

On regularity of invariant measures of multivalued stochastic differential equations

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Abstract

We prove that the invariant measure associated to a multivalued stochastic differential equation is absolutely continuous with respect to the Lebesgue measure with a density $\rho \in \mathbf{B}_{\text{loc}}^{s,p,q}$ for all $1 < p < d/(d-1)$, $0 < s < 1$ and $q \geq 1$, and $\rho \in W^{1,q}(\mathcal{O})$ for all $q > 1$ provided that $\mathcal{O} \subseteq \text{Int}(D(A))$. In particular, ρ is locally α -Hölder continuous in $\text{Int}(D(A))$ for all $\alpha < 1$.

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1. Introduction

This paper is a continuation of [13] and we are concerned with the following multivalued stochastic differential equation (MSDE in short) on the Euclidean space \mathbb{R}^d :

$$\begin{cases} dX_t + A(X_t)dt \ni b(X_t)dt + \sigma(X_t)dW_t \\ X_0 = x \in \overline{D(A)}, \end{cases} \quad (1)$$

where A is a multi-valued maximal monotone operator on \mathbb{R}^d with the domain $D(A) := \{x \in \mathbb{R}^d; A(x) \neq \emptyset\}$, $b: \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\sigma: \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ and W is a d -dimensional standard Brownian

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motion. Such an equation is called a multivalued stochastic differential equation (MSDE in short) for which the existence and uniqueness of solutions has been obtained in [10] under the usual Lipschitz and linear growth assumptions. It is then remarked that the existence and uniqueness result remains true with the log-Lipschitz instead of the Lipschitz assumption in [13], and there we have also established the existence and uniqueness of an invariant measure, of the Markov semigroup associated to (1) under standard assumptions.

The aim of the present paper is to further study the regularity of the invariant measures, namely its absolute continuity with respect to the Lebesgue measure and the smoothness of the corresponding Radom–Nikodym derivative.

For the regularity of the invariant measures of stochastic differential equations with or without reflection (SDEs or RSDEs), there have been rapidly increasing interests and numerous results have been produced in the past two decades; see [7] for an excellent survey and references therein and in particular [2,4–6,11] for subjects closely related to the problem considered in the present paper.

As far as the analysis concerning invariant measure is involved, perhaps the most significant difference between SDEs and RSDEs on one side, and MSDEs on the other side, is that there is a direct connection between the former and the PDE theory, which is not the case for the latter. For this reason, it is not possible, unlike in [2,7], to deduce necessary estimates directly from the PDE theory. Rather, we shall prove some uniform estimates on the Yosida approximation and use weak convergence arguments.

In [2], Barbu and Da Prato considered a special case, namely they studied the problem associated to the following reflected stochastic differential equation with a constant diffusion part:

$$dX_t + \partial I_{\mathcal{O}}(X_t)dt \ni b(X_t)dt + dW_t, \quad X_0 = x.$$

They proved, under adequate assumptions, that the invariant measure is absolutely continuous with respect to the Lebesgue measure with a density $\rho \in L^1(\mathbb{R}^d)$ and $\rho \in \mathbf{W}_{loc}^{1,q}(\mathcal{O}^\circ)$ for all $q > 1$. Their approach depends on the connection between reflected stochastic differential equations and elliptic Neumann problems.

Our main result is presented and proved in Section 3. Roughly speaking, we will be able to show that the invariant measure is absolutely continuous with respect to the Lebesgue measure and the density is Besov and Hölder regular under reasonable assumptions. So we extend the corresponding results of [2] to a more general context and we do not require the diffusion part to be constant.

Throughout the paper, $\mathbf{W}^{k,p}$ denotes the usual Sobolev space on \mathbb{R}^d and $\mathbf{W}_{loc}^{k,p}$ its local version, $\mathcal{C}_b(\mathbb{R}^d)$ and $\mathcal{B}_b(\mathbb{R}^d)$ denote the spaces of bounded, continuous functions and bounded, measurable functions on \mathbb{R}^d respectively. $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel σ -algebra of \mathbb{R}^d and B_m is the closed ball with radius m , centered at the origin.

2. Preliminaries

We begin by fixing some notions and notations. Let $\{X_t(x), t \geq 0, x \in \mathbb{R}^d\}$ be a family of Markov processes with state space \mathbb{R}^d and transition probability $P_t(x, E)$, $E \in \mathcal{B}(\mathbb{R}^d)$. Let $(P_t)_{t \geq 0}$ be the corresponding semigroup defined by

$$P_t f(x) := \mathbf{E}f(X_t(x)), \quad t > 0, f \in \mathcal{B}_b(\mathbb{R}^d).$$

Definition 2.1. A measure μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is called an invariant measure for $(P_t)_{t \geq 0}$ if

$$\int_{\mathbb{R}^d} P_t(x, E) \mu(dx) = \mu(E), \quad \forall t > 0, E \in \mathcal{B}(\mathbb{R}^d).$$

We will use the classical Khasminskii and Doob theorems (cf. [11]).

