

Girsanov's formula for G -Brownian motion

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Abstract

In this paper, we establish Girsanov's formula for G -Brownian motion. Peng (2007, 2008) [7,8] constructed G -Brownian motion on the space of continuous paths under a sublinear expectation called G -expectation; as obtained by Denis et al. (2011) [2], G -expectation is represented as the supremum of linear expectations with respect to martingale measures of a certain class. Our argument is based on this representation with an enlargement of the associated class of martingale measures, and on Girsanov's formula for martingales in the classical stochastic analysis. The methodology differs from that of Xu et al. (2011) [13], and applies to the multidimensional G -Brownian motion.

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1. Introduction

Motivated by risk measures and volatility uncertainty problems in finance, S. Peng introduced the notion of G -Brownian motion. Intuitively, G -Brownian motion is a Brownian motion whose variance is uncertain. While the classical Brownian motion is defined on a probability space, G -Brownian motion is defined on a sublinear expectation space, that is, the triple $(\Omega, \mathcal{H}, \mathbb{E})$, where Ω is a given set and \mathcal{H} is a vector lattice of real-valued functions on Ω containing 1, which is the domain of a sublinear expectation \mathbb{E} . G -Brownian motion is defined by using two notions concerning distributions on a sublinear expectation space: identical distributedness and

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independence. On a sublinear expectation space, the notion of distributions cannot be interpreted as that on a probability space; indeed, as introduced in [9], it also needs to be interpreted as a sublinear expectation on a class of test functions suitably chosen according to the domain \mathcal{H} .

Peng [7,8] constructed a sublinear expectation space on which the canonical process of the space $\Omega = C_0([0, \infty); \mathbb{R}^d)$ of continuous paths starting from 0 becomes a G -Brownian motion. The sublinear expectation in this space is called G -expectation. Itô's integrals with respect to G -Brownian motion and the quadratic variation process of G -Brownian motion were also defined in [7,8]. Recently, L. Denis, M. Hu and S. Peng have proved in [2] that G -expectation can be represented as the supremum of linear expectations, referred to as the upper expectation, with respect to martingale measures of a certain class.

In this paper, we derive Girsanov's formula for G -Brownian motion; when we are given a G -Brownian motion and a drift on the sublinear expectation space of Peng [7,8], we construct a new sublinear expectation space on which the G -Brownian motion with the drift is a G -Brownian motion. Through the construction, G -expectation is transformed into a weighted G -expectation. The weight has the same form as that in the classical Girsanov's formula, in which Itô's integral for G -Brownian motion and the quadratic variation process are involved. A remarkable point of the construction is that not only G -expectation but also its domain is changed. As a sublinear expectation space is the notion including the domain of a sublinear expectation, in general some care about the choice of domains is needed when changing sublinear expectations. In the course of our discussion, it is also required that the notion of distributions is appropriately defined in the new sublinear expectation space. Those are main reasons why the domain of G -expectation is changed in order to formulate Girsanov's formula for G -Brownian motion.

In the classical stochastic analysis, Girsanov's formula for Brownian motion plays a fundamental role; it is applied in many directions such as the derivation of large deviations of Schilder's [11], the construction of weak solutions to stochastic differential equations driven by Brownian motion and so on. Among them is the derivation of a variational representation for functionals of Brownian motion due to Boué–Dupuis [1], where they also showed the usefulness of the representation by applying it to prove Laplace principles for families of functionals of Brownian motion. Using the main result of the present paper, we establish in [5] a variational representation for functionals of G -Brownian motion and show that a similar application is possible under the framework of G -expectation space. Independently of our work [5], Gao [3] also obtains the representation by using our Girsanov's formula, and discusses an application to a large deviation for stochastic flows driven by G -Brownian motion.

The keys to the proof of our main result are: (i) the representation of the upper expectation for G -expectation due to Denis–Hu–Peng [2], with an enlargement of the associated class of martingale measures as given in Soner–Touzi–Zhang [12], and (ii) Girsanov's formula for martingales in the classical stochastic analysis. Our methodology is different from that of Xu–Shang–Zhang [13], in which they obtained Girsanov's formula for one-dimensional G -Brownian motion; their proof relies on the martingale characterization of one-dimensional G -Brownian motion in [14], which restricts their argument to one dimension, whereas the method we employ in this paper equally works for multidimensional G -Brownian motion. See Remark 5.8.

This paper is organized as follows. From Section 2 through Section 4, we introduce necessary notions and related results as preliminaries: the notion of distributions on a sublinear expectation space, the construction of G -expectation, stochastic integrals for G -Brownian motion, and the upper expectation for G -expectation given by Denis–Hu–Peng [2]. In Section 5, we state and prove Girsanov's formula for G -Brownian motion.

1.1. Notation

- $C_{b,\text{Lip}}(\mathbb{R}^n)$: the space of all bounded and Lipschitz continuous functions on \mathbb{R}^n .
- $C_{l,\text{Lip}}(\mathbb{R}^n)$: the space of all functions φ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y| \quad \text{for all } x, y \in \mathbb{R}^n$$

for some $C > 0, k \in \mathbb{N}$ depending on φ .

- $\mathbb{R}^{d \times d}$: all $d \times d$ real matrices.
- I_d : the $d \times d$ unit matrix.
- $|x| := \sqrt{x \cdot x}$: the norm of $x \in \mathbb{R}^n$, where \cdot is the inner product of \mathbb{R}^n .
- $\|A\| := \sqrt{\text{tr}[AA^*]}$: the norm of $A \in \mathbb{R}^{d \times d}$, where A^* is the transposed matrix of A .
- For a probability measure P , E_P denotes the expectation with respect to P .

In the sequel, unless otherwise stated, probability spaces we deal with are all assumed to be completed.

2. Sublinear expectation spaces

Following Peng [9, Chapter I], we introduce the definition of sublinear expectations and related notions.

Let Ω be a given set and \mathcal{H} a vector lattice of real functions on Ω containing 1, that is, \mathcal{H} is a linear space such that $1 \in \mathcal{H}$ and that $X \in \mathcal{H}$ implies $|X| \in \mathcal{H}$.

Definition 2.1. A functional $\mathbb{E} : \mathcal{H} \rightarrow \mathbb{R}$ is called a *sublinear expectation* if it satisfies

- (i) $\mathbb{E}[X] \leq \mathbb{E}[Y]$ if $X \leq Y$,
- (ii) $\mathbb{E}[c] = c$ for all $c \in \mathbb{R}$,
- (iii) $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$ for all $X, Y \in \mathcal{H}$,
- (iv) $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$ for all $\lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a *sublinear expectation space*.

Definition 2.2. Let $(\Omega, \mathcal{H}, \mathbb{E})$ be a sublinear expectation space. $X = (X^1, \dots, X^n)$ is called an *n-dimensional random vector*, denoted by $X \in \mathcal{H}^n$, if $X^i \in \mathcal{H}$ for each $i = 1, \dots, n$. $\{X_t; t \geq 0\}$ is called an *n-dimensional stochastic process* if for each $t \geq 0$, X_t is an *n-dimensional random vector*.

Next we introduce the notion of distributions of random variables under a sublinear expectation space. Let us consider the following sublinear expectation space:

$$\text{for all } n \in \mathbb{N} \text{ and } \varphi \in C_{l,\text{Lip}}(\mathbb{R}^n), \quad X \in \mathcal{H}^n \text{ implies } \varphi(X) \in \mathcal{H}. \quad (2.1)$$

Definition 2.3. Let X_1 and X_2 be two *n-dimensional random vectors*, and X_3 an *m-dimensional random vector* defined on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. X_1 and X_2 are called *identically distributed* if

$$\mathbb{E}[\varphi(X_1)] = \mathbb{E}[\varphi(X_2)] \quad \text{for each } \varphi \in C_{l,\text{Lip}}(\mathbb{R}^n). \quad (2.2)$$

X_3 is said to be *independent* from X_1 if

$$\mathbb{E}[\varphi(X_1, X_3)] = \mathbb{E}[\mathbb{E}[\varphi(x, X_3)]|_{x=X_1}] \quad \text{for each } \varphi \in C_{l,\text{Lip}}(\mathbb{R}^{n+m}). \quad (2.3)$$

We remark that, as in (2.1), in order to define the notion of distributions, the essential requirement for \mathcal{H} is that \mathcal{H} is closed under substitutions of its elements into functions φ of

a certain class, which may also be chosen, e.g., as $C_{b,\text{Lip}}(\mathbb{R}^n)$, the space of all bounded Lipschitz continuous functions on \mathbb{R}^n ; then in the above definition, $C_{l,\text{Lip}}(\mathbb{R}^n)$ in (2.2) and $C_{l,\text{Lip}}(\mathbb{R}^{n+m})$ in (2.3) are replaced by $C_{b,\text{Lip}}(\mathbb{R}^n)$ and $C_{b,\text{Lip}}(\mathbb{R}^{n+m})$, respectively.

