_E[0, \infty)$ .

**Proof.** Let  $P_n \stackrel{\circ}{=} \mathcal{L}(X^n)$ ,  $P \stackrel{\circ}{=} \mathcal{L}(X)$  be the laws of  $X^n, X$ ;  $\mathcal{M}' \subset C(E)$  be the collection of finite sums of functions in  $\mathcal{M}$ ; and  $\{h_k\}_{k=0}^\infty \subset \overline{C}(D_{\mathbb{R}}[0, \infty))$  s.s.p. Then, it follows by Theorem 5 that the collection  $\{h_k \circ \tilde{g}\}_{k \in \mathbb{N}, g \in \mathcal{M}'}$  s.s.p. on  $D_E[0, \infty)$ , and by hypothesis that

$$\int_{D_E[0, \infty)} \prod_{i=1}^m h^i(\tilde{g}^i(x)) P_n(dx) \rightarrow \int_{D_E[0, \infty)} \prod_{i=1}^m h^i(\tilde{g}^i(x)) P(dx)$$

for all  $g^i \in \mathcal{M}', h^i \in \{h_k\}_{k=0}^\infty$ , and  $m \in \mathbb{N}$ . Therefore, it follows by Theorem 6(a) that  $X^n \Rightarrow X$  on  $D_E[0, \infty)$ .  $\square$

Clearly, the use of the homeomorphism in Theorem 6 simplified the development of Theorem 10 compared to Ethier and Kurtz [5, Theorem 3.9.1 and Corollary 3.9.2], which uses compactness techniques and requires completeness. We also got around their boundedness assumption in assuming only that  $\mathcal{M} \subset C(E)$  instead of  $\mathcal{M} \subset \overline{C}(E)$ .

**Remark 3.** An alternative, direct proof of Theorem 10 may be of interest. Suppose we choose  $\{g_k\}_{k=0}^\infty \subset \mathcal{M}$  that s.s.p., set  $G(x) = (g_0(x), g_1(x), \dots)$  and let

$$\begin{aligned} Y^n &= G(X^n), \quad Y = G(X) \\ Y^{n,k} &= (g_0(X^n), \dots, g_k(X^n), 0, 0, \dots) \\ Y^k &= (g_0(X), \dots, g_k(X), 0, 0, \dots). \end{aligned}$$

Then, for fixed  $k$ ,  $Y^{n,k} \Rightarrow Y^k$  in  $D_{\mathbb{R}^\infty}[0, \infty)$  by assumption. Next, letting  $r(x, y) = \sum_{k=0}^\infty \frac{|x_k - y_k| \wedge 1}{2^k}$  be the metric on  $\mathbb{R}^\infty$ , one finds that

$$\sup_{n \in \mathbb{N}} \sup_{t \geq 0} r(Y_t^{n,k}, Y_t^n) \quad \text{and} \quad \sup_{t \geq 0} r(Y_t^k, Y_t)$$

converge to zero in probability as  $k \rightarrow \infty$ . It follows by Theorem 4.2 of [2] that  $Y^n \Rightarrow Y$  in  $D_{\mathbb{R}^\infty}[0, \infty)$  and in  $D_{G(E)}[0, \infty)$  with the relative topology by the Portmanteau theorem and the fact that  $P(Y^n \in D_{G(E)}[0, \infty) \cap O) = P(Y^n \in O)$  for all open  $O \subset D_{\mathbb{R}^\infty}[0, \infty)$ . Thus,  $X^n = G^{-1}(Y^n) \Rightarrow G^{-1}(Y) = X$ .

Now, we can use Theorem 6(a) to generalize and simplify the development of Theorem 3.4.5(a) in [5] as well as Kallianpur and Xiong [7, Theorem 2.1.4]. We note that (b) of the following theorem applies when  $\mathcal{M}$  is a countable collection of measurable functions whilst (e) accommodates an uncountable collection but the collection must be continuous.

**Theorem 11.** Let  $(E, \mathcal{T})$  be a topological space;  $P, Q$  be Borel probability measures;  $\mathcal{M} \subset \mathcal{B}(E)$  be closed under multiplication; and

$$\int_E g dQ = \int_E g dP \quad \forall g \in \mathcal{M}.$$

Then,

- (a)  $Q = P$  on  $\mathcal{B}(\mathcal{T}^{\mathcal{M}})$ , where  $\mathcal{T}^{\mathcal{M}}$  is the topology with basis  $\mathbb{B}^{\mathcal{M}}$  is defined as in (1);
- (b)  $Q = P$  if  $(E, \mathcal{T})$  is consistent with a Polish space,  $\mathcal{M}$  s.p. and  $\mathcal{M}$  is countable;
- (c)  $Q = P$  if  $\mathcal{M}$  s.s.p. and is countable;
- (d)  $Q = P$  if  $P, Q$  are regular and  $\mathcal{M} \subset \overline{\mathcal{C}}(E)$  s.p.; and
- (e)  $Q = P$  if  $(E, \mathcal{T})$  is consistent with a Polish space and  $\mathcal{M} \subset \overline{\mathcal{C}}(E)$  s.p.