Theorem 2.2. If P_t is strong Feller and irreducible, then it possesses at most one invariant measure and this measure is ergodic.

Theorem 2.3. If P_t is strong Feller and irreducible and μ is the invariant measure for it, then it is strongly mixing and

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t P_s \varphi(x) ds = \int \varphi(y) \mu(dy), \quad \forall x \in \mathbb{R}^d, \varphi \in C_b(\mathbb{R}^d).$$

We also give some notions and properties of multivalued maximal monotone operators.

Definition 2.4. Given a multi-valued operator A from \mathbb{R}^d to $2^{\mathbb{R}^d}$, define:

$$D(A) := \{x \in \mathbb{R}^d : A(x) \neq \emptyset\},$$

$$\text{Gr}(A) := \{(x, y) \in \mathbb{R}^{2d} : x \in \mathbb{R}^d, y \in A(x)\}.$$

(1) A multivalued operator A is called monotone if

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0, \quad \forall (x_1, y_1), (x_2, y_2) \in \text{Gr}(A).$$

(2) A monotone operator A is called maximal monotone if and only if

$$(x_1, y_1) \in \text{Gr}(A) \Leftrightarrow \langle y_1 - y_2, x_1 - x_2 \rangle \geq 0, \quad \forall (x_2, y_2) \in \text{Gr}(A).$$

Some useful facts and properties of the multivalued maximal monotone operator A are listed in the proposition below. For more details, we refer to [10].

Proposition 2.5. Let A be a multi-valued maximal monotone operator on \mathbb{R}^d . Then:

- (i) $\text{Int}(D(A))$ and $\overline{D(A)}$ are convex subsets of \mathbb{R}^d and $\text{Int}(D(A)) = \text{Int}(\overline{D(A)})$.
- (ii) For each $x \in D(A)$, $A(x)$ is a closed and convex subset of \mathbb{R}^d . Let $A^\circ(x) := \text{proj}_{A(x)}(0)$ be the minimal section of A , where $\text{proj}_{\mathcal{O}}$ denotes the projection on a closed convex subset $\mathcal{O} \subset \mathbb{R}^d$ and $\text{proj}_{\emptyset}(0) = \infty$. Then

$$x \in D(A) \Leftrightarrow |A^\circ(x)| < +\infty.$$
- (iii) The resolvent operator $J_n := (1 + \frac{1}{n}A)^{-1}$ is a single-valued and contractive operator defined on \mathbb{R}^d and valued in $D(A)$.
- (iv) The Yosida approximation $A_n := n(I - J_n)$ is a single-valued, maximal monotone and Lipschitz continuous function with Lipschitz constant n . Moreover, for every $x \in D(A)$, as $n \nearrow \infty$,

$$A_n(x) \rightarrow A^\circ(x)$$

and

$$|A_n(x)| \nearrow |A^\circ(x)| \quad \text{if } x \in D(A),$$

$$|A_n(x)| \nearrow +\infty \quad \text{if } x \notin D(A).$$

The following definition and proposition are taken from [10].

Definition 2.6. A pair of continuous and (\mathcal{F}_t) -adapted processes (X, H) is called a solution of (1) if

- (i) $X_0 = x$ and for all $t \geq 0$, $X_t \in \overline{D(A)}$ a.s.;
- (ii) H is of locally finite variation and $H_0 = 0$ a.s.;
- (iii) $dX_t = b(X_t)dt + \sigma(X_t)dW_t - dH_t$, $0 \leq t < \infty$, a.s.;
- (iv) For any continuous and (\mathcal{F}_t) -adapted functions (α, β) with $(\alpha_t, \beta_t) \in \text{Gr}(A)$, $\forall t \in [0, +\infty)$, the measure $\langle X_t - \alpha_t, dH_t - \beta_t dt \rangle \geq 0$.

Proposition 2.7. Let A be a multivalued maximal monotone operator, (X, H) and (X', H') be continuous functions with $X, X' \in \overline{D(A)}$, H, H' being of finite variation. Let (α, β) be continuous functions satisfying

$$(\alpha_t, \beta_t) \in \text{Gr}(A), \quad \forall t \geq 0.$$

If

$$\begin{aligned} \langle X_t - \alpha_t, dH_t - \beta_t dt \rangle &\geq 0, \\ \langle X'_t - \alpha_t, dH'_t - \beta_t dt \rangle &\geq 0, \end{aligned}$$

then

$$\langle X_t - X'_t, dH_t - dH'_t \rangle \geq 0.$$

3. Main result

We make the following assumptions.

(H1) $0 \in \text{Int}(D(A))$.