3. G-Brownian motion and G-expectation

Following Peng [7,8], we introduce the construction of G -Brownian motion and related notions.

Throughout the paper, we fix $T > 0$ and denote by Θ a fixed non-empty, bounded and closed subset of $\mathbb{R}^{d \times d}$. Let $\Omega := C_0([0, T]; \mathbb{R}^d)$ be the space of all \mathbb{R}^d -valued continuous functions $(\omega_t)_{t \in [0, T]}$ with $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^1, \omega^2) := \max_{t \in [0, T]} |\omega_t^1 - \omega_t^2|.$$

For each $t \in [0, T]$, we also set $\Omega_t := \{\omega_{\cdot \wedge t} : \omega \in \Omega\}$. We denote by $\mathcal{B}(\Omega)$ (resp. $\mathcal{B}(\Omega_t)$) the Borel σ -algebra on Ω (resp. Ω_t).

3.1. G-Brownian motion and G-expectation

For each $\varphi \in C_{b,\text{Lip}}(\mathbb{R}^d)$, we denote by $u_\varphi \in C([0, T] \times \mathbb{R}^d)$ the unique viscosity solution of the following nonlinear partial differential equation called the G -heat equation:

$$\begin{cases} \frac{\partial u}{\partial t} - G(D^2 u) = 0 & \text{in } (0, T) \times \mathbb{R}^d, \\ u|_{t=0} = \varphi & \text{in } \mathbb{R}^d, \end{cases} \quad (3.1)$$

where $D^2 u$ is the Hessian matrix of u and

$$G(A) := \sup_{\gamma \in \Theta} \left\{ \frac{1}{2} \text{tr}[\gamma \gamma^* A] \right\}$$

for a $d \times d$ symmetric real matrix A ; for the existence and uniqueness of a viscosity solution of (3.1), refer to Appendix C, Section 3 in [9].

Remark 3.1. If there exists a constant $\sigma_0 > 0$ such that $\gamma \gamma^* \geq \sigma_0 I_d$ for all $\gamma \in \Theta$, then (3.1) has a unique $C^{1,2}$ -solution.

Let B be the canonical process of Ω . For each $t \in [0, T]$, we denote by $C_{b,\text{Lip}}(\Omega_t)$ the set of all bounded Lipschitz cylinder functions on Ω_t :

$$C_{b,\text{Lip}}(\Omega_t) := \{\varphi(B_{t_1}, \dots, B_{t_n}) : n \in \mathbb{N}, t_1, \dots, t_n \in [0, t], \varphi \in C_{b,\text{Lip}}((\mathbb{R}^d)^n)\},$$

and we write $C_{b,\text{Lip}}(\Omega) \equiv C_{b,\text{Lip}}(\Omega_T)$ simply. We can construct a consistent sublinear expectation \mathbb{E} on $C_{b,\text{Lip}}(\Omega)$ such that

- for all $0 \leq s < t \leq T$ and $\varphi \in C_{b,\text{Lip}}(\mathbb{R}^d)$,

$$\mathbb{E}[\varphi(B_t - B_s)] = \mathbb{E}[\varphi(B_{t-s})] = u_\varphi(t - s, 0),$$

- for all $n \in \mathbb{N}$, $0 \leq t_1 < \dots < t_n \leq T$ and $\varphi \in C_{b,\text{Lip}}((\mathbb{R}^d)^n)$,

$$\mathbb{E}[\varphi(B_{t_1}, \dots, B_{t_n})] = \mathbb{E}[\varphi_1(B_{t_1}, \dots, B_{t_{n-1}})],$$

where $\varphi_1(x_1, \dots, x_{n-1}) := \mathbb{E}[\varphi(x_1, \dots, x_{n-1}, B_{t_n}^{t_{n-1}} + x_{n-1})]$ with $B_t^s := B_t - B_s$ for $0 \leq s \leq t \leq T$.

For $t_{k-1} \leq t < t_k$, the related conditional expectation of $\varphi(B_{t_1}, \dots, B_{t_n})$ on $C_{b,\text{Lip}}(\Omega_t)$ is defined by

$$\mathbb{E}_t[\varphi(B_{t_1}, \dots, B_{t_n})] := \varphi_{n-k}(B_{t_1}, \dots, B_{t_{k-1}}, B_t),$$

where $\varphi_{n-k}(x_1, \dots, x_{k-1}, x_k) = \mathbb{E}[\varphi(x_1, \dots, x_{k-1}, B_{t_k}^t + x_k, \dots, B_{t_n}^t + x_k)]$.

Let $\mathcal{L}_G^1(\Omega_t)$ be the completion of $C_{b,\text{Lip}}(\Omega_t)$ under the norm $\mathbb{E}[|\cdot|]$, and we write $\mathcal{L}_G^1(\Omega) \equiv \mathcal{L}_G^1(\Omega_T)$ simply. We can extend $\mathbb{E}[\cdot]$ (resp. $\mathbb{E}_t[\cdot]$) to a unique sublinear expectation (resp. a conditional sublinear expectation) on $\mathcal{L}_G^1(\Omega)$. It is called G -expectation (resp. conditional G -expectation).

Definition 3.2. A stochastic process B on $(\Omega, \mathcal{L}_G^1(\Omega), \mathbb{E})$ is called a G -Brownian motion if

(i) $B_0 = 0$,

(ii) for all $0 \leq s < t \leq T$ and $\varphi \in C_{b,\text{Lip}}(\mathbb{R}^d)$,

$$\mathbb{E}[\varphi(B_t - B_s)] = \mathbb{E}[\varphi(B_{t-s})] = u_\varphi(t - s, 0),$$

(iii) for all $n \in \mathbb{N}$, $0 \leq t_1 < \dots < t_n \leq T$ and $\varphi \in C_{b,\text{Lip}}((\mathbb{R}^d)^n)$,

$$\mathbb{E}[\varphi(B_{t_1}, \dots, B_{t_n})] = \mathbb{E}[\varphi_1(B_{t_1}, \dots, B_{t_{n-1}})],$$

where $\varphi_1(x_1, \dots, x_{n-1}) := \mathbb{E}[\varphi(x_1, \dots, x_{n-1}, B_{t_n} - B_{t_{n-1}} + x_{n-1})]$.

Note that (ii) means $B_t - B_s$ and B_{t-s} are identically distributed, and that (iii) means $B_{t_n} - B_{t_{n-1}}$ is independent from $(B_{t_1}, \dots, B_{t_{n-1}})$. From the above definition, we can see that on the sublinear expectation space $(\Omega, \mathcal{L}_G^1(\Omega), \mathbb{E})$, the canonical process is a G -Brownian motion.

3.2. Itô's integral for G -Brownian motion

For each $p \geq 1$, we denote by $\mathcal{L}_G^p(\Omega_t)$ the completion of $C_{b,\text{Lip}}(\Omega_t)$ under $\mathbb{E}[|\cdot|^p]^{1/p}$. Let

$$M_G^{p,0}(\Omega) := \left\{ \sum_{k=0}^{n-1} \xi_k \mathbb{1}_{[t_k, t_{k+1})} : n \in \mathbb{N}, 0 = t_0 < t_1 < \dots < t_n = T, \xi_k \in \mathcal{L}_G^p(\Omega_{t_k}) \right\},$$

and let $M_G^p(\Omega)$ be the completion of $M_G^{p,0}(\Omega)$ under $(\int_0^T \mathbb{E}[|\cdot|^p] dt)^{1/p}$.