**Remark 4.** To be precise, we follow Dudley [4, p. 174] for our definition of regularity.

**Proof.** Suppose  $\mathcal{N} \triangleq \{g_k\}_{k=0}^\infty$  s.p. and  $\rho$  is defined as in (2). Then, we find by an identity-map change of variables that

$$\int_{(E, \rho)} g dQ = \int_{(E, \rho)} g dP \quad \forall g \in \mathcal{N}$$

and  $\mathcal{N}$  s.s.p. on  $(E, \rho)$  so  $Q = P$  on the Borel sets of  $(E, \mathcal{T}^{\mathcal{N}})$  by Theorem 6(b), where  $\mathcal{T}^{\mathcal{N}}$  is defined as in the sentence containing Eq. (1). Therefore, for (a), we take  $\mathcal{N} \triangleq \{g_k\}_{k=0}^\infty \subset \mathcal{M}$ , turn  $\rho^{\mathcal{N}}$  as defined as in (2) into a metric on the equivalence classes  $\widehat{E}$  of points in  $E$ , set  $\widehat{Q} = Q\widehat{T}^{-1}$ , where  $\widehat{T}$  maps a point into its equivalence class, and find  $Q = P$  on  $\mathcal{B}(E, \rho^{\mathcal{N}}) \triangleq \mathcal{T}(\{y \in E : \rho^{\mathcal{N}}(x, y) < \varepsilon\} : x \in E, \varepsilon > 0\})$  for pseudometric  $\rho^{\mathcal{N}}$ . Since  $\mathcal{N}$  was arbitrary, we have that  $Q = P$  on  $\mathbb{B}^{\mathcal{M}}$  and hence on  $\mathcal{B}(\mathcal{T}^{\mathcal{M}})$ . Now, (b) and (c) follow respectively by Lemma 3(a) and (b).

Next, for (d), the fact that  $\overline{\mathcal{C}}(E)$  s.p. implies that  $(E, \mathcal{T}^{\mathcal{M}})$  is Hausdorff so compacts are closed. We let  $K \subset E$  be any compact, take  $\varepsilon > 0$  and note by regularity that there is a compact  $\widehat{K} \subset K^c$  such that  $P(\widehat{K}) > P(K^c) - \varepsilon$ . Since  $K^c$  is open in  $\mathcal{T}^{\mathcal{M}}$ , we can find  $G_x \in \mathbb{B}^{\mathcal{M}}$  such that  $x \in G_x \subset K^c$  for each  $x \in \widehat{K}$ . By compactness, there is a finite collection  $\{G_{x_i}\}_{i=1}^n$  such that  $\widehat{K} \subset \bigcup_{i=1}^n G_{x_i} \subset K^c$ . Thus, using (a), one finds that

$$\begin{aligned} P(K) &= 1 - P(K^c) > 1 - P(\widehat{K}) - \varepsilon \\ &\geq 1 - P\left(\bigcup_{i=1}^n G_{x_i}\right) - \varepsilon \\ &= 1 - Q\left(\bigcup_{i=1}^n G_{x_i}\right) - \varepsilon \\ &\geq 1 - Q(K^c) - \varepsilon \end{aligned}$$

and  $P(K) \geq Q(K)$ . (d) follows by symmetry. Finally, (e) follows from (d) and Ulam's theorem on the regularity of probability measures on Polish spaces.  $\square$

Our generalization of [5, Theorem 3.4.5(a)] was from Polish to topological spaces. While we did require that  $\mathcal{M}$  be countable in (c), this is no restriction when  $\mathcal{T}$  has a countable basis. Part (d) of our result also generalizes Kallianpur and Xiong [7, Theorem 2.1.4] since  $\mathcal{M} \triangleq \{d(x, \cdot) \wedge 1 : d \in \mathcal{D}\}$  are continuous, bounded functions that s.p. for any class of pseudometrics

$\mathcal{D}$  that generate the topology and their notion of a Radon probability measure is handled by our use of regularity. Moreover, we can extend parts (b) and (e) to the context of Lusin spaces.

