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On the small-time behaviour of Lévy-type processes

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Abstract

We show some Chung-type lim inf law of the iterated logarithm results at zero for a class of (pure-jump) Feller or Lévy-type processes. This class includes all Lévy processes. The norming function is given in terms of the symbol of the infinitesimal generator of the process. In the Lévy case, the symbol coincides with the characteristic exponent.

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1. Introduction

We study the short-time behaviour of a class of one-dimensional Feller processes $(X_t)_{t\geqslant 0}$. To do so we identify suitable norming functions u, v, w such that the following Chung-type LIL (law of the iterated logarithm) assertions hold $\mathbb{P}^x$-almost surely:

$$\varliminf_{t\to 0} \frac{\sup_{0\leqslant s\leqslant t} |X_s - x|}{u^{-1}(x, t/\log|\log t|)} = C(x), \tag{1}$$

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$$\varlimsup_{t\to 0} \frac{\sup_{0\leqslant s\leqslant t} |X_s - x|}{v(t,x)} = 0 \quad \text{or } = +\infty, \tag{2}$$

$$\varliminf_{t\to 0} \frac{|X_t - x|}{w(t,x)} = \gamma(x) > 0 \quad \text{or } = +\infty. \tag{3}$$

Assertions of this kind are classical for Brownian motion, the corresponding results for Lévy processes are due to Dupuis [6] and Aurzada, Döring and Savov [1]. The class of Feller processes considered in this paper includes Lévy processes and extends the results of these authors. We will characterize the norming functions with the help of the symbol of the infinitesimal generator of the Feller process. In the case of a Lévy process this becomes a rather simple criterion in terms of the characteristic exponent of the process.

Lévy processes. A (real-valued) Lévy process $(X_t)_{t\geqslant 0}$ is a stochastic process with stationary and independent increments and càdlàg (right continuous with finite left limits) sample paths. The transition function is uniquely determined through the characteristic function which is of the following form:

$$\lambda_t(x,\xi) := \mathbb{E}^x e^{i\xi(X_t - x)} = \mathbb{E}^0 e^{i\xi X_t} = e^{-t\psi(\xi)}, \quad t \geqslant 0,\ \xi \in \mathbb{R}.$$

The *characteristic exponent* $\psi : \mathbb{R} \to \mathbb{C}$ is given by the *Lévy–Khintchine formula*

$$\psi(\xi) = il\xi + \tfrac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{iy\xi} + iy\xi \mathbb{1}_{(0,1]}(|y|)\right) \nu(dy) \tag{4}$$

and the *Lévy triplet* (l, σ^2, ν) where ν is a measure on $\mathbb{R}\setminus\{0\}$ such that $\int_{y\neq 0}(1\wedge y^2)\,\nu(dy) < \infty$, and $l \in \mathbb{R}$, $\sigma \geqslant 0$. The characteristic exponent is also the symbol of the infinitesimal generator A of the Lévy process:

$$Au(x) = -\psi(D)u(x) := -\int_{\mathbb{R}} e^{ix\xi}\hat{u}(\xi)\,\psi(\xi)\,d\xi, \quad u \in C_c^\infty(\mathbb{R}),$$

where $\hat{u}(\xi) = (2\pi)^{-1}\int_{\mathbb{R}} u(x)e^{-ix\xi}\,dx$ denotes the Fourier transform of u.

Feller processes. The generator of a Lévy process has *constant coefficients*: it does not depend on the state space variable x. This is due to the fact that a Lévy process is spatially homogeneous which means that the transition semigroup $P_t u(x) = \mathbb{E}^x u(X_t) = \mathbb{E}u(X_t + x)$ is given by convolution operators. We are naturally led to *Feller processes* if we give up spatial homogeneity.

Definition 1. A (one-dimensional) *Feller process* is a real-valued Markov process $(X_t)_{t\geqslant 0}$ whose transition semigroup $P_t u(x) := \mathbb{E}^x u(X_t),\ u \in B_b(\mathbb{R})$, is a *Feller semigroup*, i.e.

(a) P_t is Markovian: if $u \in B_b(\mathbb{R})$, $u \geqslant 0$ then $P_t u \geqslant 0$ and $P_t 1 = 1$;
(b) P_t maps $C_\infty(\mathbb{R}) := \left\{u \in C(\mathbb{R}) : \lim_{|x|\to\infty} u(x) = 0\right\}$ into itself;
(c) P_t is a strongly continuous contraction semigroup in $(C_\infty(\mathbb{R}), \|\cdot\|_\infty)$.

Every Lévy process is a Feller process.

Write $(A, D(A))$ for the generator of the Feller semigroup. If $C_c^\infty(\mathbb{R}) \subset D(A)$, then

$$Au(x) = -p(x,D)u(x) := -\int_{\mathbb{R}} e^{ix\xi}\hat{u}(\xi)\,p(x,\xi)\,d\xi, \quad u \in C_c^\infty(\mathbb{R}),$$

see e.g. [10, Vol. 1, Theorem 4.5.21, p. 360]; this means that A is a pseudo differential operator whose symbol $p : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is such that for every fixed x the function $\xi \mapsto p(x, \xi)$ is the characteristic exponent of a Lévy process

$$p(x,\xi) = il(x)\xi + \tfrac{1}{2}\sigma^2(x)\xi^2 + \int_{\mathbb{R}\setminus\{0\}} \big(1 - e^{i\xi y} + i\xi y \mathbb{1}_{(0,1]}(|y|)\big)\,\nu(x, dy). \tag{5}$$

The Lévy triplet $(l(x), \sigma^2(x), \nu(x, dy))$ now depends on the state space, i.e. the generator is an operator with variable 'coefficients'. A Feller process $(X_t)_{t\geqslant 0}$ is said to be a *Lévy-type process*, if the symbol of the generator admits the representation (5). Typical examples are elliptic diffusions where the symbol (in one dimension) is of the form $p(x,\xi) = \frac{1}{2}\sigma^2(x)\xi^2$ and stable-like processes where $p(x,\xi) = |\xi|^{\alpha(x)}$ with $0 < \alpha_0 \leqslant \alpha(x) \leqslant \alpha_1 < 2$ is Lipschitz continuous, cf. [2]. For further details we refer to [10] or [11], or [4] as an up-to-date standard reference.

The symbol $p(x,\xi)$ plays very much the same role as the characteristic exponent of a Lévy process and it is possible to use $p(x,\xi)$ to describe the path behaviour of a Feller process, for example [19,11] or [21]. Note however that, due to the lack of spatial homogeneity, $p(x,\xi)$ is not the exponent of the characteristic function, i.e.

$$\lambda_t(x,\xi) = \mathbb{E}^x e^{i(X_t - x)\xi} \neq e^{-tp(x,\xi)}.$$

A brief overview of LIL-type results. For a general Lévy process the first result is due to Khintchine [13], cf. [12] for the Brownian LIL. Khintchine provides a necessary and sufficient criterion for a positive increasing function $u : (0,\epsilon) \to (0,\infty)$ to be the upper function for a one-dimensional Lévy process $(X_t)_{t\geqslant 0}$ without Gaussian component:

$$\varlimsup_{t\to 0} \frac{|X_t|}{u(t)} \leqslant c \quad \mathbb{P}^0\text{-a.s. if, and only if,} \quad \int_{0+} \frac{\mathbb{P}^0\{|X_t| > cu(t)\}}{t}\,dt < \infty. \tag{6}$$

As usual, we indicate by $\int_{0+} \dots$ that the integral converges at the origin. For a Brownian motion this result is sharp with $u(t) = \sqrt{t \log|\log t|}$.

Khintchine's result is generalized by the following integral test due to Savov [18]. Let $N(t) := \int_{|x|>t} \nu(dx)$ and $b(t)$ be a function which satisfies some mild growth assumptions. Then

$$\int_0^1 N(b(t))dt < \infty \quad \text{or} \ = +\infty \Longrightarrow \varlimsup_{t\to 0} \frac{|X_t|}{b(t)} = \lambda(b) \quad \text{or} \ = +\infty.$$

The first Chung-type LIL for (n-dimensional α-stable) Lévy processes is due to Taylor [22]. If $0 < \alpha < n$ and if the transition density satisfies $p_t(0) > 0$, then (1) holds with $u^{-1}(x,t) = t^{1/\alpha}$ and $C(x) = C$. Pruitt and Taylor [17] extended this result for Lévy processes with independent stable components. Based on [9], Fristedt and Pruitt [7,8] prove a LIL for subordinators (one-sided increasing Lévy processes), where the upper function is determined by the Laplace exponent of the process. Dupuis [6] extends these results for *symmetric* Lévy processes, with $u^{-1}(t) := 1/\psi^U(1/t)$, where $\psi^U(\xi) = \int_{y\neq 0} \min\{1, |\xi u|^2\}\nu(du)$. Using a different approach, this result was independently rediscovered by Aurzada–Döring–Savov [1]. Further, more refined results of type (1) for (non-symmetric) Lévy processes are due to Wee [23,24].