For every $h \in (M_G^2(\Omega))^d$, we denote

$$\int_0^t h_s \cdot dB_s = \sum_{i=1}^d \int_0^t h_s^i dB_s^i.$$

Here each summand denotes Itô's integral with respect to the i -th coordinate B^i of G -Brownian motion B , which is defined as an element of $\mathcal{L}_G^2(\Omega_t)$. For every $i, j = 1, \dots, d$, the mutual variation of B^i and B^j

$$\langle B^i, B^j \rangle_t := B_t^i B_t^j - \int_0^t B_s^i dB_s^j - \int_0^t B_s^j dB_s^i$$

is also defined since $B^i, B^j \in M_G^2(\Omega)$. We denote by $\langle B \rangle_t := (\langle B^i, B^j \rangle_t)_{1 \leq i, j \leq d}$, $0 \leq t \leq T$, the quadratic variation of B . For each $\eta \in (M_G^1(\Omega))^d$, we can define

$$\int_0^t (d\langle B \rangle_s \eta_s) := \left(\sum_{j=1}^d \int_0^t \eta_s^j d\langle B^i, B^j \rangle_s \right)_{1 \leq i \leq d}$$

as an element of $(\mathcal{L}_G^1(\Omega_t))^d$. Noting that $\eta^1 \eta^2 \in M_G^1(\Omega)$ for any $\eta^1, \eta^2 \in M_G^2(\Omega)$, we also set, for each $h \in (M_G^2(\Omega))^d$,

$$\int_0^t h_s \cdot (d\langle B \rangle_s h_s) := \sum_{i,j=1}^d \int_0^t h_s^i h_s^j d\langle B^i, B^j \rangle_s.$$

3.3. G -martingales

Now we introduce the notion of G -martingales.

Definition 3.3. A process $X = \{X_t; 0 \leq t \leq T\}$ is called a G -martingale if for each $0 \leq s \leq t \leq T$, we have $X_t \in \mathcal{L}_G^1(\Omega_t)$ and

$$\mathbb{E}_s[X_t] = X_s \quad \text{in } \mathcal{L}_G^1(\Omega_s).$$

We call X a *symmetric G -martingale* if both X and $-X$ are G -martingales.

For $h \in (M_G^2(\Omega))^d$, for example, Itô's integral process $\int_0^\cdot h_s \cdot dB_s$ is a symmetric G -martingale. $\int_0^\cdot h_s \cdot (d\langle B \rangle_s h_s) - \int_0^\cdot 2G(h_s, h_s^*) ds$ is a G -martingale, but in general, not a symmetric G -martingale (see [8] Example 50).

4. An upper expectation for G -expectation

We introduce a representation of G -expectation as an upper expectation proved by Denis–Hu–Peng [2].

Let W be a standard d -dimensional Brownian motion under a probability measure P on Ω , and let \mathbb{F}^W be the filtration generated by W :

$$\mathcal{F}_t^W := \sigma(W_u, 0 \leq u \leq t) \vee \mathcal{N}, \quad \mathbb{F}^W := \{\mathcal{F}_t^W; t \geq 0\},$$

where \mathcal{N} is the collection of all P -null subsets. For a given bounded and closed set $\Theta \subset \mathbb{R}^{d \times d}$, let

$$\mathcal{A}_{0,T}^\Theta := \{\text{all } \Theta\text{-valued } \mathbb{F}^W\text{-progressively measurable processes on the interval } [0, T]\}.$$

We identify two elements $\theta, \theta' \in \mathcal{A}_{0,T}^\Theta$ if they are equivalent:

$$\theta_t(\omega) = \theta'_t(\omega) \quad dt \times P\text{-a.e. } (t, \omega) \in [0, T] \times \Omega.$$

The quotient set of $\mathcal{A}_{0,T}^\Theta$ by this equivalence relation is still denoted by the same symbol $\mathcal{A}_{0,T}^\Theta$. For each $\theta \in \mathcal{A}_{0,T}^\Theta$, let P_θ be the law of the process $\{\int_0^t \theta_s dW_s; 0 \leq t \leq T\}$. Now we define the *capacity* $c: \mathcal{B}(\Omega) \rightarrow [0, 1]$ by

$$c(A) := \sup_{\theta \in \mathcal{A}_{0,T}^\Theta} P_\theta(A) \quad \text{for } A \in \mathcal{B}(\Omega).$$

We introduce the capacity-related terminology.

- A property holds *quasi-surely* (q.s.) if it holds outside a set A with $c(A) = 0$.
- A mapping $X: \Omega \rightarrow \mathbb{R}$ is said to be *quasi-continuous* (q.c.) if for all $\varepsilon > 0$, there exists an open set O with $c(O) < \varepsilon$ such that $X|_{O^c}$ is continuous.

- We say that $X : \Omega \rightarrow \mathbb{R}$ has a *q.c. version* if there exists a q.c. function $Y : \Omega \rightarrow \mathbb{R}$ with $X = Y$ q.s.

For $t \in [0, T]$, we denote by $L^0(\Omega_t)$ the space of all $\mathcal{B}(\Omega_t)$ -measurable real-valued functions. For $t = T$, we simply write $L^0(\Omega)$. For each $X \in L^0(\Omega)$ such that $E_{P_\theta}[X]$ exists for all $\theta \in \mathcal{A}_{0,T}^\Theta$, we set

$$\bar{\mathbb{E}}[X] := \sup_{\theta \in \mathcal{A}_{0,T}^\Theta} E_{P_\theta}[X].$$

The following theorem plays a key role in the formulation and proof of Girsanov's formula.

Theorem 4.1 ([2] Theorem 54). *It holds that*

$$\begin{aligned} \mathcal{L}_G^1(\Omega_t) &= \{X \in L^0(\Omega_t) : X \text{ has a q.c. version, } \lim_{n \rightarrow \infty} \bar{\mathbb{E}}[|X| \mathbb{1}_{\{|X| > n\}}] = 0\}, \\ \mathbb{E}[X] &= \bar{\mathbb{E}}[X] \quad \text{for all } X \in \mathcal{L}_G^1(\Omega). \end{aligned}$$

5. Main result

In this section, we first characterize symmetric G -martingales. We then state and prove the main result of this paper, Girsanov's formula for G -Brownian motion. We also explore a condition that plays a similar role to Novikov's condition in the classical stochastic analysis.

5.1. A characterization of symmetric G -martingales

We start with a lemma that characterizes conditional G -expectations. For each $\theta \in \mathcal{A}_{0,T}^\Theta$ and $t \in [0, T]$, set

$$\mathcal{A}(t, \theta) := \{\theta' \in \mathcal{A}_{0,T}^\Theta : \theta' = \theta \text{ on } [0, t]\},$$

where the identity between θ' and θ is to be understood as

$$\theta'_s(\omega) = \theta_s(\omega) \quad ds \times P\text{-a.e. } (s, \omega) \in [0, t] \times \Omega.$$

Lemma 5.1. *For each $\theta \in \mathcal{A}_{0,T}^\Theta$, $X \in \mathcal{L}_G^1(\Omega)$ and $t \in [0, T]$, it holds that*

$$\mathbb{E}_t[X] = \operatorname{ess\,sup}_{\theta' \in \mathcal{A}(t, \theta)} E_{P_{\theta'}}[X | \mathcal{F}_t] \quad P_\theta\text{-a.s.}, \quad (5.1)$$

where $\{\mathcal{F}_t; 0 \leq t \leq T\}$ is the natural filtration of B .

As noted in the proof of Proposition 3.4 of Soner–Touzi–Zhang [12], the validity of (5.1) for $X \in C_{b,\text{Lip}}(\Omega)$ follows from [2]; the assertion for $X \in \mathcal{L}_G^1(\Omega)$ is then seen to hold by approximations as done in the proof of their proposition. Since there seems to be an inadequacy in its approximating argument and the family $\{P_\theta : \theta \in \mathcal{A}_{0,T}^\Theta\}$ of probability measures is strictly smaller than the one in their proposition, we give a proof of this lemma for the sake of self-containedness of the paper.