(H2) b is continuous and for any $x, y \in \mathbb{R}^d$, there exists a constant $\lambda_1 > 0$ such that

$$2\langle x - y, b(x) - b(y) \rangle + \|\sigma(x) - \sigma(y)\|^2 \leq \lambda_1 |x - y|^2.$$

(H3) There exists a $\lambda_2 > 0$ and an $m \in \mathbb{N}$ such that for every $x \in \mathbb{R}^d$,

$$|b(x)| + \|\sigma(x)\| \leq \lambda_2(1 + |x|^m).$$

(H4) $\exists \lambda_0 > 0$ such that

$$\lambda_0 I < \sigma \sigma^* < \lambda_0^{-1} I.$$

(H5) There exist constants $\lambda_3 > 0$, $\lambda_4 \geq 0$ such that for all $x \in \mathbb{R}^d$

$$2\langle x, b(x) \rangle + \|\sigma(x)\|_{\text{HS}}^2 \leq -\lambda_3 |x|^2 + \lambda_4.$$

(H6) For $a := \frac{1}{2} \sigma \sigma^*$, $a_{ij} \in \mathbf{W}_{loc}^{2,p}$, $p > 1$ and a_{ij} is α -Hölder continuous, $0 < \alpha < 1$.

Denote by $X_t(x)$ the solution to Eq. (1) with initial value x . Set

$$\begin{aligned} P_t f(x) &= \mathbf{E} f(X_t(x)), \quad f \in \mathcal{B}_b(\mathbb{R}^d), \\ P_t(x, B) &= \mathbf{P}(X_t(x) \in B), \quad B \in \mathcal{B}(\mathbb{R}^d). \end{aligned}$$

We have proved in [13] that under the assumptions above, there exists a unique invariant measure, say μ , for $(P_t)_{t \geq 0}$.

Consider the Yosida approximation of Eq. (1):

$$X_t^n = x + \int_0^t b(X_s^n) ds + \int_0^t \sigma(X_s^n) dW_s - \int_0^t A_n(X_s^n) ds, \quad (2)$$

where A_n is the Yosida approximation of the operator A . According to Proposition 2.5, A_n is Lipschitz continuous with the constant n . Hence there exists a unique solution to Eq. (2), denoted by $X_t^n(x)$. Moreover, it has been proved in [10] that

$$\lim_{n \rightarrow \infty} X_t^n(x) = X_t(x) \quad (3)$$

in distribution for all $x \in D(A)$ and $t \geq 0$.

Then the transition semigroup $(P_t)_{t \geq 0}$ associated to $(X_t^n(x))_{x \in \mathbb{R}^d, t \geq 0}$ is given by

$$P_t^n f(x) = \mathbf{E}[f(X_t^n(x))], \quad f \in \mathcal{C}_b(\mathbb{R}^d).$$

By (H1)–(H5) and the Lipschitz continuity of A_n , according to [6,7], for every n , $(P_t^n)_{t \geq 0}$ has a unique invariant probability measure μ_n and P_t^n has a unique extension to a \mathcal{C}_0 contraction semigroup in $L^2(\mathbb{R}^d, \mu_n)$. Its infinitesimal generator is given by

$$L_n f(x) = a_{ij}(x) \partial_i \partial_j f(x) + \partial_i f(x) [b(x) - A_n(x)]_i, \quad \forall f \in \mathcal{C}_b^2(\mathbb{R}^d),$$

where $\partial_i f := \frac{\partial}{\partial x_i} f(x)$ and $a = \frac{1}{2} \sigma \sigma^*$. Moreover, the invariant measure μ_n is absolutely continuous with respect to the Lebesgue measure dx with the Radon–Nikodym derivative $\rho_n \in \mathbf{W}_{loc}^{1,q}(\mathbb{R}^d)$ for all $q > 1$ and, in particular, ρ_n is locally Hölder continuous. Here, for a domain $D \subset \mathbb{R}^d$, $\mathbf{W}^{1,q}(D)$ is the usual Sobolev space of functions and $\mathbf{W}_{loc}^{1,q}(D)$ denotes the space of all such f that $f \in \mathbf{W}^{1,q}(D')$ for every D' satisfying $\overline{D'} \subset D$.

Furthermore, we have the following estimate on the solution to Eq. (2).

Proposition 3.1. Assume that (H1) and (H4)–(H5) hold. Then for every $p \geq 1$,

$$\begin{aligned} \sup_n \int |y|^p \mu_n(dy) &< \infty, \\ \sup_n \int |A_n(y)| \mu_n(dy) &< \infty. \end{aligned}$$

Proof. According to [10, Lemma 5.4], there exist $\alpha > 0$ and $\kappa \geq 0$, independent of n , such that

$$\langle x, A_n(x) \rangle \geq \alpha |A_n(x)| - \kappa |x| - \alpha \kappa.$$

Hence by Itô's formula, we have

$$\begin{aligned} |X_t^n(x)|^p &= |x|^p + p \int_0^t |X_s^n(x)|^{p-2} \langle X_s^n(x), b(X_s^n(x)) \rangle ds \\ &\quad + \frac{1}{2} \int_0^t \sum_{i,j} f_{ij}(X_s^n(x)) (\sigma \sigma^*(X_s^n(x)))_{ij} ds \\ &\quad + p \int_0^t |X_s^n(x)|^{p-2} \langle X_s^n(x), \sigma(X_s^n(x)) \rangle dW_s \\ &\quad - p \int_0^t |X_s^n(x)|^{p-2} \langle X_s^n(x), A_n(X_s^n(x)) \rangle ds \\ &\leq |x|^p + \frac{p(p-1)}{2} \int_0^t |X_s^n(x)|^{p-2} (-\lambda_3 |X_s^n(x)|^2 + \lambda_4) ds \end{aligned}$$

