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Consecutive minors for Dyson's Brownian motions

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Abstract

In 1962, Dyson (1962) introduced dynamics in random matrix models, in particular into GUE (also for $\beta = 1$ and 4), by letting the entries evolve according to independent Ornstein–Uhlenbeck processes. Dyson shows the spectral points of the matrix evolve according to non-intersecting Brownian motions. The present paper shows that the interlacing spectra of two consecutive principal minors form a Markov process (diffusion) as well. This diffusion consists of two sets of Dyson non-intersecting Brownian motions, with a specific interaction respecting the interlacing. This is revealed in the form of the generator, the transition probability and the invariant measure, which are provided here; this is done in all cases: $\beta = 1, 2, 4$. It is also shown that the spectra of three consecutive minors ceases to be Markovian for $\beta = 2, 4$.

MSC: primary 60B20; 60G55; secondary 60J65; 60J10

Keywords: Dyson's Brownian motion; Diffusions on spectra of minors; Markov process

1. Introduction

In 1962, Dyson [8] introduced dynamics in random matrix models, in particular into GUE, by letting the entries evolve according to independent Ornstein–Uhlenbeck processes. According to Dyson, the spectral points of the matrix evolve according to non-intersecting Brownian motions.

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The present paper addresses the question whether taking two consecutive principal minors leads to a diffusion on the two interlacing spectra of the minors, taken together. This is so! The diffusion is given by the Dyson diffusion for each of the spectra, augmented with a strong coupling term, which is responsible for a very specific interaction between the two sets of spectral points, to be explained in this paper. However the motion induced on the spectra of three consecutive minors is non-Markovian, for generic initial conditions. A further question: is the motion of two interlacing spectra a determinantal process? We believe this is not the case; but determinantal processes appear upon looking at a different space–time directions. These issues are addressed in another paper by the authors.

During the last few years, the question of interlacing spectra for GUE-minors have come up in many different contexts. In a recent paper, Johansson and Nordenstam [15], based on domino tilings results of Johansson [14], show that domino tilings of Aztec diamonds provide a good discrete model for the consecutive eigenvalues of GUE-minors. In an effort to put some dynamics in the domino tiling model, Nordenstam [23] then shows that the shuffling algorithm for domino tilings is a discrete version of an interlacing of two Dyson Brownian motions, introduced and investigated by Jon Warren [28]; see also [4]. Recently Gorin and Shkolnikov [11] have introduced a new multilevel β-Dyson process, which generalizes Warren's process, for which the Markov property holds for k consecutive spectra. One might have suspected that the Warren process would coincide with the diffusion on the spectra of two consecutive principal minors. They are different!

Non-intersecting paths and interlaced processes (random walks and continuous processes) have been investigated by several authors in many different interesting directions; see e.g. [22,12,14,13,26,16,24,21,6,16,17,2], just to name a few. In particular, in [26,2], partial differential equations were derived for the Dyson process and related processes.

The plan of this paper is the following. We state precisely all the results in Section 2. Some useful matrix equalities are derived in Section 3 which are used in Section 4 to derive transition densities for the various processes considered. Stochastic differential equations are derived in Sections 5 and 6. The fact that the spectra of three consecutive minors are not Markovian for generic initial conditions is demonstrated in Section 7.

There is a companion paper by the same authors aiming at determining the kernel for the point process related to the Dyson Brownian minor process along space-like paths [1].

For RSK, percolation theory and nonintersecting paths, see Chapter 10 and for Laguerre, Jacobi and tridiagonal ensembles, see Chapter 3 in [10]. In the concluding remarks of [7], M. Defosseux mentions, without proof, that the minor process for Hermitian matrices is not Markovian for more than 3 consecutive minors; see also [6]. In [5], it is shown that for a discrete non-commutative analogue of the Dyson Brownian motion (quantum random walk), Markovianess is established for consecutive minors and non-Markovianess for three consecutive minors.

2. The Ornstein–Uhlenbeck process and Dyson's Brownian motion

Consider the space $\mathcal{H}_n^{(\beta)}$ of $n \times n$ matrices B, with entries $B_{k\ell} \in \mathbb{R}, \mathbb{C}, \mathbb{H}$ ($\beta = 1, 2, 4$) satisfying the symmetry conditions

$$B_{k\ell} = B_{\ell k}^{*}. \tag{2.1}$$

Any element $z \in \mathbb{R}, \mathbb{C}, \mathbb{H}$ admits a decomposition $z = z^{[0]} + \sum_{\ell=1}^{\beta-1} z^{[\ell]} e_\ell$, with e_i's satisfying

$$\begin{aligned} &e_1^2 = e_2^2 = e_3^2 = -1, \qquad e_1 e_2 = -e_2 e_1 = e_3, \\ &e_1 e_3 = -e_3 e_1 = -e_2, \qquad e_2 e_3 = -e_2 e_3 = e_1. \end{aligned} \tag{2.2}$$

Note that the square bracket superscript ${}^{[\ell]}$ will, throughout this paper, refer to the coordinates of an element $z \in \mathbb{R}, \mathbb{C}, \mathbb{H}$. The conjugate * of an element $z \in \mathbb{R}, \mathbb{C}, \mathbb{H}$ and its norm are given by

$$z^* = z^{[0]} - \sum_{\ell=1}^{\beta-1} z^{[\ell]} e_\ell, \qquad |z|^2 = zz^* = \sum_{\ell=0}^{\beta-1} z^{[\ell]2},$$

with z admitting a polar decomposition $z = |z|u$, with $|u|^2 = \sum_{\ell=0}^{\beta-1} u^{[\ell]2} = 1$. The matrices $B \in \mathcal{H}_n^{(\beta)}$, as in (2.1), correspond to:

$$\mathcal{H}_n^{(\beta)} = \begin{cases} \text{real symmetric } n \times n \text{ matrices, for } \beta = 1 \\ \text{complex Hermitian } n \times n \text{ matrices, for } \beta = 2 \\ \text{self-dual Hermitian } n \times n \text{ “quaternionic” matrices, for } \beta = 4 \end{cases}$$

with the compact groups of vol $(\mathcal{U}_n^{(\beta)}) = 1$,

$$\mathcal{U}_n^{(\beta)} := \begin{cases} O(n), & \beta = 1 \\ \mathcal{U}(n), & \beta = 2 \\ \mathrm{Symp}(n), & \beta = 4, \end{cases} \tag{2.3}$$

acting on it by conjugation.

For $\beta = 4$, it is well known that the quaternionic entries z can be represented as follows

$$z = z^{[0]} + \sum_{\ell=1}^{\beta-1} z^{[\ell]} e_\ell \longmapsto \hat{z} = \begin{pmatrix} z^{[0]} + i z^{[1]} & z^{[2]} + i z^{[3]} \\ -z^{[2]} + i z^{[3]} & z^{[0]} - i z^{[1]} \end{pmatrix}. \tag{2.4}$$

So, the $n \times n$ quaternionic matrices B can be turned into $2n \times 2n$ self-dual Hermitian matrices $\hat{B}$, of which the real spectrum is doubly degenerate. Here, we shall define the n distinct eigenvalues as the spectrum of B. Unless stated otherwise we shall be working with the $n \times n$ quaternionic matrices, rather than the $2n \times 2n$ Hermitian matrices. Also, when working with matrices having quaternionic entries, the *trace* will be defined in the usual way, that is as the sum of the diagonal entries of the $n \times n$-matrix.

The *determinant* of a matrix $B \in \mathcal{H}_n^{(4)}$ is given in terms of the $2n \times 2n$ matrix $\widehat{B}$ (as defined in (2.4)), by the following procedure: first define the skew-symmetric $2n \times 2n$ matrix $\mathbb{B}$ by the following product:

$$\mathbb{B} := \widehat{B} \cdot \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes I_n \right],$$

and then “det B” is defined as

$$\det B := \mathrm{Pfaff}\,(\mathbb{B}) = (\det(\mathbb{B}))^{1/2} = \sum_p (-1)^{n-\ell} \prod_1^\ell B_{\alpha\beta} B_{\beta\gamma} \ldots B_{\delta\alpha}, \tag{2.5}$$

where p is any permutation of the indices $(1, 2, \ldots, n)$ consisting of ℓ exclusive cycles of the form $(\alpha \to \beta \to \gamma \to \cdots \to \alpha)$; see Mehta [19]. In particular, this means that

$$\det(\lambda I - B) = \prod_1^n (\lambda - \lambda_i), \qquad \operatorname{spec} B = \{\lambda_1, \ldots, \lambda_p\}, \tag{2.6}$$

with the λ_i being the double eigenvalues of $\mathbb{B}$.

The following normalization constant $Z_{n,\beta}^{-1}$ will come back over and over again:

$$Z_{n,\beta}^{-1} := 2^{-\frac{n}{2}} \left(\frac{\beta}{\pi}\right)^{N_{n,\beta}}, \quad \text{with } N := N_{n,\beta} := \frac{n}{2} + \frac{\beta}{4} n(n-1). \tag{2.7}$$

Dyson's idea was to let the free parameters of the matrix evolve according to the SDE of the Ornstein–Uhlenbeck (OU) process:

$$\begin{aligned} dB_{ii} &= -B_{ii} dt + \sqrt{\frac{2}{\beta}} db_{ii}, \quad i = 1, \ldots, n \\ dB_{ij}^{[\ell]} &= -B_{ij}^{[\ell]} dt + \frac{1}{\sqrt{\beta}} db_{ij}^{[\ell]}, \quad 1 \le i < j \le n \text{ and } \ell = 0, \ldots, \beta - 1, \end{aligned} \tag{2.8}$$

where db_{ii}, for $i = 1, \ldots, n$, and $db_{ij}^{[\ell]}$, for $1 \le i < j \le n$ and $\ell = 0, \ldots, \beta-1$, are independent, standard Brownian motions; for notation, see (2.2). Since the Ornstein–Uhlenbeck diffusions are independent, the Dyson process on the matrix B has a generator, which is just the sum of the OU-processes above:

$$\mathcal{A}_{\mathrm{Dys}} := \sum_{i=1}^n \left(\frac{1}{\beta} \frac{\partial^2}{\partial B_{ii}^2} - B_{ii} \frac{\partial}{\partial B_{ii}} \right) + \sum_{1 \le i < j \le n} \sum_{\ell=0}^{\beta-1} \left(\frac{1}{2\beta} \frac{\partial^2}{\partial B_{ij}^{[\ell]2}} - B_{ij}^{[\ell]} \frac{\partial}{\partial B_{ij}^{[\ell]}} \right), \tag{2.9}$$

with transition probability, setting $c := e^{-t}$ and using the constant (2.7),

$$\begin{aligned} \mathbb{P}[B_t \in dB \mid B_0 = \bar{B}] =: {} & p(t, \bar{B}, B)\, dB \\ &= \frac{Z_{n,\beta}^{-1}}{(1-c^2)^{N_{n,\beta}}} e^{-\frac{\beta}{2(1-c^2)} \operatorname{Tr}(B - c\bar{B})^2}\, dB, \end{aligned} \tag{2.10}$$

where dB is the product measure over all the independent parameters B_{ii}, $B_{ij}^{[\ell]}$. The transition probability (2.10) satisfies the Fokker–Planck equation

$$\frac{\partial p}{\partial t} = \mathcal{A}_{\mathrm{Dys}}^{\top} p,$$

with

$$\mathcal{A}_{\mathrm{Dys}}^{\top} = \frac{2}{\beta} \left(\frac{1}{2} \sum_{i=1}^n \frac{\partial}{\partial B_{ii}} h^\beta \frac{\partial}{\partial B_{ii}} \frac{1}{h^\beta} + \frac{1}{4} \sum_{1 \le i < j \le n} \sum_{\ell=0}^{\beta-1} \frac{\partial}{\partial B_{ij}^{[\ell]}} h^\beta \frac{\partial}{\partial B_{ij}^{[\ell]}} \frac{1}{h^\beta} \right), \tag{2.11}$$

with a delta-function initial condition, $p(t, \bar{B}, B)|_{t=0} = \delta(\bar{B}, B)$, and with invariant measure (density)

$$\lim_{t \to \infty} p(t, \overline{B}, B) = Z_{n,\beta}^{-1} (h(B))^\beta, \quad \text{with } h := h(B) := e^{-\frac{1}{2} \operatorname{Tr} B^2}. \tag{2.12}$$

Dyson discovered in [8] the surprising fact that the process restricted to $\operatorname{spec}(B) := \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is Markovian as well. This is the content of Dyson's celebrated theorem (Theorem 2.1).

Before stating the main theorem, we define diagonal matrices $X = \operatorname{diag}(x_1, \ldots, x_n)$ and $Y = \operatorname{diag}(y_1, \ldots, y_n)$, vectors $w, v \in \mathbb{R}^n, \mathbb{C}^n, \mathbb{H}^n$ and the inner-product $\langle w, v\rangle = \sum_1^n w_i v_i^*$. Then consider the integral

$$G_n^{(\beta)}(X, Y; w, v) := \int_{\mathcal{U}_n^{(\beta)}} dU e^{(\operatorname{Tr} XUYU^{-1} + 2\operatorname{Re}\langle w, Uv\rangle)}, \tag{2.13}$$

and its integrand

$$\mathcal{G}_n^{(\beta)}(U; X, Y; w, v) := e^{(\operatorname{Tr} XUYU^{-1} + 2\operatorname{Re}\langle w, Uv\rangle)}. \tag{2.14}$$

For $w = v = 0$, this is the more familiar integral

$$F_n^{(\beta)}(X, Y) := G_n^{(\beta)}(X, Y; 0, 0) = \int_{\mathcal{U}_n^{(\beta)}} dU e^{\operatorname{Tr} XUYU^{-1}},$$

which for $\beta = 2$ gives the Harish–Chandra–Itzykson–Zuber formula:

$$F_n^{(2)}(X, Y) = \frac{\det[e^{x_i y_j}]_{1\le i,j\le n}}{\Delta_n(x)\Delta_n(y)} \prod_{r=1}^{n-1} r!, \quad \text{with } \Delta_n(x) = \prod_{j>i}(x_j - x_i). \tag{2.15}$$

Does the integral (2.13) admit such a representation? *This is an open problem.*

For future use, we introduce the function

$$\Phi_n(\lambda) = e^{-\frac{1}{2}\sum_1^n \lambda_i^2} |\Delta_n(\lambda)|. \tag{2.16}$$

Also throughout the paper, λ denotes the vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ and sometimes the diagonal matrix $\lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, with $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. This will be clear from the context.