Proof of Lemma 5.1. By Lemma 44 of [2] and by the upper expectation representation for G -expectation (Theorem 4.1), we see that

$$\mathbb{E}[\varphi(x, B_T^t)]|_{x=\zeta} = \operatorname{ess\,sup}_{\theta \in \mathcal{A}_{0,T}^\Theta} E_P[\varphi(\zeta, B_T^{t,\theta}) | \mathcal{F}_t^W] \quad P\text{-a.s.}$$

for all $t \in [0, T]$, $m \in \mathbb{N}$, $\varphi \in C_{b,\text{Lip}}(\mathbb{R}^{m+d})$ and $\zeta \in L^2(\Omega, \mathcal{F}_t^W, P; \mathbb{R}^m)$. Here and below we write

$$B_t^{s,\theta} = \int_s^t \theta_u dW_u \quad \text{for } 0 \leq s \leq t \leq T.$$

Then, repeating the same argument as in the proof of Theorem 45 of [2], we see inductively that

$$\mathbb{E}[\varphi(x, B_{s_1}^t, B_{s_2}^{s_1}, \dots, B_{s_k}^{s_{k-1}})]|_{x=\zeta} = \text{ess sup}_{\theta \in \mathcal{A}_{0,T}^\theta} E_P[\varphi(\zeta, B_{s_1}^{t,\theta}, B_{s_2}^{s_1,\theta}, \dots, B_{s_k}^{s_{k-1},\theta}) | \mathcal{F}_t^W] \quad (5.2)$$

P -a.s. for all $t \in [0, T]$, $k, m \in \mathbb{N}$, $t \leq s_1 < \dots < s_k \leq T$, $\varphi \in C_{b,\text{Lip}}(\mathbb{R}^m \times (\mathbb{R}^d)^k)$ and $\zeta \in L^2(\Omega, \mathcal{F}_t^W, P; \mathbb{R}^m)$. Now we fix $\theta \in \mathcal{A}_{0,T}^\theta$ and $t \in [0, T]$ arbitrarily. We take $X = \varphi(B_{t_1}, \dots, B_{t_n}) \in C_{b,\text{Lip}}(\Omega)$ with a partition $0 = t_0 \leq t_1 < \dots < t_n = T$, and let $i = 0, 1, \dots, n-1$ be such that $t \in [t_i, t_{i+1})$. If we set

$$\varphi_1(x_1, \dots, x_i, x) := \mathbb{E}[\varphi(x_1, \dots, x_i, B_{t_{i+1}}^t + x, \dots, B_{t_n}^t + x)]$$

for $(x_1, \dots, x_i, x) \in (\mathbb{R}^d)^{i+1}$, then we have by (5.2)

$$\varphi_1(B_{t_1}^{0,\theta}, \dots, B_{t_i}^{0,\theta}, B_t^{0,\theta}) = \text{ess sup}_{\theta' \in \mathcal{A}(t,\theta)} E_P[\varphi(B_{t_1}^{0,\theta'}, \dots, B_{t_n}^{0,\theta'}) | \mathcal{F}_t^W] \quad P\text{-a.s.}$$

Let $U \in \mathcal{F}_t$ be arbitrary and set $V = \{B_t^{0,\theta} \in U\} \in \mathcal{F}_t^W$. Then

$$\begin{aligned} E_{P_\theta}[\mathbb{1}_U \varphi_1(B_{t_1}, \dots, B_{t_i}, B_t)] &= E_P[\mathbb{1}_V \varphi_1(B_{t_1}^{0,\theta}, \dots, B_{t_i}^{0,\theta}, B_t^{0,\theta})] \\ &= E_P[\mathbb{1}_V \text{ess sup}_{\theta' \in \mathcal{A}(t,\theta)} E_P[\varphi(B_{t_1}^{0,\theta'}, \dots, B_{t_n}^{0,\theta'}) | \mathcal{F}_t^W]] \\ &= \sup_{\theta' \in \mathcal{A}(t,\theta)} E_P[\mathbb{1}_V \varphi(B_{t_1}^{0,\theta'}, \dots, B_{t_n}^{0,\theta'})] \\ &= \sup_{\theta' \in \mathcal{A}(t,\theta)} E_{P_{\theta'}}[\mathbb{1}_U \varphi(B_{t_1}, \dots, B_{t_n})], \end{aligned}$$

where we used Yan's commutation theorem (see, e.g., [6] Theorem a3) for the third line. Using Yan's commutation theorem again, and noting $P_\theta = P_{\theta'}$ on \mathcal{F}_t for $\theta' \in \mathcal{A}(t, \theta)$, we see that this is further rewritten as

$$E_{P_\theta}[\mathbb{1}_U \text{ess sup}_{\theta' \in \mathcal{A}(t,\theta)} E_{P_{\theta'}}[\varphi(B_{t_1}, \dots, B_{t_n}) | \mathcal{F}_t]].$$

As $\varphi_1(B_{t_1}, \dots, B_{t_i}, B_t) = \mathbb{E}_t[\varphi(B_{t_1}, \dots, B_{t_n})]$ by definition, it follows that

$$\mathbb{E}_t[\varphi(B_{t_1}, \dots, B_{t_n})] = \text{ess sup}_{\theta' \in \mathcal{A}(t,\theta)} E_{P_{\theta'}}[\varphi(B_{t_1}, \dots, B_{t_n}) | \mathcal{F}_t] \quad P_\theta\text{-a.s.}$$

Therefore (5.1) is proved for $X \in C_{b,\text{Lip}}(\Omega)$.

Now for $X \in L_G^1(\Omega)$, we take a sequence $\{X_n\}_{n=1}^\infty \subset C_{b,\text{Lip}}(\Omega)$ such that

$$\mathbb{E}[|X - X_n|] \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

For each $\theta \in \mathcal{A}_{0,T}^\Theta$,

$$\begin{aligned} & E_{P_\theta}[|\mathbb{E}_t[X] - \operatorname{ess\,sup}_{\theta' \in \mathcal{A}(t,\theta)} E_{P_{\theta'}}[X|\mathcal{F}_t]|] \\ & \leq E_{P_\theta}[|\mathbb{E}_t[X] - \mathbb{E}_t[X_n]|] + E_{P_\theta}[|\operatorname{ess\,sup}_{\theta' \in \mathcal{A}(t,\theta)} E_{P_{\theta'}}[X|\mathcal{F}_t] - \operatorname{ess\,sup}_{\theta' \in \mathcal{A}(t,\theta)} E_{P_{\theta'}}[X_n|\mathcal{F}_t]|] \\ & =: \mathbf{I}_n + \mathbf{II}_n. \end{aligned}$$

It is easily seen that $\mathbf{I}_n \leq \mathbb{E}[|X - X_n|]$. Also for \mathbf{II}_n , we have

$$\begin{aligned} \mathbf{II}_n & \leq E_{P_\theta}[\operatorname{ess\,sup}_{\theta' \in \mathcal{A}(t,\theta)} E_{P_{\theta'}}[|X - X_n||\mathcal{F}_t|]] \\ & = \sup_{\theta' \in \mathcal{A}(t,\theta)} E_{P_{\theta'}}[|X - X_n|] \\ & \leq \mathbb{E}[|X - X_n|], \end{aligned}$$

where the equality follows from Yan's commutation theorem and the identity $P_\theta = P_{\theta'}$ on \mathcal{F}_t for $\theta' \in \mathcal{A}(t, \theta)$. Therefore both \mathbf{I}_n and \mathbf{II}_n converge to 0 as $n \rightarrow \infty$, which yields (5.1) for $X \in \mathcal{L}_G^1(\Omega)$. \square

As a consequence of Lemma 5.1, we have the following characterization of symmetric G -martingales.

Proposition 5.2. $X = \{X_t; 0 \leq t \leq T\}$ is a symmetric G -martingale on $(\Omega, \mathcal{L}_G^1(\Omega), \mathbb{E})$ if and only if $X_t \in \mathcal{L}_G^1(\Omega_t)$ for all $t \in [0, T]$ and X is a P_θ -martingale for each $\theta \in \mathcal{A}_{0,T}^\Theta$.

Proof. We start with the if part. The condition that $X_t \in \mathcal{L}_G^1(\Omega_t)$, $t \in [0, T]$, means that X is a process on $(\Omega, \mathcal{L}_G^1(\Omega), \mathbb{E})$. If X is also a P_θ -martingale for each $\theta \in \mathcal{A}_{0,T}^\Theta$, we have, for $0 \leq s \leq t \leq T$,

$$X_s = \operatorname{ess\,sup}_{\theta' \in \mathcal{A}(s,\theta)} E_{P_{\theta'}}[X_t|\mathcal{F}_s] \quad P_\theta\text{-a.s.}$$

By Lemma 5.1, it follows that $X_s = \mathbb{E}_s[X_t] \quad P_\theta\text{-a.s.}$ and that

$$\mathbb{E}[|\mathbb{E}_s[X_t] - X_s|] = 0.$$

Similarly, we have $\mathbb{E}_s[-X_t] = -X_s$ in $\mathcal{L}_G^1(\Omega_s)$ and hence X is a symmetric G -martingale.