2. A Chung-type lim inf LIL for Feller processes

Consider a one-dimensional Feller process $(X_t)_{t\geqslant 0}$ with symbol $p(x,\xi)$ of the form (5). Throughout we assume:

$$\begin{aligned}&C_c^\infty(\mathbb{R}) \text{ is in the domain of the infinitesimal generator;}\\ &x\mapsto p(x,\xi) \text{ is continuous and has no diffusion part} : \sigma^2\equiv 0;\\ &\textit{sector condition}: \ \exists c_0\in(0,\infty) \quad \forall x,\xi\in\mathbb{R}: |\mathrm{Im}\,p(x,\xi)|\leqslant c_0\mathrm{Re}\,p(x,\xi).\end{aligned} \tag{A1}$$

Define the function

$$p^{\cup}(x,\xi) := \int_{y\neq 0} \min\{|\xi y|^2,\ 1\}\,\nu(x,dy); \tag{7}$$

this is a natural generalization of a typical 'truncated second moments' function which appears naturally in the context of limit theorems, see e.g. [6, p. 46] and [16]. It is not hard to see that $|p(x,\xi)| \leqslant 2p^{\cup}(x,\xi)$ and $p^{\cup}(x,2\xi)\leqslant 4p^{\cup}(x,\xi)$ for all $x,\xi\in\mathbb{R}$. We will also need the following regularity assumptions:

$$\exists\kappa(x)>1 \quad \forall R\leqslant 1: \sup_{|x-y|\leqslant 2R} p^{\cup}\big(y,\tfrac{1}{R}\big) \leqslant \kappa(x) \inf_{|x-y|\leqslant 3R} p^{\cup}\big(y,\tfrac{1}{R}\big); \tag{A2}$$

$$\begin{aligned}&\exists t_0\in(0,1)\ \exists q=q(x)\in(0,1)\ \forall R\in(0,1],\ y\in B(x,R),\\ &\quad t\in[0,t_0]: \mathbb{P}^y(X_t<y)\leqslant q.\end{aligned} \tag{A3}$$

For example (A3) holds (even with equality) with $q=1/2$ if $\lambda_t(x,\xi)=\mathbb{E}^x e^{i\xi(X_t-x)}$ is real-valued. For Lévy processes which are not compound Poisson processes (A3) follows from $\lim_{t\to 0}\mathbb{P}(X_t>0)=\rho\in(0,1)$ (Spitzer's condition); see [5, Chapter 7] for the necessary and sufficient conditions in the Lévy case. Set

$$u\equiv u(x,R) := \frac{1}{\inf_{|x-y|\leqslant 3R} p^{\cup}\big(y,\frac{1}{R}\big)}, \qquad R\in(0,1], \tag{8}$$

and denote by $u^{-1}(x,\rho) := \inf\{r : u(x,r)\geqslant\rho\}$ the generalized inverse of $R\mapsto u(x,R)$.

We can now state the main result of this section.

Theorem 2. *Let $(X_t)_{t\geqslant 0}$ be a one-dimensional Feller process with symbol $p(x,\xi)$ satisfying (A1)–(A3). Then there exists a constant $C(x)>0$ such that*

$$\varliminf_{t\to 0} \frac{\sup_{0\leqslant s\leqslant t}|X_s-x|}{u^{-1}(x,t/\log|\log t|)} = C(x) \quad (\mathbb{P}^x\text{-}a.s.) \tag{9}$$

where u^{-1} is the generalized inverse of the function $R\mapsto u(x,R)$ defined in (8).

Before we prove Theorem 2 let us consider an example.

Example 3. Take $\nu(x,dy)=\frac{1}{4}\,\alpha(x)(2-\alpha(x))\,|y|^{-1-\alpha(x)}\,dy$, where $\alpha:\mathbb{R}\to[\alpha_0,\alpha_1]\subset(0,2)$ is continuously differentiable, with uniformly bounded derivative. Clearly, (A1) holds. A direct calculation shows that $p^{\cup}(x,\xi)=|\xi|^{\alpha(x)}$.

We will now check (A2). Pick $R \in (0, 1]$. Since α is continuously differentiable, we have

$$\sup_{|x-y|\leqslant 3R} \alpha(y) - \inf_{|x-y|\leqslant 3R} \alpha(y) = \max_{z,y\in B(x,3R)} |\alpha(y) - \alpha(z)| \leqslant 6R \max_{y\in B(x,3R)} |\alpha'(y)|,$$

implying

$$\frac{\sup_{|x-y|\leqslant 2R} p^{\mathsf{U}}(y, 1/R)}{\inf_{|x-y|\leqslant 3R} p^{\mathsf{U}}(y, 1/R)} \leqslant \frac{\sup_{|x-y|\leqslant 3R} p^{\mathsf{U}}(y, 1/R)}{\inf_{|x-y|\leqslant 3R} p^{\mathsf{U}}(y, 1/R)} \leqslant \left(\frac{1}{R}\right)^{6R \max_{y\in B(x,3R)} |\alpha'(y)|}$$
$$= \left(\frac{1}{R^R}\right)^{6 \max_{y\in B(x,3R)} |\alpha'(y)|}$$

for all $R \in (0, 1]$. Thus, (A2) holds with any function $\kappa(x) \geqslant 64^{\max_{y\in B(x,1)} |\alpha'(y)|}$.

Let us check (A3). In [15, Theorem 5.1], see also [14], it is proved that for $\alpha \in C_b^1(\mathbb{R})$, there exists a Feller process $(X_t)_{t\geqslant 0}$ corresponding to the characteristic triplet $\big(0, 0, \frac{1}{4}\,\alpha(x)(2 - \alpha(x))\,|y|^{-1-\alpha(x)}\,dy\big)$. Moreover, this process has a transition density $p(t, y, z)$, and

$$p(t, y, z) = p_{\alpha(y)}(t, y - z)[1 + O(1) \min\{1, (1 + |\log t|)|y - z|\} + O(t^{\delta})] + \frac{O(t)}{1 + |y - z|^{\alpha_0+1}}$$

for the symmetric $\alpha(y)$-stable transition density $p_{\alpha(y)}(t, y - z)$ and some $\delta \in (0, 1)$; the big-O terms refer to $t \to 0$ and do not depend on y, z. Using the scaling property we have the equality $p_{\alpha(y)}(t, y - z) = t^{-1/\alpha(y)} p_{\alpha(y)}(1, t^{-1/\alpha(y)}(y - z))$ and the unimodality of the stable law we get

$$\mathbb{P}^y(X_t > y) \geqslant \int_{y\leqslant z\leqslant y+\epsilon t^{1/\alpha(y)}} p(t, y, z)\, dz \geqslant \epsilon p_{\alpha(y)}(1, \epsilon)(1 + O(1)) + O(t^{1/\alpha(y)})$$

which proves (A3).

Next we calculate the rate of convergence. Since $\alpha(y)$ is continuously differentiable, we may assume that on $(x - 3R, x + 3R)$ there is a local minimum at x, say. Then for R small enough $\min_{|x-y|\leqslant 3R} \alpha(y) = \alpha(x)$, and so

$$u(x, R) = \left(\inf_{|x-y|\leqslant 3R} p^{\mathsf{U}}(y, R^{-1})\right)^{-1} = \sup_{|x-y|\leqslant 3R} R^{\alpha(y)} = R^{\min_{|x-y|\leqslant 3R} \alpha(y)} = R^{\alpha(x)}.$$

Consequently, $u^{-1}(x, \rho) = \rho^{1/\alpha(x)}$.