$$\begin{aligned}
& + p \int_0^t |X_s^n(x)|^{p-2} \langle X_s^n(x), \sigma(X_s^n(x)) \rangle dW_s \\
& + p \int_0^t |X_s^n(x)|^{p-2} (-\alpha |A_n(X_s^n(x))| + \kappa |X_s^n(x)| + \alpha \kappa) ds,
\end{aligned}$$

where

$$f_{ij}(z) := p|z|^{p-2}\delta_{ij} + p(p-2)|z|^{p-4}z_i z_j.$$

By Young's inequality, we get

$$\begin{aligned}
|X_t^n(x)|^p & \leq |x|^p - \frac{p(p-1)\lambda_3}{4} \int_0^t |X_s^n(x)|^p ds \\
& + C(p, \lambda_3, \lambda_4, \alpha, \kappa)t + p \int_0^t |X_s^n(x)|^{p-2} \langle X_s^n(x), \sigma(X_s^n(x)) \rangle dW_s.
\end{aligned}$$

Thus

$$\mathbf{E}|X_t^n(x)|^p \leq |x|^p + C(p, \lambda_3, \lambda_4, \alpha, \kappa)t - \frac{p(p-1)\lambda_3}{4} \int_0^t \mathbf{E}|X_s^n(x)|^p ds,$$

which immediately yields

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbf{E}|X_s^n(x)|^p ds \leq \frac{4}{\lambda_3 p(p-1)} C(p, \lambda_3, \lambda_4, \alpha, \kappa) < \infty.$$

For $m > 0$, set $\varphi_m(y) = m \wedge |y|^p$. Then by [Theorem 2.3](#), we have for every x ,

$$\begin{aligned}
\int \varphi_m(y) \mu_n(dy) & = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t P_s \varphi_m(x) ds \\
& = \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbf{E} \varphi_m(X_s^n(x)) ds \\
& \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbf{E} |X_s^n|^p ds \\
& \leq \frac{4}{\lambda_3 p(p-1)} C(p, \lambda_3, \lambda_4, \alpha, \kappa) < \infty
\end{aligned}$$

which, letting $m \rightarrow \infty$, leads to

$$\int |y|^p \mu_n(dy) < C_p.$$

Hence the desired result follows since C_p is independent of n .

In particular, by taking $p = 2$ in the above calculus, we have

$$\begin{aligned}
|X_t^n|^2 & = |x|^2 + 2 \int_0^t \langle X_s^n, b(X_s^n) \rangle ds + 2 \int_0^t \langle X_s^n, \sigma(X_s^n) dW_s \rangle \\
& - 2 \int_0^t \langle X_s^n, dK_s^n \rangle + \int_0^t \|\sigma(X_s^n)\|^2 ds \\
& \leq |x|^2 + 2 \int_0^t \langle X_s^n, b(X_s^n) \rangle ds + 2 \int_0^t \langle X_s^n, \sigma(X_s^n) dW_s \rangle \\
& + \int_0^t \|\sigma(X_s^n)\|^2 ds - \alpha \int_0^t |A_n(X_s^n)| ds + \kappa \int_0^t |X_s^n|^2 ds + \alpha \kappa t.
\end{aligned}$$

Thus

$$\alpha \mathbf{E} \int_0^t |A_n(X_s^n)| ds \leq |x|^2 + \mathbf{E} \int_0^t (-\lambda_3 |X_s^n|^2 + \lambda_4) ds + \kappa \mathbf{E} \int_0^t |X_s^n| ds + \alpha \kappa t.$$

Consequently,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \mathbf{E} \int_0^t |A_n(X_s^n)| ds \leq C_0 < \infty,$$

which in turn implies by the same reasoning as above that

$$\int |A_n(y)| \mu_n(dy) < \infty. \quad \square$$

To proceed, we will need the following materials. For more details, we refer to [12].

Let Γ be the Green function associated to $a_{ij}(x) \partial_{ij}^2$. That is

$$\Gamma(x, y) = \begin{cases} c_d B(x, y)^{\frac{2-d}{2}} [\det(a^{ij}(y))]^{-\frac{1}{2}}, & d > 2, \\ c_2 \log[B(x, y)]^{-\frac{1}{2}} [\det(a^{ij}(y))]^{\frac{1}{2}}, & d = 2, \end{cases}$$

where $c_d = [(d-2)\omega_d]^{-1}$ for $d > 2$, ω_d being the surface area of the unit sphere of \mathbb{R}^d , $c_2 = 2\pi$, $(a^{ij}) = (a_{ij})^{-1}$ and

$$B(x, y) := (x - y)'(a^{ij}(y))(x - y).$$

It is well known that

$$\frac{\partial^k}{\partial x_i} \Gamma(x, y) = O(|x - y|^{2-k-d}), \quad k = 0, 1, 2, \quad i = 1, \dots, d. \quad (4)$$