Conversely, if X is a symmetric G -martingale, then $X_t \in \mathcal{L}_G^1(\Omega_t)$ for all $t \in [0, T]$. Since X is a G -martingale,

$$0 = \mathbb{E}[|\mathbb{E}_s[X_t] - X_s|] = \sup_{\theta \in \mathcal{A}_{0,T}^\Theta} E_{P_\theta}[|\mathbb{E}_s[X_t] - X_s|].$$

Therefore, for every $\theta \in \mathcal{A}_{0,T}^\Theta$, we have by Lemma 5.1,

$$X_s = \mathbb{E}_s[X_t] = \operatorname{ess\,sup}_{\theta' \in \mathcal{A}(t,\theta)} E_{P_{\theta'}}[X_t|\mathcal{F}_s] \geq E_{P_\theta}[X_t|\mathcal{F}_s] \quad P_\theta\text{-a.s.}$$

Similarly, we deduce $X_s \leq E_{P_\theta}[X_t|\mathcal{F}_s] P_\theta\text{-a.s.}$ from that $-X$ is a G -martingale. Hence, X is a P_θ -martingale for each $\theta \in \mathcal{A}_{0,T}^\Theta$. \square

5.2. Girsanov's formula for G -Brownian motion

Let $h \in (M_G^2(\Omega))^d$. We define, for $0 \leq t \leq T$,

$$D_t := \exp \left(\int_0^t h_s \cdot dB_s - \frac{1}{2} \int_0^t h_s \cdot (d\langle B \rangle_s h_s) \right), \quad (5.3)$$

$$\hat{B}_t := B_t - \int_0^t (d\langle B \rangle_s h_s),$$

and we set

$$\hat{C}_{b,\text{Lip}}(\Omega) := \{\varphi(\hat{B}_{t_1}, \dots, \hat{B}_{t_n}) : n \in \mathbb{N}, t_1, \dots, t_n \in [0, T], \varphi \in C_{b,\text{Lip}}((\mathbb{R}^d)^n)\}.$$

As $\hat{B}_t \in (\mathcal{L}_G^1(\Omega_t))^d$ for each $t \in [0, T]$, we may deduce from [Theorem 4.1](#) that $\hat{C}_{b,\text{Lip}}(\Omega)$ is a subspace of $\mathcal{L}_G^1(\Omega)$.

Girsanov's formula for G -Brownian motion is stated as follows.

Theorem 5.3. Assume that there exists $\sigma_0 > 0$ such that

$$\gamma\gamma^* \geq \sigma_0 \text{Id} \quad \text{for all } \gamma \in \Theta, \quad (5.4)$$

and that D is a symmetric G -martingale on $(\Omega, \mathcal{L}_G^1(\Omega), \mathbb{E})$. Define a sublinear expectation $\hat{\mathbb{E}}$ by

$$\hat{\mathbb{E}}[X] := \mathbb{E}[XD_T] \quad \text{for } X \in \hat{C}_{b,\text{Lip}}(\Omega). \quad (5.5)$$

Let $\hat{\mathcal{H}}$ be the completion of $\hat{C}_{b,\text{Lip}}(\Omega)$ under the norm $\hat{\mathbb{E}}[|\cdot|]$, and extend $\hat{\mathbb{E}}$ to a unique sublinear expectation on $\hat{\mathcal{H}}$. Then the process $\{\hat{B}_t; 0 \leq t \leq T\}$ is a G -Brownian motion on the sublinear expectation space $(\Omega, \hat{\mathcal{H}}, \hat{\mathbb{E}})$.

We remark that the uniform nondegeneracy of Θ is also assumed in [\[13\]](#).

We prove [Theorem 5.3](#) in the next subsection. Before we proceed to the proof, there are several things we must verify. The first thing is the well-definedness of the right-hand side of (5.5), which is immediate from [Theorem 4.1](#) since D_T is in $\mathcal{L}_G^1(\Omega)$ by assumption and X is a bounded element of $\mathcal{L}_G^1(\Omega)$. The second is that the functional $\hat{\mathbb{E}}$ defined by (5.5) is indeed a sublinear expectation. As the assumption on D also yields $\mathbb{E}[D_T] = -\mathbb{E}[-D_T] = 1$, this functional possesses the property (ii) in [Definition 2.1](#). The other three properties follow readily from the definition. The last thing to be verified prior to the proof of [Theorem 5.3](#) is that $\{\hat{B}_t; 0 \leq t \leq T\}$ is a stochastic process on $(\Omega, \hat{\mathcal{H}}, \hat{\mathbb{E}})$, which we will check in the next lemma. For a fixed $\theta \in \mathcal{A}_{0,T}^\Theta$, set

$$Q_\theta(A) := E_{P_\theta}[\mathbb{1}_A D_T] \quad \text{for } A \in \mathcal{B}(\Omega). \quad (5.6)$$

Note that, by [Theorem 4.1](#), we have

$$\hat{\mathbb{E}}[X] = \sup_{\theta \in \mathcal{A}_{0,T}^\Theta} E_{Q_\theta}[X] \quad (5.7)$$

for all $X \in \hat{C}_{b,\text{Lip}}(\Omega)$.

Lemma 5.4. For all $t \in [0, T]$, we have $\hat{B}_t \in \hat{\mathcal{H}}^d$. Therefore \hat{B} is a stochastic process on $(\Omega, \hat{\mathcal{H}}, \hat{\mathbb{E}})$.

Proof. Fix $i = 1, \dots, d$ and take an arbitrary $\theta \in \mathcal{A}_{0,T}^\Theta$. By definition, the i -th coordinate B^i of the canonical process B is a P_θ -martingale. Note that the process D is also a P_θ -martingale by

Proposition 5.2 and satisfies the following relation with B^i and \hat{B}^i :

$$\hat{B}_t^i = B_t^i - \int_0^t \frac{d\langle D, B^i \rangle_s}{D_s}.$$

Therefore, by Girsanov's formula, \hat{B}^i is a local martingale under Q_θ and

$$\langle \hat{B}^i \rangle_t = \langle B^i \rangle_t \quad \text{for all } t \in [0, T], \quad Q_\theta\text{-a.s. and } P_\theta\text{-a.s.}$$

By definition, $\langle B^i \rangle_T$ under P_θ is identical in law with $\int_0^T (\theta_s \theta_s^*)^{ii} ds$, where $(\theta_s \theta_s^*)^{ii}$ is the (i, i) -entry of the matrix $\theta_s \theta_s^*$. We thus deduce that, by the boundedness of Θ , there exists a constant $C > 0$ depending only on Θ such that

$$\langle \hat{B}^i \rangle_T \leq CT \quad Q_\theta\text{-a.s.} \quad (5.8)$$

Moreover, by the time-change formula due to Dambis–Dubins–Schwarz (see, e.g., [4] Theorem 3.4.6), there exists a standard Brownian motion β under Q_θ such that

$$\hat{B}_t^i = \beta_{\langle \hat{B}^i \rangle_t} \quad \text{for all } t \in [0, T], \quad Q_\theta\text{-a.s.}$$

Combining these, we have, for some $p > 1$ (actually, for all $p > 1$),

$$\sup_{\theta \in \mathcal{A}_{0,T}^\Theta} E_{Q_\theta} [|\hat{B}_t^i|^p] \leq \sup_{\theta \in \mathcal{A}_{0,T}^\Theta} E_{Q_\theta} [\max_{0 \leq t \leq CT} |\beta_t|^p] = E_P [\max_{0 \leq t \leq CT} |W_t|^p] < \infty, \quad (5.9)$$

where W is a one-dimensional Brownian motion under a probability measure P .

Now define the sequence $\{\varphi_n(\hat{B}_t^i)\}_{n=1}^\infty \subset \hat{C}_{b,\text{Lip}}(\Omega)$ through

$$\varphi_n(x) := (x \wedge n) \vee (-n) \quad \text{for } x \in \mathbb{R}.$$

This approximates \hat{B}_t^i under the norm $\hat{\mathbb{E}}[\cdot]$. Indeed, by (5.9)

$$\sup_{\theta \in \mathcal{A}_{0,T}^\Theta} E_{Q_\theta} [|\hat{B}_t^i - \varphi_n(\hat{B}_t^i)|] \leq \sup_{\theta \in \mathcal{A}_{0,T}^\Theta} E_{Q_\theta} [|\hat{B}_t^i| \mathbb{1}_{\{|\hat{B}_t^i| > n\}}] \rightarrow 0 \quad (n \rightarrow \infty). \quad (5.10)$$

By noting that, from (5.7), $\hat{\mathcal{H}}$ can be seen as the completion of $\hat{C}_{b,\text{Lip}}(\Omega)$ under the norm $\sup_{\theta \in \mathcal{A}_{0,T}^\Theta} E_{Q_\theta} [\|\cdot\|]$, (5.10) shows $\hat{B}_t^i \in \hat{\mathcal{H}}$. \square

In the proof of **Theorem 5.3**, it will also be required that \hat{B} is a true martingale under Q_θ , which follows immediately from (5.8) and Corollary IV.1.25 of [10]. We state it in the lemma.