Assume now that x is not a local minimum of α. Then x is either a local maximum, or α is decreasing (respectively, increasing) on $[x - 3R, x + 3R]$. In both cases the minimum is attained at one of the endpoints. Without loss of generality we assume that the minimum is attained at the point $x - 3R$. Thus,

$$u(x, R) = R^{\alpha(x-3R)}$$

and as $\alpha_0 \leqslant \alpha(x) \leqslant \alpha_1$, we have

$$c_0\rho^{1/\alpha_0} \leqslant u^{-1}(x, \rho) \leqslant c_1\rho^{1/\alpha_1}. \tag{10}$$

Since α is continuously differentiable we get, using a Taylor expansion,

$$\alpha(x - R) = \alpha(x) - R\alpha'(x - \theta R), \tag{11}$$

where $\theta = \theta(x, R) \in (0, 1)$. Note that the function $R \mapsto R^{\alpha(x-3R)}$ is continuous and tends to 0 as $R \to 0$, implying that for sufficiently small ρ the equation $R^{\alpha(x-3R)} = \rho$ admits a solution; thus,

$$u^{-1}(x, \rho) = \min\{R : R^{\alpha(x-3R)} = \rho\}.$$

By (11) the function $u^{-1}(x, \rho)$ satisfies the equation

$$u^{-1}(x, \rho) = \rho^{1/(\alpha(x)-3u^{-1}(x,\rho)\alpha'(x-3\theta u^{-1}(x,\rho)))}. \tag{12}$$

Therefore, by (12) we have

$$\begin{aligned}\lim_{\rho\to 0} \frac{\rho^{1/\alpha(x)}}{u^{-1}(x, \rho)} &= \lim_{\rho\to 0} \exp\Big\{\Big(\frac{1}{\alpha(x)} - \frac{1}{\alpha(x) - 3u^{-1}(x, \rho)\alpha'(x - 3\theta u^{-1}(x, \rho))}\Big) \ln \rho\Big\}\\ &= 1,\end{aligned}$$

where we used the fact that $\alpha(x) \in [\alpha_0, \alpha_1]$ and, because of (10), $u^{-1}(x, \rho) \ln \rho \to 0$ as $\rho \to 0$. This gives (9) with $u^{-1}(x, \rho) = \rho^{1/\alpha(x)}$.

For the proof of Theorem 2 we need several auxiliary results in order to estimate the probability that $(X_t)_{t\geqslant 0}$ exits a ball of radius $r > 0$ within time $t > 0$.

Lemma 4 is the key to derive the LIL. We record it in a form which is convenient for our purposes, and refer to [19] for the original version, as well as to its improvement (with a simplified proof) [21, Proposition 4.3]. A close inspection of the arguments in [21] reveals that one does not need the 'bounded coefficients' assumption $\sup_{x\in\mathbb{R},|\xi|\leqslant 1} |p(x, \xi)| < \infty$.

Lemma 4. *Let $(X_t)_{t\geqslant 0}$ be a one-dimensional Feller process with symbol $p(x, \xi)$ satisfying* **(A1)**. *Then for all $t > 0$ and $R > 0$ we have*

$$\mathbb{P}^x\Big(\sup_{0\leqslant s\leqslant t} |X_s - x| \geqslant R\Big) \leqslant c\, t \sup_{|x-y|\leqslant R} p^{\mathsf{U}}\big(y, \tfrac{1}{R}\big), \tag{13}$$

$$\mathbb{P}^x\Big(\sup_{0\leqslant s\leqslant t} |X_s - x| < R\Big) \leqslant c\left(t \inf_{|x-y|\leqslant R} p^{\mathsf{U}}\big(y, \tfrac{1}{R}\big)\right)^{-1}. \tag{14}$$

The constant $c \geqslant 1$ depends only on the sector constant c_0, but not on x.

First we extend (14).

Lemma 5. *Under the assumptions of Lemma 4, we find for $n \geqslant 2$ and all $t, R > 0$*

$$\mathbb{P}^x\Big(\sup_{s\leqslant nt} |X_s - x| < R\Big) \leqslant (4c)^n \left(t \inf_{|x-y|\leqslant 3R} p^{\mathsf{U}}\big(y, \tfrac{1}{R}\big)\right)^{-n}. \tag{15}$$

Proof. Set, for simplicity, $X_t^* := \sup_{0\leqslant s\leqslant t} |X_s - x|$. We use induction and the Markov property.

$$\begin{aligned}\mathbb{P}^x(X_{nt}^* < R) &\leqslant \mathbb{E}^x\left(\mathbb{1}_{\{X_{(n-1)t}^* < R\}} \mathbb{1}_{\{\sup_{0\leqslant s\leqslant t} |X_{(n-1)t+s} - X_{(n-1)t}| < 2R\}}\right)\\ &= \mathbb{E}^x\left(\mathbb{1}_{\{X_{(n-1)t}^* < R\}} \mathbb{E}^{X_{(n-1)t}}\big[\mathbb{1}_{\{X_t^* < 2R\}}\big]\right)\\ &\leqslant \sup_{|x-y|\leqslant R} \mathbb{E}^y\big[\mathbb{1}_{\{X_t^* < 2R\}}\big]\mathbb{P}^x(X_{(n-1)t}^* < R).\end{aligned}$$

Using induction and the fact that $p^{\cup}(x, 2\xi) \leqslant 4p^{\cup}(x, \xi)$, we derive

$$\begin{aligned}\mathbb{P}^x(X^*_{nt} < R) &\leqslant c \sup_{|x-y|\leqslant R} \Big(t \inf_{|z-y|\leqslant 2R} p^{\cup}\big(z, \tfrac{1}{2R}\big)\Big)^{-1} (4c)^{n-1} \\ &\qquad \times \Big(t \inf_{|x-z|\leqslant 3R} p^{\cup}\big(z, \tfrac{1}{R}\big)\Big)^{-n+1} \\ &\leqslant (4c)^n \Big(t \inf_{|x-y|\leqslant 3R} p^{\cup}\big(y, \tfrac{1}{R}\big)\Big)^{-n}. \quad \Box\end{aligned}$$

Remark 6. Let $u(x, R)$ be as in (8). Then (15) becomes for any $\gamma > 1$

$$\mathbb{P}^x\Big(\sup_{s\leqslant n\cdot(4\gamma c)u(x,R)} |X_s - x| < R\Big) \leqslant \gamma^{-n}, \quad R > 0.$$

Lemma 7. *Suppose that the assumptions of Theorem 2 are satisfied. Denote by c the constant appearing in Lemma 4, by $\kappa(x)$ the constant from* (**A2**)*, and $\gamma = \gamma(x) > \max\big\{1, \frac{\kappa(x)}{4(1-q)}\big\}$, where $q = q(x)$ is the constant from* (**A3**)*. Then there exist constants $p_{1,\gamma}(x)$, $p_{2,\gamma}(x) \in (0, 1)$ such that for all $m \geqslant 1$*

$$p_{2,\gamma}(x)^{m+1} \leqslant \mathbb{P}^x\Big(\sup_{s\leqslant mu(x,R)} |X_s - x| \leqslant R\Big) \leqslant p_{1,\gamma}(x)^m. \tag{16}$$

Proof. Set $X^*_t := \sup_{r\leqslant t} |X_r - x|$ and $X^*_{s,t} := \sup_{s\leqslant r\leqslant t} |X_r - X_s|$. First we prove

$$C_2^n \leqslant \mathbb{P}^x(X^*_{nu(x,R)/(16\kappa(x)\gamma c)} \leqslant R) \quad \text{and} \quad \mathbb{P}^x(X^*_{n\cdot(4\gamma c)u(x,R)} \leqslant R) \leqslant C_1^n, \tag{17}$$

where $n \geqslant 1$ and $C_1, C_2 > 0$ are some constants. The upper bound follows from Lemma 5 and Remark 6 with $C_1 = 1/\gamma$, where $\gamma > 1$ is arbitrary, independent of x.