Denote by $\mathbf{B}^{s,p,q}(D)$ the Besov space over D , i.e.,

$$\mathbf{B}^{s,p,q}(D) = (\mathbf{L}^p(D), \mathbf{W}^{m,p}(D))_{s/m,q;K},$$

where the right hand side stands for the intermediate space produced by the K -methods (c.f., e.g., [1,3]). The norm in $\mathbf{B}^{s,p,q}(D)$ is given by

$$\|f\|_{s,p,q} := \left(\int_0^1 (\varepsilon^{-s} K(\varepsilon; f))^q \frac{d\varepsilon}{\varepsilon} \right)^{1/q},$$

where

$$K(\varepsilon; f) := \inf\{\|f_1\|_p + \varepsilon\|f_2\|_{1,p} : f = f_1 + f_2, f_1 \in \mathbf{L}^p(D), f_2 \in \mathbf{W}^{1,p}(D)\}.$$

We have

Lemma 3.2. *Let D be a bounded domain and $f \in L^1(D)$. Then*

$$g(y) := \int_D \Gamma(x, y) f(x) dx, \quad h(y) := \int_D \partial_i \Gamma(x, y) f(x) dx$$

are in the Besov spaces $\mathbf{B}^{s,p,1}(D)$ for $1 < p < d/(d-1)$, $s \in (0, 2)$ and $s \in (0, 1)$ respectively and

$$\|g\|_{s,p,1} \leq C \|f\|_1, \quad s \in (0, 2), \quad p \in \left(1, \frac{d}{d-1}\right),$$

$$\|h\|_{s,p,1} \leq C\|f\|_1, \quad s \in (0, 1), \quad p \in \left(1, \frac{d}{d-1}\right),$$

where C depends only on s and p . Here and below $\partial_i \Gamma(x, y) := \frac{\partial}{\partial x_i} \Gamma(x, y)$.

Proof. We shall prove the second conclusion, the first one being similar. Let

$$\Gamma_\varepsilon(x, y) := \begin{cases} -\frac{\omega_d}{2}(x_j - y_j)a^{ij}(y)B(x, y)^{-\frac{d}{2}}[\det(a^{ij}(y))]^{-\frac{1}{2}}, & \text{if } B(x, y) \geq \varepsilon^2, \\ -\frac{\omega_d}{2}(x_j - y_j)a^{ij}(y)\varepsilon^{-d}[\det(a^{ij}(y))]^{-\frac{1}{2}}, & \text{if } B(x, y) \leq \varepsilon^2. \end{cases}$$

$$h_\varepsilon(y) := \int_D \Gamma_\varepsilon(x, y)f(x)dx.$$

Then it is easy to see

$$|h_\varepsilon(y) - h(y)| \leq C \int_D 1_{(0,\varepsilon)}(B(x, y)) \left(\varepsilon^{-d} - B(x, y)^{-\frac{d}{2}} \right) a^{ij}(x_j - y_j) \|f(x)\| dx.$$

Hence by the Minkowski inequality,

$$\begin{aligned} \|h_\varepsilon - h\|_p &\leq C \int_D \|1_{(0,\varepsilon)}(B(x, \cdot)) \left(\varepsilon^{-d} - B(x, \cdot)^{-\frac{d}{2}} \right) a^{ij}(x_j - \cdot)_p \|_p \|f(x)\| dx \\ &\leq C\varepsilon \|f\|_1 \left(\int_0^\varepsilon |\varepsilon^{-d} - r^{-d}|^p r^{p+d-1} dr \right)^{1/p} < \infty, \\ &\quad \text{for } 1 < p < d/(d-1). \end{aligned}$$

On the other hand, since $\Gamma_\varepsilon(x, y)$ is Lipschitz in y and

$$\frac{\partial}{\partial y_k} \Gamma_\varepsilon(x, y) \leq \begin{cases} C|B(x, y)|^{-\frac{d}{2}}, & |B(x, y)| \geq \varepsilon^2, \\ C\varepsilon^{-d}, & |B(x, y)| \leq \varepsilon^2. \end{cases}$$

$h_\varepsilon(y)$ is Lipschitz and since D is bounded,

$$\begin{aligned} \left| \frac{\partial}{\partial y_k} h_\varepsilon(y) \right| &\leq \left\| \frac{\partial}{\partial y_k} \Gamma_\varepsilon(\cdot, y) \right\|_1 \int_D |f(x)| dx \\ &\leq C \int_\varepsilon^1 r^{-1} dr + C \int_0^\varepsilon \varepsilon^{-1} dr \\ &\leq C(1 - \log \varepsilon). \end{aligned}$$

Hence

$$\left\| \frac{\partial}{\partial y_k} h_\varepsilon \right\|_p \leq -C \log \varepsilon.$$