Lemma 5.5. *For each $\theta \in \mathcal{A}_{0,T}^\Theta$, the process $\{\hat{B}_t; 0 \leq t \leq T\}$ is a Q_θ -martingale.*

5.3. Proof of Theorem 5.3

A probability measure P on $(\Omega, \mathcal{B}(\Omega))$ is called a *martingale measure* if the canonical process B is a martingale with respect to \mathbb{F}^B under P , where \mathbb{F}^B is the filtration generated by B :

$$\mathcal{F}_t^B := \sigma(B_u, 0 \leq u \leq t) \vee \mathcal{N}, \quad \mathbb{F}^B := \{\mathcal{F}_t^B; 0 \leq t \leq T\},$$

where \mathcal{N} is the collection of all P -null subsets. Let \mathcal{P} be the family of all martingale measures P satisfying

$$\frac{d\langle B \rangle_t^P}{dt} \in \{\gamma \gamma^* : \gamma \in \Theta\}, \quad \text{a.e. } t \in [0, T], \quad P\text{-a.s.},$$

where $\langle B \rangle^P$ is the quadratic variation process of B under P . First we prove the following lemma.

Lemma 5.6. *For all $X \in C_{b,\text{Lip}}(\Omega)$,*

$$\mathbb{E}[X] = \sup_{P \in \mathcal{P}} E_P[X].$$

In the case that the set $\Theta \subset \mathbb{R}^{d \times d}$ has a form $\{\gamma \in \mathbb{R}^{d \times d} : \sigma_0 I_d \leq \gamma \gamma^* \leq \sigma_1 I_d\}$ for some constants $0 < \sigma_0 \leq \sigma_1$, this lemma follows readily from Proposition 3.4 in [12]. Notice that the proof below does not use any structures of Θ other than uniform nondegeneracy (5.4).

Proof of Lemma 5.6. Since $\{P_\theta : \theta \in \mathcal{A}_{0,T}^\Theta\} \subset \mathcal{P}$, it is clear that $\mathbb{E}[X] = \bar{\mathbb{E}}[X] \leq \sup_{P \in \mathcal{P}} E_P[X]$. We check the reverse inequality

$$\mathbb{E}[X] \geq \sup_{P \in \mathcal{P}} E_P[X]. \quad (5.11)$$

For each $n \in \mathbb{N}$, we set the statement $\mathfrak{p}(n)$ as follows:

$$\begin{aligned} \mathfrak{p}(n) : & \text{ For all } 0 \leq t_1 < \cdots < t_n \leq T, \quad \text{and} \quad \varphi \in C_{b,\text{Lip}}((\mathbb{R}^d)^n), \\ & \sup_{P \in \mathcal{P}} E_P[\varphi(B_{t_1}, \dots, B_{t_n})] \leq \mathbb{E}[\varphi(B_{t_1}, \dots, B_{t_n})] \quad \text{holds.} \end{aligned}$$

We show (5.11) by induction with respect to n .

(i) First we let $n = 1$, and v be the solution of the following G -heat equation:

$$\begin{cases} -\frac{\partial v}{\partial t} - G(D^2 v) = 0 & \text{in } (0, t_1) \times \mathbb{R}^d, \\ v|_{t=t_1} = \varphi & \text{in } \mathbb{R}^d. \end{cases}$$

Note that $v \in C^{1,2}((0, t_1) \times \mathbb{R}^d)$ by assumption (5.4) (see Remark 3.1). For all $P \in \mathcal{P}$, it follows from Itô's formula that P -a.s.

$$\begin{aligned} \varphi(B_{t_1}) &= v(t_1, B_{t_1}) \\ &= v(0, 0) + \int_0^{t_1} (Dv)(t, B_t) \cdot dB_t \\ &\quad + \int_0^{t_1} \left(-G((D^2 v)(t, B_t)) dt + \frac{1}{2} \text{tr}[(D^2 v)(t, B_t) d\langle B \rangle_t^P] \right) \\ &\leq v(0, 0) + \int_0^{t_1} (Dv)(t, B_t) \cdot dB_t. \end{aligned}$$

Taking the expectation under P , we have $E_P[\varphi(B_{t_1})] \leq v(0, 0) = \mathbb{E}[\varphi(B_{t_1})]$. Hence

$$\sup_{P \in \mathcal{P}} E_P[\varphi(B_{t_1})] \leq \mathbb{E}[\varphi(B_{t_1})].$$

(ii) We now assume that $\mathfrak{p}(n)$ is true for some $n \in \mathbb{N}$. Take $0 \leq t_1 < \cdots < t_n < t_{n+1} \leq T$ and $\varphi \in C_{b,\text{Lip}}((\mathbb{R}^d)^{n+1})$ to be arbitrary. By the definition of conditional G -expectations, it holds that

$$\mathbb{E}_{t_n}[\varphi(B_{t_1}, \dots, B_{t_n}, B_{t_{n+1}})] = v(t_n, B_{t_n}; B_{t_1}, \dots, B_{t_n}),$$

where $v(t, x; x_1, \dots, x_n) \in C^{1,2}((t_n, t_{n+1}) \times \mathbb{R}^d)$ is the solution of the following G -heat equation:

$$\begin{cases} -\frac{\partial v}{\partial t} - G(D_x^2 v) = 0 & \text{in } (t_n, t_{n+1}) \times \mathbb{R}^d, \\ v(t_{n+1}, x; x_1, \dots, x_n) = \varphi(x_1, \dots, x_n, x), & x \in \mathbb{R}^d. \end{cases}$$

Under each $P \in \mathcal{P}$, we apply Itô's formula (see [Remark 5.7](#)) to $v(t_{n+1}, B_{t_{n+1}}; B_{t_1}, \dots, B_{t_n})$ to obtain P -a.s.

$$\begin{aligned} \varphi(B_{t_1}, \dots, B_{t_n}, B_{t_{n+1}}) &= v(t_{n+1}, B_{t_{n+1}}; B_{t_1}, \dots, B_{t_n}) \\ &= v(t_n, B_{t_n}; B_{t_1}, \dots, B_{t_n}) \\ &\quad + \int_{t_n}^{t_{n+1}} (D_x v)(t, B_t; B_{t_1}, \dots, B_{t_n}) \cdot dB_t \\ &\quad + \int_{t_n}^{t_{n+1}} \left(-G((D_x^2 v)(t, B_t; B_{t_1}, \dots, B_{t_n})) dt \right. \\ &\quad \left. + \frac{1}{2} \text{tr}[(D_x^2 v)(t, B_t; B_{t_1}, \dots, B_{t_n}) d\langle B \rangle_t^P] \right) \\ &\leq v(t_n, B_{t_n}; B_{t_1}, \dots, B_{t_n}) \\ &\quad + \int_{t_n}^{t_{n+1}} (D_x v)(t, B_t; B_{t_1}, \dots, B_{t_n}) \cdot dB_t. \end{aligned} \quad (5.12)$$

Taking the expectation under P , we have

$$E_P[\varphi(B_{t_1}, \dots, B_{t_n}, B_{t_{n+1}})] \leq E_P[v(t_n, B_{t_n}; B_{t_1}, \dots, B_{t_n})].$$

Therefore

$$\sup_{P \in \mathcal{P}} E_P[\varphi(B_{t_1}, \dots, B_{t_n}, B_{t_{n+1}})] \leq \sup_{P \in \mathcal{P}} E_P[v(t_n, B_{t_n}; B_{t_1}, \dots, B_{t_n})].$$

Notice that the function $(x_1, \dots, x_n) \mapsto v(t_n, x_n; x_1, \dots, x_n)$ belongs to $C_{b, \text{Lip}}((\mathbb{R}^d)^n)$. By the assumption that $\mathfrak{p}(n)$ is true, we have

$$\begin{aligned} \sup_{P \in \mathcal{P}} E_P[v(t_n, B_{t_n}; B_{t_1}, \dots, B_{t_n})] &\leq \mathbb{E}[v(t_n, B_{t_n}; B_{t_1}, \dots, B_{t_n})] \\ &= \mathbb{E}[\varphi(B_{t_1}, \dots, B_{t_n}, B_{t_{n+1}})]. \end{aligned}$$

So $\mathfrak{p}(n+1)$ is also true, and hence we complete the induction argument. \square

Remark 5.7. The second equality in (5.12) may be seen in the following manner: for each $i = 1, \dots, n$, define the process M^i on $[t_n, t_{n+1}]$ by

$$M_t^i := B_{t_i}, \quad t_n \leq t \leq t_{n+1},$$

and set $M_t := (t, B_t, M_t^1, \dots, M_t^n)$. Clearly $\{M_t; t_n \leq t \leq t_{n+1}\}$ is an \mathbb{F}^B -semimartingale. We may write $v(M_t)$ for $v(t, B_t; B_{t_1}, \dots, B_{t_n})$, to which Itô's formula applies to yield the desired equality.

