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Permanental vectors

Hana Kogan[a], Michael B. Marcus[b,*]

[a] City College of CUNY, 10031, United States
[b] City College and the CUNY Graduate Center, 10023-2750, United States

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Abstract

A permanental vector is a generalization of a vector with components that are squares of the components of a Gaussian vector, in the sense that the matrix that appears in the Laplace transform of the vector of Gaussian squares is not required to be either symmetric or positive definite. In addition, the power of the determinant in the Laplace transform of the vector of Gaussian squares, which is $-1/2$, is allowed to be any number less than zero.

It was not at all clear what vectors are permanental vectors. In this paper, we characterize all permanental vectors in R^3_+ and give applications to permanental vectors in R^n_+ and to the study of permanental processes.

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1. Introduction

A β-permanental vector $\theta := \{\theta_1, \dots, \theta_n\}$, is an R^n_+ valued random variable with Laplace transform

$$E\left(\exp\left(-\sum_{i=1}^n \alpha_i\theta_i\right)\right) = \frac{1}{|I+\alpha\Gamma|^\beta}, \tag{1.1}$$

* Corresponding author. Tel.: +1 6317252764; fax: +1 6316042275.
E-mail address: mbmarcus@optonline.net (M.B. Marcus).

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where I is the $n \times n$ identity matrix, α is the diagonal matrix with $\alpha_{i,i} = \alpha_i, \alpha_i \in R_+, 1 \le i \le n$, and $\Gamma = \{\Gamma(i,j)\}_{i,j=1}^n$ is an $n \times n$ matrix, $\beta > 0$ and $|I + \alpha\Gamma| > 0$ for all $\alpha \in R_+^n$. This last requirement implies that $|\Gamma| \ge 0$. (For a matrix A we use $|A|$ to denote the determinant of A.)

The fundamental question here is: for what $n \times n$ matrices is the right-hand side of (1.1) a Laplace transform? There are very well known cases in which the right-hand side of (1.1) is a Laplace transform. When $\beta = 1/2$ and Γ is symmetric and positive definite,

$$\theta = (G_1^2/2, \ldots, G_n^2/2), \tag{1.2}$$

where $(G_1, \ldots, G_n)$ is a Gaussian random variable with covariance Γ. (We sometimes refer to a vector like θ as a vector of Gaussian squares.)

The innovation in the question posed here, introduced by Vere-Jones [9], is that we consider all $\beta > 0$ and do not require that Γ is symmetric or positive definite.

Before we attempt to answer this question it is important to note that the matrix Γ is not unique. If D is any diagonal matrix with non-zero entries we have

$$|I + \alpha\Gamma| = |I + \alpha D\Gamma D^{-1}| = |I + \alpha D\Gamma^T D^{-1}|, \tag{1.3}$$

for all diagonal matrices α. The matrix Γ is said to be diagonally equivalent to Γ' if $\Gamma' = D\Gamma D^{-1}$ for some diagonal matrix D with non-zero entries. For a very large class of irreducible matrices Γ, it is known that the class of diagonally equivalent matrices are the only sources of non-uniqueness; see [6].

Sometimes one can take D to have diagonal entries ± 1. Such matrices are called signature matrices. It is obvious that if S is a signature matrix then $S = S^{-1}$. We also note that (1.3) may hold with $D = I$, the identity matrix even when $\Gamma \ne \Gamma'$. For example, if Γ and Γ' are $n \times n$ matrices with the same diagonal elements and all zeros below the diagonal, then (1.3) holds with $D = I$. In this case we say that Γ and Γ' are effectively equivalent. (We also note that we sometimes refer to Γ as a kernel for θ.)

Bapat and Griffiths [1,3] (see, also [7, Chapter 13]) completely describe the vectors of Gaussian squares for which, (1.1) is a Laplace transform for all $\beta > 0$. They do this in solving a classical problem posed by Lévy: when is a vector of Gaussian squares infinitely divisible? The answer is the following proposition.

Proposition 1.1. *A vector of Gaussian squares is infinitely divisible if and only if the covariance matrix Γ is diagonally equivalent to an M-matrix.*

A matrix $A = \{a_{i,j}\}$, is said to be an M matrix if

(1) $a_{i,j} \le 0$ for all $i \ne j$;
(2) A is nonsingular and $A^{-1} \ge 0$, (i.e., all the entries of A^{-1} are greater than or equal to zero).

Strictly speaking, knowing that a vector of Gaussian squares is infinitely divisible only asserts the existence of the Laplace transform for $\beta = k/(2n)$, for all integers $k, n \ge 1$. However the proof of Proposition 1.1 shows that (1.1) holds for all $\beta > 0$.

There are permanental vectors with kernels that are not diagonally equivalent to symmetric matrices. Eisenbaum and Kaspi, [2, Lemma 4.2] recognize that the Bapat–Griffiths necessary and sufficient condition for infinite divisibility in the case of symmetric kernels also works for non-symmetric kernels.

It is well known that positive definite symmetric matrices may or may not have an inverse that is diagonally equivalent to an M-matrix. On the other hand when $\beta = 1/2$, (1.1) is the Laplace transform of a vector of Gaussian squares. Based on these observations we divide the class of kernels Γ of permanental vectors into three categories.

1. Γ is diagonally equivalent to a symmetric positive definite matrix.
2. Γ^{-1} is diagonally equivalent to an M matrix.
3. Γ is not in class 1 or class 2.

Note that we use the expression positive definite to include what is sometimes called positive semi-definite. Also, we emphasize that classes 1 and 2 are not disjoint.

There is ample reason to think that there is an abundance of examples of kernels in class 3. One should be able to take a symmetric positive definite matrix not in class 2, and alter its off diagonal elements very slightly. One then might expect that the altered matrix would be in class 3. We worked for a long time to find an example of a kernel of a permanental vector in class 3, but were not successful. We then set out to fully characterize 3×3 matrices that are kernels of permanental vectors. The main result of this paper is the following theorem which states that for permanental vectors in R^3_+, class 3 is empty.

Theorem 1.1. *A 3×3 matrix that is the kernel of a permanental vector in R^3_+, belongs to class* 1*or class* 2*, or both.*

This result also applies to permanental vectors in R^n_+ in the sense that if $\theta = \{\theta_1, \dots, \theta_n\}$ is a permanental vector in R^n_+ then any three components of θ is a permanental vector in R^3_+.

Another consequence of Theorem 1.1 is that if a kernel of a permanental vector in R^3_+ is not diagonally equivalent to a kernel with positive entries then it is the kernel of a vector of Gaussian squares.

A permanental process $\{P(t), t \in T\}$ is a stochastic process with finite dimensional distributions that are permanental vectors. Eisenbaum and Kaspi study permanental processes in [2]. Roughly speaking, they show that the potential density of a Markov process is the kernel of a permanental process. (When this is the case we say that the permanental process is associated with the Markov process.) In fact they show that permanental processes are the missing link that allows the Dynkin Isomorphism Theorem to be extended to the local times of Markov processes that are not symmetric, [2, Corollary 3.5]. There are several intimate connections between permanental processes with a kernel that is the potential density of a Markov process and the Markov process itself.

Permanental processes are introduced by Vere-Jones in [9]. In [9, Proposition 4.5], he gives necessary and sufficient conditions for (1.1) to be the Laplace transform of the vector $(\theta_1, \dots, \theta_n)$ in terms of the modified resolvent matrix

$$\Gamma_r := \Gamma(I + r\Gamma)^{-1}, \tag{1.4}$$

where $r \geq 0$, and Γ is the matrix in (1.1).

Proposition 1.2 (*[9, Proposition 4.5]*)**.** *For* (1.1) *to represent the Laplace transform of a non-negative random vector it is necessary and sufficient that for all $r \geq 0$*

(i) *Γ_r exists and is β-positive definite;*
(ii) $\det(I + r\Gamma) > 0$.

Furthermore, given Γ_r, Proposition 1.2, (i) may hold for some values of β but not for others. (Item (ii) is equivalent to: all the real, non-zero, eigenvalues of Γ are positive.)

There is no point in giving the very complicated definition of β-positive definite here. One can refer to [9] or to [2,5] where it is repeated. It seems almost impossible to verify Proposition 1.2(i) unless all the entries of the matrix Γ_r are greater than or equal to zero, in which case (i) holds for all $\beta > 0$. In [2, Theorem 3.1], Eisenbaum and Kaspi point out that this is the case when Γ is associated with a Markov process and that Proposition 1.2(ii), also holds for these kernels.

If all the entries of the matrix Γ_r are not greater than or equal to zero verifying Proposition 1.2 (i) necessitates examining an infinite sequence of increasingly larger matrices derived from Γ_r. Otherwise we know no sufficient condition for the existence of a permanental vector that might be in class 3. One is given in [9, Proposition 4.6], but it is not correct. We discuss this in Remark 5.1.