Thus

$$K(\varepsilon, h) \leq -C\varepsilon \log \varepsilon.$$

Consequently,

$$\int_0^1 [\varepsilon^{-s} K(\varepsilon, h)] \frac{d\varepsilon}{\varepsilon} \leq -C \int_0^1 \varepsilon^{-s} \varepsilon \log \varepsilon \frac{d\varepsilon}{\varepsilon} < \infty,$$

as desired. \square

Remark 3.3. From the proof, we can actually get for any $q > 1$,

$$\int_0^1 [\varepsilon^{-s} K(\varepsilon, h)]^q \frac{d\varepsilon}{\varepsilon} \leq C \int_0^1 (\varepsilon^{-s} \varepsilon \log \varepsilon)^q \frac{d\varepsilon}{\varepsilon} < \infty.$$

Proposition 3.4. The sequence $\{\rho_n\}$ is bounded in $\mathbf{B}_{loc}^{s,p,1}$, $1 < p < d/(d-1)$, $0 < s < 1$.

Proof. Let $\psi \in C_0^\infty(B_R)$ be such that

$$\psi(x) = 1, \quad \text{for } |x| \leq R_1 < R.$$

By (H2)–(H3), for any n ,

$$\rho_n \in \mathbf{W}_{loc}^{1,p}(\mathbb{R}^d), \quad p > 1,$$

and is a weak solution to the equation below (cf. Remark 2.14, [6]):

$$a_{ij} \partial_{ij}^2 \rho_n + 2 \partial_i a_{ij} \partial_j \rho_n - \operatorname{div}((b - A_n) \rho_n) + \partial_{ij}^2 a_{ij} \rho_n = 0, \quad (5)$$

where the standard summation rule is applied for repeated indices. A direct calculation then gives:

$$\begin{aligned} a_{ij} \partial_{ij}^2 (\psi \rho_n) &= -2 \psi \partial_i a_{ij} \partial_j \rho_n + \psi \operatorname{div}((b - A_n) \rho_n) - \psi \partial_{ij}^2 a_{ij} \rho_n \\ &\quad + 2 a_{ij} \partial_i \psi \partial_j \rho_n + \rho_n a_{ij} \partial_{ij}^2 \psi. \end{aligned}$$

By the representation formula for general elliptic operators (see, e.g., [12, Chap. 5, Section 6]),

$$\begin{aligned} \rho_n(y) \psi(y) &= - \int \Gamma(x, y) (a_{ij} \partial_{ij}^2 (\rho_n \psi)(x) + \partial_j a_{ij} \partial_i (\rho_n \psi) + \partial_{ij}^2 a_{ij} \rho_n \psi) dx \\ &= - \int \Gamma(x, y) [-2 \psi \partial_i a_{ij} \partial_j \rho_n + \psi \operatorname{div}((b - A_n) \rho_n) \\ &\quad - \psi \partial_{ij}^2 a_{ij} \rho_n + 2 a_{ij} \partial_i \psi \partial_j \rho_n + \rho_n a_{ij} \partial_{ij}^2 \psi + \partial_j a_{ij} \partial_i (\rho_n \psi) + \partial_{ij}^2 a_{ij} \rho_n \psi] dx \\ &:= - \sum_{i=1}^7 \int \Gamma(x, y) f_i(x) dx. \end{aligned}$$

Integration by parts gives

$$\begin{aligned} \int \Gamma(x, y) f_1(x) dx &= 2 \int \partial_j [\Gamma(x, y) \psi(x) \partial_i a_{ij}(x)] \rho_n(x) dx \\ &= 2 \int \partial_j \Gamma(x, y) \psi(x) \partial_i a_{ij}(x) \rho_n(x) dx \\ &\quad + 2 \int \Gamma(x, y) \partial_j [\psi(x) \partial_i a_{ij}(x)] \rho_n(x) dx. \end{aligned}$$

By Proposition 3.1 and (H5)–(H6),

$$\sup_n \{\|\psi \partial_i a_{ij} \rho_n\|_1 + \|\partial_j [\psi \partial_i a_{ij}] \rho_n\|_1\} < \infty,$$

Hence by Lemma 3.2, we have

$$\int \Gamma(x, y) f_1(x) dx \in \mathbf{B}^{s,p,1}(B_R), \quad 1 < p < d/(d-1), 0 < s < 1,$$

uniformly.