Now we are in a position to prove [Theorem 5.3](#).

Proof of Theorem 5.3. It is sufficient to show that for all $k \in \mathbb{N}$, $t_1, \dots, t_k \in [0, T]$, and $\varphi \in C_{b, \text{Lip}}((\mathbb{R}^d)^k)$,

$$\hat{\mathbb{E}}[\varphi(\hat{B}_{t_1}, \dots, \hat{B}_{t_k})] = \mathbb{E}[\varphi(B_{t_1}, \dots, B_{t_k})].$$

Indeed, it is obvious that \hat{B} satisfies Definition 3.2(i). If we obtain the above equation, it then follows from the right-hand side that \hat{B} satisfies Definition 3.2(ii), (iii) under $\hat{\mathbb{E}}$. Note that, since $\hat{\mathcal{H}}$ is the completion of $\hat{C}_{b, \text{Lip}}(\Omega)$, identical distributedness and independence on $(\Omega, \hat{\mathcal{H}}, \hat{\mathbb{E}})$ can be, as those on $(\Omega, \mathcal{L}_G^1(\Omega), \mathbb{E})$ are, checked through test functions of the class consisting of bounded, Lipschitz cylinder functionals (see Definition 2.3 and the comment given just after it).

For simplicity, we write $\varphi(B)$ and $\varphi(\hat{B})$ for $\varphi(B_{t_1}, \dots, B_{t_k})$ and $\varphi(\hat{B}_{t_1}, \dots, \hat{B}_{t_k})$, respectively.

(i) First we show that $\mathbb{E}[\varphi(B)] \leq \hat{\mathbb{E}}[\varphi(\hat{B})]$.

It is enough to show the following:

$$\text{for all } \Theta\text{-valued simple processes } \theta \text{ on } [0, T], \quad E_{P_\theta}[\varphi(B)] \leq \hat{\mathbb{E}}[\varphi(\hat{B})]. \quad (5.13)$$

To see this, we fix $\theta \in \mathcal{A}_{0,T}^\Theta$. Then, for all $\varepsilon > 0$, there exists a Θ -valued simple process θ^ε on $[0, T]$ such that

$$E_P \left[\int_0^T \|\theta_s^\varepsilon - \theta_s\|^2 ds \right] < \varepsilon^2$$

(see, e.g., [4] Problem 3.2.5). Therefore, if (5.13) holds, we have

$$\begin{aligned} E_{P_\theta}[\varphi(B)] &\equiv E_{P_\theta}[\varphi(B_{t_1}, \dots, B_{t_k})] \\ &\leq E_{P_{\theta^\varepsilon}}[\varphi(B_{t_1}, \dots, B_{t_k})] \\ &\quad + C_\varphi E_P \left[\left(\sum_{i=1}^k \sum_{j=1}^d \left| \sum_{l=1}^d \int_0^{t_i} (\theta_s - \theta_s^\varepsilon)^{jl} dW_s^l \right|^2 \right)^{1/2} \right] \\ &\leq \hat{\mathbb{E}}[\varphi(\hat{B}_{t_1}, \dots, \hat{B}_{t_k})] + C_\varphi E_P \left[\sum_{i=1}^k \sum_{j=1}^d \left| \sum_{l=1}^d \int_0^{t_i} (\theta_s - \theta_s^\varepsilon)^{jl} dW_s^l \right|^2 \right]^{1/2} \\ &\leq \hat{\mathbb{E}}[\varphi(\hat{B})] + C_\varphi \sqrt{k} E_P \left[\int_0^T \|\theta_s - \theta_s^\varepsilon\|^2 ds \right]^{1/2} \\ &\leq \hat{\mathbb{E}}[\varphi(\hat{B})] + C_\varphi \sqrt{k} \varepsilon, \end{aligned}$$

where C_φ is a Lipschitz constant of φ and $(\theta_s - \theta_s^\varepsilon)^{jl}$ is the (j, l) -entry of $\theta_s - \theta_s^\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get

$$E_{P_\theta}[\varphi(B)] \leq \hat{\mathbb{E}}[\varphi(\hat{B})].$$

Now we show (5.13). Let θ be given in the form

$$\theta_t = \eta_0 \mathbb{1}_{[t_0, t_1]}(t) + \eta_1(W) \mathbb{1}_{(t_1, t_2]}(t) + \dots + \eta_{n-1}(W) \mathbb{1}_{(t_{n-1}, t_n]}(t) \quad (5.14)$$

for $0 \leq t \leq T$, where $0 = t_0 < t_1 < \dots < t_n = T$ is a partition of $[0, T]$, $\eta_0 \in \Theta$, and $\eta_i(\omega) \equiv \eta_i(\omega_t, t \leq t_i)$, $\omega \in \Omega$, is a Θ -valued measurable functional on Ω for $i = 1, \dots, n-1$. We now define the sequence of random variables $\{\tilde{\eta}_i\}_{i=1}^{n-1}$ and the simple process $\tilde{\theta} = \{\tilde{\theta}_t; 0 \leq t \leq T\}$ as

follows:

$$\begin{cases} \tilde{\eta}_0 := \eta_0, \\ \tilde{\eta}_1 := \eta_1 \left(W_t - \int_0^t \tilde{\theta}_s^* h_s^{(\tilde{\theta})} ds, t \leq t_1 \right), & \tilde{\theta}_t := \tilde{\eta}_0, t_0 \leq t \leq t_1, \\ \vdots & \vdots \\ \tilde{\eta}_{n-1} := \eta_{n-1} \left(W_t - \int_0^t \tilde{\theta}_s^* h_s^{(\tilde{\theta})} ds, t \leq t_{n-1} \right), & \tilde{\theta}_t := \tilde{\eta}_{n-1}, t_{n-1} < t \leq t_n, \end{cases}$$

where $h_s^{(\tilde{\theta})} := h_s(\int_0^t \tilde{\theta}_u dW_u)$. As the right-hand side of (5.14) is given as a functional of W , we denote it by $\theta_t(W)$ with a slight abuse of notation. Then, from the above construction of $\tilde{\theta}$, for all $0 \leq t \leq T$,

$$\tilde{\theta}_t = \theta_t \left(W - \int_0^t \tilde{\theta}_s^* h_s^{(\tilde{\theta})} ds \right).$$

Set

$$\begin{aligned} \tilde{W}_t &:= W_t - \int_0^t \tilde{\theta}_s^* h_s^{(\tilde{\theta})} ds, \quad 0 \leq t \leq T, \\ D_T^{(\tilde{\theta})} &:= \exp \left(\int_0^T \tilde{\theta}_t^* h_t^{(\tilde{\theta})} \cdot dW_t - \frac{1}{2} \int_0^T h_t^{(\tilde{\theta})} \cdot (\tilde{\theta}_t \tilde{\theta}_t^* h_t^{(\tilde{\theta})}) dt \right), \\ \tilde{P}(A) &:= E_P[\mathbb{1}_A D_T^{(\tilde{\theta})}], \quad A \in \mathcal{F}_T^W. \end{aligned}$$

Since, by Girsanov's formula, \tilde{W} is a Brownian motion under \tilde{P} , we have

$$\begin{aligned} E_{P_{\tilde{\theta}}}[\varphi(B)] &= E_{\tilde{P}} \left[\varphi \left(\int_0^T \theta_s(\tilde{W}) d\tilde{W}_s \right) \right] \\ &= E_P \left[\varphi \left(\int_0^T \tilde{\theta}_s dW_s - \int_0^T \tilde{\theta}_s \tilde{\theta}_s^* h_s^{(\tilde{\theta})} ds \right) D_T^{(\tilde{\theta})} \right] \\ &= E_{P_{\tilde{\theta}}} \left[\varphi \left(B - \int_0^T (d\langle B \rangle_s h_s) \right) D_T \right] \\ &\leq \mathbb{E} \left[\varphi \left(B - \int_0^T (d\langle B \rangle_s h_s) \right) D_T \right] = \hat{\mathbb{E}}[\varphi(\hat{B})], \end{aligned}$$

which shows (5.13).