Let us establish the lower bound. The crux of the matter is now the behaviour of $p^{\cup}(x, 1/R)$ with respect to the variable x. Recall that $p^{\cup}(x, \xi)$ satisfies (**A2**) with some constant $\kappa(x)$. Then we get from (13) and (**A2**) for any z such that $|z - x| < R$

$$\begin{aligned}\mathbb{P}^z(X^*_{u(x,R)/(4c\gamma)} \geqslant R) &\leqslant \frac{1}{4\gamma} u(x, R) \sup_{|z-y|\leqslant R} p^{\cup}\big(y, \tfrac{1}{R}\big) \\ &\leqslant \frac{1}{4\gamma} u(x, R) \sup_{|x-y|\leqslant 2R} p^{\cup}\big(y, \tfrac{1}{R}\big) \leqslant \frac{\kappa(x)}{4\gamma}.\end{aligned}$$

Taking $\gamma \equiv \gamma(x) > \max\big\{1, \frac{\kappa(x)}{4(1-q)}\big\}$, we find for all z with $|z - x| < R$

$$\mathbb{P}^z(X^*_{u(x,R)/(4c\gamma)} < R) \geqslant 1 - \frac{\kappa(x)}{4\gamma} > q, \quad R > 0. \tag{18}$$

Let $T := u(x, R)/(4\gamma c)$ be fixed. Observe that $\{X^*_{nT} \leqslant 2R\} \supset \bigcap_{k=0}^{n-1} A_k$, where

$$A_k := \left\{ X^*_{kT,(k+1)T} \leqslant R,\ X_{(k+1)T} - X_{kT} \in \begin{cases}[0, R], & \text{if } X_{kT} < x,\\ [-R, 0], & \text{if } X_{kT} \geqslant x.\end{cases} \right\}$$

In other words, if $X_{kT} < x$, then at the next end-point $(k + 1)T$ the process is above X_{kT}, but within the ball $B(X_{kT}, R)$, and if $X_{kT} \geqslant x$, then at time $(k + 1)T$ the process is below X_{kT} but still in the ball $B(X_{kT}, R)$. Denote $\mathcal{F}_k := \sigma\{X_s, s \leqslant Tk\}$. Then

$$\begin{aligned}\mathbb{E}^x\left[\mathbb{1}_{A_{n-1}}|\mathcal{F}_{n-1}\right] &= \mathbb{P}^{X_{(n-1)T}}(A_0)\\ &= \mathbb{P}^{X_{(n-1)T}}\left(X_T^* \leqslant R, X_T - X_0 \in [0,R]\right)\mathbb{1}_{\{X_{(n-1)T}<x\}}\\ &\quad + \mathbb{P}^{X_{(n-1)T}}\left(X_T^* \leqslant R, X_T - X_0 \in [-R,0]\right)\mathbb{1}_{\{X_{(n-1)T}\geqslant x\}}\\ &\geqslant \inf_{x-R\leqslant z<x} \mathbb{P}^z\left(X_T^* \leqslant R, X_T - X_0 \in [0,R]\right).\end{aligned}$$

Without loss of generality we may assume that **(A3)** holds with $t_0 = 1$; otherwise we would just get a further multiplicative factor. By **(A3)** we have

$$\mathbb{P}^z\left(X_T^* \leqslant R, X_T - X_0 \in [-R,0]\right) \leqslant \mathbb{P}^z\left(X_T - X_0 < 0\right) \leqslant q,$$

uniformly in $z \in B(x,R)$ and $T \in [0,1]$. Using (18) with $z \in B(x,R)$ we get

$$\begin{aligned}\mathbb{P}^z\left(X_T^* \leqslant R, X_T - X_0 \in [0,R]\right) &= \mathbb{P}^z\left(X_T^* \leqslant R\right) - \mathbb{P}^z\left(X_T^* \leqslant R, X_T - X_0 \in [-R,0]\right)\\ &\geqslant \mathbb{P}^z\left(X_T^* \leqslant R\right) - q \geqslant C_2,\end{aligned}$$

where $C_2 := 1 - \frac{\kappa(x)}{4\gamma} - q > 0$ by our choice of γ. Thus,

$$\mathbb{E}^x\left[\mathbb{1}_{A_{n-1}}|\mathcal{F}_{n-1}\right] \geqslant C_2.$$

Note that $\prod_{k=0}^{n-2}\mathbb{1}_{A_k}$ is $\mathcal{F}_{n-1}$-measurable, and by the Markov property,

$$\begin{aligned}\mathbb{E}^x\left(\prod_{k=0}^{n-1}\mathbb{1}_{A_k}\right) &= \mathbb{E}^x\left(\mathbb{E}^x\left[\prod_{k=0}^{n-1}\mathbb{1}_{A_k}\middle|\mathcal{F}_{n-1}\right]\right) = \mathbb{E}^x\left(\prod_{k=0}^{n-2}\mathbb{1}_{A_k}\mathbb{E}^x\left[\mathbb{1}_{A_{n-1}}\middle|\mathcal{F}_{n-1}\right]\right)\\ &= \mathbb{E}^x\left(\prod_{k=0}^{n-2}\mathbb{1}_{A_k}\mathbb{P}^{X_{(n-1)T}}(A_{n-1})\right) \geqslant C_2\mathbb{E}^x\left(\prod_{k=0}^{n-2}\mathbb{1}_{A_k}\right).\end{aligned}$$

With **(A2)** and the fact that $p^{\cup}(x,2\xi) \leqslant 4p^{\cup}(x,\xi)$ we see

$$\begin{aligned}\inf_{|x-y|\leqslant 3R} p^{\cup}(y,1/R) &\leqslant \inf_{|x-y|\leqslant 2R} p^{\cup}(y,1/R) \leqslant \sup_{|x-y|\leqslant 2R} p^{\cup}(y,1/R)\\ &\leqslant \kappa(x)\inf_{|x-y|\leqslant 6R} p^{\cup}(y,1/R) \leqslant 4\kappa(x)\inf_{|x-y|\leqslant 6R} p^{\cup}(y,1/(2R)),\end{aligned}$$

which implies

$$u(x,2R) \leqslant 4\kappa(x)u(x,R). \tag{19}$$

Thus, by induction (recall that $T = u(x,R)/(4\gamma c)$)

$$\begin{aligned}\mathbb{P}^x(X^*_{nu(x,2R)/(16\kappa(x)\gamma c)} \leqslant 2R) \geqslant \mathbb{P}^x(X^*_{nu(x,R)/(4\gamma c)} \leqslant 2R) &= \mathbb{P}^x(X^*_{nT} \leqslant 2R)\\ &\geqslant \mathbb{E}^x\left[\prod_{k=0}^{n-1}\mathbb{1}_{A_k}\right] \geqslant C_2^n.\end{aligned}$$

Finally, we show how (16) follows from (17). Put $m := \lfloor n(4\gamma c)\rfloor + 1$ ($\lfloor x\rfloor$ denotes the largest integer smaller or equal to $x \in \mathbb{R}$); then $n\cdot(4\gamma c) \leqslant m \leqslant n\cdot(4\gamma c)+1$, implying

$$\mathbb{P}^x(X^*_{mu(x,R)} \leqslant R) \leqslant \mathbb{P}^x(X^*_{n(4\gamma c)u(x,R)} \leqslant R) \leqslant C_1^n = \left(C_1^{\frac{n}{m}}\right)^m \leqslant C_1^{\frac{m}{4\gamma c+1}} =: p^m_{1,\gamma}(x).$$

For the lower bound we set $m := \lfloor n/(16\gamma\kappa(x)c)\rfloor$. Then $\frac{n}{16\gamma\kappa(x)c} - 1 \leqslant m \leqslant \frac{n}{16\gamma\kappa(x)c}$, and

$$\mathbb{P}^x(X^*_{mu(x,R)} \leqslant R) \geqslant \mathbb{P}^x(X^*_{nu(x,R)/(16\gamma\kappa(x)c)} \leqslant R) \geqslant C_2^n = \left(C_2^{\frac{n}{m+1}}\right)^{m+1}$$
$$\geqslant C_2^{(m+1)16\gamma\kappa(x)c} =: p_{2,\gamma}^{m+1}(x). \quad \square$$

Remark 8. Note that the constants $p_{1,\gamma}$, $p_{2,\gamma}$ in Lemma 7 depend on the variable x through $\kappa(x)$ and $\gamma(x) > \max\left\{1, \frac{\kappa(x)}{4(1-q(x))}\right\}$. Without loss of generality we can choose the function $\gamma(x)$ such that $\inf_x \gamma(x) > \max\left\{1, \frac{\sup_x \kappa(x)}{\inf_x(1-q(x))}\right\}$; under this condition we have $\inf_x p_{2,\gamma}(x) > 0$.

Proof of Theorem 2. Fix $x \in \mathbb{R}$ and write $\tau^x(a) := \inf\{s \geqslant 0 : X_s - x \notin [-a, a]\}$ for the first exit time of the process $(X_t)_{t\geqslant 0}$ with $X_0 = x$. Then we can follow the arguments from [6].