Similarly, by integration by parts,

$$\begin{aligned} - \int \Gamma(x, y) f_2(x) dx &= \int \partial_i (\Gamma(x, y) \psi(x)) (b - A_n)_i \rho_n dx \\ &= \int \partial_i \Gamma(x, y) \psi(x) (b - A_n)_i \rho_n dx + \int \Gamma(x, y) \partial_i \psi(x) (b - A_n)_i \rho_n dx. \end{aligned}$$

By Proposition 3.1,

$$\begin{aligned} \sup_n \int |\psi(b - A_n)_i \rho_n| dx &\leq C \sup_n \int |(b - A_n)_i| \rho_n dx < \infty, \\ \sup_n \int |\partial_i \psi(b - A_n)_i \rho_n| dx &\leq C \sup_n \int |(b - A_n)_i| \rho_n dx < \infty. \end{aligned}$$

Thus by Lemma 3.2 again, we have uniformly

$$\int \Gamma(x, y) f_2(x) dx \in \mathbf{B}^{s,p,1}(B_R), \quad 1 < p < d/(d-1).$$

Other terms can be treated in the same way and the Proposition is thus proven. \square

Theorem 3.5. *The invariant measure μ has a density $\rho \in \mathbf{B}_{\text{loc}}^{s,p,q}(\text{Int}(D(A)))$ for $p \in (1, d/(d-1))$, $s \in (0, 1)$ and all $q \geq 1$. Moreover, if \mathcal{O} is an open set with compact closure in $\text{Int}(D(A))$, then $\rho \in \mathbf{W}^{1,q}(\mathcal{O})$ for all $q > 1$. In particular, $\mathbf{B}_{\text{loc}}^{s,p,q}(\text{Int}(D(A))) \subset \cap_q \mathbf{W}_{\text{loc}}^{1,q}(\text{Int}(D(A)))$ and ρ is locally α -Hölder continuous on \mathcal{O} for all $\alpha < 1$.*

Proof. Fix $1 < p < d/(d-1)$. For every $R > 0$, since $\{\rho_n\}$ is bounded in $\mathbf{B}^{s,p,1}(B_R)$, there exists a subsequence n_k such that ρ^{n_k} converges weakly to some ρ in $\mathbf{B}^{s,p,1}(B_R)$. Hence a function ρ is well defined on \mathbb{R}^d which we now show to be the density of μ . In fact, let f be a bounded continuous function on \mathbb{R}^d . Since

$$X_t^n(x) \xrightarrow{d} X_t(x), \quad \forall t \geq 0, x \in \overline{D(A)},$$

we have

$$\mathbf{E}[f(X_t^n(x))] \rightarrow \mathbf{E}[f(X_t(x))], \quad \forall t \geq 0, x \in \overline{D(A)}.$$

Now denoting by B_m the ball centered at 0 with radius m , we have

$$\begin{aligned} &\int \mathbf{E}[f(X_t^n(x))] \rho_n(x) dx - \int \mathbf{E}[f(X_t(x))] \rho(x) dx \\ &= \int_{B_m} \mathbf{E}[f(X_t^n(x))] \rho_n(x) dx - \int_{B_m} \mathbf{E}[f(X_t(x))] \rho(x) dx \\ &\quad + \int_{B_m^c} \mathbf{E}[f(X_t^n(x))] \rho_n(x) dx - \int_{B_m^c} \mathbf{E}[f(X_t(x))] \rho(x) dx. \end{aligned}$$

Since

$$\begin{aligned} \left| \int_{B_m^c} \mathbf{E}[f(X_t^n(x))] \rho_n(x) dx \right| &\leq C \int_{B_m^c} \rho_n(x) dx, \\ \left| \int_{B_m^c} \mathbf{E}[f(X_t(x))] \rho(x) dx \right| &\leq C \int_{B_m^c} \rho(x) dx, \end{aligned}$$

by Proposition 3.1, we have

$$\lim_{m \rightarrow \infty} \sup_n \left[\left| \int_{B_m^c} \mathbf{E}[f(X_t^n(x))] \rho_n(x) dx \right| + \left| \int_{B_m^c} \mathbf{E}[f(X_t(x))] \rho(x) dx \right| \right] = 0.$$

On the other hand,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{B_m} \mathbf{E}[f(X_t(x))] \rho(x) dx - \int_{B_m} \mathbf{E}[f(X_t^n(x))] \rho_n(x) dx \\ & \leq \lim_{n \rightarrow \infty} \left| \int_{B_m} \mathbf{E}[f(X_t(x))] (\rho(x) - \rho_n(x)) dx \right| \\ & \quad + \lim_{n \rightarrow \infty} \left(\int_{B_m} |\mathbf{E}[f(X_t(x))] - \mathbf{E}[f(X_t^n(x))]|^{p'} dx \right)^{1/p'} \left(\int_{B_m} |\rho_n(x)|^p dx \right)^{1/p} = 0, \end{aligned}$$

where $p' = p/(p-1)$. By letting first $n \rightarrow \infty$ and then $m \rightarrow \infty$, we obtain

$$\begin{aligned} \int \mathbf{E}[f(X(t, x))] \rho(x) dx &= \lim_{n \rightarrow \infty} \int \mathbf{E}[f(X_n(t, x))] \rho_n(x) dx \\ &= \lim_{n \rightarrow \infty} \int f(x) \rho_n(x) dx \\ &= \int f(x) \rho(x) dx. \end{aligned}$$

Consequently, $\rho(x)dx$ is an invariant measure for $(P_t)_{t \geq 0}$. By the uniqueness of the invariant measure, we have $\mu(dx) = \rho(x)dx$.