The main statement of the paper will be contained in Theorem 2.3. But first, we discuss Dyson's process; in the following theorem, formulae (2.17), (2.19) and (2.21) are due to Dyson [8].

Theorem 2.1 (*Dyson Process*)**.** *The Dyson process restricted to its spectrum* $\operatorname{spec}(B) = \lambda = (\lambda_1, \ldots, \lambda_n)$ *is Markovian with SDE given by:*

$$d\lambda_i = \left(-\lambda_i + \sum_{j\ne i} \frac{1}{\lambda_i - \lambda_j}\right) dt + \sqrt{\frac{2}{\beta}} db_{ii}, \quad i = 1, \ldots, n. \tag{2.17}$$

Its transition probability,[1] *with* $c := e^{-t}$,

$$\mathbb{P}[\lambda_t \in d\lambda | \lambda_0 = \bar\lambda] = p_\lambda(t, \bar\lambda, \lambda)\, d\lambda_1 \cdots d\lambda_n$$

[1] $C_{n,\beta}^{-1}$ is the norming constant for the Gaussian ensemble for general β, as obtained from the Selberg formula (see Mehta [20], formula (3.3.10)), (see (2.7) for $N_{n,\beta}$)

$$C_{n,\beta}^{-1} = (2\pi)^{-\frac{n}{2}} \beta^{N_{n,\beta}} \prod_{j=1}^n \left(\frac{\Gamma\left(1+\frac{\beta}{2}\right)}{\Gamma\left(1+\frac{\beta j}{2}\right)}\right).$$

$$= \frac{C_{n,\beta}^{-1}}{(1-c^2)^{N_{n,\beta}}} e^{-\frac{\beta}{2(1-c^2)}\sum_1^n(\lambda_i^2+c^2\bar{\lambda}_i^2)} \times F_n^{(\beta)}\left(\frac{\beta c}{1-c^2}\lambda, \bar{\lambda}\right)|\Delta_n(\lambda)|^\beta \prod_1^n d\lambda_i, \tag{2.18}$$

satisfies the Dyson diffusion equation, with delta-function initial condition $(p_\lambda|_{t=0} = \delta(\lambda, \bar{\lambda}))$ *(forward equation)*

$$\frac{\partial p_\lambda}{\partial t} = \mathcal{A}_\lambda^\top p_\lambda, \quad \text{with } \mathcal{A}_\lambda^\top := \frac{1}{\beta}\sum_{i=1}^n \frac{\partial}{\partial\lambda_i}(\Phi_n(\lambda))^\beta \frac{\partial}{\partial\lambda_i}\frac{1}{(\Phi_n(\lambda))^\beta}. \tag{2.19}$$

The generator is

$$\mathcal{A}_\lambda = \sum_{i=1}^n\left(\frac{1}{\beta}\frac{\partial^2}{\partial\lambda_i^2} + \left(-\lambda_i + \sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j}\right)\frac{\partial}{\partial\lambda_i}\right), \tag{2.20}$$

and the invariant measure of the Dyson process on B*, projected onto* spec (B)*, is given by the* GOE (n), GUE (n), GSE (n) *measure for* $\beta = 1, 2, 4$ *respectively, with* $\Phi_n(\lambda)$ *as in* (2.16)*:*

$$C_{n,\beta}^{-1}(\Phi_n(\lambda))^\beta d\lambda_1\cdots d\lambda_n. \tag{2.21}$$

For completeness we shall prove (2.18), (2.21) in Section 4 and (2.17), (2.19) in Section 5.

Throughout the paper, $B^{(n-1)}, B^{(n-2)}, \ldots$ will denote the principal minors of B of sizes $n-1, n-2, \ldots$. Note that this superscript is *different from the square bracket superscripts referred to in* (2.2). It is remarkable that the Dyson process is not only Markovian upon restriction to the spectrum of any *single* principal minor $B, B^{(n-1)}, B^{(n-2)}, \ldots$, of sizes $n, n-1, n-2, \ldots$, but also upon restriction to any *two consecutive* principal minors, in particular,

$$(\text{spec } B, \text{spec } B^{(n-1)}) := (\lambda, \mu) := ((\lambda_1, \ldots, \lambda_n), (\mu_1, \ldots, \mu_{n-1})),$$

with intertwining property

$$\lambda_1 \le \mu_1 \le \lambda_2 \le \mu_2 \le \cdots \le \lambda_{n-1} \le \mu_{n-1} \le \lambda_n.$$

We denote by $\mathcal{A}_\lambda$ and $\mathcal{A}_\mu$ the generators of the consecutive spectra spec B and spec $B^{(n-1)}$, as defined in (2.19). Define the characteristic polynomials of the two consecutive minors B and $B^{(n-1)}$,

$$P_n(z) = \prod_{\alpha=1}^n(z-\lambda_\alpha), \qquad P_{n-1}(z) = \prod_{\beta=1}^{n-1}(z-\mu_\beta), \tag{2.22}$$

and the Vandermonde determinants

$$\begin{aligned}\Delta_n(\lambda) &:= \prod_{j>i}(\lambda_j-\lambda_i) \ge 0,\\ \Delta_n(\lambda,\mu) &:= \prod_{i=1}^n\prod_{j=1}^{n-1}(\lambda_i-\mu_j) = \prod_1^n P_{n-1}(\lambda_i) = \prod_1^{n-1} P_n(\mu_i),\end{aligned} \tag{2.23}$$

with $\Delta_n(\lambda,\mu)(-1)^{\frac{n(n-1)}{2}} \ge 0$ because of the intertwining.

In order to state Theorem 2.3, we need the following property of any matrix $B \in \mathcal{H}_n^{(\beta)}$:

Lemma 2.2 (*Conjugation to Bordered Matrices*). *Not only can B be conjugated by a matrix $U^{(n)} \in \mathcal{U}_n^{(\beta)}$ (see (2.3)), in the standard way, such that*

$$(U^{(n)})^{-1} B U^{(n)} = \operatorname{diag}(\lambda_1, \dots, \lambda_n),$$

but also by a matrix of the form $\begin{pmatrix} U^{(n-1)} & 0 \\ 0 & 1 \end{pmatrix}$, with $U^{(n-1)} \in \mathcal{U}_{n-1}^{(\beta)}$, to yield a bordered matrix B_{bord}:

$$\begin{pmatrix} U^{(n-1)} & 0 \\ 0 & 1 \end{pmatrix} B \begin{pmatrix} U^{(n-1)} & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \mu_1 & 0 & \cdots & 0 & r_1 u_1 \\ 0 & \mu_2 & \cdots & 0 & r_2 u_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mu_{n-1} & r_{n-1} u_{n-1} \\ r_1 u_1^* & r_2 u_2^* & \cdots & r_{n-1} u_{n-1}^* & r_n \end{pmatrix} =: B_{\text{bord}} \tag{2.24}$$

with $|u_i| = 1$ (angular variables) and with $r_i \geq 0$ for $1 \leq i \leq n-1$ and r_n, given by

$$r_k^2 := -\frac{P_n(\mu_k)}{P'_{n-1}(\mu_k)} \geq 0, \quad 1 \leq k \leq n-1, \qquad r_n := \sum_1^n \lambda_i - \sum_1^{n-1} \mu_i. \tag{2.25}$$

The conjugation in (2.24) transforms the last column v of B into the last column of the bordered matrix B_{bord} (except for the last entry); i.e.,

$$U^{(n-1)} v = (r_1 u_1, \dots, r_{n-1} u_{n-1})^\top, \quad \text{and} \quad B_{nn} = r_n, \\ \text{with } v := (B_{1,n}, \dots, B_{n-1,n})^\top. \tag{2.26}$$

These facts, (2.24)–(2.26), will be discussed and shown in Section 3.

The main statement of the paper is the analogue of Theorem 2.1 for the case of the spectra of two consecutive minors.

Theorem 2.3 (*Spectra of Two Consecutive Minors*). *The Dyson process on B restricted to*

$$(\operatorname{spec} B, \operatorname{spec} B^{(n-1)}) = (\lambda, \mu) := ((\lambda_1, \dots, \lambda_n), (\mu_1, \dots, \mu_{n-1}))$$

is a diffusion $(\lambda(t), \mu(t))$ as well, with the following SDE:

$$\begin{aligned} d\lambda_\alpha &= \left(-\lambda_\alpha + \sum_{\varepsilon \neq \alpha} \frac{1}{\lambda_\alpha - \lambda_\varepsilon} \right) dt \\ &\quad + \sqrt{\frac{2}{\beta}} \frac{P_{n-1}(\lambda_\alpha)}{P'_n(\lambda_\alpha)} \left(\sum_{1 \leq i < j \leq n-1} \frac{\sqrt{2}\, r_i r_j \, d\tilde{b}_{ij}}{(\lambda_\alpha - \mu_i)(\lambda_\alpha - \mu_j)} \right. \\ &\quad \left. + \sum_{i=1}^{n-1} \frac{r_i^2 db_{ii}}{(\lambda_\alpha - \mu_i)^2} + \sum_{i=1}^{n-1} \frac{\sqrt{2}\, r_i d\tilde{b}_{in}}{\lambda_\alpha - \mu_i} + db_{nn} \right), \end{aligned} \tag{2.27}$$

$$d\mu_\gamma = \left(-\mu_\gamma + \sum_{\varepsilon \neq \gamma} \frac{1}{\mu_\gamma - \mu_\varepsilon} \right) dt + \sqrt{\frac{2}{\beta}} db_{\gamma\gamma},$$

in terms of independent standard Brownian motions $\{db_{ii}, d\tilde{b}_{ij}\}_{1\le i<j\le n}$. *Its transition probability*[2] *is given by:*

$$\begin{aligned}
&p_{\lambda\mu}\left(t, (\bar\lambda,\bar\mu), (\lambda,\mu)\right) d\lambda\, d\mu \\
&\quad = \mathbb{P}[(\lambda_t,\mu_t)\in(d\lambda, d\mu) \mid (\lambda_0,\mu_0) = (\bar\lambda,\bar\mu)] \\
&\quad = \frac{\hat Z_{n,\beta}^{-1}}{(1-c^2)^N} e^{-\frac{\beta}{2(1-c^2)}\sum_1^n(\lambda_i^2+c^2\bar\lambda_i^2)} \int_{(S^{\beta-1})^{2(n-1)}} \prod_1^{n-1} d\Omega^{(\beta-1)}(u_i) d\Omega^{(\beta-1)}(\bar u_i) \\
&\qquad \times G_{n-1}^{(\beta)}\left(\frac{\beta c}{1-c^2}\mu, \bar\mu; \frac{\beta c}{1-c^2}(r_i u_i)_1^{n-1}, (\bar r_i \bar u_i)_1^{n-1}\right) \\
&\qquad \times e^{\frac{\beta c r_n \bar r_n}{1-c^2}} |\Delta_n(\lambda)\Delta_{n-1}(\mu)|\, |\Delta_n(\lambda,\mu)|^{\left(\frac{\beta}{2}-1\right)} \prod_1^n d\lambda_i \prod_1^{n-1} d\mu_j,
\end{aligned} \tag{2.28}$$

where the r_i*'s are given by* (2.25)*. It is also a solution of the following forward diffusion equation, with delta-function initial condition*

$$\frac{\partial p_{\lambda\mu}}{\partial t} = \mathcal{A}^\top p_{\lambda\mu}, \quad \text{with } \mathcal{A}^\top := \mathcal{A}_\lambda^\top + \mathcal{A}_\mu^\top + \mathcal{A}_{\lambda\mu}^\top, \tag{2.29}$$

where

$$\mathcal{A}_{\lambda\mu}^\top := -\frac{2}{\beta}\sum_{i=1}^n\sum_{j=1}^{n-1}\frac{\partial}{\partial\lambda_i}\frac{\partial}{\partial\mu_j}\left(\frac{1}{(\lambda_i-\mu_j)^2}\frac{P_{n-1}(\lambda_i)}{P_n'(\lambda_i)}\frac{P_n(\mu_j)}{P_{n-1}'(\mu_j)}\right) \tag{2.30}$$

and where $\mathcal{A}_\lambda^\top$ *and* $\mathcal{A}_\mu^\top$ *are defined by* (2.19)*. The Dyson process restricted to* (λ,μ) *has invariant measure, (see* (2.23)*),*

$$\hat Z_{n,\beta}^{-1}\left(\operatorname{vol}(S^{\beta-1})\right)^{2(n-1)} e^{-\frac{\beta}{2}\sum_1^n\lambda_i^2} |\Delta_n(\lambda)\Delta_{n-1}(\mu)||\Delta_n(\lambda,\mu)|^{\frac{\beta}{2}-1}\prod_1^n d\lambda_i\prod_1^{n-1}d\mu_i. \tag{2.31}$$

The SDE (2.27) and generator (2.29) are computed in Section 6 while the expressions for transition density (2.28) and invariant measure (2.31) are proved in Section 4.

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[2] In the formula (2.28), $S^{\beta-1}\subset\mathbb{R}^\beta$ is the $\beta-1$-dimensional sphere with induced uniform measure $d\Omega^{(\beta-1)}$. The constant reads

$$\hat Z_{n,\beta}^{-1} = \frac{\beta^{N_{n,\beta}}\left(\Gamma\left(1+\frac{\beta}{2}\right)\right)^{n-1}}{(2\pi)^{\frac{n}{2}}\pi^{\frac{\beta}{2}(n-1)}\prod_{j=1}^{n-1}\Gamma\left(1+\frac{\beta j}{2}\right)(\operatorname{vol}(S^{\beta-1}))^{n-1}},$$

and $\operatorname{vol}(S^k) = 2\pi\prod_{i=1}^{k-1}\left(2\int_0^{\pi/2}(\cos\theta)^i d\theta\right)$ for $k\ge 2$, $\operatorname{vol}(S^0)=1$ and $\operatorname{vol}(S^1)=2\pi$, which is proved by induction on k; so $\operatorname{vol}(S^2)=4\pi$ and $\operatorname{vol}(S^3)=2\pi^2$.