Step 1. Using (16) we can prove, similar to [6], that there exists a constant $\xi \in (0, \infty)$ such that

$$\mathbb{P}^x\left(\sup_{2a_{2m}\leqslant a\leqslant 2a_m} \frac{\tau^x(a)}{u(x,a)\log|\log u(x,a)|} < \xi\right) \leqslant \exp\left(-m^{1/4}\right), \quad m \geqslant 1, \tag{20}$$

where $a_m = a_m(x)$ is the solution to $u(x, a_m) = e^{-m^2}$; clearly, $\lim_{m\to\infty} a_m = 0$. Indeed: let

$$c_1 := \left(3\sup_x |\log p_{2,\gamma}(x)|\right)^{-1}, \qquad \lambda_m := 2c_1 \log m, \qquad u_k := \lambda_k u(x, a_k),$$

$$\sigma_k := \sum_{i=k}^{\infty} u_i,$$

and consider the sets

$$G_k := \{X^*_{\sigma_k,\sigma_{k-1}} > a_k\}, \qquad H_k := \{X^*_{\sigma_k} > a_k\}, \qquad D_k := \{X^*_{\sigma_{k-1}} > 2a_k\},$$

where $X^*_{u,v} := \sup_{u\leqslant s<v}|X_s - X_u|$, and $k \geqslant 2$. By Remark 8 we can assume that $c_1 > 0$. By our choice of the sequence $(a_k)_{k\geqslant 1} = (a_k(x))_{k\geqslant 1}$, there exists a constant $c_2 > 0$ such that

$$\frac{\sigma_k}{u_k} = \sum_{j=1}^{\infty} \frac{u_{k+j}}{u_k} = \sum_{j=1}^{\infty} \frac{\lambda_{k+j}u(x, a_{k+j})}{\lambda_k u(x, a_k)} = \sum_{j=1}^{\infty} \frac{\log(k+j)e^{-(k+j)^2}}{\log k e^{-k^2}} \leqslant c_2 e^{-2k}. \tag{21}$$

Using the Markov property and the lower bound in Lemma 7, we get

$$\begin{aligned}\mathbb{P}^x(G_k) \leqslant \sup_z \mathbb{P}^z\Big(\sup_{0\leqslant s\leqslant u_k}|X_s - z| > a_k\Big) &= \sup_z\Big(1 - \mathbb{P}^z\Big(\sup_{0\leqslant s\leqslant \lambda_k u(x,a_k)}|X_s - z| \leqslant a_k\Big)\Big)\\
&\leqslant \sup_z\big(1 - p_{2,\gamma}(z)^{\lambda_k+1}\big)\\
&\leqslant 1 - \inf_z p_{2,\gamma}(z)e^{-2c_1\sup_z|\log p_{2,\gamma}(z)|\cdot\log m}\\
&= 1 - c_3 m^{-2/3} \leqslant \exp\big(-c_3 m^{-2/3}\big),\end{aligned} \tag{22}$$

where $c_3 := \inf_z p_{2,\gamma}(z) > 0$, cf. Remark 8. Now (13), (**A2**) and (21) yield

$$\mathbb{P}^x(X^*_{\sigma_k} > a_k) = \mathbb{P}^x\Big(X^*_{\frac{\sigma_k}{u(x,a_k)}\cdot u(x,a_k)} > a_k\Big) \leqslant c\frac{\kappa(x)\sigma_k}{u(x,a_k)} = c\frac{\kappa(x)\lambda_k\sigma_k}{u_k}$$
$$\leqslant c_4(x)\log k e^{-2k}, \tag{23}$$

where $c_4(x) = 2cc_1c_2\kappa(x)$.

Define $A_m := \bigcap_{k=m}^{2m} D_k$. Since $D_k \subset G_k \cup H_k$, $k \geqslant 1$, we have

$$A_m \subset \Big(\bigcap_{k=m}^{2m} G_k\Big) \cup \Big(\bigcap_{k=m}^{2m} H_k\Big) \subset \Big(\bigcap_{k=m}^{2m} G_k\Big) \cup \Big(\bigcup_{k=m}^{2m} H_k\Big).$$

Therefore, by (22), m applications of the Markov property, and (23) we have

$$\begin{aligned}
\mathbb{P}^x(A_m) &\leqslant \mathbb{P}^x\Big(\bigcap_{k=m}^{2m} G_k\Big) + \mathbb{P}^x\Big(\bigcup_{k=m}^{2m} H_k\Big)\\
&\leqslant \prod_{k=m}^{2m} \exp\big(-c_2 k^{-2/3}\big) + c_4(x)\sum_{k=m}^{2m} e^{-2k}\log k\\
&\leqslant \exp\big(-c_3 m(2m)^{-2/3}\big) + c_4(x)e^{-2m}\log(2m)\sum_{k=0}^{m} e^{-2k}\\
&\leqslant \exp\big(-c_3 2^{-2/3} m^{1/3}\big) + c_5(x)e^{-2m}\log(2m),
\end{aligned}$$

where $c_5(x) = c_4(x)e/(e-1)$. Therefore, there is some $m_0 = m_0(x)$ such that $\mathbb{P}(A_m) \leqslant e^{-m^{1/4}}$ for all $m \geqslant m_0$. Finally,

$$\begin{aligned}
e^{-m^{1/4}} \geqslant \mathbb{P}^x(A_m) = \mathbb{P}^x\Big(\bigcap_{k=m}^{2m}\Big\{\frac{X^*_{\sigma_{k-1}}}{2a_k} > 1\Big\}\Big) &= \mathbb{P}^x\Big(\inf_{m\leqslant k\leqslant 2m}\frac{X^*_{\sigma_{k-1}}}{2a_k} > 1\Big)\\
&= \mathbb{P}^x\Big(\sup_{m\leqslant k\leqslant 2m}\frac{\tau^x(2a_k)}{\sigma_{k-1}} < 1\Big)
\end{aligned}$$

where we used the definition of the first exit time $\tau^x(a)$ introduced at the beginning of the proof. As $\sigma_{k-1} = \sum_{j=k-1}^{\infty} u_j \geqslant u_k = 2c_1 u(x,a_k)\log k$, from the very definition of $u(x,a_k)$ we see that for $m \geqslant m_0$, where m_0 is large enough,

$$\begin{aligned}
\exp\big(-m^{1/4}\big) &\geqslant \mathbb{P}^x\Big(\sup_{m\leqslant k\leqslant 2m}\frac{\tau^x(2a_k)}{2c_1 u(x,a_k)\log k} < 1\Big)\\
&= \mathbb{P}^x\Big(\sup_{m\leqslant k\leqslant 2m}\frac{\tau^x(2a_k)}{c_1 u(x,a_k)\log|\log u(x,a_k)|} < 1\Big)\\
&\geqslant \mathbb{P}^x\Big(\sup_{2a_{2m}\leqslant a\leqslant 2a_m}\frac{\tau^x(a)}{c_1 u(x,a/2)\log|\log u(x,a/2)|} < 1\Big)\\
&\geqslant \mathbb{P}^x\Big(\sup_{2a_{2m}\leqslant a\leqslant 2a_m}\frac{\tau^x(a)}{u(x,a)\log|\log u(x,a)|} < \xi\Big),
\end{aligned}$$

with $\xi = \frac{c_1}{4\kappa(x)}$. In the last inequality we used (19), the fact that in the above inequalities $a > 0$ is small enough, and that by monotonicity of the function $a \mapsto u(x,a/2)$ we have $\log|\log u(x,a)| \leqslant \log|\log u(x,a/2)|$.

Step 2. From (20) we conclude with the Borel–Cantelli lemma that

$$\overline{\lim_{a\to 0}} \frac{\tau^x(a)}{u(x,a)\log|\log u(x,a)|} \geqslant \xi \quad (\mathbb{P}^x\text{-a.s.}).$$

Let ℓ_k be given by $u(x,\ell_k) = e^{-k}$, $k \geqslant 1$, and set $b := -4/\log p_{1,\gamma}$, cf. (16). By definition, the functions $u(x,a)$ and $\tau^x(a)$ are monotone increasing in the variable a, and we have that $u(x,\ell_{k+1}) = e^{-1}u(x,\ell_k)$. Therefore,

$$\begin{aligned} B_k &:= \left\{ \sup_{\ell_{k+1}\leqslant a\leqslant \ell_k} \frac{\tau^x(a)}{u(x,a)\log|\log u(x,a)|} \geqslant b \right\} \\ &\subset \left\{ \tau^x(\ell_k) \geqslant b u(x,\ell_{k+1}) \log|\log u(x,\ell_{k+1})| \right\} \\ &= \left\{ \tau^x(\ell_k) \geqslant b e^{-1} u(x,\ell_k) \log|\log u(x,\ell_{k+1})| \right\}, \end{aligned}$$

implying, by the upper estimate in (16), that $\mathbb{P}(B_k) \leqslant \exp\big(-4e^{-1}\log(k+1)\big) = (k+1)^{-4/e}$. Thus,

$$\overline{\lim_{\ell\to 0}} \frac{\tau^x(\ell)}{u(x,\ell)\log|\log u(x,\ell)|} \in [\xi, b]. \tag{24}$$