There is a potentially important application of Proposition 1.2 if one can figure out how to verify (i). If the kernel of a permanental vector is in class 2, (1.1) is a Laplace transform for all $\beta > 0$. If the kernel of a permanental vector is in class 1 and not in class 2, then we only know that (1.1) is a Laplace transform for $\beta = 1/2$, and trivially, for all $\beta = k/2$, for integers $k \geq 1$. Possibly there exist other values of $\beta > 0$ for which (1.1) is a Laplace transform. Applying Proposition 1.2, which depends on β, would answer this question.

There are many other interesting applications of Theorem 1.1. The next result answers a question that started our interest in 3-dimensional permanental vectors. We point out in the beginning of this Introduction that the univariate marginals of a 1/2-permanental process are squares of normal random variables. It also follows from (1.1) that pairs (θ_i, θ_j), of a 1/2-permanental process, are equal in law to $(G_i^2/2, G_j^2/2)$, where (G_i, G_j) is a Gaussian vector with covariance matrix

$$\widetilde{\Gamma} = \begin{bmatrix} \Gamma(i,i) & (\Gamma(i,j)\Gamma(j,i))^{1/2} \\ (\Gamma(i,j)\Gamma(j,i))^{1/2} & \Gamma(j,j) \end{bmatrix}. \tag{1.5}$$

(See [8, Lemma 3.1].) It follows from this that

$$E(\theta_i) = \frac{\Gamma(i,i)}{2} \quad \text{and} \quad \operatorname{cov}\{\theta_i, \theta_j\} = \frac{\Gamma(i,j)\Gamma(j,i)}{2}. \tag{1.6}$$

Therefore, if

$$\Gamma(i,j)\Gamma(j,i) = 0 \quad \forall\, 1 \leq i \neq j \leq n, \tag{1.7}$$

the components of a 1/2-permanental process are pairwise independent.

Actually, (1.7) is a necessary and sufficient condition for the components of any β-permanental process to be pairwise independent. This is because in this case the determinant of $\widetilde{\Gamma}$ is a product of its diagonal elements and the right-hand side of (1.1)

$$|I + \alpha\Gamma|^{\beta} = |I + \alpha_i \Gamma(i,i)|^{\beta} |I + \alpha_j \Gamma(j,j)|^{\beta}. \tag{1.8}$$

We also know from [9, bottom of page 135] that for any β permanental process

$$\operatorname{cov}\{\theta_i, \theta_j\} = \beta\Gamma(i,j)\Gamma(j,i). \tag{1.9}$$

If Γ is symmetric and positive definite and $\theta = (G_1^2/2, \ldots, G_n^2/2)$, where $(G_1, \ldots, G_n)$ is a Gaussian random variable with covariance Γ, with $\Gamma(i,j) = \Gamma(j,i) = 0$, then the components of θ are independent. We asked ourselves the following question: "for a general β-permanental vector θ, that is not the square of a Gaussian vector, does (1.7) imply that the components of θ are independent?" The answer is yes. We prove the following theorem.

Theorem 1.2. *Let θ be an n-dimensional β-permanental vector with pairwise independent components. Then the components of θ are independent.*

It is clear that when $\beta = 1/2$ Theorem 1.2 implies that the only permanental vectors with independent components are those with components that are squares of independent Gaussian random variables.

The next result deals with a function that appears in sufficient conditions for the continuity of permanental processes in [8],

$$d(x, y) = \left(\Gamma(x, x) + \Gamma(y, y) - 2\left(\Gamma(x, y)\Gamma(y, x)\right)^{1/2}\right)^{1/2}. \tag{1.10}$$

If $\Gamma(x, y) = \Gamma(y, x)$ is the covariance of the Gaussian vector $\{G(t), t \in T\}$, then

$$d(x, y) = \left(E\left(G(x) - G(y)\right)^2\right)^{1/2}, \tag{1.11}$$

which is a metric on T. However, if $\Gamma(x, y)$ is the kernel of a permanental process and $\Gamma(x, y) \neq \Gamma(y, x)$ it was not clear whether or not $d(x, y)$ is a metric on T. We can now say that even in this case $d(x, y)$ is a metric on T.

Corollary 1.1. *Let $\{P(t), t \in T\}$ be a permanental process with kernel $\Gamma(x, y)$. The function $\{d(x, y), x, y \in T\}$ in (1.10) is a metric on T.*

In Section 2, we give many properties that are necessary for an $n \times n$ matrix to be the kernel of a permanental vector. In Section 3, we obtain an interesting property of the eigenvalues of 3×3 positive definite symmetric matrices that plays a critical role in the proof of Theorem 1.1. The proof of Theorem 1.1 uses completely different methods when the off diagonal elements of the kernel are all negative or all positive. These cases are considered separately in Sections 4 and 5. Sections 6 and 7 give, respectively, the proofs of Theorem 1.2 and Corollary 1.1.

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2. Preliminaries

If θ is a permanental vector in R^n_+ then any subset of its components, say of p components, is a permanental vector in R^p_+. For $p = 2$, the Laplace transform of the vector $\{\theta_i, \theta_j\}$ takes the form

$$E\left(\exp\left(-\frac{1}{2}\left(\alpha_i\theta_i + \alpha_j\theta_j\right)\right)\right) = \frac{1}{|I + \alpha\Gamma|^\beta} = (1 + \alpha_i\Gamma(i, i)) + \alpha_j\Gamma(j, j) + \alpha_i\alpha_j\left(\Gamma(i, i)\Gamma(j, j) - \Gamma(i, j)\Gamma(j, i)\right)^{-\beta}. \tag{2.1}$$

Taking $\alpha_i = \alpha_j$ sufficiently large, this implies that

$$\Gamma(i, i)\Gamma(j, j) - \Gamma(i, j)\Gamma(j, i) \geq 0. \tag{2.2}$$

If we also set $\alpha_j = 0$ in (1.1) we see that for any $i \in n$

$$\Gamma(i, i) \geq 0. \tag{2.3}$$

In addition, by [9, Proposition 3.8], for any pair $i, j \in T$

$$\Gamma(i, j)\Gamma(j, i) \geq 0. \tag{2.4}$$

In the next lemma, we show that there are many transformations of kernels of permanental processes that give other kernels of permanental processes.

Lemma 2.1. *Let A be a kernel of a β-permanental vector $\theta = (\theta_1, \dots, \theta_n)$. Let U_1 and U_2 be diagonal matrices with non-zero diagonal entries $u_i^{(j)}$, $i = 1, \dots, n$, $j = 1, 2$, for U_1 and U_2 respectively, with the property that $u_i^{(1)} u_i^{(2)} > 0$, $i = 1, \dots, n$. Then $U_1 A U_2$ is the kernel of the β-permanental vector $(u_1^{(1)} u_1^{(2)} \theta_1, \dots, u_n^{(1)} u_n^{(2)} \theta_n)$.*

Proof. Since θ is an R^n_+ valued random variable so is $(u_1^{(1)} u_1^{(2)} \theta_1, \dots, u_n^{(1)} u_n^{(2)} \theta_n)$. The Laplace transform of $(u_1^{(1)} u_1^{(2)} \theta_1, \dots, u_n^{(1)} u_n^{(2)} \theta_n)$ is

$$\begin{aligned}
E\left(\exp\left(-\sum_{i=1}^n \alpha_i (u_i^{(1)} u_i^{(2)} \theta_i)\right)\right) &= E\left(\exp\left(-\sum_{i=1}^n (\alpha_i u_i^{(1)} u_i^{(2)}) \theta_i\right)\right) \\
&= |I + (\alpha U_2 U_1) A|^{-\beta} \\
&= |U_2 (I + \alpha (U_1 A U_2)) U_2^{-1}|^{-\beta} \\
&= |I + \alpha (U_1 A U_2)|^{-\beta}. \quad \square
\end{aligned} \tag{2.5}$$

Example 2.1. We note two cases. Let U be a strictly positive diagonal matrix.

1. When $\gamma + \gamma' = 0$

$$E\left(\exp\left(-\sum_{i=1}^n \alpha_i \theta_i\right)\right) = |I + \alpha U^{\gamma} A U^{-\gamma}|^{-\beta}. \tag{2.6}$$

2. When $\gamma + \gamma' = 1$

$$E\left(\exp\left(-\sum_{i=1}^n \alpha_i u_i \theta_i\right)\right) = |I + \alpha U^{\gamma} A U^{(1-\gamma)}|^{-\beta}. \tag{2.7}$$

In particular,

$$E\left(\exp\left(-\sum_{i=1}^n \alpha_i u_i \theta_i\right)\right) = |I + \alpha U^{1/2} A U^{1/2}|^{-\beta}. \tag{2.8}$$

Remark 2.1. It is easy to see that for a', b', c' strictly positive, the two matrices

$$\begin{pmatrix} 1 & a' & c' \\ a' & 1 & b' \\ c' & b' & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -a' & -c' \\ -a' & 1 & b' \\ -c' & b' & 1 \end{pmatrix} \tag{2.9}$$

are diagonally equivalent to each other. Similarly

$$\begin{pmatrix} 1 & -a' & c' \\ -a' & 1 & b' \\ c' & b' & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -a' & -c' \\ -a' & 1 & -b' \\ -c' & -b' & 1 \end{pmatrix} \tag{2.10}$$

are diagonally equivalent to each other. It should also be clear that these observations hold if any two of the three pairs of entries in (2.9) are taken to be negative and if any pair of entries in (2.10) is taken to be negative.