Note that it is an immediate consequence of Theorem 2.1 that the generator $\mathcal{A}_{\mathrm{Dys}}$, defined in (2.9), acting on the λ_i and μ_i, has the form

$$\mathcal{A}_{\mathrm{Dys}}(\lambda_i) = \mathcal{A}_\lambda(\lambda_i) \quad \text{and} \quad \mathcal{A}_{\mathrm{Dys}}(\mu_i) = \mathcal{A}_\mu(\mu_i), \tag{2.32}$$

where $\mathcal{A}_\lambda$ and $\mathcal{A}_\mu$ are defined by (2.20).

Whereas all statements in this paper hold for $\beta = 1, 2, 4$, a part of it can be extended to general $\beta > 0$, as will be shown in Section 6, after the proof of Theorem 2.3:

Corollary 2.4. *For general $\beta > 0$, the SDE (2.27), in terms of the independent standard Brownian motions $\{db_{ii}, d\tilde{b}_{ij}\}_{1\le i<j\le n}$, defines a diffusion, whose generator is given by the same Eqs. (2.29), and whose invariant measure is given by (2.31). Moreover, this diffusion restricted to the λ_i's (or to the μ_i's) is the standard Dyson Brownian motion (2.17).*

The following corollary shows that the μ_i's in $\lambda_i \le \mu_i \le \lambda_{i+1}$ are repelled by the boundary and *fluctuate in unison* with the boundary points, when they get close.

Corollary 2.5 (*Gap Behavior*)**.** *The nonnegative gaps $\mu_i - \lambda_i$ and $\lambda_{i+1} - \mu_i$ for $1 \le i \le n-1$ satisfy, in the notation of (2.27),*

$$\begin{aligned} d(\mu_i - \lambda_i) &= F_i(\lambda, \mu)dt + \sqrt{\mu_i - \lambda_i} \sum_{1\le k\le \ell\le n} \alpha_{k\ell} d\tilde{b}_{k\ell} \\ d(\lambda_{i+1} - \mu_i) &= \hat{F}_i(\lambda, \mu)dt + \sqrt{\lambda_{i+1} - \mu_i} \sum_{1\le k\le \ell\le n} \hat{\alpha}_{k\ell} d\tilde{b}_{k\ell} \end{aligned} \tag{2.33}$$

with

$$\begin{cases} \text{some } \alpha_{k\ell} = \mathcal{O}(1) \quad \text{for } \mu_i \simeq \lambda_i \quad \text{and} \quad \text{some } \hat{\alpha}_{ij} = \mathcal{O}(1) \quad \text{for } \mu_i \simeq \lambda_{i+1}. \\ F_i(\lambda, \mu)|_{\mu_i = \lambda_i} > 0, \qquad \hat{F}_i(\lambda, \mu)|_{\mu_i = \lambda_{i+1}} > 0. \end{cases}$$

This is to be compared with the Warren process [28], which also describes two intertwined Dyson processes λ and μ, but with an entirely different interaction: namely the μ_i's near the boundaries of the intervals $[\lambda_i, \lambda_{i+1}]$ behave like the absolute value of one-dimensional Brownian motion near the origin.

As we saw, the Dyson process on B, restricted to the spectrum of one principal minor or the spectra of two consecutive minors leads to two Markov processes. Opposed to that, we prove the spectra of three consecutive minors is not Markovian, at least for $\beta = 2$ and 4; we suspect it is true for $\beta = 1$ as well; that would require a different proof.

Theorem 2.6 (*Spectra of Three Consecutive Minors*)**.** *For $\beta = 2$ and 4, the restriction of the Dyson process restricted to the following data*

$$(\text{spec } B, \text{spec } B^{(n-1)}, \text{spec } B^{(n-2)}) := (\lambda, \mu, \nu)$$

is not *Markovian for generic initial conditions on B, i.e., the joint spectra of any three neighboring set of minors of B are not Markovian.*

This statement will be proved in Section 7.

3. Some matrix identities

This section will be devoted to proving Lemma 2.2; in the course of doing so we shall also prove Lemma 3.1:

Lemma 3.1.

$$\sum_1^{n-1} r_i^2 + \frac{r_n^2}{2} = \frac{1}{2}\left(\sum_1^n \lambda_i^2 - \sum_1^{n-1} \mu_i^2\right) \quad \textit{and} \quad \prod_1^{n-1} r_i^2 = \frac{|\Delta_n(\lambda,\mu)|}{\Delta_{n-1}^2(\mu)}. \tag{3.1}$$

One also has the (often used) identities

$$\sum_{i=1}^{n-1} \frac{r_i^2}{\lambda_\ell - \mu_i} + r_n - \lambda_\ell = 0 \quad \textit{and} \quad \frac{P_n'(\lambda_\ell)}{P_{n-1}(\lambda_\ell)} = \sum_{i=1}^{n-1}\left(\frac{r_i}{\lambda_\ell - \mu_i}\right)^2 + 1. \tag{3.2}$$

Finally, one has, for fixed $(\mu_1, \ldots, \mu_{n-1})$ *and fixed* $(u_1, \ldots, u_{n-1})$*,*

$$\prod_1^{n-1} dr_j^2\, dr_n = (-1)^{n-1} \frac{\Delta_n(\lambda)}{\Delta_{n-1}(\mu)} \prod_1^n d\lambda_i. \tag{3.3}$$

Proof of Lemma's 2.2 and 3.1. From the form of the matrix B_{bord} as in (2.24), one checks (see (2.23) and also the formula (2.5) for the determinant in the quaternionic case)

$$\prod_1^n (\lambda_i - z) = \det(B_{\text{bord}} - zI) = \prod_1^{n-1}(\mu_i - z)\left(\sum_{i=1}^{n-1} \frac{r_i^2}{z - \mu_i} + r_n - z\right),$$

from which it follows that[3]

$$\begin{aligned} -\sum_{i=1}^{n-1} \frac{r_i^2}{z-\mu_i} - r_n + z &= \frac{P_n(z)}{P_{n-1}(z)} \\ &= z - (\sigma_1(\lambda) - \sigma_1(\mu)) - \frac{1}{z}\Big(\sigma_1(\lambda)\sigma_1(\mu) + \sigma_2(\mu) \\ &\qquad - \sigma_2(\lambda) - \sigma_1^2(\mu)\Big) + O\left(\frac{1}{z^2}\right). \end{aligned} \tag{3.4}$$

Then taking residues in formula (3.4) yields the first formulae (2.25) and thus the formula for $\prod_1^{n-1} r_i^2$ in (3.1). Comparing the coefficients of z^0 and the z^{-1} on both sides of (3.4) yields the first formula of (3.1). Setting $z = \lambda_\ell$ in the expression (3.4) and its derivative with regard to z implies the two sets of n identities (3.2), in view of the definition (2.22) of P_n. Formula (3.3) amounts to computing the Jacobian determinant of the transformation from $\lambda_1, \ldots, \lambda_n$ to $r_1, \ldots, r_n$; to do this, take the differential of the first of the n expressions appearing in (3.2) (as functions of $\lambda_1, \ldots, \lambda_n$ and $r_1, \ldots, r_n$), keeping the μ_i's fixed and use the second of the expressions (3.2):

$$\begin{aligned} 0 &= \sum_{i=1}^{n-1} \frac{dr_i^2}{\lambda_\ell - \mu_i} + dr_n - \left(1 + \sum_{i=1}^{n-1} \frac{r_i^2}{(\lambda_\ell - \mu_i)^2}\right) d\lambda_\ell \\ &= \sum_{i=1}^{n-1} \frac{dr_i^2}{\lambda_\ell - \mu_i} + dr_n - \frac{P_n'(\lambda)}{P_{n-1}(\lambda)} d\lambda_\ell, \end{aligned}$$

[3] The $\sigma_k(\lambda)$ are the k-th symmetric polynomials: $\sigma_1(\lambda) = \sum_i \lambda_i$, $\sigma_2(\lambda) = \sum_{i<j} \lambda_i\lambda_j$, etc...; the same for $\sigma_k(\mu)$.

which in matrix form reads

$$\Gamma \begin{pmatrix} dr_1^2 \\ dr_2^2 \\ \vdots \\ dr_{n-1}^2 \\ dr_n \end{pmatrix} = \operatorname{diag}\left(\frac{P_n'(\lambda_1)}{P_{n-1}(\lambda_1)}, \dots, \frac{P_n'(\lambda_n)}{P_{n-1}(\lambda_n)}\right) \begin{pmatrix} d\lambda_1 \\ d\lambda_2 \\ \vdots \\ d\lambda_{n-1} \\ d\lambda_n \end{pmatrix},$$

where (by Cauchy's determinantal formula)

$$\Gamma := \begin{pmatrix} \frac{1}{\lambda_1-\mu_1} & \frac{1}{\lambda_1-\mu_2} & \cdots & \frac{1}{\lambda_1-\mu_{n-1}} & 1 \\ \frac{1}{\lambda_2-\mu_1} & \frac{1}{\lambda_2-\mu_2} & \cdots & \frac{1}{\lambda_2-\mu_{n-1}} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{\lambda_n-\mu_1} & \frac{1}{\lambda_n-\mu_2} & \cdots & \frac{1}{\lambda_n-\mu_{n-1}} & 1 \end{pmatrix},$$

$$\text{with } \det\Gamma = (-1)^{(n-1)(\frac{n}{2}+1)} \frac{\Delta_n(\lambda)\Delta_{n-1}(\mu)}{\Delta_n(\lambda,\mu)}. \tag{3.5}$$

Formula (3.5) for the determinant follows from the observation that $\det\Gamma$ has homogeneous degree $1-n$ and vanishes when $\Delta_n(\lambda)\Delta_{n-1}(\mu)$ does and blows up (simply) when and only when $\Delta_n(\lambda,\mu)$ vanishes. Thus we have

$$\frac{\partial(r_1^2,\dots,r_{n-1}^2,r_n)}{\partial(\lambda_1,\dots,\lambda_n)} = \prod_1^n \frac{P_n'(\lambda_i)}{P_{n-1}(\lambda_i)} (\det\Gamma)^{-1} = (-1)^{n-1}\frac{\Delta_n(\lambda)}{\Delta_{n-1}(\mu)}.$$

This concludes the proof of Lemma's 2.2 and 3.1. □

4. Transition probabilities

A quick review of the Ornstein–Uhlenbeck process (see Feller [9] and Uhlenbeck–Ornstein [27]): it is a diffusion on $\mathbb{R}$, given by the one-dimensional SDE,

$$dx = -\rho x\,dt + \frac{1}{\sqrt{\beta}}\,db, \tag{4.1}$$

and it has transition probability ($c := e^{-\rho t}$)

$$\begin{aligned}\mathbb{P}[x_t \in dx \mid x_0 = \bar{x}] &=: p_{\mathrm{OU}}(t;\bar{x},x)\,dx \\ &= \left(\frac{\rho\beta}{\pi(1-c^2)}\right)^{1/2} \exp\left(-\frac{\rho\beta(x-c\bar{x})^2}{1-c^2}\right)dx.\end{aligned}$$

The transition probability is a solution of the forward (diffusion) equation, with δ-function initial condition[4]

$$\frac{\partial p_{\mathrm{OU}}}{\partial t} = \left(\frac{1}{2\beta}\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}(-\rho x)\right)p_{\mathrm{OU}} = \frac{1}{2\beta}\left(\frac{\partial}{\partial x}\phi_{\rho\beta}(x)\frac{\partial}{\partial x}\frac{1}{\phi_{\rho\beta}(x)}\right)p_{\mathrm{OU}}, \tag{4.2}$$

[4] The backward equation becomes the heat equation with $(x,t) \mapsto (xe^{\rho t}, \frac{1-e^{2\rho t}}{2\rho})$.

and invariant measure (density)

$$\phi_{\rho\beta}(x) = \sqrt{\frac{\rho\beta}{\pi}} e^{-\rho\beta x^2} = \lim_{t\to\infty} p_{\mathrm{OU}}(t; \bar{x}, x).$$

Proof of Transition Probabilities (2.10), (2.18) and (2.28). (i) *The Fokker–Planck equation for the transition probability of the Dyson process*. The Dyson process consists of running the free parameters of the matrix $B \in \mathcal{H}_n^{(\beta)}$, as in (2.1), according to independent Ornstein–Uhlenbeck processes, with $\rho = 1$, the diagonal with $\beta \to \beta/2$ and the off-diagonal parameters with $\beta \to \beta$. Remembering the definition (2.7) of $N = N_{n,\beta}$ and the definition of the trace (after (2.4)), one has, setting $c = e^{-t}$, and using (2.8), (4.1) and (4.2), the transition probability for the Dyson process is given by[5]

$$\begin{aligned} p(t, \bar{B}, B) &= \prod_{i=1}^{n} p_{\mathrm{OU}}(t; \bar{B}_{ii}, B_{ii}) \prod_{1\le i<j\le n} \prod_{\ell=0}^{\beta-1} p_{\mathrm{OU}}(t; \bar{B}_{ij}^{[\ell]}, B_{ij}^{[\ell]}) \\ &= \prod_{i=1}^{n} \left(\frac{e^{-\beta \frac{(B_{ii}-c\bar{B}_{ii})^2}{2(1-c^2)}}}{(2\pi(1-c^2)/\beta)^{\frac{1}{2}}} \right) \prod_{1\le i<j\le n} \prod_{\ell=0}^{\beta-1} \left(\frac{e^{-\beta \frac{(B_{ij}^{[\ell]}-c\bar{B}_{ij}^{[\ell]})^2}{(1-c^2)}}}{(\pi(1-c^2)/\beta)^{\frac{1}{2}}} \right) \\ &= \frac{1}{2^{n/2}\left(\frac{\pi}{\beta}(1-c^2)\right)^{N_{n,\beta}}} e^{-\frac{\beta}{2(1-c^2)} \mathrm{Tr}\,(B-c\overline{B})^2} \\ &= \frac{Z_{n,\beta}^{-1}}{(1-c^2)^{N_{n,\beta}}} e^{-\frac{\beta}{2(1-c^2)} \mathrm{Tr}\,(B-c\overline{B})^2}, \end{aligned} \tag{4.3}$$

yielding (2.10), while $\lim_{t\to\infty} p(t, \overline{B}, B) = Z_{n,\beta}^{-1}(h(B))^{\beta}$ is immediate, showing (2.12). Moreover, from (4.2), one computes for $p(t; \bar{B}, B)$,

$$\begin{aligned} \frac{\partial}{\partial t} p(t; \bar{B}, B) &= \frac{2}{\beta} \Bigg[\frac{1}{2} \sum_{i=1}^{n} \frac{\partial}{\partial B_{ii}} \phi_{\beta/2}(B_{ii}) \frac{\partial}{\partial B_{ii}} \frac{1}{\phi_{\beta/2}(B_{ii})} \\ &\quad + \frac{1}{4} \sum_{1\le i<j\le n} \sum_{\ell=0}^{\beta-1} \frac{\partial}{\partial B_{ij}^{[\ell]}} \phi_{\beta}(B_{ij}^{[\ell]}) \frac{\partial}{\partial B_{ij}^{[\ell]}} \frac{1}{\phi_{\beta}(B_{ij}^{[\ell]})} \Bigg] p(t; \bar{B}, B) \\ &= \frac{2}{\beta} \Bigg[\frac{1}{2} \sum_{i=1}^{n} \frac{\partial}{\partial B_{ii}} h(B) \frac{\partial}{\partial B_{ii}} \frac{1}{h(B)} \\ &\quad + \frac{1}{4} \sum_{1\le i<j\le n} \sum_{\ell=0}^{\beta-1} \frac{\partial}{\partial B_{ij}^{[\ell]}} h(B) \frac{\partial}{\partial B_{ij}^{[\ell]}} \frac{1}{h(B)} \Bigg] p(t; \bar{B}, B) \end{aligned}$$

with

$$h(B) = \text{constant} \times \prod_{i=1}^{n} \phi_{\beta/2}(B_{ii}) \prod_{\substack{1\le i<j\le n \\ 0\le \ell\le \beta-1}} \phi_{\beta}(B_{ij}^{[\ell]}),$$

proving (2.11).