**Corollary 12.** *Let  $E$  be a Lusin space;  $P, Q$  be Borel probability measures; and*

$$\int_E g dQ = \int_E g dP \quad \forall g \in \mathcal{M},$$

where  $\mathcal{M} \subset B(E)$  be closed under multiplication and s.p. Then,  $Q = P$  if (a)  $\mathcal{M}$  is countable, or (b)  $\mathcal{M} \subset \overline{C}(E)$ .

**Proof.** Suppose that  $S$  is a Polish space and  $J : S \rightarrow E$  is a continuous bijection such that  $J^{-1}$  is measurable. Then, we have that  $\{g \circ J : g \in \mathcal{M}\}$  s.p. on  $S$ , so by a change of variables

$$\int_E g(x) Q(dx) = \int_S g(J(u)) QJ(du), \quad \int_E g(x) P(dx) = \int_S g(J(u)) P J(du)$$

and either part (b) or (e) of [Theorem 11](#),

$$Q(J(A)) = P(J(A)) \quad \forall A \in \mathcal{B}(S).$$

However, this implies that  $Q = P$  on  $E$  since  $J$  is a Borel isomorphism.  $\square$

While the theory of probability measures on Lusin spaces is very important, it is quite a difficult area. Our corollary here and our earlier result for weak convergence on Lusin spaces will hopefully make some problems on these spaces slightly easier.

**Remark 5.** Apropos of the separability type requirement in [Theorem 11](#), Billingsley [2] notes in his Appendix III that it is unknown if there are metric space probabilities having nonseparable support. If this is impossible, then separability is not required in Alternative (b) of [Theorem 11](#). Otherwise, there would be distinct  $P, Q$  (at least one non-Radon) on a nonseparable metric space  $E$  such that  $\int_E g dQ = \int_E g dP$  for all  $g$  in a *strongly* separating class  $\mathcal{M}$ . Indeed, by the proof of Theorem 2 in Appendix III of [2], the existence of a probability with nonseparable support implies the existence of a probability on a discrete space that has no point masses. Now, suppose that  $P_1, P_2$  are two such probabilities on disjoint discrete spaces  $A_1, A_2$ ;  $E = A_1 \cup A_2$  and

$$P \doteq \begin{cases} P_1 & \text{on } A_1 \\ 0 & \text{on } A_2 \end{cases} \quad \text{and} \quad Q \doteq \begin{cases} 0 & \text{on } A_1 \\ P_2 & \text{on } A_2. \end{cases}$$

Letting  $\mathcal{M} \subset \overline{C}(E)$  be the algebra generated by the indicator functions of singletons, one finds that small enough open balls give rise to the singletons and  $\mathcal{M}$  s.s.p. However,  $P \neq Q$  and neither have point masses in  $E$ , so  $\int_E g dQ = \int_E g dP = 0$  for  $g \in \mathcal{M}$ .

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## References

- [1] A.G. Bhatt, R.L. Karandikar, Weak convergence to a Markov process: the martingale approach, *Probab. Theory Related Fields* 96 (1993) 335–351.
- [2] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York, 1968.

- [3] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York, 1999.
- [4] R.M. Dudley, *Real Analysis and Probability*, Wadsworth & Brooks/Cole, Pacific Grove, CA, 1989.
- [5] S. Ethier, T. Kurtz, *Markov Processes, Characterization and Convergence*, Wiley, New York, 1986.
- [6] A. Jakubowski, On the Skorokhod topology, *Ann. Inst. Henri Poincaré* 22 (1986) 263–285.
- [7] G. Kallianpur, J. Xiong, *Stochastic Differential Equations in Infinite Dimensional Spaces*, in: IMS Lecture Notes—Monograph Series, vol. 26, Institute of Mathematical Statistics, 1995.
- [8] P.A. Meyer, W.A. Zheng, Tightness criteria for laws of semimartingales, *Ann. Inst. Henri Poincaré B* 20 (1984) 353–372.
- [9] I. Mitoma, Tightness of probabilities on  $C([0, 1]; S')$  and  $D([0, 1]; S')$ , *Ann. Probab.* 11 (1983) 989–999.
- [10] J.R. Munkres, *Topology, A First Course*, Prentice-Hall, Englewood Cliffs, New Jersey, 1975.
- [11] K.R. Parthasarathy, *Probability Measures on Metric Spaces*, Academic Press, New York, 1967.