Because of the observations in the previous paragraph, when we consider whether a 3×3 matrix is the kernel of a permanental vector we need only consider those matrices with all positive off-diagonal elements or all negative off-diagonal elements. (We consider 0 to be both positive and negative.)

Consider the matrix

$$E = \begin{pmatrix} 1 & a_1 & c_2 \\ a_2 & 1 & b_1 \\ c_1 & b_2 & 1 \end{pmatrix}. \tag{2.11}$$

The next elementary lemma is very useful. We leave the proof to the reader.

Lemma 2.2. *If the off diagonal elements of the matrix E in (2.11) are either all strictly positive or all strictly negative then E is diagonally equivalent to*

$$E' = \begin{pmatrix} 1 & a & c \\ a & 1 & b_1' \\ c & b_2' & 1 \end{pmatrix}, \tag{2.12}$$

where $a^2 = a_1 a_2$, $c^2 = c_1 c_2$ and b_1' and b_2' are such that $b_1' b_2' = b_1 b_2$. In addition, the signs of a, c, b_1', b_2' are the same as the signs of $a_1, a_2, c_1, c_2, b_1, b_2$.

By (2.4), the kernel Γ of a permanental vector has the property that $\Gamma(i, j)\Gamma(j, i) \geq 0$. Therefore, if E is the kernel of a permanental vector in R^3_+, a_1, a_2 are either both positive or both negative, and similarly for b_1, b_2 and c_1, c_2.

Lemma 2.3. *Suppose that the matrix E is the kernel of a permanental vector in R^3_+. Then if*

$$a_1 b_1 c_1 = a_2 b_2 c_2 \tag{2.13}$$

it is diagonally equivalent to the kernel

$$\mathcal{E} = \begin{pmatrix} 1 & \pm a & \pm c \\ \pm a & 1 & \pm b \\ \pm c & \pm b & 1 \end{pmatrix}, \tag{2.14}$$

where $a = (a_1 a_2)^{1/2}$, $b = (b_1 b_2)^{1/2}$ and $c = (c_1 c_2)^{1/2}$ and in which $\{\mathcal{E}\}_{1,2} = a$ if a_1 is positive and $\{\mathcal{E}\}_{1,2} = -a$ if a_1 is negative, and similarly with respect to b_1 and c_1.

In particular this lemma holds when both sides of (2.13) are equal to zero.

Obviously, $\mathcal{E}$ is the covariance of a Gaussian vector.

Proof. When (2.13) holds and $a_1 b_1 c_1 \neq 0$

$$a_1 b_1 c_1 + a_2 b_2 c_2 = 2(a_1 a_2 b_1 b_2 c_1 c_2)^{1/2}. \tag{2.15}$$

It is easy to see that E and $\mathcal{E}$ are diagonally equivalent. It is also easy to see that E and $\mathcal{E}$ are diagonally equivalent if, say, $b_1 = b_2 = 0$.

Finally, it is also easy to see that if $a_1 = b_2 = 0$, E is effectively equivalent to

$$\mathcal{E}' = \begin{pmatrix} 1 & 0 & (c_1 c_2)^{1/2} \\ 0 & 1 & 0 \\ (c_1 c_2)^{1/2} & 0 & 1 \end{pmatrix}. \quad \square \tag{2.16}$$

We also use the following lemma which is [5, Lemma 4.5].

Lemma 2.4. *Let*

$$\mathcal{A} = \begin{pmatrix} u & a & c \\ a & v & b \\ c & b & w \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} u & a_1 & c_2 \\ a_2 & v & b_1 \\ c_1 & b_2 & w \end{pmatrix} \tag{2.17}$$

where $a_1 a_2 = a^2, b_1 b_2 = b^2, c_1 c_2 = c^2$. Suppose that $\mathcal{A} \geq 0$. If $\mathcal{B}^{-1}$ is an M matrix then $\mathcal{A}^{-1}$ is an M matrix.

Remark 2.2. By definition an M-matrix is invertible. Therefore $|\mathcal{A}| > 0$. We also note that if $\mathcal{B}^{-1}$ is an M matrix then for any diagonal matrix D with strictly positive entries, $(D\mathcal{B}D^{-1})^{-1}$ is an M-matrix.

The next observation is used often in this paper.

Lemma 2.5. *Let $\Phi(\alpha_1, \dots, \alpha_n)$ be the Laplace transform of an R^n_+ valued random variable. For any $1 < k < n$ set $\alpha_j = u_j$, where $u_j \geq 0, k \leq j \leq n$. Then*

$$\Phi_{(n,k)}(\alpha_1, \dots, \alpha_k) = \frac{\Phi(\alpha_1, \dots, \alpha_k, u_{k+1}, \dots, u_n)}{\Phi(0, \dots, 0, u_{k+1}, \dots, u_n)} \tag{2.18}$$

is the Laplace transform of an R^k_+ valued random variable.

Furthermore, if $\Phi(\alpha_1, \dots, \alpha_n)$ is the Laplace transform of an n-dimensional permanental vector, $\Phi_{(n,k)}(\alpha_1, \dots, \alpha_k)$ is the Laplace transform of a k-dimensional permanental vector.

Proof. Since $\Phi(\alpha_1, \dots, \alpha_n)$ is a completely monotone function on R^n_+ it follows that $\Phi_{(n,k)}(\alpha_1, \dots, \alpha_k)$ is a completely monotone function on R^k_+, satisfying $\Phi_{(n,k)}(0, \dots, 0) = 1$. Therefore, it is the Laplace transform of an R^k_+ valued random variable.

(This is very well known when $k = 1$. Lacking a suitable reference for general k, we note that it follows from the Extended Continuity Theorem for probability measures on R^k_+, [4, Theorem 5.22], and the argument in the proof of [7, Lemma 13.2.2], applied to $\Phi_{(n,k)}(\alpha_1, \dots, \alpha_k)$, not its logarithm.)

Now suppose that $\Phi(\alpha_1, \dots, \alpha_n)$ is the Laplace transform of an n-dimensional permanental vector. This implies that

$$\Phi(\alpha_1, \dots, \alpha_n) = \frac{1}{|I + \alpha\Gamma|^\beta} \tag{2.19}$$

for some $n \times n$ matrix Γ, and diagonal matrix α as in (1.1). We first prove the second statement in the lemma for $k = n - 1$. Consider

$$\Phi(\alpha_1, \dots, \alpha_{n-1}, u_n) \tag{2.20}$$

and the corresponding matrix $I + \widetilde{\alpha}\Gamma$, where $\widetilde{\alpha} = (\alpha_1, \dots, \alpha_{n-1}, u_n)$.

We now show that

$$|I + \widetilde{\alpha}\Gamma| = (1 + u_n)|I + \alpha^{(n-1)}\Gamma^{(n-1)}|, \tag{2.21}$$

where $\alpha^{(n-1)}$ is the $(n-1) \times (n-1)$ diagonal matrix with diagonal entries $(a_1, \dots, \alpha_{n-1})$ and $\Gamma^{(n-1)}$ is an $(n-1) \times (n-1)$ matrix with entries that are functions of the entries of Γ and u_n.

Since

$$\Phi(0,\dots,0,\dots,0,u_n)=\frac{1}{|1+u_n|^{\beta}}, \tag{2.22}$$

the equality in (2.21) gives (2.18) when $k=n-1$.