[5] With constant $Z_{n,\beta}^{-1} = 2^{-n/2}(\frac{\pi}{\beta})^{-N_{n,\beta}}$.

(ii) *The transition probability* (2.18) *and the invariant measure for the* λ_t *process*. Set

$$\lambda = (\lambda_1, \dots, \lambda_n) \quad \text{and} \quad \bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_n).$$

By the Weyl integration formula, given $B = U\lambda U^{-1}$ and initial condition $\bar{B} = \bar{U}\bar{\lambda}\bar{U}^{-1}$, express $dB = d(U\lambda U^{-1})$ in formula (2.10) in terms of spectral and angular variables $dB = Z_{n,\beta} C_{n,\beta}^{-1} |\Delta_n(\lambda)|^\beta dU \prod_1^n d\lambda_i$, with Haar measure dU on $\mathcal{U}_n^{(\beta)}$ normalized such that $\mathrm{vol}(\mathcal{U}_n^{(\beta)}) = 1$, with $C_{n,\beta}^{-1}$ defined in footnote 1. This yields, using the transition probability (4.3),

$$\begin{aligned} \mathbb{P}[B_t \in dB \mid B_0 = \bar{B}] &= p(t; \bar{B}, B) dB \\ &= \frac{C_{n,\beta}^{-1}}{(1-c^2)^N} e^{-\frac{\beta}{2(1-c^2)} \sum_1^n (\lambda_i^2 + c^2 \bar{\lambda}_i^2)} e^{\frac{\beta c}{1-c^2} \operatorname{Tr} U\lambda U^{-1} \bar{U}\bar{\lambda}\bar{U}^{-1}} \\ &\quad \times |\Delta_n(\lambda)|^\beta dU \prod_1^n d\lambda_i. \end{aligned} \tag{4.4}$$

Note that the constant $C_{n,\beta}^{-1}$ is compatible with the fact that for $t \to \infty$ this transition probability tends to the GUE-probability; see below.

We now compute the transition probability $p_\lambda(t; \bar{\lambda}, \lambda) d\lambda$ for the spectrum of the Dyson process; this will be a model to compute the transition probability for the (λ_t, μ_t)-process. So, defining

$$\mathcal{H}_n^{(\beta)}(\lambda) := \left\{ B \in \mathcal{H}_n^{(\beta)} \mid \operatorname{spec} B = \lambda \right\},$$

consider

$$\begin{aligned} p_\lambda(t; \bar{\lambda}, \lambda) d\lambda &= \mathbb{P}(\lambda_t \in d\lambda | \lambda_0 = \bar{\lambda}) \\ &= \int_{B_t \in \mathcal{H}_n^{(\beta)}(\lambda)} \int_{\bar{B} \in \mathcal{H}_n^{(\beta)}(\bar{\lambda})} \mathbb{P}\left[B_t \in dB \mid B_0 = \bar{B} \right] \\ &\quad \times \mathbb{P}\left[B_0 \in d\bar{B} | \operatorname{spec}(B_0) = \bar{\lambda} \right] \\ &= \int_{\left(\mathcal{U}_n^{(\beta)}\right)^2} \frac{C_{n,\beta}^{-1}}{(1-c^2)^N} e^{-\frac{\beta}{2(1-c^2)} \sum_1^n (\lambda_i^2 + c^2 \bar{\lambda}_i^2)} e^{\frac{\beta c}{1-c^2} \operatorname{Tr} U\lambda U^{-1} \bar{U}\bar{\lambda}\bar{U}^{-1}} \\ &\quad \times |\Delta_n(\lambda)|^\beta dU d\bar{U} \prod_1^n d\lambda_i \\ &= \frac{C_{n,\beta}^{-1}}{(1-c^2)^{N_{n,\beta}}} e^{-\frac{\beta}{2(1-c^2)} \sum_1^n (\lambda_i^2 + c^2 \bar{\lambda}_i^2)} F_n^{(\beta)}\left(\frac{\beta c}{1-c^2} \lambda, \bar{\lambda} \right) |\Delta_n(\lambda)|^\beta \prod_1^n d\lambda_i, \end{aligned}$$

using (4.4) above, using the following conditional probability formula:

$$\mathbb{P}\left[B_0 \in d\bar{B} | \operatorname{spec}(B_0) = \bar{\lambda} \right] = \frac{\mathbb{P}\left[B_0 \in d\bar{B}, \operatorname{spec}(B_0) \in d\bar{\lambda} \right]}{\mathbb{P}\left[\operatorname{spec}(B_0) \in d\bar{\lambda} \right]} = d\bar{U}$$

and finally using the integration (2.13),

$$\int_{\mathcal{U}_n^{(\beta)}} e^{\frac{\beta c}{1-c^2} \operatorname{Tr} U\lambda U^{-1} \bar{U}\bar{\lambda}\bar{U}^{-1}} dU = \int_{\mathcal{U}_n^{(\beta)}} e^{\frac{\beta c}{1-c^2} \operatorname{Tr} \lambda U \bar{\lambda} U^{-1}} dU = F_n^{(\beta)}\left(\frac{\beta c}{1-c^2} \lambda, \bar{\lambda} \right),$$

thus yielding (2.18).

Letting $t \to \infty$, (equivalently $c \to 0$) in (4.4) proves formula (2.21) for the invariant measure, taking into account that $F_n^{(\beta)}(0, Y) = \mathrm{vol}(\mathcal{U}_n^{\beta}) = 1$.

(iii) *Proof of the transition probability* (2.28) *and the invariant measure for the* (λ_t, μ_t) *process.* The proof of (2.28) in Theorem 2.3 proceed along similar lines. First observe the identity

$$\begin{aligned}
&\mathbb{P}[B_t \in dB \mid (\mathrm{spec}(B_0), \mathrm{spec}(B_0^{(n-1)})) = (\bar{\lambda}, \bar{\mu})] \\
&= \int_{\bar{B} \in \mathcal{H}_n^{(\beta)}(\bar{\lambda}, \bar{\mu})} \mathbb{P}\left[B_t \in dB \mid B_0 = \bar{B}\right] \\
&\times \mathbb{P}\left[B_0 \in d\bar{B} | (\mathrm{spec}(B_0), \mathrm{spec}(B_0^{(n-1)})) = (\bar{\lambda}, \bar{\mu})\right]
\end{aligned} \tag{4.5}$$

with

$$\mathcal{H}_n^{(\beta)}(\bar{\lambda}, \bar{\mu}) = \mathcal{H}_n^{(\beta)} \cap \left\{ \left(\mathrm{spec}(B), \mathrm{spec}(B^{(n-1)})\right) = (\bar{\lambda}, \bar{\mu}) \right\}.$$

Next we compute the two probabilities in the integrand of the integral (4.5):

(a) The first integrand equals $p(t; \bar{B}, B)dB$, as in (4.3). Since Lebesgue measure dB is the product measure over all the free parameters, one will express dB as the product of Lebesgue measure $dB^{(n-1)}$ on the $(n-1) \times (n-1)$ minor and the measure $\prod_{\substack{1 \le i \le n-1 \\ 0 \le \ell \le \beta-1}} dB_{in}^{[\ell]} dB_{nn}$ on the last row and column, remembering the expression (2.7) for $N_{n,\beta}$, thus giving,

$$\begin{aligned}
\mathbb{P}[B_t \in dB | B_0 = \bar{B}] &= \frac{Z_{n,\beta}^{-1}}{(1-c^2)^{N_{n,\beta}}} e^{-\frac{\beta}{2(1-c^2)} \mathrm{Tr}\,(B - c\bar{B})^2} dB \\
&= \frac{Z_{n-1,\beta}^{-1}}{(1-c^2)^{N_{n-1,\beta}}} e^{-\frac{\beta}{2(1-c^2)} \mathrm{Tr}\,(B^{(n-1)} - c\bar{B}^{(n-1)})^2} dB^{(n-1)} \\
&\times \frac{Z_{n,\beta}^{-1} Z_{n-1,\beta}}{(1-c^2)^{(N_{n,\beta} - N_{n-1,\beta})}} e^{-\frac{\beta}{(1-c^2)} \left[\sum_{\substack{1 \le i \le n-1 \\ 0 \le \ell \le \beta-1}} (B_{in}^{[\ell]} - c\bar{B}_{in}^{[\ell]})^2 + \frac{1}{2}(B_{nn} - c\bar{B}_{nn})^2 \right]} \\
&\times \prod_{i=1}^{n-1} \prod_{\ell=0}^{\beta-1} dB_{in}^{[\ell]} dB_{nn}.
\end{aligned} \tag{4.6}$$

As mentioned prior to (4.4), one can set in (4.6),

$$dB^{(n-1)} = Z_{n-1,\beta} C_{n-1,\beta}^{-1} |\Delta_{n-1}(\mu)|^{\beta} dU^{(n-1)} \prod_{i=1}^{n-1} d\mu_i. \tag{4.7}$$

In (2.24), it was shown that upon conjugation by an appropriate matrix $U^{(n-1)} \in \mathcal{U}_{n-1}^{(\beta)}$, the matrix B could be transformed into the bordered matrix B_{bord}, as in (2.24) and (2.26), with $(r_1 u_1, \ldots, r_{n-1} u_{n-1})^\top = U^{(n-1)} v$ and $|u_i| = 1$. Using the same inner-product as in the formula just preceding (2.13), but for $n-1$-vectors, and using the associated norm $\| \ \|$, one finds, using the above,

$$\begin{aligned}
\sum_{\substack{1 \le i \le n-1 \\ 0 \le \ell \le \beta-1}} \left(B_{in}^{[\ell]} - c\bar{B}_{in}^{[\ell]}\right)^2 &= \|v - c\bar{v}\|^2 \\
&= \|U^{(n-1)}(v - c\bar{v})\|^2 \\
&= \|U^{(n-1)} v\|^2 + c^2 \|U^{(n-1)} \bar{v}\|^2 - 2c\mathrm{Re}\,\langle U^{(n-1)} v,\ U^{(n-1)} \bar{v} \rangle
\end{aligned}$$

$$
\begin{aligned}
&= \|U^{(n-1)}v\|^2 + c^2\|U^{(n-1)}\bar{v}\|^2 \\
&\quad - 2c\mathrm{Re}\,\langle U^{(n-1)}v,\ (U^{(n-1)}(\bar{U}^{(n-1)})^{-1})(\bar{U}^{(n-1)}\bar{v})\rangle \\
&= \sum_1^{n-1} r_i^2 + c^2\sum_1^{n-1}\bar{r}_i^2 - 2c\mathrm{Re}\,\langle (r_1u_1,\dots,r_{n-1}u_{n-1})^\top, \\
&\quad (U^{(n-1)}(\bar{U}^{(n-1)})^{-1})(\bar{r}_1\bar{u}_1,\dots,\bar{r}_{n-1}\bar{u}_{n-1})^\top\rangle. \qquad (4.8)
\end{aligned}
$$

Given that $U^{(n-1)}$ is fixed and that $\det(U^{(n-1)}) = 1$, and since the expressions u_i in (2.24) have $|u_i| = 1$, the differential below can be written in terms of a product of differentials $d(r_iu_i)$ along $S^{\beta-1} \subset \mathbb{R}^\beta$, expressed in polar coordinates, thus yielding differentials involving the r_i's and volume elements on the unit sphere $S^{\beta-1}$:

$$
\begin{aligned}
\prod_{i=1}^{n-1}\prod_{\ell=0}^{\beta-1} dB_{in}^{[\ell]} &= \prod_{i=1}^{n-1} dv_i = \prod_{i=1}^{n-1} d((U^{(n-1)})^{-1}(r_1u_1,\dots,r_{n-1}u_{n-1})^\top)_i \\
&= |\det(U^{(n-1)})^{-\beta}|\prod_1^{n-1} d(r_iu_i) \\
&= \prod_1^{n-1} r_i^{\beta-1}\,dr_i\,d\Omega_i^{(\beta-1)}(u_i) \quad \text{and} \quad dB_{nn} = dr_n. \qquad (4.9)
\end{aligned}
$$