The expression on the left-hand side of (24) belongs to $\mathcal{F}_{0+}$. By the Blumenthal 0–1 law the σ-algebra $\mathcal{F}_{0+}$ is trivial, implying that there exists a constant C such that

$$\overline{\lim_{a\to 0}} \frac{\tau^x(a)}{u(x,a)\log|\log u(x,a)|} = C \quad (\mathbb{P}^x\text{-a.s.}). \tag{25}$$

This constant is the supremum of all ξ such that (20) holds. On the other hand,

$$\frac{1}{2}C \leqslant \frac{\tau^x(a_k)}{u(x,a_k)\log|\log u(x,a_k)|} \leqslant 2C, \quad k \geqslant k_0,$$

for any sequence $\{a_k\}_{k\geqslant 0}$ such that $a_k \to 0$ as $k \to \infty$. Here k_0 might possibly depend on the choice of the sequence $(a_k)_{k\geqslant 0}$. By the very definition of the first exit time $\tau^x(a)$, the above estimate implies

$$2\tilde{C} \geqslant \frac{1}{a_k} \sup_{0\leqslant s\leqslant u(x,a_k)\log|\log u(x,a_k)|} |X_s - x| \geqslant \frac{1}{2}\tilde{C}, \quad k \geqslant k_0,$$

where $\tilde{C} > 0$ is some constant. Thus,

$$\underline{\lim_{a\to 0}} \frac{1}{a} \sup_{0\leqslant s\leqslant u(x,a)\log|\log u(x,a)|} |X_s - x| = C' \quad (\mathbb{P}^x\text{-a.s.})$$

for some constant $0 < C' < \infty$. Substituting $a := u^{-1}(x,t)$, we get (9). $\square$

3. On the upper bound

In this section we prove (2), that is we give conditions which ensure that there is a norming function $v(t,x)$ with $\overline{\lim}_{t\to 0} \sup_{0\leqslant s\leqslant t} |X_s - x|/v(t,x) = 0 \mathbb{P}^x$-a.s.; for a Lévy process we also obtain conditions ensuring $\overline{\lim}_{t\to 0} \sup_{0\leqslant s\leqslant t} |X_s - x|/v(t,x) = \infty$ $\mathbb{P}^0$-a.s. For this we adapt Khintchine's criterion (6).

Proposition 9. *Let $(X_t)_{t\geqslant 0}$ be a one-dimensional Feller process with symbol $p(x,\xi)$, satisfying* (A1). *If $v(x,t)\geqslant 0$ is a function such that $t\mapsto v(x,t)$ is monotone increasing for every x and*

$$\int_{0+}\sup_{|y-x|\leqslant v(x,t)} p^{\cup}\big(y, \tfrac{1}{v(x,t)}\big)\,dt<\infty, \tag{26}$$

then

$$\lim_{t\to 0}\frac{\sup\limits_{0\leqslant s\leqslant t}|X_s-x|}{v(x,t)}=0\quad (\mathbb{P}^x\text{-a.s.}). \tag{27}$$

Proof. Under our assumptions, the process $(X_t)_{t\geqslant 0}$ satisfies the maximal inequality (13). As before, we write $X_t^*:=\sup_{0\leqslant s\leqslant t}|X_s-x|$ to simplify notation.

We will use the (easy direction of the) Borel–Cantelli lemma. Fix some $h\ll 1$ and set $t_k:=h/2^k$. Pick $\theta_k\in[t_{k+1},t_k)$. Since $v(x,t)$ is increasing in t, we have

$$\mathbb{P}^x\big(X_{\theta_k}^*>v(x,\theta_k)\big)\leqslant \mathbb{P}^x\big(X_{\theta_k}^*>v(x,t_{k+1})\big)\leqslant c\,\theta_k\sup_{|y-x|\leqslant v(x,t_{k+1})}p^{\cup}\left(y,\tfrac{1}{v(x,t_{k+1})}\right).$$

Because of $\theta_k\leqslant t_k=2t_{k+1}$ we see

$$\sum_{k=1}^{\infty}\mathbb{P}^x\big(X_{\theta_k}^*>v(x,\theta_k)\big)<\infty.$$

By the Borel–Cantelli lemma, $\mathbb{P}^x\big(X_{\theta_k}^*\leqslant v(x,\theta_k)\text{ for finally all }k\geqslant 1\big)=1$, implying

$$\varlimsup_{t\to 0}\frac{X_t^*}{v(x,t)}\leqslant 1\quad(\mathbb{P}^x\text{-a.s.}). \tag{28}$$

From the definition of $p^{\cup}(y,\xi)$, we find $p^{\cup}(y,\xi/\lambda)\leqslant\lambda^{-2}p^{\cup}(y,\xi)$ for all $0<\lambda<1$. Thus, (26) implies

$$\int_{0+}\sup_{|y-x|\leqslant\lambda v(x,t)}p^{\cup}\big(y,\tfrac{1}{\lambda v(x,t)}\big)\,dt\leqslant\frac{1}{\lambda^2}\int_{0+}\sup_{|y-x|\leqslant v(x,t)}p^{\cup}\big(y,\tfrac{1}{v(x,t)}\big)\,dt<\infty.$$

Because of (28) we get

$$\frac{1}{\lambda}\cdot\varlimsup_{t\to 0}\frac{X_t^*}{v(x,t)}=\varlimsup_{t\to 0}\frac{X_t^*}{\lambda v(x,t)}\leqslant 1\quad(\mathbb{P}^x\text{-a.s.}).$$

Letting $\lambda\to 0$ gives (27). $\square$

Example 10. Suppose that $0<c\leqslant p(y,\xi)/p(x,\xi)\leqslant C<\infty$ for all $\xi\in\mathbb{R}$, $|x-y|\leqslant r$ where $r\ll 1$ is sufficiently small. Then it is enough to check the convergence of the integral

$$\int_{0+}p^{\cup}\left(x,\tfrac{1}{v(x,t)}\right)dt<\infty.$$

This integral converges, e.g., for functions $v(x,t)$ of the type $v(x,t)=\frac{1}{\chi\left(x,\frac{1}{t\ell_{\epsilon,n}(t)}\right)}$, where $\chi(x,\cdot):=[p^{\cup}(x,\cdot)]^{-1}$ is the inverse of $p^{\cup}(x,\cdot)$, and

$$\ell_{\epsilon,n}(t)=|\log t|\cdot|\log|\log t||\cdot\ldots\cdot\big(\underbrace{\log|\log|\ldots|\log t|\ldots|}_{n}\big)^{1+\epsilon}$$

for some $\epsilon>0$ and $n\geqslant 1$.

Example 11. Consider the stable-like Lévy measure from Example 3. Since $p^{\mathsf{U}}(x,\xi) = |\xi|^{\alpha(x)}$, we have an explicit representation of the function $v(x,t)$ from the previous Example 10:

$$v(x,t) = \left(t\ell_{\epsilon,n}(t)\right)^{1/\alpha(x)}. \tag{29}$$

Therefore, the integral (26) becomes

$$\int_{0+} \sup_{|x-y|\leqslant v(x,t)} \left(\frac{1}{t\ell_{\epsilon,n}(t)}\right)^{\frac{\alpha(y)}{\alpha(x)}} dt < \infty. \tag{30}$$

Note that $v(x,t) \to 0$ as $t \to 0$. Since α is continuously differentiable, we can take t so small that in the interval $(x - v(t,x), x + v(t,x))$ there is at most one extremum of $\alpha(y)$. If α has a local maximum at x, then the integrand in (30) is equal to $\left(t\ell_{\epsilon,n}(t)\right)^{-1}$. Otherwise, x may be a local minimum, or $\alpha'(y) > 0$ (respectively, < 0) on $(x - v(t,x), x + v(t,x))$. In both cases the maximum of α is attained at the end-points of the interval, say, at $x - v(t,x)$. Using a Taylor expansion, we have

$$\alpha(x - v(x,t)) \leqslant \alpha(x) + |\alpha'(x - \theta v(x,t))|\, v(t,x),$$

where $\theta = \theta(t,x) \in (0,1)$, implying

$$\begin{aligned}\left(\frac{1}{t\ell_{\epsilon,n}(t)}\right)^{\frac{\alpha(x-v(x,t))}{\alpha(x)}} &\leqslant \frac{1}{t\ell_{\epsilon,n}(t)}\left(\frac{1}{t\ell_{\epsilon,n}(t)}\right)^{\frac{|\alpha'(x-\theta v(t,x))|}{\alpha(x)}\, v(x,t)}\\ &= \frac{1}{t\ell_{\epsilon,n}(t)}\left(\frac{1}{v(x,t)^{v(x,t)}}\right)^{|\alpha'(x-\theta v(t,x))|} \leqslant \frac{C(x)}{t\ell_{\epsilon,n}(t)}\end{aligned}$$

for small $t > 0$, where we used that $v(x,t)^{v(x,t)} \geqslant 1/2$.