To obtain (2.21), we note the matrix $I+\widetilde{\alpha}\Gamma$ has the same determinant as the matrix obtained from it by subtracting $\Gamma(n,j)\frac{u_n}{(1+u_n)}$ times the n-th column from the j-th column, for each $1\le j\le n-1$. Call this matrix S. Note that $S(n,j)=0$, $j=1,\dots,n-1$ and $S(n,n)=1+u_n$. Let S' denote the matrix obtained by dividing the last row of S by $1+u_n$. We have

$$|I+\widetilde{\alpha}\Gamma|=(1+u_n)|S'|. \tag{2.23}$$

To be more specific the entries of S' are

$$\begin{aligned} &S'(i,j)=\delta_{i,j}+\alpha_i\left(\Gamma(i,j)-\frac{u_n\Gamma(i,n)\Gamma(n,j)}{1+u_n}\right),\quad 1\le i,j\le n-1;\\ &S'(n,j)=0,\quad 1\le j\le n-1;\\ &S'(n,n)=1. \end{aligned} \tag{2.24}$$

It is obvious that we can write

$$|S'|=|I+\alpha^{(n-1)}\Gamma^{(n-1)}|, \tag{2.25}$$

where $\Gamma^{(n-1)}$ is the matrix with components

$$\left(\Gamma(i,j)-\frac{u_n\Gamma(i,n)\Gamma(n,j)}{1+u_n}\right),\quad 1\le i,j\le n-1 \tag{2.26}$$

and $\alpha^{(n-1)}=(\alpha_1,\alpha_2,\dots\alpha_{n-1})$. We now have

$$\Phi_{(n,n-1)}(\alpha_1,\dots,\alpha_{n-1})=\frac{1}{|I+\alpha^{(n-1)}\Gamma^{(n-1)}|^{\beta}}. \tag{2.27}$$

Repeating the argument above we can show that

$$\begin{aligned}\Phi_{(n,n-2)}(\alpha_1,\dots,\alpha_{n-2}) &= \frac{\Phi_{(n,n-1)}(\alpha_1,\dots,\alpha_{n-2},u_{n-1})}{\Phi_{(n,n-1)}(0,\dots,0,u_{n-1})}\\ &= \frac{\Phi(\alpha_1,\dots,\alpha_{n-2},u_{n-1},u_n)}{\Phi(0,\dots,0,u_{n-1},u_n)},\end{aligned} \tag{2.28}$$

since

$$\Phi_{(n,n-1)}(\alpha_1,\dots,\alpha_{n-2},u_{n-1})=\frac{\Phi(\alpha_1,\dots,\alpha_{n-2},u_{n-1},u_n)}{\Phi(0,\dots,0,u_n)} \tag{2.29}$$

and

$$\Phi_{(n,n-1)}(0,\ldots,0,u_{n-1}) = \frac{\Phi(0,\ldots,0,u_{n-1},u_n)}{\Phi(0,\ldots,0,u_n)}. \tag{2.30}$$

Thus we get (2.18) for $k = n-2$.

Continuing in this way we get (2.18) for arbitrary $1 \le k \le n-1$. □

We use the following necessary condition in the proof of Theorem 1.1. It is a direct consequence of [9, Propositions 3.8 and 4.5 with $\sigma = 0$]. We provide a direct proof for the convenience of the reader.

Lemma 2.6. *Let $A = \{A_{i,j}\}_{i,j=1}^n$, be an $n \times n$ matrix. If A is a kernel of a β-permanental vector then A and all matrices obtained from A by multiplying its rows by non-negative numbers have a positive eigenvalue of maximum modulus.*

Proof. Let $\theta = (\theta_1,\ldots,\theta_n)$ be a permanental vector with kernel A. The matrices obtained by multiplying the rows of A by non-negative numbers have the form UA where U is a diagonal matrix with non-negative entries $u_1,\ldots,u_n$. Note that

$$E\left(\exp\left(-\sum_{i=1}^n \alpha_i u_i \theta_i\right)\right) = |I + \alpha UA|^{-\beta}. \tag{2.31}$$

Let z be a complex number and set

$$\begin{aligned} f(z) &= E\left(\exp\left(z\sum_{i=1}^n u_i\theta_i\right)\right) \\ &= \sum_{k=0}^\infty z^k \frac{E\left(\sum_{i=1}^n u_i\theta_i\right)^k}{k!}. \end{aligned} \tag{2.32}$$

By (2.31)

$$f(z) = |I - zUA|^\beta = \prod_{p=1}^n (1 - z\lambda_p)^{-\beta}, \tag{2.33}$$

where λ_p, $1 \le p \le n$, are the eigenvalues of UA.

Since

$$E\left(\exp\left(-\lambda\theta_i\right)\right) = \frac{1}{|I + \lambda A_{i,i}|^\beta}, \tag{2.34}$$

we see that

$$E(\theta_{x_i}^k) = (A_{i,i}^k)\widetilde{\Gamma}(\beta + k), \tag{2.35}$$

where

$$\widetilde{\Gamma}(\beta+k) = (\beta+k-1)(\beta+k-2)\cdots\beta. \tag{2.36}$$

It follows that

$$E\left(\sum_{i=1}^{n} u_i\theta_i\right)^k \le \max_{1\le i\le n} u_i^k n^k \max_{1\le i\le n} E(\theta_i^k)$$
$$= \max_{1\le i\le n} u_i^k n^k \max_{1\le i\le n} A_{i,i}^k \widetilde{\Gamma}(\beta+k). \tag{2.37}$$

Since

$$\frac{\widetilde{\Gamma}(\beta+k)}{k!} \le \left(\frac{\beta}{k}+1\right)\left(\frac{\beta}{k-1}+1\right)\cdots\left(\frac{\beta}{2}+1\right)\beta, \tag{2.38}$$

there exists a number b, such that

$$\mathcal{B}_k := \frac{E\left(\sum_{i=1}^{n} u_i\theta_i\right)^k}{k!} \le b^k. \tag{2.39}$$

This implies that the series in (2.32) has a positive radius of convergence, which we denote by R. By (2.32) and (2.33)

$$f(z) = \sum_{k=0}^{\infty} \mathcal{B}_k z^k = \prod_{p=1}^{n}(1-z\lambda_p)^{-\beta} \tag{2.40}$$

for $|z| < R$. Let $v > 0$ and note that $\lim_{v\to R} f(v) = \infty$ as the sum of a series with positive terms.

Since the terms $\mathcal{B}_k$ are positive, when $v = |z|$ we have

$$f(v) = \sum_{k=0}^{\infty} \mathcal{B}_k |z|^k \ge \left|\sum_{k=0}^{\infty} \mathcal{B}_k z^k\right|. \tag{2.41}$$

This shows that if $f(z)$ has a singularity at z_0, then $f(v_0) = \infty$, for $v_0 = |z_0|$.

By (2.40) this implies that

$$\max_{1\le p\le n} \lambda_p = \frac{1}{v_0}. \quad \square \tag{2.42}$$

3. Eigenvalues of 3 × 3 positive definite symmetric matrices

Lemma 3.1. *Let H be the real symmetric matrix*

$$H = \begin{pmatrix} 1 & a & c \\ a & 1 & b \\ c & b & 1 \end{pmatrix} \tag{3.1}$$

and let ρ be a real diagonal matrix with entries (ρ_1, ρ_2, ρ_3), where

$$\rho_1 = \frac{b}{b-ac}, \qquad \rho_2 = \frac{c}{c-ab}, \quad \text{and} \quad \rho_3 = \frac{a}{a-bc}. \tag{3.2}$$

Assume none of the denominators in (3.2) are zero. Then

$$|\rho H - \lambda I| = (\lambda-1)^2(\rho_1\rho_2\rho_3|H| - \lambda). \tag{3.3}$$

In particular, $\lambda = 1$ is an eigenvalue of ρH of multiplicity 2.

Proof.

$$|\rho H - I| = \begin{vmatrix} \frac{ac}{b-ac} & \frac{ab}{b-ac} & \frac{bc}{b-ac} \\ \frac{ac}{c-ab} & \frac{ab}{c-ab} & \frac{bc}{c-ab} \\ \frac{ac}{a-bc} & \frac{ab}{a-bc} & \frac{bc}{a-bc} \end{vmatrix}. \tag{3.4}$$

It is easy to see that the second and third row of this determinant are equal to a (different) multiple of the first row. This shows that $\lambda = 1$ is an eigenvalue of ρH of multiplicity 2.

Since the product of the eigenvalues must equal $|\rho H|$ we get (3.3). □

Remark 3.1. When $\det H \geq 0$, or equivalently, when H is positive definite, H is the covariance of a Gaussian vector, say, (ξ_1, ξ_2, ξ_3). When $\rho \geq 0$, it follows from Lemma 2.1 that ρH is diagonally equivalent to the covariance matrix of the Gaussian vector

$$\left(\rho_1^{1/2}\xi_1, \rho_2^{1/2}\xi_2, \rho_3^{1/2}\xi_3\right). \tag{3.5}$$

4. 3 × 3 matrices with negative off diagonal elements

In this section, we prove Theorem 1.1 for matrices with negative off diagonal elements.

In the next lemma, we consider the eigenvalues of a 3×3 matrix with negative off diagonal elements.