Thus all together, setting (4.7)–(4.9) in (4.6), we have shown that

$$
\begin{aligned}
&\mathbb{P}[B_t \in dB | B_0 = \bar{B}] \\
&= \frac{C_{n-1,\beta}^{-1}}{(1-c^2)^{N_{n-1,\beta}}} e^{-\frac{\beta}{2(1-c^2)}\sum_{i=1}^{n-1}(\mu_i^2 + c^2\bar{\mu}_i^2)} \\
&\quad \times |\Delta_{n-1}(\mu)|^\beta e^{\frac{\beta c}{1-c^2}\mathrm{Tr}\left(((U^{(n-1)}(\bar{U}^{(n-1)})^{-1})^{-1}\mu U^{(n-1)}(\bar{U}^{(n-1)})^{-1}\bar{\mu}\right)} dU^{(n-1)}\prod_1^{n-1} d\mu_i \\
&\quad \times \frac{Z_{n,\beta}^{-1}Z_{n-1,\beta}}{(1-c^2)^{(N_{n,\beta}-N_{n-1,\beta})}} e^{-\frac{\beta}{(1-c^2)}\left(\left(\sum_1^{n-1} r_i^2 + \frac{1}{2}r_n^2\right) + c^2\left(\sum_1^{n-1}\bar{r}_i^2 + \frac{1}{2}\bar{r}_n^2\right)\right)} \\
&\quad \times e^{\frac{2\beta c}{1-c^2}\mathrm{Re}\,\langle (r_iu_i)_1^{n-1}, U^{(n-1)}(\bar{U}^{(n-1)})^{-1}(\bar{r}_i\bar{u}_i)_1^{n-1}\rangle} e^{\frac{\beta c}{1-c^2}r_n\bar{r}_n}\,dr_n\prod_1^{n-1} r_i^{\beta-1}\,dr_i\,d\Omega_i^{(\beta-1)}(u_i) \\
&= \frac{Z_{n,\beta}^{-1}Z_{n-1,\beta}}{(1-c^2)^{N_{n,\beta}}C_{n-1,\beta}} e^{-\frac{\beta}{2(1-c^2)}\sum_1^n(\lambda_i^2 + c^2\bar{\lambda}_i^2)} \\
&\quad \times e^{\frac{\beta c}{1-c^2}r_n\bar{r}_n}|\Delta_n(\lambda)\Delta_{n-1}(\mu)|\,|\Delta_{n-1}(\lambda,\mu)|^{\frac{\beta}{2}-1} \\
&\quad \times \mathcal{G}_{n-1}^{(\beta)}\left(U^{(n-1)}(\bar{U}^{(n-1)})^{-1};\ \frac{\beta c}{1-c^2}\mu, \bar{\mu};\ \frac{\beta c}{1-c^2}(r_iu_i)_1^{n-1}, (\bar{r}_i\bar{u}_i)_1^{n-1}\right) \\
&\quad \times dU^{(n-1)}\prod_1^{n-1} d\Omega_i^{(\beta-1)}(u_i)\prod_1^{n-1} d\mu_i\prod_1^n d\lambda_i. \qquad (4.10)
\end{aligned}
$$

In the last equality we have used identities (3.1) and (3.3) and the definition (2.14) of $\mathcal{G}_{n-1}^{(\beta)}$.

(b) The second probability in (4.5) takes on the following value:

$$\mathbb{P}\left[B_0 \in d\bar{B} | (\mathrm{spec}(B_0), \mathrm{spec}(B_0^{(n-1)})) = (\bar{\lambda}, \bar{\mu})\right] = d\bar{U}^{(n-1)} \frac{\prod_1^{n-1} d\Omega_i^{(\beta-1)}(\bar{u}_i)}{(\mathrm{vol}\,(S^{\beta-1}))^{n-1}}. \tag{4.11}$$

Indeed, the probability (4.10), when $t \to \infty$ (which amounts to letting $c \to 0$), tends to the invariant measure for the Dyson Brownian motion; instead of the usual representation in the variables $(\lambda_i, U^{(n)})$, this gives the expression of the GUE-probability in the variables $(\lambda_i, \mu_j, u_k, U^{(n-1)})$:

$$\begin{aligned}\mathbb{P}[B \in dB] &= Z_{n,\beta}^{-1} e^{-\frac{\beta}{2}\mathrm{Tr}\, B^2} dB \\ &= \lim_{t\to\infty} \mathbb{P}[B_t \in dB | B_0 = \bar{B}] \\ &= f_{n,\beta}(\lambda, \mu) dU^{(n-1)} \prod_1^{n-1} d\Omega_i^{(\beta-1)}(u_i) \prod_1^{n-1} d\mu_i \prod_1^{n} d\lambda_i \end{aligned} \tag{4.12}$$

with (using $\mathcal{G}_{n-1}^{(\beta)}(U; 0, \bar{\mu}; 0, (\bar{r}_i \bar{u}_i)_1^{n-1}) = 1$)

$$f_{n,\beta}(\lambda, \mu) := \frac{Z_{n,\beta}^{-1} Z_{n-1,\beta}}{C_{n-1,\beta}} e^{-\frac{\beta}{2}\sum_1^n \lambda_i^2} |\Delta_n(\lambda)\Delta_{n-1}(\mu)|\, |\Delta_{n-1}(\lambda, \mu)|^{\frac{\beta}{2}-1}. \tag{4.13}$$

This also shows that $\mathcal{H}_n^{(\beta)}(\lambda, \mu) \simeq \mathcal{U}_{n-1}^{(\beta)} \times \left(S^{(\beta-1)}\right)^{n-1}$. Using (4.12), one checks that the conditional probability equals

$$\begin{aligned}&\mathbb{P}\left[B_0 \in d\bar{B} | (\mathrm{spec}(B_0), \mathrm{spec}(B_0^{(n-1)})) = (\bar{\lambda}, \bar{\mu})\right] \\ &= \frac{\mathbb{P}\left[B_0 \in d\bar{B}, (\mathrm{spec}(B_0), \mathrm{spec}(B_0^{(n-1)})) \in (d\bar{\lambda}, d\bar{\mu})\right]}{\mathbb{P}\left[(\mathrm{spec}(B_0), \mathrm{spec}(B_0^{(n-1)})) \in (d\bar{\lambda}, d\bar{\mu})\right]} \\ &= \frac{f_{n,\beta}(\bar{\lambda}, \bar{\mu}) d\bar{U}^{(n-1)} \prod_1^{n-1} d\Omega_i^{(\beta-1)}(\bar{u}_i) \prod_1^{n-1} d\bar{\mu}_i \prod_1^{n} d\bar{\lambda}_i}{f_{n,\beta}(\bar{\lambda}, \bar{\mu}) \prod_1^{n-1} d\bar{\mu}_i \prod_1^{n} d\bar{\lambda}_i \int_{\mathcal{U}_{n-1}^{(\beta)} \times (S^{(\beta-1)})^{n-1}} d\bar{U}^{(n-1)} \prod_1^{n-1} d\Omega_i^{(\beta-1)}(\bar{u}_i)} \\ &= \frac{d\bar{U}^{(n-1)} \prod_1^{n-1} d\Omega_i^{(\beta-1)}(\bar{u}_i)}{\left(\mathrm{vol}\,(S^{(\beta-1)})\right)^{n-1}}, \end{aligned} \tag{4.14}$$

confirming expression (4.11). Setting (4.10) and (4.11) in (4.5) and using the integral (2.13) and the identification just after (4.13), one computes:

$$\begin{aligned}&p_{\lambda,\mu}(t; (\bar{\lambda}, \bar{\mu}), (\lambda, \mu)) d\lambda d\mu \\ &= \int_{B_t \in \mathcal{H}_n^{(\beta)}(\lambda,\mu)} \int_{\bar{B} \in \mathcal{H}_n^{(\beta)}(\bar{\lambda},\bar{\mu})} \mathbb{P}\left[B_t \in dB \mid B_0 = \bar{B}\right] \\ &\quad \times \mathbb{P}\left[B_0 \in d\bar{B} | (\mathrm{spec}(B_0), \mathrm{spec}(B_0^{(n-1)})) = (\bar{\lambda}, \bar{\mu})\right]\end{aligned}$$

$$= \int_{\left(\mathcal{U}_{n-1}^{(\beta)} \times (S^{(\beta-1)})^{n-1}\right)^2} \frac{Z_{n,\beta}^{-1} Z_{n-1,\beta} C_{n-1,\beta}^{-1}}{(1-c^2)^{N_{n,\beta}} \mathrm{vol}\,(S^{(\beta-1)})^{n-1}} e^{-\frac{\beta}{2(1-c^2)} \sum_{i=1}^{n-1} (\lambda_i^2 + c^2 \bar{\lambda}_i^2)} e^{\frac{\beta c}{1-c^2} r_n \bar{r}_n}$$
$$\times \mathcal{G}_{n-1}^{(\beta)} \left(U^{(n-1)} (\bar{U}^{(n-1)})^{-1}; \frac{\beta c}{1-c^2} \mu, \bar{\mu}; \frac{\beta c}{1-c^2} (r_i u_i)_1^{n-1}, (\bar{r}_i \bar{u}_i)_1^{n-1} \right)$$
$$\times dU^{(n-1)} d\bar{U}^{(n-1)} |\Delta_n(\lambda) \Delta_{n-1}(\mu)| |\Delta_{n-1}(\lambda, \mu)|^{\frac{\beta}{2}-1}$$
$$\times \prod_1^{n-1} d\Omega_i^{(\beta-1)}(u_i) \prod_1^{n-1} d\Omega_i^{(\beta-1)}(\bar{u}_i) \prod_1^{n-1} d\mu_i \prod_1^{n} d\lambda_i$$
$$= \frac{\hat{Z}_{n,\beta}^{-1}}{(1-c^2)^{N_{n,\beta}}} e^{-\frac{\beta}{2(1-c^2)} \sum_{i=1}^{n-1} (\lambda_i^2 + c^2 \bar{\lambda}_i^2)} e^{\frac{\beta c}{1-c^2} r_n \bar{r}_n} |\Delta_n(\lambda) \Delta_{n-1}(\mu)| |\Delta_{n-1}(\lambda, \mu)|^{\frac{\beta}{2}-1} d\mu d\lambda$$
$$\times \int_{(S^{(\beta-1)})^{2n-2}} G_{n-1}^{(\beta)} \left(\frac{\beta c}{1-c^2} \mu, \bar{\mu}; \frac{\beta c}{1-c^2} (r_i u_i)_1^{n-1}, (\bar{r}_i \bar{u}_i)_1^{n-1} \right)$$
$$\prod_1^{n-1} d\Omega_i^{(\beta-1)}(u_i) d\Omega_i^{(\beta-1)}(\bar{u}_i), \tag{4.15}$$

where we used the translation invariance of $dU^{(n-1)}$ and vol $(\mathcal{U}_{n-1}^{(\beta)}) = 1$; also one checks the value of the constant $\hat{Z}_{n,\beta}^{-1}$ to be the one given in footnote 2. This establishes formula (2.28) for the transition probability of the (λ_t, μ_t)-process.

The statements concerning the invariant measures, (2.21) and (2.31) follow immediately from (2.18), (2.13), (2.28), by letting $t \to \infty$ in the transition probability. This concludes the proof of the formulae for the transition probabilities (2.10), (2.18) and invariant measure (2.31), appearing in Theorems 2.1 and 2.3. □

Remark 4.1. The diffusion Eq. (2.29), which will be established in Section 5, can also be used to confirm the form of the invariant measure, at least for $\beta = 2$. On general grounds, the density of the invariant measure, namely

$$I_{\lambda\mu}(\lambda, \mu) := C e^{-\frac{\beta}{2} \sum_1^n \lambda_i^2} |\Delta_n(\lambda) \Delta_{n-1}(\mu)| \, |\Delta_n(\lambda, \mu)|^{\frac{\beta}{2}-1}, \tag{4.16}$$

is a null vector of the forward equation, i.e.

$$\mathcal{A}^\top I_{\lambda\mu}(\lambda, \mu) = \left(\mathcal{A}_\lambda^\top + \mathcal{A}_\mu^\top + \mathcal{A}_{\lambda\mu}^\top\right) I_{\lambda\mu}(\lambda, \mu) = 0,$$

with $\mathcal{A}$ defined in (2.29). For $\beta = 2$, more is true; namely

$$\mathcal{A}_\lambda^\top(\lambda) I_{\lambda\mu} = \frac{n(n-1)}{2} I_{\lambda\mu}, \qquad \mathcal{A}_\mu^\top(\mu) I_{\lambda\mu} = \frac{n(n-1)}{2} I_{\lambda\mu}. \tag{4.17}$$

Once this is shown, it follows that

$$\mathcal{A}_{\lambda\mu}^\top I_{\lambda\mu}(\lambda, \mu) = -n(n-1) I_{\lambda\mu}(\lambda, \mu).$$

So it suffices to prove (4.17). First observe that $\Delta_n(\lambda)$ and $\Delta_{n-1}(\mu)$ are harmonic functions, i.e.

$$\sum_1^n \left(\frac{\partial}{\partial \lambda_i}\right)^2 \Delta_n(\lambda) = 0, \qquad \sum_1^{n-1} \left(\frac{\partial}{\partial \mu_i}\right)^2 \Delta_{n-1}(\mu) = 0,$$

and also homogeneous functions so that acted upon by the Euler operators,

$$\sum_1^n \lambda_i \frac{\partial}{\partial \lambda_i} \Delta_n(\lambda) = \frac{n(n-1)}{2} \Delta_n(\lambda),$$
$$\sum_1^{n-1} \mu_i \frac{\partial}{\partial \mu_i} \Delta_{n-1}(\mu) = \frac{(n-1)(n-2)}{2} \Delta_{n-1}(\mu).$$