Thus, in this case Proposition 9 holds true with $v(x,t)$ as in (29).

Let us show the counterpart of Proposition 9, i.e. a condition for $\overline{\lim}_{t\to 0} |X_t - x|/v(x,t) \geqslant C$. For this we have to use the direction of the Borel–Cantelli lemma that requires independence. Therefore, we have to restrict ourselves to Lévy processes. The following proposition appears, with a different proof based on fluctuation identities already in [18, Proposition 2.1]; this proof required growth assumptions on $v(t)$ as $t \to 0$ and $t \to \infty$. We refer also to [5, Chapter 10] for further results on the asymptotic behaviour (in probability and almost surely) of $X_t/b(t)$, where $b(t) > 0$, $b(t) \to 0$ as $t \to 0$.

Proposition 12. *Let $(X_t)_{t\geqslant 0}$ be a pure jump Lévy process with Lévy triplet $(0,0,\nu)$ and $v(t)$ be a positive increasing function. If*

$$\int_{0+} \nu\left\{y : |y| \geqslant 2Cv(t)\right\} dt = \infty \quad \textit{for some } C > 0, \tag{31}$$

then

$$\overline{\lim_{t\to 0}} \frac{\sup_{0\leqslant s\leqslant t} |X_s|}{v(t)} \geqslant \overline{\lim_{t\to 0}} \frac{|X_t|}{v(t)} \geqslant \frac{C}{3} \quad (\mathbb{P}^0\text{-}a.s.). \tag{32}$$

Proof. Applying Etemadi's inequality, cf. Billingsley [3, Theorem 22.5], we get

$$3\mathbb{P}\left\{|X_t| \geqslant \tfrac{C}{3}v(t)\right\} \geqslant \mathbb{P}\left\{\sup_{0\leqslant s\leqslant t}|X_s| \geqslant Cv(t)\right\} \geqslant 1 - e^{-t\nu\{y:|y|\geqslant 2Cv(t)\}}. \tag{33}$$

Let now $v(t)$ be such that (31) holds true. There are two possible cases.

Case 1: $\overline{\lim}_{t\to 0} t\nu\{y : |y| \geqslant 2Cv(t)\} = 0$. Using the inequality $1 - e^{-x} \geqslant c_1 x$ for small $x > 0$, we get with (31)

$$\int_{0+} \frac{1}{t}\,\mathbb{P}\left\{|X_t| \geqslant \tfrac{C}{3}v(t)\right\} dt \geqslant c_1 \int_{0+} \nu\left\{y : |y| \geqslant 2Cv(t)\right\} dt = \infty.$$

Case 2: $\underline{\lim}_{t\to 0} t\nu\{y : |y| \geqslant 2Cv(t)\} = c_2 > 0$. Then

$$\underline{\lim}_{t\to 0}\left(1 - e^{-t\nu\{y:|y|\geqslant 2Cv(t)\}}\right) = 1 - e^{-\underline{\lim}_{t\to 0} t\nu\{y:|y|\geqslant 2Cv(t)\}} = 1 - e^{-c_2} \in (0, 1].$$

Thus, there exists t_0 small enough such that $\mathbb{P}\left\{|X_t| \geqslant \frac{C}{3}v(t)\right\} \geqslant c_3 > 0$ for all $t \in (0, t_0]$ and we have automatically $\int_{0+} \frac{1}{t}\,\mathbb{P}\left\{|X_t| \geqslant \frac{C}{3}v(t)\right\} dt = \infty$. $\square$

4. LIL results via the symbol of the process

In this section we obtain a Chung-type lim inf LIL (3) for a Feller process $(X_t)_{t\geqslant 0}$. We will see that the growth of the norming function $w(x, t)$ is determined by the symbol $p(x, \xi)$ of the process. This result extends, in particular, Proposition 12 and holds for Lévy-type processes. For Lévy processes more precise results are known, but the easy argument used in the proof and the simple form of the norming function may nevertheless be of interest even in this case, see Remark 17.

Throughout we assume that **(A1)** holds with the following stronger version of the sector condition,

$$\exists\, c_0 \in [0, 1) \quad \forall x, \xi \in \mathbb{R} : |\operatorname{Im} p(x, \xi)| \leqslant c_0 \operatorname{Re} p(x, \xi).$$

The restriction $c_0 < 1$ means that we exclude the situation when the drift can dominate the overall behaviour of the process, cf. (5). For a Lévy process this implies that a bounded variation process has no drift at all.

We need a further assumption: there exists a monotone increasing function g such that

$$g(\xi) \leqslant \operatorname{Re} p(x, \xi) \leqslant C_p(1 + |\xi|^2), \quad x \in \mathbb{R},\ |\xi| \geqslant 1. \tag{\textbf{A4}}$$

We also need the following estimate for the characteristic function $\lambda_t(x, \xi) = \mathbb{E}^x e^{i\xi(X_t - x)}$ which is due to [21, Proposition 2.4]:

$$\sup_{x\in\mathbb{R}} |\lambda_t(x, \xi)| \leqslant \exp\left[-\delta t \inf_{x\in\mathbb{R}} \operatorname{Re} p(x, \xi)\right], \quad t \in [0, t_0], \qquad t_0 = t_0(\xi, \epsilon), \tag{34}$$

where $\delta = \delta(c) = 1 - c_0 - \epsilon > 0$, and $0 \leqslant c_0 < 1$ is the sector constant.

Remark 13. **(A4)** ensures that the function $t_0(\xi, \epsilon)$ is continuous in ξ. This follows from the proof of [21, Proposition 2.4]. The upper bound in **(A4)** means that the generator $A = -p(x, D)$ has *bounded coefficients*, cf. [20] for details; in fact, $C_p = 2\sup_{x\in\mathbb{R}} \sup_{|y|\leqslant 1} |p(x, \eta)|$.

Theorem 14. *Let $(X_t)_{t\geqslant 0}$ be a Feller process such that $X_0 = x$ and with the symbol $p(x,\xi)$ satisfying the conditions (**A1**) and (**A4**) with a sector constant $c_0 \in [0,1)$. Let $w(x,t)$, $t>0$, $x\in\mathbb{R}$, be a positive function which is for all x monotone decreasing as a function of t. Then we have $\mathbb{P}^x$-a.s.*

$$\varliminf_{t\to 0}\frac{|X_t - x|}{w(x,t)} = \begin{cases}\gamma(x)\in(0,\infty]\\ \infty\end{cases} \quad \textit{according to} \quad \varliminf_{t\to 0} tg\big(\tfrac{1}{w(x,t)}\big) = \begin{cases} c(x)\in(0,\infty)\\ \infty.\end{cases}$$

Proof. Take $1<a<b<\infty$. By Fubini's theorem we find, since $g(\xi)\leqslant \inf_x \operatorname{Re} p(x,\xi)$,

$$\left|\mathbb{E}^x\int_a^b e^{i\xi\,\frac{X_t-x}{w(x,t)}}\,d\xi\right| = \left|\int_a^b \lambda_t\big(x,\tfrac{\xi}{w(x,t)}\big)\,d\xi\right| \leqslant \int_a^b \exp\big[-\delta t g\big(\tfrac{\xi}{w(x,t)}\big)\big]\,d\xi$$
$$\leqslant (b-a)\exp\big[-\delta t g\big(\tfrac{1}{w(x,t)}\big)\big],$$

where we used the monotonicity of g in the last estimate. This inequality holds for all $0\leqslant t\leqslant t(a,b,\epsilon)$, $t(a,b,\epsilon) = \inf_{\xi\in[a,b]} t_0(\xi,\epsilon)$ where $t_0(\xi,\epsilon)$ is the constant from (34). Since it depends continuously on ξ, cf. Remark 13, we have $t(a,b,\epsilon)>0$. Taking the $\varlimsup_{t\to 0}$ on both sides, we get

$$\varlimsup_{t\to 0}\left|\mathbb{E}^x\int_a^b e^{i\xi\,\frac{X_t-x}{w(x,t)}}\,d\xi\right| \leqslant (b-a)\exp\Big\{-\delta\varliminf_{t\to 0}\Big[tg\big(\tfrac{1}{w(x,t)}\big)\Big]\Big\}. \tag{35}$$