Lemma 4.1. *Let A_- be the matrix*

$$A_- = \begin{pmatrix} 1 & -a'_1 & -c'_2 \\ -a'_2 & 1 & -b'_1 \\ -c'_1 & -b'_2 & 1 \end{pmatrix}, \tag{4.1}$$

with $a'_1a'_2, b'_1b'_2, c'_1c'_2$ all greater than or equal to 0 and less than or equal to 1. Assume that $\det A_- \geq 0$ and $\{A_-\}_{i,j}\{A_-\}_{j,i} \leq 1$. Then if A_- is not diagonally equivalent to a symmetric matrix there exists a diagonal matrix Φ, with strictly positive entries, such that ΦA_- has only one real eigenvalue.

Furthermore, the real part of the complex eigenvalues of ΦA_- is greater than the real eigenvalue.

Proof. We first consider the case in which $a'_1, a'_2, b'_1, b'_2, c'_1, c'_2$ are all strictly positive; let $a'_1a'_2 = a^2$, $b'_1b'_2 = b^2$ and $c'_1c'_2 = c^2$ and let Φ be a real diagonal matrix with entries (ϕ_1, ϕ_2, ϕ_3), where

$$\phi_1 = \frac{b}{b+ac}, \quad \phi_2 = \frac{c}{c+ab} \quad \text{and} \quad \phi_3 = \frac{a}{a+bc}. \tag{4.2}$$

Let K be the matrix

$$K = \begin{pmatrix} 1 & -a & -c \\ -a & 1 & -b \\ -c & -b & 1 \end{pmatrix}. \tag{4.3}$$

We show in Lemma 3.1 that

$$\begin{aligned}|\Phi K-\lambda I| &= -\lambda^3+\lambda^2(\phi_1+\phi_2+\phi_3)\\ &\quad -\lambda\left(\phi_1\phi_2(1-a^2)+\phi_2\phi_3(1-b^2)+\phi_1\phi_3(1-c^2)\right)+\phi_1\phi_2\phi_3|K|\\ &= (\lambda-1)^2(\phi_1\phi_2\phi_3|K|-\lambda). \end{aligned} \tag{4.4}$$

Since $(\phi_1+\phi_2+\phi_3)/3<1$, the second derivative of $|\Phi K-\lambda I|$ is negative at $\lambda=1$. Therefore, $|\Phi K-\lambda I|$ has a local maximum of 0 at $\lambda=1$. Consider

$$H(\lambda):=|\Phi A_- -\lambda I|. \tag{4.5}$$

By Lemma 2.2 it suffices to take A_- to be

$$\begin{pmatrix} 1 & -a & -c\\ -a & 1 & -b_1\\ -c & -b_2 & 1\end{pmatrix}, \tag{4.6}$$

where $b_1b_2=b^2$.

We have

$$|K|=1-a^2-b^2-c^2-2abc \tag{4.7}$$

and

$$|A_-|=1-a^2-b^2-c^2-ac(b_1+b_2). \tag{4.8}$$

Therefore, since $b_1b_2=b^2$ we see from the first equality in (4.4) that

$$|\Phi K-\lambda I|-|\Phi A_- -\lambda I|=\phi_1\phi_2\phi_3\left(ac(b_1+b_2-2b)\right). \tag{4.9}$$

Unless, $b_1=b_2$, the right-hand side of (4.9) is strictly positive. Since $|\Phi K-\lambda I|$ has a local maximum of 0 at $\lambda=1$, $|\Phi A_- -\lambda I|$ has a local maximum that is strictly negative at $\lambda=1$. This implies that when $b_1\neq b_2$, $|\Phi A_- -\lambda I|$ has only one real root. This is equivalent to the statement of this lemma.

We now consider the cases in which some of the entries of A_- in (4.1) are zero. Considering Lemma 2.2 we can restrict our attention to the following matrices

$$\begin{pmatrix} 1 & -a & -c\\ -a & 1 & -b'\\ -c & 0 & 1\end{pmatrix}\qquad \begin{pmatrix} 1 & -a & 0\\ -a & 1 & -b'\\ -c' & 0 & 1\end{pmatrix}\qquad \begin{pmatrix} 1 & -a' & 0\\ 0 & 1 & -b'\\ -c' & 0 & 1\end{pmatrix} \tag{4.10}$$

in which a,b,c,a',b',c' are all strictly positive. Label them, respectively, $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$.

We first show that $\mathcal{F}_1$ has only one real eigenvalue. Similar to (4.9) we have

$$\begin{aligned}|\Phi K-\lambda I|-|\Phi\mathcal{F}_1-\lambda I| &= -\phi_2\phi_3b^2(\phi_1-\lambda)-\phi_1\phi_2\phi_3\left(2abc-acb'\right)\\ &= \phi_2\phi_3b^2\lambda-\phi_1\phi_2\phi_3\left(b^2+2abc-acb'\right). \end{aligned} \tag{4.11}$$

We assume b' is fixed. One can choose $0<b<1$ such that $b^2+2abc-acb'=0$. That $b>0$ is elementary. That $b<1$ follows from the fact that $\det\mathcal{F}_1\geq 0$ implies that $acb'<1$. With this choice of b

$$|\Phi K-\lambda I|-|\Phi\mathcal{F}_1-\lambda I|=\phi_2\phi_3b^2\lambda. \tag{4.12}$$

Considering the graph of $|\Phi K - \lambda I|$ (see the last line of (4.4)), and the fact that the right-hand side of (4.12) is strictly positive for $\lambda > 0$, we see that $|\Phi \mathcal{F}_1 - \lambda I|$ has only one real root, or equivalently, that $\mathcal{F}_1$ has only one real eigenvalue.

Similar arguments show that $\mathcal{F}_2$ has only one real eigenvalue. With regard to $\mathcal{F}_2$ we have

$$\begin{aligned} &|\Phi K - \lambda I| - |\Phi \mathcal{F}_2 - \lambda I| \\ &\quad = -\phi_2\phi_3 b^2(\phi_1 - \lambda) - \phi_1\phi_3 c^2(\phi_2 - \lambda) - \phi_1\phi_2\phi_3\left(2abc - ac'b'\right) \\ &\quad = (\phi_2\phi_3 b^2 + \phi_1\phi_3 c^2)\lambda - \phi_1\phi_2\phi_3\left(b^2 + c^2 + 2abc - ac'b'\right). \end{aligned} \tag{4.13}$$

For simplicity we can take $b = c$. We can choose $1 > b > 0$ such that $2b^2 + 2ab^2 - ac'b' = 0$. With this choice of b

$$|\Phi K - \lambda I| - |\Phi \mathcal{F}_2 - \lambda I| = (\phi_2\phi_3 + \phi_1\phi_3)b^2\lambda. \tag{4.14}$$

So we can use the same argument we just used to show that $\mathcal{F}_2$ has only one real eigenvalue.

For $\mathcal{F}_3$ we consider

$$|\Phi \mathcal{F}_2 - \lambda I| - |\Phi \mathcal{F}_3 - \lambda I| = \phi_1\phi_2 a^2\lambda - \phi_1\phi_2\phi_3\left(a^2 + ab'c' - a'b'c'\right). \tag{4.15}$$

We can choose $0 < a < 1$ so that $a^2 + 2ab'c' - a'b'c' = 0$ and get

$$|\Phi \mathcal{F}_2 - \lambda I| - |\Phi \mathcal{F}_3 - \lambda I| = \phi_1\phi_2 a^2\lambda. \tag{4.16}$$

Considering the graph of $|\Phi \mathcal{F}_2 - \lambda I|$ we see that $\mathcal{F}_3$ has only one real eigenvalue. (Actually, in this case, it is easy to see that the three eigenvalues of $\mathcal{F}_3$ are $\{(abc)^{1/3}, (abc)^{1/3}\frac{-1\pm i\sqrt{3}}{2}\}$.)

This completes the proof of the assertions in the first paragraph of this lemma.