Now compute from (2.19) and (4.16) that (remember $\Phi_n(\lambda) := e^{-\frac{1}{2}\sum_1^n \lambda_i^2} |\Delta_n(\lambda)|$)

$$\begin{aligned}
\mathcal{A}_\lambda^\top(\lambda) I_{\lambda\mu} &= \frac{1}{2} \sum_1^n \frac{\partial}{\partial \lambda_i} (\Phi_n(\lambda))^2 \frac{\partial}{\partial \lambda_i} \frac{\Delta_n(\lambda)\Delta_{n-1}(\mu) e^{-\sum_1^n \lambda_j^2}}{(\Phi_n(\lambda))^2} \\
&= -\frac{\Delta_{n-1}(\mu)}{2} \sum_1^n \frac{\partial}{\partial \lambda_i} e^{-\sum_1^n \lambda_j^2} \frac{\partial}{\partial \lambda_i} \Delta_n(\lambda) \\
&= -\frac{1}{2} e^{-\sum_1^n \lambda_j^2} \Delta_{n-1}(\mu) \sum_{i=1}^n \left(\frac{\partial^2}{\partial \lambda_i^2} - 2\lambda_i \frac{\partial}{\partial \lambda_i} \right) \Delta_n(\lambda) \\
&= \frac{n(n-1)}{2} I_{\lambda\mu}
\end{aligned}$$

and also that

$$\begin{aligned}
\mathcal{A}_\mu^\top(\mu) I_{\lambda\mu} &= \frac{1}{2} \sum_1^{n-1} \frac{\partial}{\partial \mu_i} (\Phi_{n-1}(\mu))^2 \frac{\partial}{\partial \mu_i} \frac{\Delta_n(\lambda)\Delta_{n-1}(\mu) e^{-\sum_1^n \lambda_j^2}}{(\Phi_{n-1}(\mu))^2} \\
&= -\frac{1}{2} \Delta_n(\lambda) e^{-\sum_1^n \lambda_i^2} \sum_1^{n-1} \frac{\partial}{\partial \mu_i} e^{\sum_1^{n-1} \mu_i^2} \frac{\partial}{\partial \mu_i} \left(\Delta_{n-1}(\mu) e^{-\sum_1^{n-1} \mu_i^2} \right) \\
&= -\frac{1}{2} \Delta_n(\lambda) e^{-\sum_1^n \lambda_i^2} \left(\sum_1^{n-1} \left(\frac{\partial^2}{\partial \mu_i^2} - 2\mu_i \frac{\partial}{\partial \mu_i} \right) \Delta_{n-1}(\mu) \right. \\
&\qquad \left. - 2(n-1) \Delta_{n-1}(\mu) \right) \\
&= \frac{1}{2} \Delta_n(\lambda) e^{-\sum_1^n \lambda_i^2} ((n-1)(n-2) + 2(n-1)) \Delta_{n-1}(\mu) \\
&= \frac{n(n-1)}{2} I_{\lambda\mu}.
\end{aligned}$$

This ends the proof of identities (4.17).

5. Itō's lemma and Dyson's theorem

To fix notation we repeat some well known facts from stochastic calculus in a way that will be useful later. Given a diffusion $X_t \in \mathbb{R}^n$, given by the SDE[6]

$$dX_t = \sigma(X_t) db_t + a(X_t) dt, \tag{5.1}$$

[6] The subscript t in X_t and B_t will often be omitted, as it has in previous sections.

where db_t is a vector of independent standard Brownian motions, where $x, a(x) \in \mathbb{R}^n$ and $\sigma(x)$ an $n \times n$ matrix. Then the generator of this diffusion is given by

$$\mathcal{A} = \frac{1}{2}\sum_{i,j}(\sigma\sigma^\top)_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i a_i(x)\frac{\partial}{\partial x_i},$$

and, by straight forward verification,

$$\begin{aligned} &(\sigma\sigma^\top)_{ij}(x) = \mathcal{A}(x_i x_j) - x_i\mathcal{A}(x_j) - x_j\mathcal{A}(x_i) = \left(\frac{dX_i dX_j}{dt}\right)(x) \\ &a_i(x) = \mathcal{A}x_i. \end{aligned} \tag{5.2}$$

The transition density $p(t, \bar{x}, x)$ is a solution of the forward equation (in x)

$$\frac{\partial p}{\partial t} = \mathcal{A}^\top p. \tag{5.3}$$

Moreover for a function $g : \mathbb{R}^n \mapsto \mathbb{R}^p$ with $g \in \mathcal{C}^2$, the SDE for $Y_t = g(X_t)$ has the form

$$dY_k = \sum_i \frac{\partial g_k}{\partial x_i} dX_i + \frac{1}{2}\sum_{i,j}\frac{\partial^2 g_k}{\partial x_i \partial x_j} dX_i dX_j = \sum_j \left(\sum_i \frac{\partial g_k}{\partial x_i}\sigma_{ij}\right) db_j + h_k dt, \tag{5.4}$$

for $k = 1, \ldots, p$ and for some function h_k; i.e., the local martingale part only depends on first derivatives of g. This follows from the standard multiplication rules of stochastic calculus ($dtdt = 0, dtdb = 0$ and $db_i db_j = \delta_{ij} dt$):

$$dX_i\, dX_j = \left(a_i\, dt + \sum_{\ell=1}^n \sigma_{i\ell}\, db_\ell\right)\left(a_j\, dt + \sum_{k=1}^n \sigma_{jk}\, db_k\right) \tag{5.5}$$

$$= \left(\sum_{\ell=1}^n \sigma_{i\ell}\, db_\ell\right)\left(\sum_{k=1}^n \sigma_{jk}\, db_k\right) = (\sigma\sigma^\top)_{ij}\, dt. \tag{5.6}$$

More details can be found in any book on stochastic calculus, for example [18] or [25]. As a warm-up exercise, we first prove Dyson's original result, namely the formulae for the SDE and for the generator of Theorem 2.1, including some consequences.

Proof of (2.17) and (2.19) in Theorem 2.1. The Dyson process is invariant under conjugation by $U \in \mathcal{U}_n^{(\beta)}$; to be precise from (2.10),

$$p(t; U\bar{B}U^{-1}, UBU^{-1}) = p(t; \bar{B}, B).$$

Therefore, we are free, at any fixed choice of t, to reset

$$B(t) \mapsto UB(t)U^{-1}, \quad \text{for any } U \in \mathcal{U}_n^{(\beta)}.$$

At any given time t, diagonalize the matrix B to yield $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ and consider the perturbation

$$\operatorname{diag}(\lambda_1, \ldots, \lambda_n) + [dB_{ij}],$$

where one defines the $n \times n$ matrix, for $1 \le i < j \le n$,

$$[dB_{ij}] := \begin{pmatrix} \ddots & & & & \\ & 0 & \cdots & dB_{ij}^{[0]} + \sum_{1}^{\beta-1} dB_{ij}^{[\ell]} e_\ell & \\ & \vdots & \ddots & \vdots & \\ & dB_{ij}^{[0]} - \sum_{1}^{\beta-1} dB_{ij}^{[\ell]} e_\ell & \cdots & 0 & \\ & & & & \ddots \end{pmatrix} \tag{5.7}$$

and, for $i = 1, \dots, n$,

$$[dB_{ii}] := \operatorname{diag}(0, \dots, dB_{ii}, \dots, 0),$$

with, by (2.8),

$$dB_{ij}^{[\ell]}\, dB_{i'j'}^{[\ell']} = \delta_{ii'}\delta_{jj'}\delta_{\ell\ell'}\frac{dt}{\beta} \qquad dB_{ii}\, dB_{jj} = 2\delta_{ij}\tfrac{dt}{\beta} \qquad dB_{ij}^{[\ell]}\, dB_{kk} = 0. \tag{5.8}$$

Remember by Ito's formula (5.4), one only needs to keep track of at most second order changes of the arguments.

Thus for *non-diagonal perturbations* ($i \ne j$), one checks[7]

$$\begin{aligned} 0 &= \det(\operatorname{diag}(\lambda_1, \dots, \lambda_n) + [dB_{ij}] - \lambda I)|_{\lambda \mapsto \lambda_\alpha + d\lambda_\alpha} \\ &= \prod_{\substack{\ell=1 \\ \ell \ne i,j}}^{n} (\lambda_\ell - \lambda)\left((\lambda - \lambda_i)(\lambda - \lambda_j) - \sum_{\ell=0}^{\beta-1}(dB_{ij}^{[\ell]})^2\right)\Bigg|_{\lambda \mapsto \lambda_\alpha + d\lambda_\alpha} \\ &= \begin{cases} (\text{non-zero function}) \times d\lambda_\alpha, & \text{for } \alpha \ne i, j, \\ (\text{non-zero function}) \times \left((\lambda_i - \lambda_j) d\lambda_i - \sum_{\ell=0}^{\beta-1}(dB_{ij}^{[\ell]})^2\right) & \text{for } \alpha = i, \\ (\text{non-zero function}) \times \left((\lambda_j - \lambda_i) d\lambda_j - \sum_{\ell=0}^{\beta-1}(dB_{ij}^{[\ell]})^2\right) & \text{for } \alpha = j, \end{cases} \end{aligned}$$

showing that an off-diagonal perturbation of the diagonal matrix $B(t) = \operatorname{diag}(\lambda_1(t), \dots, \lambda_n(t))$ yields

$$d\lambda_i = \frac{\sum_{\ell=0}^{\beta-1}(dB_{ij}^{[\ell]})^2}{\lambda_i - \lambda_j}, \qquad d\lambda_j = \frac{\sum_{\ell=0}^{\beta-1}(dB_{ij}^{[\ell]})^2}{\lambda_j - \lambda_i}, \quad \text{and} \quad d\lambda_\alpha = 0 \quad \text{for } \alpha \ne i, j.$$

7 Remember, for $\beta = 4$, the determinant is defined in (2.6).

For *diagonal perturbations* ($i = j$), one finds

$$\begin{aligned}
&\det(\mathrm{diag}\,(\lambda_1, \ldots, \lambda_n) + [dB_{ii}] - \lambda I)|_{\lambda\mapsto\lambda_\alpha+d\lambda_\alpha} \\
&\quad = \prod_{\substack{\ell=1\\ \ell\neq i}}^{n} (\lambda_\ell - \lambda)\,(\lambda_i + dB_{ii} - \lambda)\,|_{\lambda\mapsto\lambda_\alpha+d\lambda_\alpha} \\
&\quad = \begin{cases} (\text{non-zero function}) \times d\lambda_\alpha, & \text{for } \alpha \neq i \\ (\text{non-zero function}) \times (dB_{\alpha\alpha} - d\lambda_\alpha), & \text{for } \alpha = i, \end{cases}
\end{aligned}$$

and thus

$$d\lambda_\alpha = 0 \quad \text{for } \alpha \neq i \quad \text{and} \quad d\lambda_\alpha = dB_{\alpha\alpha} \quad \text{for } \alpha = i.$$

Then summing up all the perturbations, one finds the SDE (2.17) announced in Theorem 2.1, using $dB_{ii} = -B_{ii}dt + \sqrt{\frac{2}{\beta}}db_{ii} = -\lambda_i dt + \sqrt{\frac{2}{\beta}}db_{ii}$ and formula (5.8),

$$\begin{aligned}
d\lambda_i &= \left(dB_{ii} + \sum_{j\neq i} \frac{\sum_{\ell=0}^{\beta-1} (dB_{ij}^{[\ell]})^2}{\lambda_i - \lambda_j} \right) \\
&= \left(-\lambda_i + \sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j} \right) dt + \sqrt{\frac{2}{\beta}}db_{ii}, \quad \text{for } i = 1, \ldots, n.
\end{aligned}$$

This ends the formal part of the proof of the SDE. A rigorous approach uses the ideas of Section 4.3 in Anderson, Guionnet and Zeitouni [3]. Indeed to prove the existence of a strong solution to the SDE (2.17), one introduces a cut-off of the polar part; i.e., one replaces the polar part in SDE (2.17), namely

$$\left(\sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j} \right) dt, \tag{5.9}$$

by an expression containing a uniformly Lipschitz function ϕ_R,

$$\left(\sum_{j\neq i} \phi_R(\lambda_i^R - \lambda_j^R) \right) dt, \quad \text{with } \phi_R(x) = \begin{cases} x^{-1} & \text{if } |x| \geq R^{-1} \\ R^2 x & \text{otherwise.} \end{cases} \tag{5.10}$$

The resulting SDE then admits a unique strong solution, which coincides with the solution of the SDE containing (5.9), for

$$t < \tau_R := \inf \left\{ t \text{ such that } \min_{i\neq j} |\lambda_i^R(t) - \lambda_i^R(t)| < R^{-1} \right\};$$

note this first hitting time τ_R of the cut-off is monotone increasing in R. One then shows by introducing an appropriate Lyapunov function $f(x_1, \ldots, x_n)$ to control the (stopping) time $T_M = \inf\{t \geq 0 \mid f(\lambda^R(t)) \geq M\}$, where M is a function of R. Before that stopping time T_M, the solution of the SDE with the polar part (5.9) agrees with the one containing the cut-off function. This enables one to show that $T_M \to \infty$ a.s. for $M \to \infty$. This then shows the existence and uniqueness of a strong (and weak) solution to the SDE (2.17), with initial condition in the space

$\{\lambda_1 < \cdots < \lambda_n\}$. It then requires further argumentation to show rigorously that this solution is the same as the one of the spectral problem.

Then translating the SDE into the generator of the diffusion on $(\lambda_1, \ldots, \lambda_n)$, one finds, by (5.2), that

$$\mathcal{A}_\lambda = \sum_1^n \left(\frac{1}{\beta} \frac{\partial^2}{\partial \lambda_i^2} + \left(-\lambda_i + \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \right) \frac{\partial}{\partial \lambda_i} \right),$$

and thus

$$\begin{aligned} \mathcal{A}_\lambda^\top &= \sum_1^n \left(\frac{1}{\beta} \frac{\partial^2}{\partial \lambda_i^2} - \frac{\partial}{\partial \lambda_i} \left(-\lambda_i + \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \right) \right) \\ &= \frac{1}{\beta} \sum_{i=1}^n \frac{\partial}{\partial \lambda_i} \left(\Phi_n(\lambda)\right)^\beta \frac{\partial}{\partial \lambda_i} \frac{1}{(\Phi_n(\lambda))^\beta}, \end{aligned}$$

with $\Phi_n(\lambda)$ as in (2.21), confirming formula (2.19) in Theorem 2.1. Finally $\mathcal{A}_{\mathrm{Dys}} \lambda_i = \mathcal{A}_\lambda \lambda_i$, mentioned in (2.32), follows from the fact that the generator $\mathcal{A}_{\mathrm{Dys}}$ restricted to the functions $(\lambda_1, \ldots, \lambda_n)$ equals $\mathcal{A}_\lambda$, as a consequence of (5.1) to (5.3); of course, this holds for the spectrum of every principal minor of the matrix B. □

6. SDE for the Dyson process on the spectra of two consecutive minors

In this section we prove the formulas (2.27) for the λ- and μ-SDE's, together with the generator (2.29).