(ii) Next we show that $\hat{\mathbb{E}}[\varphi(\hat{B})] \leq \mathbb{E}[\varphi(B)]$.

For each $\theta \in \mathcal{A}_{0,T}^{\Theta}$, let Q_{θ} be the measure defined by (5.6). By Lemma 5.5, \hat{B} is a Q_{θ} -martingale. Girsanov's formula also implies that

$$\langle \hat{B} \rangle = \langle B \rangle, \quad P_{\theta}\text{-a.s. and } Q_{\theta}\text{-a.s.}$$

Hence $Q_{\theta} \circ \hat{B}^{-1} \in \mathcal{P}$, where $Q_{\theta} \circ \hat{B}^{-1}(A) := Q_{\theta}(\hat{B} \in A)$ for each $A \in \mathcal{B}(\Omega)$. Then, using Lemma 5.6, we have

$$E_{P_{\theta}}[\varphi(\hat{B}) D_T] = E_{Q_{\theta} \circ \hat{B}^{-1}}[\varphi(B)] \leq \sup_{P \in \mathcal{P}} E_P[\varphi(B)] = \mathbb{E}[\varphi(B)].$$

Therefore we get

$$\hat{\mathbb{E}}[\varphi(\hat{B})] = \sup_{\theta \in \mathcal{A}_{0,T}^\theta} E_{P_\theta}[\varphi(\hat{B})D_T] \leq \mathbb{E}[\varphi(B)],$$

and complete the proof. \square

Remark 5.8. In [14], a Lévy-type characterization of one-dimensional G -Brownian motion is given, and by using that characterization, Xu–Shang–Zhang [13] obtains Girsanov’s formula for one-dimensional G -Brownian motion. On the other hand, as far as we know, such a characterization is not available in the case of multidimension. We remark that unlike the classical Brownian motion, components of multidimensional G -Brownian motion are correlated due to variance uncertainty. The advantage of our method is to appeal directly to the definition of G -Brownian motion (Definition 3.2), which enables us to deal with the multidimensional case.

We conclude this subsection with a remark on the construction of $\hat{\mathbb{E}}$.

Remark 5.9. Eq. (5.5) holds on $\hat{\mathcal{H}}$, namely

$$XD_T \in \mathcal{L}_G^1(\Omega) \quad \text{for all } X \in \hat{\mathcal{H}}.$$

To see this, it is sufficient to check the completeness of $\mathcal{L} := \{X \in \mathcal{L}_G^1(\Omega) : XD_T \in \mathcal{L}_G^1(\Omega)\}$ with respect to the norm $\hat{\mathbb{E}}[\cdot]$. Let $\{X_n\}_{n=1}^\infty \subset \mathcal{L}$ be an $\hat{\mathbb{E}}[\cdot]$ -Cauchy sequence, that is,

$$\mathbb{E}[|X_n - X_m|D_T] \rightarrow 0 \quad (n, m \rightarrow \infty).$$

This implies that $\{X_n D_T\}_{n=1}^\infty \subset \mathcal{L}_G^1(\Omega)$ is an $\mathbb{E}[\cdot]$ -Cauchy sequence. Hence, from completeness of $\mathcal{L}_G^1(\Omega)$, there exists a unique $Y \in \mathcal{L}_G^1(\Omega)$ such that

$$\mathbb{E}[|X_n D_T - Y|] \rightarrow 0 \quad (n \rightarrow \infty).$$

As $X := Y D_T^{-1}$ is in \mathcal{L} , we get $\hat{\mathbb{E}}[|X_n - X|] \rightarrow 0 \quad (n \rightarrow \infty)$. Therefore \mathcal{L} is complete under the norm $\hat{\mathbb{E}}[\cdot]$.

5.4. G -Novikov’s condition

For $h \in (M_G^2(\Omega))^d$, consider the process D defined by (5.3). In this subsection, we give a sufficient condition for D to be a symmetric G -martingale, which reads as follows: there exists $\varepsilon > 0$ such that

$$\mathbb{E} \left[\exp \left(\frac{1}{2} (1 + \varepsilon) \int_0^T h_s \cdot (d\langle B \rangle_s h_s) \right) \right] < \infty. \quad (5.15)$$

This condition may be regarded as a sublinear counterpart to the well-known Novikov’s condition in the classical stochastic analysis, and we refer to it as G -Novikov’s condition. We remark that in the one-dimensional case, this condition is the same as that imposed in [13].

Proposition 5.10. *If $h \in (M_G^2(\Omega))^d$ satisfies G -Novikov’s condition (5.15), then the process D is a symmetric G -martingale.*

Proof. Note that under the condition (5.15), the usual Novikov’s condition is fulfilled for all $\theta \in \mathcal{A}_{0,T}^\theta$:

$$E_{P_\theta} \left[\exp \left(\frac{1}{2} \int_0^T h_s \cdot (d\langle B \rangle_s h_s) \right) \right] < \infty.$$

Therefore D is a P_θ -martingale for each $\theta \in \mathcal{A}_{0,T}^\theta$. In view of Proposition 5.2, it remains to prove that $D_t \in \mathcal{L}_G^1(\Omega_t)$ for each $t \in [0, T]$.

Fix $t \in [0, T]$ and let

$$p = \frac{1 + \varepsilon}{2\sqrt{1 + \varepsilon} - 1}, \quad q = \frac{2\sqrt{1 + \varepsilon} - 1}{\sqrt{1 + \varepsilon}}.$$

Note that $p, q > 1$ and

$$p^2 q^2 = \frac{pq(pq - 1)}{q - 1} = 1 + \varepsilon. \quad (5.16)$$

Then, for all $\theta \in \mathcal{A}_{0,T}^\theta$,

$$\begin{aligned} E_{P_\theta}[(D_t)^p] &= E_{P_\theta} \left[\exp \left(\int_0^t p h_s \cdot dB_s - \frac{1}{2} \int_0^t p^2 q h_s \cdot (d\langle B \rangle_s h_s) \right) \right. \\ &\quad \times \left. \exp \left(\frac{p(pq - 1)}{2} \int_0^t h_s \cdot (d\langle B \rangle_s h_s) \right) \right] \\ &\leq E_{P_\theta} \left[\exp \left(\int_0^t pq h_s \cdot dB_s - \frac{1}{2} \int_0^t p^2 q^2 h_s \cdot (d\langle B \rangle_s h_s) \right) \right]^{1/q} \\ &\quad \times E_{P_\theta} \left[\exp \left(\frac{pq(pq - 1)}{2(q - 1)} \int_0^t h_s \cdot (d\langle B \rangle_s h_s) \right) \right]^{1-1/q}. \end{aligned}$$

By (5.15) and (5.16), we have

$$\bar{\mathbb{E}} \left[\exp \left(\frac{pq(pq - 1)}{2(q - 1)} \int_0^t h_s \cdot (d\langle B \rangle_s h_s) \right) \right] < \infty,$$

and Novikov's condition implies that the process

$$\left\{ \exp \left(\int_0^t pq h_s \cdot dB_s - \frac{1}{2} \int_0^t p^2 q^2 h_s \cdot (d\langle B \rangle_s h_s) \right); 0 \leq t \leq T \right\}$$

is a P_θ -martingale. Therefore $\bar{\mathbb{E}}[(D_t)^p] < \infty$, and hence

$$\lim_{N \rightarrow \infty} \bar{\mathbb{E}}[D_t \mathbb{1}_{\{D_t > N\}}] = 0.$$

Moreover, D_t has a q.c. version and belongs to $L^0(\Omega_t)$ since $\int_0^t h_s \cdot dB_s$ and $\int_0^t h_s \cdot (d\langle B \rangle_s h_s)$ do by their definitions. Therefore, by Theorem 4.1, we have $D_t \in \mathcal{L}_G^1(\Omega_t)$. \square

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