We now consider the assertions in the second paragraph of this lemma. Consider $H(\lambda)$. Suppose that δ is a real root of this polynomial. Dividing $H(\lambda)$ by $\lambda - \delta$ we see that the real part of the complex roots of $H(\lambda)$ is greater than δ if and only if

$$\phi_1 + \phi_2 + \phi_3 > 3\delta. \tag{4.17}$$

Suppose that $a_1', a_2', b_1', b_2', c_1', c_2'$ are all strictly positive. It follows from (4.9) and the fact that $\phi_1\phi_2\phi_3 \det A$ is a root of $|\Phi K - \lambda I|$, that $\delta < \phi_1\phi_2\phi_3 \det K$. Furthermore, since $|K| \le 1$, to prove (4.17) it suffices to show that

$$\phi_1 + \phi_2 + \phi_3 \ge 3\phi_1\phi_2\phi_3. \tag{4.18}$$

Since $\phi_1+\phi_2+\phi_3 \ge 3(\phi_1\phi_2\phi_3)^{1/3}$ and ϕ_1, ϕ_2, ϕ_3 are all less than 1 we see that (4.18) is satisfied. Thus, when $a_1', a_2', b_1', b_2', c_1', c_2'$ are all strictly positive and ΦA_- has only one real eigenvalue, the real part of the complex eigenvalues is greater than the real eigenvalue. It is easy to see that this argument also works when A_- has the form of $\mathcal{F}_1, \mathcal{F}_2$ or $\mathcal{F}_3$. □

Proof of Theorem 1.1 for matrices with negative off diagonal entries. This follows immediately from Lemmas 4.1 and 2.6. □

5. 3 × 3 matrices with positive off diagonal entries

Consider the matrix

$$A = \begin{pmatrix} 1 & a_1 & c_2 \\ a_2 & 1 & b_1 \\ c_1 & b_2 & 1 \end{pmatrix}, \tag{5.1}$$

in which all the entries are greater than or equal to zero. When $\det A > 0$,

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} 1 - b_1 b_2 & c_2 b_2 - a_1 & a_1 b_1 - c_2 \\ b_1 c_1 - a_2 & 1 - c_1 c_2 & a_2 c_2 - b_1 \\ a_2 b_2 - c_1 & a_1 c_1 - b_2 & 1 - a_1 a_2 \end{pmatrix}. \tag{5.2}$$

Proof of Theorem 1.1 for matrices with positive off diagonal entries. This follows from the next lemma. □

Lemma 5.1. *Let A, in (5.1), be the kernel of a permanental vector $\theta = (\theta_1, \theta_2, \theta_3)$. Then A is either diagonally equivalent to a symmetric positive definite matrix or A^{-1} is an M matrix.*

Furthermore, if one of the off diagonal terms of A is equal to 0, then A is diagonally equivalent to a symmetric matrix.

Proof. As in (2.19), but with Γ replaced by A, we have

$$\Phi(\alpha_1, \alpha_2, \alpha_3) = \frac{1}{|I + \alpha A|^\beta}. \tag{5.3}$$

Therefore, as in (2.27)

$$\Phi_{(3,2)}(\alpha_1, \alpha_2) = \frac{1}{|I + \alpha^{(2)} \Gamma^{(2)}|^\beta}, \tag{5.4}$$

where, by (2.26)

$$\Gamma^{(2)} = \begin{pmatrix} 1 - v c_1 c_2 & a_1 - v c_2 b_2 \\ a_2 - v b_1 c_1 & 1 - v b_1 b_2 \end{pmatrix} \tag{5.5}$$

and where $v = \frac{u_3}{1+u_3}$. It follows from Lemma 2.5 that $\Gamma^{(2)}$ is the kernel of a permanental vector. Therefore, by (2.4)

$$(a_1 - v c_2 b_2)(a_2 - v b_1 c_1) \geq 0. \tag{5.6}$$

Suppose none of the off diagonal entries of A are equal to 0. The inequality in (5.6) holds for all $v \in (0, 1)$. Therefore, either

$$a_1 \geq c_2 b_2 \quad \text{and} \quad a_2 \geq b_1 c_1 \tag{5.7}$$

or there exists a $v_0 \in (0, 1)$ such that

$$a_1 - v_0 c_2 b_2 = a_2 - v_0 c_1 b_1 = 0. \tag{5.8}$$

It follows from (5.8) that $a_1 = v_0 c_2 b_2$ and $a_2 = v_0 c_1 b_1$, or, equivalently, that

$$a_1 b_1 c_1 = a_2 b_2 c_2. \tag{5.9}$$

If (5.9) holds, it follows from Lemma 2.3 that the matrix A is diagonally equivalent to a symmetric matrix.

We repeat this argument twice considering $\Phi_{(3,2)}(\alpha_1, \alpha_3)$ and $\Phi_{(3,2)}(\alpha_2, \alpha_3)$. If (5.8) holds then we get comparable equalities when we consider $\Phi_{(3,2)}(\alpha_1, \alpha_3)$ and $\Phi_{(3,2)}(\alpha_2, \alpha_3)$. However if (5.7) holds we also get

$$b_1 \geq a_2 c_2, \qquad b_2 \geq a_1 c_1, \qquad c_1 \geq a_2 b_2 \quad \text{and} \quad c_2 \geq a_1 b_1. \tag{5.10}$$

It follows from (5.7) and (5.10) that A is an M-matrix if it is invertible, or equivalently, $|A| > 0$.

However, if $|A| = 0$, A does not have an inverse and the consideration of whether A^{-1} is an M-matrix is meaningless. Therefore we must show that when (5.7) and (5.10) hold and A is not diagonally equivalent to a symmetric positive definite matrix then $|A| > 0$.

We need only consider the case in which A is not symmetric. Without loss of generality we can consider that

$$A = \begin{pmatrix} 1 & a & c_1 \\ a & 1 & b \\ c_2 & b & 1 \end{pmatrix}, \tag{5.11}$$

where $a, b, c > 0, a^2 = a_1 a_2, b^2 = b_1 b_2, c_1 c_2 = c^2$ and $c_1 \neq c_2$. Let d be such that $c_1 + c_2 = dc$. Obviously $d > 2$.

Since $|A| = 0$ we have

$$1 - (a^2 + b^2 + c^2) + abcd = 0. \tag{5.12}$$

We consider d in (5.12) as a function of a, b, c, i.e.,

$$d(a, b, c) = \frac{(a^2 + b^2 + c^2) - 1}{abc}. \tag{5.13}$$

Note that the gradient of $d(a, b, c)$ is

$$\nabla d(a, b, c) = \frac{1}{(abc)^2} \Big(bc(1 + a^2 - b^2 - c^2), ac(1 + b^2 - a^2 - c^2), \\ ab(1 + c^2 - a^2 - b^2) \Big). \tag{5.14}$$

The inequalities in (5.7) and (5.10) hold when A is not diagonally equivalent to a symmetric matrix. (The argument we give does not require that $|A| > 0$.) When they hold we see that the components of $\nabla d(a, b, c)$ are all greater than or equal to 0. For example, $a \geq bc$ implies that

$$(1 + a^2 - b^2 - c^2) \geq (1 + (bc)^2 - b^2 - c^2) = (1 - b^2)(1 - c^2) \geq 0. \tag{5.15}$$

Note that $d(1, 1, 1) = 2$. Therefore, since a, b, c are all less than or equal to 1, $d(a, b, c) \leq 2$. This contradiction shows that here are no permanental vectors with $|A| = 0$ other than those with kernels that are diagonally equivalent to a symmetric matrix.

To show that if one of the off diagonal terms of A is equal to 0, then A is diagonally equivalent to a symmetric matrix, we consider (5.6). Suppose $a_1 = 0$, then either $a_2 = 0$ or one of b_2, c_2 is equal to 0. In these cases, it follows from Lemma 2.3 that A is effectively equivalent to a symmetric matrix. Using $\Phi(2, 3)$ and $\Phi(1, 3)$ we come to the same conclusion for all the other ways one or more of the off diagonal terms of A can be equal to 0. □

Example 5.1. It seems worthwhile to point out that there are many symmetric matrices with positive entries that have determinant 0. All 3×3 symmetric matrices of the form of D in (3.1), with, $|a| \leq 1$, $|b| \leq 1$ and $|c| \leq 1$ and with $|D| = 0$ have the form

$$\mathcal{S}_\pm(x, y) = \begin{pmatrix} 1 & \sin x & \cos y \\ \sin x & 1 & \sin(x \pm y) \\ \cos y & \sin(x \pm y) & 1 \end{pmatrix}, \tag{5.16}$$

for any x and y which $\sin x, \cos y, \sin(x+y)$ or $\sin x, \cos y, \sin(x-y)$ are greater than or equal to zero. To get this we note that $|D|=0$ implies that

$$c = ab \pm \Big((1-a^2)(1-b^2)\Big)^{1/2}. \tag{5.17}$$

If we take $a=\sin x$, and $b=\cos y$ and solve for c we get (5.16).

We now ask for what values of x and y is the adjugate, (also called the adjoint) of $\mathcal{S}_\pm(x,y)$ a singular M-matrix; (i.e. even though the matrix is not invertible, the adjugate has negative, including 0, off diagonal elements). Referring to D, and noting (5.2), this is equivalent to asking for what values of x and y are

$$c \geq ab, \quad a \geq bc, \text{ and } b \geq ac. \tag{5.18}$$

To achieve the first inequality in (5.18) we must use the plus sign in (5.17) which gives $c=\sin(x+y)$. This implies that a, which satisfies an analogue of (5.17), satisfies

$$a = bc - \Big((1-b^2)(1-c^2)\Big)^{1/2}. \tag{5.19}$$

To get the second inequality in (5.18) we can take $c=1$ and $a=b$, which can be achieved by taking $x\in[0,\pi/2]$ and $y=(\pi/2)-x$, or $a=c$ and $b=1$ which can be achieved by taking $x\in[0,\pi/2]$ and $y=0$. In either case we get matrices of the form

$$\mathcal{A}(a) = \begin{pmatrix} 1 & a & a \\ a & 1 & 1 \\ a & 1 & 1 \end{pmatrix} \quad a\in[0,1], \tag{5.20}$$

and the matrices that can be obtained from them by interchanging their rows and columns. The adjugate of $\mathcal{A}(a)$ is

$$\mathcal{A}'(a) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-a^2 & -(1-a^2) \\ 0 & -(1-a^2) & 1-a^2 \end{pmatrix}. \tag{5.21}$$

The next lemma is an analogue of Lemma 4.1 when the off diagonal elements of the kernel are all greater than or equal to zero. It also shows that the necessary condition in Lemma 2.6 is satisfied in this case.