Proof of SDE (2.27) and Generator (2.29) in Theorem 2.3. Using the same idea as in the proof of (2.17) and (2.19) in the Section 5, we choose, at time t, to conjugate the matrix B so as to have the form B_{bord} of (2.24) and let the matrix B_{bord} evolve according to the Dyson process. We will consider only the first order effects on the λ's and ignore second order effects. We will again restrict ourselves to the formal part of the argument, as in Section 5. Here again, a totally rigorous proof would require introducing a cut-off as in (5.9).

At first, we need to compute the (first order) variation of the λ_α's as a function of the (first order) variation of the entries:

Case 1: Consider the perturbation of B_{bord}, using the notation (5.7) for $[dB_{ij}]$, namely

$$B_{\mathrm{bord}} + [dB_{ij}], \quad \text{for } 1 \le i < j \le n-1.$$

Up to first order, one must compute the effect of the perturbation on each of the eigenvalues λ_α, by explicitly computing the characteristic polynomial of the bordered matrix (2.24) with the extra non-diagonal perturbation; then, by neglecting the second order terms in dB, one finds[8]:

$$\begin{aligned} 0 = \det(B_{\mathrm{bord}} + [dB_{ij}] - \lambda I)|_{\lambda \mapsto \lambda_\alpha + d\lambda_\alpha} &= \prod_{\ell=1}^{n-1} (\mu_\ell - \lambda) \\ &\times \left(\sum_1^{n-1} \frac{r_\ell^2}{\lambda - \mu_\ell} + r_n - \lambda + \frac{2 r_i r_j}{(\lambda - \mu_i)(\lambda - \mu_j)} \sum_{\ell=0}^{\beta-1} dB_{ij}^{[\ell]} (u_i u_j^*)^{[\ell]} \right) \Bigg|_{\lambda \mapsto \lambda_\alpha + d\lambda_\alpha}. \end{aligned}$$

[8] Notice that $2\,\mathrm{Re}\, dB_{ij}^* u_i u_j^* = dB_{ij}^* u_i u_j^* + (dB_{ij}^* u_i u_j^*)^* = 2 \sum_{\ell=0}^{\beta-1} dB_{ij}^{[\ell]} (u_i u_j^*)^{[\ell]}$, using $(ab)^* = b^* a^*$. Remember, for $\beta = 4$, quaternion multiplication does not commute and for the determinant of a matrix, use formula (2.5).

Setting $\lambda \mapsto \lambda_\alpha + d\lambda_\alpha$ in this expression, shows that the product $\prod_{\ell=1}^{n-1}(\mu_\ell - \lambda_\alpha - d\lambda_\alpha)$ is of the form (non-zero-function)+(function) $\times\, d\lambda_\alpha$, whereas for the second part, upon Taylor expanding in λ_α, keeping in the expression first order terms only, evaluated by (3.2), and upon noticing that the 0th-order term vanishes (again using (3.2)), we find

$$-\frac{P_n'(\lambda_\alpha)}{P_{n-1}(\lambda_\alpha)}d\lambda_\alpha + \frac{2r_ir_j}{(\lambda_\alpha-\mu_i)(\lambda_\alpha-\mu_j)}\sum_{\ell=0}^{\beta-1} dB_{ij}^{[\ell]}(u_iu_j^*)^{[\ell]} = 0.$$

Finally, adding up the first order contributions from all the perturbations $dB_{ij}^{[0]} + \sum_1^{\beta-1} dB_{ij}^{[\ell]}$, with $1 \le i < j \le n-1$, yields

$$d\lambda_\alpha = \frac{P_{n-1}(\lambda_\alpha)}{P_n'(\lambda_\alpha)} \sum_{1\le i<j\le n-1} \frac{2r_ir_j}{(\lambda_\alpha-\mu_i)(\lambda_\alpha-\mu_j)}\sum_{\ell=0}^{\beta-1} dB_{ij}^{[\ell]}(u_iu_j^*)^{[\ell]}. \tag{6.1}$$

Case 2: For the perturbation $[dB_{ii}]$, with $i = 1, \ldots, n-1$, again neglecting the second order terms,

$$\begin{aligned} 0 &= \det(B_{\text{bord}} + [dB_{ii}] - \lambda I)|_{\lambda\mapsto\lambda_\alpha+d\lambda_\alpha} \\ &= \prod_{\ell=1}^{n-1}(\mu_\ell + \delta_{\ell i}dB_{ii} - \lambda_\alpha - d\lambda_\alpha) \\ &\quad \times \left(\sum_{\ell=1}^{n-1}\frac{r_\ell^2}{\lambda_\alpha + d\lambda_\alpha - \mu_\ell - \delta_{\ell i}dB_{ii}} + (r_n + \delta_{in}dB_{ii}) - \lambda_\alpha - d\lambda_\alpha\right). \end{aligned}$$

Upon expanding this expression as a function of λ_α, μ_i up to first order, noticing as before that the first part does not matter, and using again (3.2), this leads to

$$\begin{aligned} &-\frac{P_n'(\lambda_\alpha)}{P_{n-1}(\lambda_\alpha)}d\lambda_\alpha + \frac{r_i^2}{(\lambda_\alpha-\mu_i)^2}dB_{ii} = 0 \quad \text{for } i = 1,\ldots,n-1, \\ &-\frac{P_n'(\lambda_\alpha)}{P_{n-1}(\lambda_\alpha)}d\lambda_\alpha + dB_{nn} = 0 \quad \text{for } i = n, \end{aligned}$$

and thus summing up all the contributions coming from the dB_{ii} for $i = 1, \ldots, n$, one finds

$$d\lambda_\alpha = \frac{P_{n-1}(\lambda_\alpha)}{P_n'(\lambda_\alpha)}\left(\sum_{i=1}^{n-1}\frac{r_i^2}{(\lambda_\alpha-\mu_i)^2}dB_{ii} + dB_{nn}\right). \tag{6.2}$$

Case 3: For the perturbation $[dB_{in}]$, $i = 1, \ldots, n-1$,

$$\begin{aligned} 0 &= \det(B_{\text{bord}} + [dB_{in}] - \lambda I)|_{\lambda\mapsto\lambda_\alpha+d\lambda_\alpha} \\ &= \prod_{\ell=1}^{n-1}(\mu_\ell - \lambda)\left(\sum_{k=1}^{n-1}\frac{r_k^2 + r_i(u_i\, dB_{in}^* + u_i^*\, dB_{in})\delta_{ik}}{\lambda-\mu_k} + r_n - \lambda\right)\Bigg|_{\lambda\mapsto\lambda_\alpha+d\lambda_\alpha}. \end{aligned}$$

Then, using $u_i dB_{in}^* + dB_{in}u_i^* = 2\sum_{\ell=0}^{\beta-1} u_i^{[\ell]}dB_{in}^{[\ell]}$, using formula (3.2) and finally summing up over all perturbations of the last row and column ($1 \le i \le n-1$) yields

$$d\lambda_\alpha = \frac{P_{n-1}(\lambda_\alpha)}{P_n'(\lambda_\alpha)}\sum_{i=1}^{n-1}\frac{2r_i\sum_{\ell=0}^{\beta-1}u_i^{[\ell]}dB_{in}^{[\ell]}}{\lambda_\alpha-\mu_i}. \tag{6.3}$$

Then *summing up the three contributions* (6.1)–(6.3) gives us the total first order contribution to $d\lambda_\alpha$:

$$
\begin{aligned}
d\lambda_\alpha &= \frac{P_{n-1}(\lambda_\alpha)}{P_n'(\lambda_\alpha)} \left(\sum_{1\le i<j\le n-1} \frac{2r_i r_j}{(\lambda_\alpha-\mu_i)(\lambda_\alpha-\mu_j)} \sum_{\ell=0}^{\beta-1} dB_{ij}^{[\ell]} (u_i u_j^*)^{[\ell]} \right. \\
&\quad \left. + \sum_{i=1}^{n-1} \frac{r_i^2}{(\lambda_\alpha-\mu_i)^2} dB_{ii} + dB_{nn} + \sum_{i=1}^{n-1} \frac{2r_i \sum_{\ell=0}^{\beta-1} u_i^{[\ell]} dB_{in}^{[\ell]}}{\lambda_\alpha-\mu_i} \right).
\end{aligned}
$$

We now set the SDE's (2.8) for the dB_{ii}, $dB_{ij}^{[\ell]}$ into the equation obtained above, thus yielding, by (5.4),

$$
\begin{aligned}
d\lambda_\alpha &= F_\alpha^{(n)}(\lambda)\,dt + \sqrt{\frac{2}{\beta}} \frac{P_{n-1}(\lambda_\alpha)}{P_n'(\lambda_\alpha)} \left(\sum_{1\le i<j\le n-1} \frac{\sqrt{2}\, r_i r_j}{(\lambda_\alpha-\mu_i)(\lambda_\alpha-\mu_j)} \sum_{\ell=0}^{\beta-1} (u_i u_j^*)^{[\ell]} db_{ij}^{[\ell]} \right. \\
&\quad \left. + \sum_{i=1}^{n-1} \left(\frac{r_i}{\lambda_\alpha-\mu_i} \right)^2 db_{ii} + db_{nn} + \sqrt{2} \sum_{i=1}^{n-1} \frac{r_i}{\lambda_\alpha-\mu_i} \sum_{\ell=0}^{\beta-1} u_i^{[\ell]} db_{in}^{[\ell]} \right),
\end{aligned}
$$

for some function $F_\alpha^{(n)}(\lambda)$ to be determined later. Notice that in $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, the norm $|v|$ satisfies $|vw| = |v|.\,|w|$ and $|v| = |v^*|$. Therefore, when $|u_i| = 1$, we also have $|u_i u_j^*| = 1$, implying that

$$
d\tilde{b}_{in} := \sum_{\ell=0}^{\beta-1} u_i^{[\ell]} db_{in}^{[\ell]} \quad \text{and} \quad d\tilde{b}_{ij} := \sum_{\ell=0}^{\beta-1} (u_i u_j^*)^{[\ell]} db_{ij}^{[\ell]}
$$

are both standard Brownian motions on the sphere $S^{\beta-1}$; since they are different linear combinations, they are independent standard Brownian motions, and independent of db_{ii}, $1 \le i \le n$. This is precisely formula (2.27) of Theorem 2.3, namely

$$
\begin{aligned}
d\lambda_\alpha &= F_\alpha^{(n)}(\lambda)dt + \sqrt{\frac{2}{\beta}} \frac{P_{n-1}(\lambda_\alpha)}{P_n'(\lambda_\alpha)} \left(\sum_{1\le i<j\le n-1} \frac{\sqrt{2}\, r_i r_j\, d\tilde{b}_{ij}}{(\lambda_\alpha-\mu_i)(\lambda_\alpha-\mu_j)} \right. \\
&\quad \left. + \sum_{i=1}^{n-1} \frac{r_i^2 db_{ii}}{(\lambda_\alpha-\mu_i)^2} + \sum_{i=1}^{n-1} \frac{\sqrt{2}\, r_i d\tilde{b}_{in}}{\lambda_\alpha-\mu_i} + db_{nn} \right).
\end{aligned} \tag{6.4}
$$

The SDE for the Dyson process induced on the $(n-1)\times(n-1)$ upper-left minor is given by the first formula of Theorem 2.1 with $n \mapsto n-1$ and $\lambda \mapsto \mu$, yielding the formula in (2.27). Therefore the product of the SDEs in (2.27), together with identity (2.25) yields

$$
\frac{d\lambda_i d\mu_j}{dt} = \frac{2}{\beta} \frac{P_{n-1}(\lambda_i)}{P_n'(\lambda_i)} \left(\frac{r_j}{\lambda_i-\mu_j} \right)^2 = -\frac{2}{\beta} \frac{1}{(\lambda_i-\mu_j)^2} \frac{P_{n-1}(\lambda_i)P_n(\mu_j)}{P_n'(\lambda_i)P_{n-1}'(\mu_j)}. \tag{6.5}
$$

Moreover,

$$
\frac{d\lambda_\alpha d\lambda_\gamma}{dt} = \mathcal{A}_{\mathrm{Dys}}(\lambda_\alpha\lambda_\gamma) - \lambda_\alpha \mathcal{A}_{\mathrm{Dys}}(\lambda_\gamma) - \lambda_\gamma \mathcal{A}_{\mathrm{Dys}}(\lambda_\alpha)
$$

$$
\begin{aligned}
&= \mathcal{A}_\lambda(\lambda_\alpha\lambda_\gamma) - \lambda_\alpha \mathcal{A}_\lambda(\lambda_\gamma) - \lambda_\gamma \mathcal{A}_\lambda(\lambda_\alpha) \\
&= 2\left(\text{coefficient of } \frac{\partial^2}{\partial\lambda_\alpha\lambda_\beta} \text{ in } \mathcal{A}_\lambda\right) \\
&= \frac{2}{\beta}\delta_{\alpha\gamma}
\end{aligned}
$$

and similarly,

$$
\frac{d\mu_i d\mu_j}{dt} = \frac{2}{\beta}\delta_{ij}. \tag{6.6}
$$

These identities can also be computed from the expressions (6.4) of $d\lambda_\alpha$ in terms of the λ_i, μ_j, as done in the remark below. From Ito's formula (5.4), it then follows that

$$
\begin{aligned}
&(d\lambda_1, \ldots, d\lambda_n, d\mu_1, \ldots, d\mu_{n-1}) \\
&\quad = (\mathcal{A}_{\text{Dys}}\lambda_1, \ldots, \mathcal{A}_{\text{Dys}}\lambda_n, \mathcal{A}_{\text{Dys}}\mu_1, \ldots, \mathcal{A}_{\text{Dys}}\mu_{n-1})dt + \sigma(\lambda,\mu)db_t,
\end{aligned}
$$

where, according to (2.32) and (2.20),

$$
\begin{aligned}
\mathcal{A}_{\text{Dys}}(\lambda_i) &= \mathcal{A}_\lambda(\lambda_i) = -\lambda_i + \sum_{j\neq i}\frac{1}{\lambda_i - \lambda_j}, \\
\mathcal{A}_{\text{Dys}}(\mu_i) &= \mathcal{A}_\mu(\mu_i) = -\mu_i + \sum_{j\neq i}\frac{1}{\mu_i - \mu_j},
\end{aligned}
$$

establishing the form of $F_\alpha^{(n)}(\lambda)$ in (6.4), thus yielding (2.27). Identities (2.27), (6.5) and (6.6), together with Ito's formula (5.4), then establish the formula (2.30) for $\mathcal{A}_{\lambda\mu}$.[9] □