Recall that for any $u \in C_0^\infty(\text{Int}(D(A)))$,

$$\int_{\text{Int}(D(A))} a_{ij} \partial_{ij}^2 u \rho_n(x) dx + \int_{\text{Int}(D(A))} (b - A_n)_i \partial_i u \rho_n(x) dx = 0.$$

Choose a compact set A such that $\text{supp}(u) \subset A \subset \text{Int}(D(A))$ and $\mu(\partial A) = 0$. Then

$$\int_A a_{ij} \partial_{ij}^2 u \rho_n(x) dx + \int_A (b - A_n)_i \partial_i u \rho_n(x) dx = 0. \quad (6)$$

Let

$$\begin{aligned} v_0(dx) &:= |A|^{-1} dx, \\ v_n(dx) &:= \frac{\rho_n(x) dx}{\int_A \rho_n(x) dx}, \\ v(dx) &:= \frac{\rho(x) dx}{\int_A \rho(x) dx}. \end{aligned}$$

Then $v_n \rightarrow v$ weakly. Define

$$R_n := \int_A \left(\log \frac{dv_n}{dv_0} \right) dv_n.$$

By Lemma 3.2, we have

$$\sup_n R_n < \infty.$$

Also note that A^o is bounded on Λ by Brézis [9, Proposition 2.9]. Hence by Boué and Dupuis [8, Lemma 2.8], one can let $n \rightarrow \infty$ in (6) to obtain

$$\int_{\Lambda} a_{ij} \partial_{ij}^2 u \rho(x) dx + \int_{\Lambda} (b - A^o)_i \partial_i u \rho(x) dx = 0.$$

Consequently,

$$\int_{\text{Int}(D(A))} a_{ij} \partial_{ij}^2 u \rho(x) dx + \int_{\text{Int}(D(A))} (b - A^o)_i \partial_i u \rho(x) dx = 0. \quad (7)$$

Let $\mathcal{O} \subset \text{Int}(D(A))$ with compact closure in $\text{Int}(D(A))$ and let $\delta \in \mathcal{C}_0^\infty(\mathcal{O})$. Then by integration by parts and (H4),

$$\begin{aligned} \int_{\mathcal{O}} (a_{ij}) \nabla(\delta \rho) \cdot \nabla u dx &= \int_{\mathcal{O}} a_{ij} \partial_i (\delta \rho) \partial_j u dx \\ &= - \int_{\mathcal{O}} a_{ij} \partial_{ij}^2 u \delta \rho dx - \int_{\mathcal{O}} \partial_i a_{ij} \partial_j u \delta \rho dx \\ &= - \int_{\mathcal{O}} a_{ij} \partial_{ij}^2 (u \delta) \rho dx + \int_{\mathcal{O}} a_{ij} \partial_{ij}^2 \delta u \rho dx \\ &\quad + 2 \int_{\mathcal{O}} a_{ij} \partial_i u \partial_j \delta \rho dx - \int_{\mathcal{O}} \partial_i a_{ij} \partial_j (\delta u) \rho dx + \int_{\mathcal{O}} \partial_i a_{ij} \partial_j \delta u \rho dx \\ &= \int_{\mathcal{O}} [(b - A^o)_i - \partial_j a_{ij}] \partial_i (\delta u) \rho dx \\ &\quad + \int_{\mathcal{O}} (a_{ij} \partial_{ij}^2 \delta + \partial_i a_{ij} \partial_j \delta) \rho u dx + 2 \int_{\mathcal{O}} a_{ij} \partial_i \delta \partial_j \rho u dx. \end{aligned}$$

Thus by (H2), (H4) and Poincaré's inequality,

$$\left| \int_{\mathcal{O}} (a_{ij}) \nabla(\delta \rho) \cdot \nabla u dx \right| \leq C \|\nabla u\|_{L^{p'}(\mathcal{O}; dx)}, \quad \forall u \in \mathcal{C}_0^\infty(\mathcal{O}).$$

Therefore $(a_{ij}) \nabla(\delta \rho) \in L^p(\mathcal{O})$ (cf. [1, Chapter 3]), which yields that $\nabla(\delta \rho) \in L^p(\mathcal{O})$ by the uniform ellipticity of (a_{ij}) . Hence $\delta \rho \in \mathbf{W}_0^{1,p}(\mathcal{O})$ and furthermore $\rho \in \mathbf{W}^{1,p}(\mathcal{O}_1)$ for any domain \mathcal{O}_1 with $\bar{\mathcal{O}}_1 \subset \mathcal{O}$, which implies that $\rho \in L^{p_1}(\mathcal{O}_1)$ for $1/p_1 = 1/p - 1/d$. Continuing this process, we get that

$$\rho \in \mathbf{W}^{1,q}(\mathcal{O}'), \quad \forall q > 1$$

for any domain \mathcal{O}' satisfying $\bar{\mathcal{O}}' \subset \text{Int}(D(A))$. Now the last conclusion follows from the well-known Sobolev embedding theorem. \square

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