Lemma 5.2. *Let A_+ be the matrix*

$$A_+ = \begin{pmatrix} 1 & a_1' & c_2' \\ a_2' & 1 & b_1' \\ c_1' & b_2' & 1 \end{pmatrix}, \tag{5.22}$$

in which $a_1', a_2', b_1', b_2', c_1', c_2'$ are all greater than or equal to zero, with $a_1'a_2', b_1'b_2', c_1'c_2'$ all less than or equal to 1. If A_+ is not diagonally equivalent to a symmetric matrix and A_+^{-1} is an M-matrix, there exists a diagonal matrix Φ, with strictly positive diagonal entries, such that ΦA_+ has only one real eigenvalue.

Furthermore, the real part of the complex eigenvalues of ΦA_+ is less than the real eigenvalue.

Proof of Lemma 5.2. Assume, to begin, that $a_1', a_2', b_1', b_2', c_1', c_2'$ are all strictly positive. We use the notation of the proof of Lemma 3.1 except that we consider the associated symmetric matrix K_+ to be the matrix

$$K_+ = \begin{pmatrix} 1 & a & c \\ a & 1 & b \\ c & b & 1 \end{pmatrix}, \tag{5.23}$$

where $a_1'a_2' = a^2$, $b_1'b_2' = b^2$ and $c_1'c_2' = c^2$. It follows from Remark 2.2 that $\det K_+ > 0$. As in the proof of Lemma 3.1

$$\begin{aligned} \det(\Phi K_+ - \lambda I) &= -\lambda^3 + \lambda^2(\phi_1 + \phi_2 + \phi_3) \\ &\quad - \lambda\left(\phi_1\phi_2(1-a^2) + \phi_2\phi_3(1-b^2) + \phi_1\phi_3(1-c^2)\right) \\ &\quad + \phi_1\phi_2\phi_3 \det K \\ &= (\lambda - 1)^2(\phi_1\phi_2\phi_3 \det K_+ - \lambda). \end{aligned} \tag{5.24}$$

Note that

$$\frac{d^2}{d\lambda^2}\det(\Phi K_+ - \lambda I) = -6\lambda + 2(\phi_1 + \phi_2 + \phi_3). \tag{5.25}$$

Using the fact that

$$\phi_1 + \phi_2 + \phi_3 > 3, \tag{5.26}$$

we see that $\det(\Phi K_+ - \lambda I)$ has a local minimum of 0 at $\lambda = 1$.

Consider

$$\det\left(\Phi A_+ - \lambda I\right). \tag{5.27}$$

By Lemma 2.2 it suffices to take A_+ to be

$$\begin{pmatrix} 1 & a & c \\ a & 1 & b_1 \\ c & b_2 & 1 \end{pmatrix}, \tag{5.28}$$

where $b_1 b_2 = b^2$. We have

$$\det K_+ = 1 - a^2 - b^2 - c^2 + 2abc \tag{5.29}$$

and

$$\det A_+ = 1 - a^2 - b^2 - c^2 + ac(b_1 + b_2). \tag{5.30}$$

Since $b_1 b_2 = b^2$ we see from the first equality in (5.24) that

$$\det(\Phi A_+ - \lambda I) - \det(\Phi K_+ - \lambda I) = ac(b_1 + b_2 - 2b). \tag{5.31}$$

Since $\det(\Phi K_+ - \lambda I)$ has a local minimum of 0 at $\lambda = 1$, when $b_1 \neq b_2$, $\det(\Phi A_+ - \lambda I)$ has a local minimum that is strictly positive at $\lambda = 1$. This implies that when $b_1 \neq b_2$, $\det(\Phi A_+ - \lambda I)$ has only one real root. This is equivalent to the statement of this lemma.

Suppose now that one of the terms $a'_1, a'_2, b'_1, b'_2, c'_1, c'_2$ is equal to zero. Let us say it is a'_1. Then since

$$A_+^{-1} = \frac{1}{\det A_+} \begin{pmatrix} 1 - b'_1 b'_2 & c'_2 b'_2 - a'_1 & a'_1 b'_1 - c'_2 \\ b'_1 c'_1 - a'_2 & 1 - c'_1 c'_2 & a'_2 c'_2 - b'_1 \\ a'_2 b'_2 - c'_1 & a'_1 c'_1 - b'_2 & 1 - a'_1 a'_2 \end{pmatrix}, \tag{5.32}$$

we see that either b'_2 or c'_2 must be zero. Suppose it is b'_2. Then A_+ in (5.28) has the form

$$\widetilde{A}_+ = \begin{pmatrix} 1 & 0 & c'_2 \\ a'_2 & 1 & b'_1 \\ c'_1 & 0 & 1 \end{pmatrix}, \tag{5.33}$$

and therefore it is equivalent to the symmetric matrix

$$\overline{A}_+ = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ c & 0 & 1 \end{pmatrix}. \tag{5.34}$$

We now show that when ΦA_+ has only one real eigenvalue, the real part of the complex eigenvalues is less than the real eigenvalue. Let δ denote the real eigenvalue. Referring to (4.17) we see that we must show that

$$\phi_1 + \phi_2 + \phi_3 < 3\delta. \tag{5.35}$$

It follows from (5.31) that the roots of $\det(\Phi A_+ - \lambda I)$ are greater than the roots of $\det(\Phi K_+ - \lambda I)$. Therefore, by (3.3), $\delta > \phi_1\phi_2\phi_3 \det K_+$. Thus to obtain (5.35) it suffices to show that

$$\phi_1 + \phi_2 + \phi_3 \le 3\phi_1\phi_2\phi_3 \det K_+. \tag{5.36}$$

By (5.25) the second derivative of $|\Phi K_+ - \lambda I|$ is negative when $\lambda > (\phi_1 + \phi_2 + \phi_3)/3$. Considering the graph of $|\Phi K_+ - \lambda I|$ we see that this implies that the single real root of $|\Phi K_+ - \lambda I|$ is greater than $(\phi_1 + \phi_2 + \phi_3)/3$; hence we have (5.35). □

Remark 5.1 (*[9, Proposition 4.6]*)**.** States that a necessary condition for a kernel Γ to define a permanental vector in (1.1), for all $\beta > 0$, is that Γ and all matrices obtained from Γ by multiplying its rows by non-negative real numbers, have only real non-negative eigenvalues. This is not correct even for symmetric matrices since kernels of Gaussian squares that are in class 1 and not in class 2 satisfy this condition but are not kernels of permanental vectors for all $\beta > 0$. In fact, considering Lemmas 4.1 and 5.2, a correct statement is: if a 3×3 matrix Γ is the kernel of a permanental vector, and all matrices obtained from it by multiplying its rows by non-negative real numbers have only real non-negative eigenvalues, then Γ is diagonally equivalent to a symmetric positive definite matrix.

6. Permanental vectors with pairwise independent components

Proof of Theorem 1.2. We first consider the case in which $\theta \in R^3_+$. Let Γ be the kernel of θ. It is enough to consider the case in which all the diagonal elements of Γ are equal to one. Since θ has pairwise independent components we know that (1.6) holds. In this case either

$\det \Gamma = \Gamma(1,1)\Gamma(2,2)\Gamma(3,3) = 1$ or else

$$\Gamma = \begin{pmatrix} 1 & 0 & a \\ b & 1 & 0 \\ 0 & c & 1 \end{pmatrix}, \tag{6.1}$$

with $abc \neq 0$, or Γ^T is equal to this matrix. (It is obvious that Γ must contain three zeros off the diagonal. Any configuration other than (6.1) or its transpose has determinant equal to 1.)

Suppose the off diagonal elements of Γ are positive. It is obvious from (5.2) that Γ^{-1}, if it exists, is not an M-matrix. Therefore, by Theorem 1.1, Γ is diagonally equivalent to a symmetric matrix, which implies that $abc = 0$. If the off diagonal elements of Γ are negative Theorem 1.1 again implies that Γ is diagonally equivalent to a symmetric matrix, which again implies that $abc = 0$. Therefore Theorem 1.2 holds for $\theta \in R^3_+$.