Remark 6.1. Note that the identities (6.6) can be computed as well from the SDE (6.4), using residue calculations:

$$
\begin{aligned}
\frac{d\lambda_\alpha d\lambda_\gamma}{dt} &= \frac{2}{\beta}\left(\frac{P_{n-1}(\lambda_\alpha)}{P_n'(\lambda_\alpha)}\right)\left(\frac{P_{n-1}(\lambda_\gamma)}{P_n'(\lambda_\gamma)}\right) \\
&\quad\times\left(\sum_{1\le i<j\le n-1}\frac{2r_i^2r_j^2}{(\lambda_\alpha-\mu_i)(\lambda_\alpha-\mu_j)(\lambda_\gamma-\mu_i)(\lambda_\gamma-\mu_j)}\right. \\
&\quad\left. + \sum_{i=1}^{n-1}\frac{r_i^4}{(\lambda_\alpha-\mu_i)^2(\lambda_\gamma-\mu_i)^2} + \sum_{i=1}^{n-1}\frac{2r_i^2}{(\lambda_\alpha-\mu_i)(\lambda_\gamma-\mu_i)} + 1\right) \\
&= \frac{2}{\beta}\delta_{\alpha\gamma}
\end{aligned}
$$

$$
\frac{d\mu_i d\mu_j}{dt} = \frac{2}{\beta}\delta_{ij}. \tag{6.7}
$$

We now turn to the proof of Corollaries 2.4 and 2.5:

[9] As an alternative way, (2.32) and (6.5) suffice to establish (2.30), with (6.5) needed to establish the coupling $\mathcal{A}_{\lambda\mu}$.

Proof of Corollary 2.4. Note, using logarithmic derivatives, that

$$\big(\text{Invariant measure (2.31)}\big)\beta\mathcal{A}^{\top}\big(\text{Invariant measure (2.31)}\big)^{-1}$$

is a quadratic polynomial in β, which by Theorem 2.3 vanishes for $\beta = 1, 2, 4$ and thus it vanishes identically in β. That the process restricted to λ or μ is the standard Dyson process follows from the form of the generator $\mathcal{A}^{\top}$. □

Proof of Corollary 2.5. In order to study the stochastic behavior of $\mu_\alpha - \lambda_\alpha$ and $\lambda_{\alpha+1} - \mu_\alpha$ when μ_α gets close to λ_α or $\lambda_{\alpha+1}$, one rewrites the Brownian part of $d(\lambda_\alpha - \mu_\alpha)$ as follows:

$$\begin{aligned}
&\sqrt{\frac{\beta}{2}}\,\text{Brownian part of } d(\lambda_\alpha - \mu_\alpha) \\
&\quad = \frac{P_{n-1}(\lambda_\alpha)}{P_n'(\lambda_\alpha)}\left(\sum_{1\le i<j\le n-1}\frac{\sqrt{2}\,r_i r_j\, d\tilde{b}_{ij}}{(\lambda_\alpha-\mu_i)(\lambda_\alpha-\mu_j)} + \sum_{\substack{i=1\\ i\neq\alpha}}^{n-1}\frac{r_i^2 db_{ii}}{(\lambda_\alpha-\mu_i)^2}\right.\\
&\qquad \left. + \sum_{i=1}^{n-1}\frac{\sqrt{2}\,r_i d\tilde{b}_{in}}{\lambda_\alpha-\mu_i} + db_{nn}\right) + \left(\frac{P_{n-1}(\lambda_\alpha)}{P_n'(\lambda_\alpha)}\frac{r_\alpha^2}{(\lambda_\alpha-\mu_\alpha)^2} - 1\right) db_{\alpha\alpha}. \qquad (6.8)
\end{aligned}$$

At first notice that for $\mu_\alpha \simeq \lambda_\alpha$, one has, using the expression (2.25) for r_k^2,

$$P_{n-1}(\lambda_\alpha) = \mathcal{O}(\mu_\alpha - \lambda_\alpha), \qquad r_\alpha = \mathcal{O}(\sqrt{\mu_\alpha - \lambda_\alpha}) \quad \text{and} \quad r_i = \mathcal{O}(1) \quad \text{for } i \neq \alpha,$$

from which one deduces that the first line on the right hand side of (6.8) has order $\mathcal{O}(\sqrt{\mu_\alpha - \lambda_\alpha})$. Using again (2.25), the second line of (6.8) multiplied with $P_n'(\lambda_\alpha)P_{n-1}'(\mu_\alpha)$ reads:

$$\begin{aligned}
&P_{n-1}'(\mu_\alpha)P_n'(\lambda_\alpha)\left(\frac{P_{n-1}(\lambda_\alpha)}{P_n'(\lambda_\alpha)}\frac{r_\alpha^2}{(\lambda_\alpha-\mu_\alpha)^2} - 1\right)\\
&= \left(\frac{P_{n-1}(\lambda_\alpha) - P_{n-1}(\mu_\alpha)}{\lambda_\alpha - \mu_\alpha}\right)\left(\frac{P_n(\mu_\alpha) - P_n(\lambda_\alpha)}{\mu_\alpha - \lambda_\alpha}\right) - P_{n-1}'(\mu_\alpha)P_n'(\lambda_\alpha)\\
&= \mathcal{O}(\mu_\alpha - \lambda_\alpha).
\end{aligned}$$

Then

$$\begin{aligned}
\left.\frac{dt\text{-part of } d(\mu_\alpha - \lambda_\alpha)}{dt}\right|_{\mu_\alpha=\lambda_\alpha} &= \left.\left(\lambda_\alpha - \mu_\alpha + \sum_{\substack{1\le j\le n-1\\ j\neq\alpha}}\frac{1}{\mu_\alpha - \mu_j} - \sum_{\substack{1\le j\le n\\ j\neq\alpha}}\frac{1}{\lambda_\alpha - \lambda_j}\right)\right|_{\mu_\alpha=\lambda_\alpha}\\
&= \sum_{\substack{1\le j\le n-1\\ j\neq\alpha}}\frac{\mu_j - \lambda_j}{(\lambda_\alpha - \mu_j)(\lambda_\alpha - \lambda_j)} + \frac{1}{\lambda_n - \lambda_\alpha} > 0,
\end{aligned}$$

which follows from the inequalities (for $1 \le j \le n-1$),

$$\mu_j - \lambda_j \ge 0, \qquad (\lambda_\alpha - \mu_j)(\lambda_\alpha - \lambda_j) \ge 0 \quad \text{and} \quad \lambda_n - \lambda_\alpha > 0.$$

This proves the first relation (2.33), while the second one is done in a similar way. □

7. The eigenvalues of three consecutive minors

In this section we shall prove Theorem 2.6, which affirms that for the Dyson process the joint spectra of any three consecutive minors is not Markovian, although the Markovianess of the spectra holds for any one or any two consecutive minors.

Note that given an Itô diffusion $X_t \in \mathbb{R}^n$, with stochastic differential equation $dX_t = a(X_t)dt + \sigma(X_t)db_t$, as in (5.1), and generator $\mathcal{A}$, the process restricted to $Y_i = \varphi_i(X)$, $1 \le i \le \ell$ is not Markovian (at least for generic initial conditions) if the generator fails to preserve the field of functions $\mathcal{F}(Y)$ generated by the $(Y_1, \dots, Y_\ell) := (\varphi_1(X), \dots, \varphi_\ell(X))$, i.e.

$$\mathcal{A}\mathcal{F}(Y) \not\subseteq \mathcal{F}(Y), \tag{7.1}$$

and provided the diffusion does not hit the Y-boundary of the domain.

Proof of Theorem 2.6. In order to show the non-Markovianess of

$$\Gamma := (\lambda, \mu, \nu) = (\text{spec } B, \text{spec } B^{(n-1)}, \text{spec } B^{(n-2)})$$

it suffices to find a function, such that the function, obtained by applying the Dyson-generator to it, is not a function of (λ, μ, ν). We pick a function of the product form $xy = g(\Gamma)h(\Gamma)$, where

$$x := g(\Gamma) := \sum_{i=1}^{n-2} B_{ii} \quad \text{and} \quad y := h(\Gamma) := \det B$$

are two independent functions. Then, according to formula (5.2)

$$\mathcal{A}_{\text{Dys}} xy = \frac{dx dy}{dt} + x\mathcal{A}_{\text{Dys}} y + y\mathcal{A}_{\text{Dys}} x. \tag{7.2}$$

Since x and $\mathcal{A}_{\text{Dys}} x$ are functions of ν only and since y and $\mathcal{A}_{\text{Dys}} y$ are functions of λ only, $x\mathcal{A}_{\text{Dys}} y + y\mathcal{A}_{\text{Dys}} x$ is a function of (λ, ν) only. Therefore, to establish non-Markovianess of (λ, μ, ν), it suffices to show that $\frac{dxdy}{dt}$ is not only a function of (λ, μ, ν). Since, by Itô's formula (5.4),

$$\begin{aligned} dx\, dy &= \sum_1^{n-2} dB_{ii} \left(\sum_1^n \frac{\partial \det B}{\partial B_{jj}} dB_{jj} + \sum_{1 \le i < j \le n} \sum_{\ell=0}^{\beta-1} \frac{\partial \det B}{\partial B_{ij}^{[\ell]}} dB_{ij}^{[\ell]} \right) \\ &= \frac{2}{\beta} dt \sum_1^{n-2} \frac{\partial \det B}{\partial B_{ii}} = \frac{2}{\beta} \sum_{i=1}^{n-2} \det(\text{minor}_{ii}(B)) dt, \end{aligned}$$

it suffices to show that the right hand side is not a function of (λ, μ, ν) only. Here minor_{ii} denotes removing row i and column i of the matrix.

For example in the case $\beta = 2, n = 3$, this amounts to showing that the determinant of the lower-right 2×2 principal minor of B is not a function of (λ, μ, ν) only; to do this, it is convenient to reparametrize the matrix as

$$B = \begin{pmatrix} B_{11} & \rho_3 e^{i\eta_3} & \rho_2 e^{-i\eta_2} \\ \rho_3 e^{-i\eta_3} & B_{22} & \rho_1 e^{i\eta_1} \\ \rho_2 e^{i\eta_2} & \rho_1 e^{-i\eta_1} & B_{33} \end{pmatrix}.$$

Using the following formulae

$$B_{11} = \nu_1,$$

$$\begin{aligned}
&B_{22} = \mu_1 + \mu_2 - \nu_1,\\
&B_{33} = \lambda_1 + \lambda_2 + \lambda_3 - \mu_1 - \mu_2\\
&\rho_3^2 = (\mu_2 - \nu_1)(\nu_1 - \mu_1),
\end{aligned}$$

the lower-right 2×2 principal minor of B reads

$$\begin{aligned}
\det(\mathrm{minor}_{11}(B)) &= B_{22}B_{33} - \rho_1^2\\
&= (\mu_1 + \mu_2 - \nu_1)(\lambda_1 + \lambda_2 + \lambda_3 - \mu_1 - \mu_2) - \rho_1^2.
\end{aligned}$$

One observes that

$$\begin{aligned}
0 &= \det B - \lambda_1\lambda_2\lambda_3\\
&= B_{11}B_{22}B_{33} - \lambda_1\lambda_2\lambda_3 - \sum_{i=1}^{3} \rho_i^2 B_{ii} + 2\rho_1\rho_2\rho_3 \cos(\eta_1 + \eta_2 + \eta_3)\\
&= F_1(\lambda, \mu, \nu) - \rho_1^2\nu_1 - \rho_2^2(\mu_1 + \mu_2 - \nu_1)\\
&\quad + 2\rho_1\rho_2\sqrt{(\mu_2 - \nu_1)(\nu_1 - \mu_1)} \cos(\eta_1 + \eta_2 + \eta_3)
\end{aligned}$$

and

$$\begin{aligned}
0 &= \mathrm{Tr}\, B^2 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) = \sum_{i=1}^{3} B_{ii}^2 + 2\sum_{i=1}^{3} \rho_i^2 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\\
&= F_2(\lambda, \mu, \nu) + 2(\rho_1^2 + \rho_2^2),
\end{aligned}$$

where $F_i(\lambda, \mu, \nu)$ are functions of the spectral data (λ, μ, ν). Upon solving these two equations in ρ_1 and ρ_2, one notices that, in particular, ρ_1 is a function of $\cos(\eta_1 + \eta_2 + \eta_3)$ and the spectral data (λ, μ, ν), hence showing that $\det(\mathrm{minor}_{11}(B))$ is not a function of (λ, μ, ν) only; thus the same is true for $\mathcal{A}_{\mathrm{Dys}}xy$. This proves that $\mathcal{A}_{\mathrm{Dys}}xy$ does not belong to the field of functions depending on (λ, μ, ν).

More generally, by a perturbation argument about $B^{(n-1)} = \mathrm{diag}\,(\mu_1, \ldots, \mu_{n-1})$, one shows similarly that

$$\sum_{i=1}^{n-2} \det(\mathrm{minor}_{ii}(B)) \notin \mathcal{F}(\lambda, \mu, \nu),$$

for $\beta = 2$ and 4.

Finally, the boundary of the process (λ, μ, ν) is given by the subvariety where some of the μ_i's hit the λ_j's or the ν_k's; that is when $P_n(\mu_i) = 0$ or $P_{n-1}(\nu_k) = 0$ for some $1 \le i \le n-1$ or for some $1 \le k \le n-2$; $r_j^2(\mu, \nu) = 0$ for $1 \le j \le n-2$. From Corollary 2.5, on sees that the process never reaches that boundary. This ends the proof of Theorem 2.6. $\square$

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