Case 1. Assume that $\varliminf_{t\to 0} tg(1/w(x,t)) = c(x)>0$. Then

$$\varlimsup_{t\to 0}\left|\mathbb{E}^x\int_a^b e^{i\xi\,\frac{X_t-x}{w(x,t)}}\,d\xi\right| \leqslant (b-a)e^{-\delta c(x)}.$$

On the other hand, using $|z|\geqslant|\operatorname{Re} z|$, we derive

$$\left|\int_a^b e^{i\xi\,\frac{X_t-x}{w(x,t)}}\,d\xi\right| \geqslant \left|\int_a^b \cos\left(\xi\,\frac{X_t-x}{w(x,t)}\right)d\xi\right| \geqslant \int_a^b \cos\left(\xi\,\frac{X_t-x}{w(x,t)}\right)d\xi.$$

Suppose that the claim does not hold, and $\varliminf_{t\to 0}|X_t - x|/w(x,t) = 0$. Without loss of generality, we can choose a and b such that $\cos\big(\xi\,\frac{X_t-x}{w(x,t)}\big)>0$ for $a<\xi<b$. Since cos is bounded below by -1, we can apply Fatou's lemma and get

$$\varliminf_{t\to 0}\left|\mathbb{E}^x\int_a^b e^{i\xi\,\frac{X_t-x}{w(x,t)}}\,d\xi\right| \geqslant \mathbb{E}^x\left(\int_a^b \varliminf_{t\to 0}\cos\left(\xi\,\frac{X_t-x}{w(x,t)}\right)d\xi\right) = b-a.$$

Thus, we arrive at $1\leqslant e^{-\delta c(x)}$, which is wrong, since $c(x)$ is strictly positive.

Case 2. Assume that $\varliminf_{t\to 0} tg(1/w(x,t)) = \infty$. From (35) and the fact that $|\operatorname{Re} z|\leqslant|z|$ we see

$$0 = \varlimsup_{t\to 0}\left|\int_a^b \cos\left(\xi\,\frac{X_t-x}{w(x,t)}\right)d\xi\right|. \tag{36}$$

Assume that there exists a sequence of positive real numbers $(t_n)_{n\geqslant 0}$ with $\lim_{n\to\infty} t_n = 0$ and $\lim_{n\to\infty}|X_{t_n} - x|/w(x,t_n) = c<\infty$. Since $1<a<b<\infty$ are arbitrary, we can chose the interval $[a,b]$ in such a way that

$$\int_a^b \cos\left(\xi\,\frac{X_{t_n}-x}{w(x,t_n)}\right)d\xi > \epsilon > 0 \quad \text{for all } n\geqslant 1$$

and some $\epsilon = \epsilon(c)>0$. This contradicts (36) and the proof is finished. $\square$

Example 15. Let $\nu(x, dy)$ be the kernel from Example 3. In this case the symbol $p(x, \xi)$ related to the kernel $\nu(x, dy)$ via (5) can be calculated explicitly, i.e. $p(x, \xi) = b(x)|\xi|^{\alpha(x)}$, where $b(x) > 0$ is some constant. In this case $g(\xi) = c|\xi|^{\alpha_0}$, where $\alpha_0 = \min_x \alpha(x)$. Taking $w(t) = t^{1/\alpha}$ we see that $\underline{\lim}_{t\to 0} tg(1/w(t)) = 1$, implying that $\underline{\lim}_{t\to 0} t^{-1/\alpha}|X_t - x| \in (0, \infty]$. On the other hand, taking $w(t) = t^{1/\beta}$, where $\beta < \alpha$, we arrive at $\underline{\lim}_{t\to 0} t^{-1/\beta}|X_t - x| = \infty$. Thus, Theorem 14 gives a rough picture of the behaviour of the process.

Remark 16. With a careful analysis it is possible to capture the influence of starting point x in Example 15. Let us sketch the (very technical) argument. (13) and (**A4**) entail, in particular, that $\inf_x \mathbb{P}(\tau_\epsilon > 0) = 1$, if τ_ϵ is the moment of occurrence of the first jump bigger than ϵ. Thus, $\underline{\lim}_{t\to 0} \frac{|X_t - x|}{w(x,t)} \overset{\text{a.s.}}{=} \underline{\lim}_{t\to 0} \frac{|\tilde{X}_t - x|}{w(x,t)}$, where $\tilde{X}$ is the Feller process which arises if we remove all large jumps from X. For the symbol this means to replace $\nu(x, dy)$ in (5) by $\mathbb{1}_{B_\epsilon(0)}(y)\nu(x, dy)$. In particular, all estimates for the symbol remain valid. This allows to localize the proof of Theorem 14 using $\inf_{y\in B_{R+\epsilon}(x)} \operatorname{Re} p(y, \xi)$ instead of $g(\xi) = \inf_{y\in\mathbb{R}} \operatorname{Re} p(y, \xi)$. In Example 15 we will thus get indices $\alpha_0 = \inf_{y\in B_{R,\epsilon}(x)} \alpha(y)$ and $\alpha_1 = \sup_{y\in B_{R,\epsilon}(x)} \alpha(y)$, and then letting $R, \epsilon \to 0$ gives a local version.

Remark 17. If the constant $c(x)$ appearing in the statement of the preceding theorem is uniformly bounded away from zero, i.e. $\inf_x c(x) = c > 0$, then $\inf_x \gamma(x) = \gamma > 0$. Indeed, assume that $\gamma = 0$. Taking $\sup_x$ on both sides of (35) we get in the same way as above that

$$(b-a)e^{-c\delta} \geqslant \sup_x \overline{\lim_{t\to 0}} \left| \mathbb{E}^x \int_a^b e^{i\xi \frac{X_t - x}{w(x,t)}} \, d\xi \right| \geqslant \mathbb{E}^x \left(\int_a^b \inf_x \underline{\lim_{t\to 0}} \cos\left(\xi \frac{X_t - x}{w(x,t)}\right) d\xi \right) = b - a,$$

which contradicts to the assumption $c > 0$.

Remark 18. If $(X_t)_{t\geqslant 0}$ is a symmetric Lévy process with exponent $\psi(\xi) \geqslant g(\xi) \geqslant 0$ and a monotone increasing function g, Theorem 14 reads

$$\underline{\lim_{t\to 0}} \frac{|X_t|}{w(t)} = \begin{cases} \gamma \in (0,\infty), \\ \infty \end{cases} \quad \text{according to} \quad \underline{\lim_{t\to 0}} \, tg\left(\tfrac{1}{w(t)}\right) = \begin{cases} c > 0, \\ \infty. \end{cases}$$

Indeed: now we can take $a = 0$ and $b = 1$ and get

$$\left| \mathbb{E}\left[\frac{e^{i\frac{X_t}{w(t)}} - 1}{X_t/w(t)} \right] \right| = \left| \mathbb{E}\left[\int_0^1 e^{-i\xi \frac{X_t}{w(t)}} \, d\xi \right] \right| = \mathbb{E}\left[\int_0^1 e^{-i\xi \frac{X_t}{w(t)}} \, d\xi \right] = \int_0^1 e^{-t\psi(\frac{\xi}{w(t)})} \, d\xi.$$

Assume in Case 1 of the proof of Theorem 14 that the lower limit $\underline{\lim}_{t\to 0} tg(1/w(t)) = c \in (0, \infty)$ and that $\underline{\lim}_{t\to 0} |X_t|/w(t) = \infty$. Let $(t_n)_{n\geqslant 0}$ be a sequence decreasing to 0 such that $\lim_{n\to\infty} t_n g(1/w(t_n)) = c$. Then

$$\lim_{n\to\infty} \left| \mathbb{E}\left[\frac{e^{i\frac{X_{t_n}}{w(t_n)}} - 1}{X_{t_n}/w(t_n)} \right] \right| = \lim_{n\to\infty} \int_0^1 e^{-t_n\psi(\frac{\xi}{w(t_n)})} \, d\xi = \int_0^1 \lim_{n\to\infty} e^{-t_n\psi(\frac{\xi}{w(t_n)})} \, d\xi \geqslant \lim_{n\to\infty} e^{-t_n g(\frac{1}{w(t_n)})} = e^{-c}.$$

From the elementary estimate $|e^{i\xi} - 1| \leqslant |\xi|$ we see that the expression on the left tends to 0, and we have reached a contradiction also in this case. The rest of the proof applies literally.

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