We next consider a generalization of (6.1) to $n \times n$ matrices, $\widetilde{\Gamma}_n, n \geq 3$. These are matrices for which

$$|I + \alpha \widetilde{\Gamma}_n| = \prod_{i=1}^n (1+\alpha_i) + \mathcal{C}(\widetilde{\Gamma}_n) \prod_{i=1}^n \alpha_i, \tag{6.2}$$

in which $\mathcal{C}(\widetilde{\Gamma}_n)$ is a real valued function of the components of $\widetilde{\Gamma}_n$, independent of $\alpha_1, \ldots, \alpha_n$. We have the following lemma.

Lemma 6.1. *Let $\theta \in R^n_+, n \geq 3$, be a β-permanental vector, with pairwise independent components and kernel $\widetilde{\Gamma}_n$. Then $\mathcal{C}(\widetilde{\Gamma}_n) = 0$, in which case the components of θ are independent.*

Proof. We show in the beginning of this section that this lemma is true when $n = 3$. For $n > 3$ the Laplace transform of θ is

$$\Phi(\alpha_1, \ldots, \alpha_n) = \left| \prod_{i=1}^n (1+\alpha_i) + \mathcal{C}(\widetilde{\Gamma}_n) \prod_{i=1}^n \alpha_i \right|^{-\beta}. \tag{6.3}$$

Let $\alpha_i = 1$ for all $4 \leq i \leq n$. By Lemma 2.5

$$\begin{aligned} \widetilde{\Phi}(\alpha_1, \alpha_2, \alpha_3) &:= \frac{\Phi(\alpha_1, \alpha_2, \alpha_3, 1, \ldots, 1)}{\Phi(0,0,0,1,\ldots,1)} \\ &= \left| (1+\alpha_1)(1+\alpha_2)(1+\alpha_3) + \frac{\mathcal{C}(\widetilde{\Gamma}_n)}{2^{n-3}} \alpha_1\alpha_2\alpha_3 \right|^{-\beta} \end{aligned} \tag{6.4}$$

is a Laplace transform of a random variable in R^3_+. Furthermore, the form of the right-hand side of (6.4) shows that $(\theta_1, \theta_2, \theta_3)$ is a permanental vector with a kernel of the form of (6.1), (or its transpose). We show in the beginning of this section that for such a vector we must have $\mathcal{C}(\widetilde{\Gamma}_n) = 0$. □

Proof of Theorem 1.2 continued. The proof is by induction. We show in the beginning of this section that Theorem 1.2 holds for $n = 3$. Let $n \geq 3$ and assume that the theorem holds for all $m < n$. Let θ be a β-permanental process in R^n_+ with pairwise independent components and kernel G. This theorem follows from Lemma 6.1 once we show that the determinant $|I + \alpha G|$ for this process has the form of (6.2).

Suppose it does not. Then it must contain terms of the form

$$|I+\gamma G'|\prod_{j=1}^{k}(1+\alpha_{i_j}G(i_j,i_j)), \tag{6.5}$$

where $1\le k<n$ and $(i_1,\dots,i_k)$ is a proper subset of $(1,\dots,n)$ and G' is the $(n-k)\times(n-k)$ matrix obtained by removing the i_1-th, $\dots$, i_k-th row and column from G. Furthermore, I is the $(n-k)\times(n-k)$ identity matrix and γ is the diagonal matrix with $\left(\gamma_{i,i}=\alpha_{j_i}\right)$, $1\le i\le n-k$, where $(\alpha_{j_1},\dots,\alpha_{j_{n-k}})$ are the elements of $(\alpha_1,\dots,\alpha_n)$ that are not in $(\alpha_{i_1},\dots,\alpha_{i_k})$. This should be obvious, (6.5) is simply the general form of the term in $|I+\alpha G|$ that contains $\prod_{j=1}^{k}(1+\alpha_{i_j}G(i_j,i_j))$.

By the hypothesis of this theorem $|I+\gamma G'|^{-\beta}$ is the Laplace transform of a permanental process in R_+^{n-k} with pairwise independent components. By the induction hypotheses these components are independent. Therefore

$$|I+\gamma G'|=\prod_{i=1}^{n-k}(1+\alpha_{j_i}G(j_i,j_i)). \tag{6.6}$$

This shows that the determinant $|I+\alpha G|$ for this process has the form of (6.2). □

7. Proof of Corollary 1.1

It is easy to see what we must show. For all $x,y,z\in T$

$$d(x,y)\le d(x,z)+d(y,z). \tag{7.1}$$

This follows if we can show that the determinant of

$$\widehat{\Gamma}=\begin{pmatrix}\Gamma(x,x) & (\Gamma(x,y)\Gamma(y,x))^{1/2} & (\Gamma(x,z)\Gamma(z,x))^{1/2}\\ (\Gamma(x,y)\Gamma(y,x))^{1/2} & \Gamma(y,y) & (\Gamma(y,z)\Gamma(z,y))^{1/2}\\ (\Gamma(x,z)\Gamma(z,x))^{1/2} & (\Gamma(y,z)\Gamma(z,y))^{1/2} & \Gamma(z,z)\end{pmatrix} \tag{7.2}$$

is greater than or equal to zero. This follows because, by (2.2), the 2×2 principal minor of $\widehat{\Gamma}$ is greater than or equal to zero. Therefore, if $|\widehat{\Gamma}|\ge 0$, $\widehat{\Gamma}$ is positive definite and hence the covariance of a Gaussian vector in R^3. Considering (1.11) we get (7.1).

By Lemma 2.4 the kernel of the permanental vector $(P(x),P(y),P(z))$ is

$$\Gamma=\begin{pmatrix}\Gamma(x,x) & \Gamma(x,y) & \Gamma(x,z)\\ \Gamma(y,x) & \Gamma(y,y) & \Gamma(y,z)\\ \Gamma(z,x) & \Gamma(z,y) & \Gamma(z,z)\end{pmatrix}. \tag{7.3}$$

Since Γ is the kernel of a permanental vector we have $|\Gamma|\ge 0$. We must show that

$$|\Gamma|\ge 0 \quad \text{implies that } |\widehat{\Gamma}|\ge 0. \tag{7.4}$$

One of the idiosyncrasies that we must take into account is that the off diagonal elements of $\widehat{\Gamma}$ are always greater than or equal to zero whereas the off diagonal elements of Γ may be negative.

Suppose $\Gamma\ge 0$ and $|\Gamma|\ge 0$. By Theorem 1.1 and Lemma 2.3 either Γ^{-1} is an M-matrix or else Γ is diagonally equivalent to a symmetric positive definite matrix. In the first case, by Lemma 2.4 and Remark 2.2, $|\widehat{\Gamma}|>0$. In the second case, it is easy to see from (2.15) that $|\Gamma|=|\widehat{\Gamma}|$ so we also have $|\widehat{\Gamma}|>0$.

When Γ has negative off diagonal terms we consider two cases. The first is that Γ is diagonally equivalent to a matrix with positive off diagonal terms, say Γ', i.e., $\Gamma = D\Gamma' D^{-1}$. In this case $|\Gamma| = |D\Gamma' D^{-1}|$ and the matrix in (7.2) is the same for Γ and Γ'. Therefore, the argument in the previous paragraph shows that $|\widehat{\Gamma}| > 0$.

Now, suppose Γ has negative off diagonal terms and it is not diagonally equivalent to a matrix with positive off diagonal terms. Relabel the matrix Γ'. Without loss of generality we can assume

$$\Gamma' = \begin{pmatrix} \Gamma(x,x) & -\Gamma(x,y) & -\Gamma(x,z) \\ -\Gamma(y,x) & \Gamma(y,y) & -\Gamma(y,z) \\ -\Gamma(z,x) & -\Gamma(z,y) & \Gamma(z,z) \end{pmatrix}. \tag{7.5}$$

By Lemma 3.1 Γ' is diagonally equivalent to a symmetric matrix. By Lemma 2.3, $|\Gamma'|$ is equal to

$$\begin{vmatrix} \Gamma(x,x) & -(\Gamma(x,y)\Gamma(y,x))^{1/2} & -(\Gamma(x,z)\Gamma(z,x))^{1/2} \\ -(\Gamma(x,y)\Gamma(y,x))^{1/2} & \Gamma(y,y) & -(\Gamma(y,z)\Gamma(z,y))^{1/2} \\ -(\Gamma(x,z)\Gamma(z,x))^{1/2} & -(\Gamma(y,z)\Gamma(z,y))^{1/2} & \Gamma(z,z) \end{vmatrix}. \tag{7.6}$$

Therefore, since Γ' is the determinant of a permanental vector $|\Gamma'| \geq 0$. This implies that $|\widehat{\Gamma}| \geq 0$ in this case also. $\square$

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