Available online at www.sciencedirect.com
ScienceDirect
Stochastic Processes and their Applications 126 (2016) 1885–1900
stochastic processes and their applications
www.elsevier.com/locate/spa

On the functional CLT for stationary Markov chains started at a point

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Received 18 March 2015; received in revised form 1 December 2015; accepted 1 December 2015
Available online 17 December 2015

Dedicated to the memory of Mikhail Gordin

Abstract

We present a general functional central limit theorem started at a point also known under the name of quenched. As a consequence, we point out several new classes of stationary processes, defined via projection conditions, which satisfy this type of asymptotic result. One of the theorems shows that if a Markov chain is stationary ergodic and reversible, this result holds for bounded additive functionals of the chain which have a martingale coboundary in $\mathbb{L}_1$ representation. Our results are also well adapted for strongly mixing sequences providing for this case an alternative, shorter approach to some recent results in the literature.

MSC: primary 60F05; 60F17; secondary 60G10; 60G42; 60J05

Keywords: Functional central limit theorem; Quenched convergence; Functions of Markov chains; Martingale approximation; Reversible Markov chains

1. Introduction and results

In this paper we address the question of the validity of functional limit theorem for processes started at a point for almost all starting points. These types of results are also known under the

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http://dx.doi.org/10.1016/j.spa.2015.12.001
0304-4149/

name of quenched limit theorems or almost sure conditional invariance principles. The quenched functional CLT is more general than the usual one and it is very important for analyzing random processes in random environment, Markov chain Monte Carlo procedures and the discrete Fourier transform (see [30,31,2]). On the other hand there are numerous examples of processes satisfying the functional CLT but failing to satisfy the quenched CLT. Some examples were constructed by Volný and Woodroofe [35] and for the discrete Fourier transforms by Barrera [1]. This is the reason why it is desirable to point out classes of processes satisfying a quenched CLT. Special attention will be devoted to reversible Markov chains and several open problems will be pointed out. Reversible Markov chains have applications to statistical mechanics and to Metropolis Hastings algorithms used in Monte Carlo simulations. The methods of proof we used are based on martingale techniques combined with results from ergodic theory.

The field of limit theorems for stationary stochastic processes is closely related to Markov operators and dynamical systems. All the results for stationary sequences can be translated in the language of Markov operators and vice-versa. In this paper we shall mainly use the Markov operator language and also indicate the connection with stationary processes.

We assume that $(\xi_n)_{n\in\mathbb{Z}}$ is a stationary Markov chain defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a measurable state space $(S, \mathcal{A})$, with marginal distribution $\pi(A) = \mathbb{P}(\xi_0 \in A)$ and regular conditional distribution for ξ_1 given ξ_0, denoted by $Q(x, A) = \mathbb{P}(\xi_1 \in A|\xi_0 = x)$. Let Q also denote the Markov operator acting via $(Qf)(x) = \int_S f(s)Q(x, ds)$. Next, for $p \geq 1$, let $\mathbb{L}_p^0(\pi)$ be the set of measurable functions on S such that $\int |f|^p d\pi < \infty$ and $\int f d\pi = 0$. For some function $f \in \mathbb{L}_2^0(\pi)$, let

$$X_i = f(\xi_i), \qquad S_n = S_n(f) = \sum_{i=1}^{n} X_i. \tag{1}$$

Denote by $\mathcal{F}_k$ the σ-field generated by ξ_i with $i \leq k$. For any integrable random variable X we denote by $\mathbb{E}_k(X) = \mathbb{E}(X|\mathcal{F}_k)$ the conditional expectation of X given $\mathcal{F}_k$. With this notation, $\mathbb{E}_0(X_1) = (Qf)(\xi_0) = \mathbb{E}(X_1|\xi_0)$. We denote by $\|X\|_p$ the norm in $\mathbb{L}_p = \mathbb{L}_p(\Omega, \mathcal{F}, \mathbb{P})$. The integral on the space $(S, \mathcal{A}, \pi)$ will be denoted by $\mathbb{E}_\pi$. So, $\mathbb{E}f(\xi_0) = \mathbb{E}_\pi f$.

The Markov chain is usually constructed in a canonical way on $\Omega = S^\infty$ endowed with sigma algebra $\mathcal{A}^\infty$, and ξ_n is the nth projection on S. The shift $T : \Omega \to \Omega$ is defined by $\xi_n(T\omega) = \xi_{n+1}(\omega)$ for every integer n.

For any probability measure υ on $\mathcal{A}$ the law of $(\xi_n)_{n\in\mathbb{Z}}$ with transition operator Q and initial distribution υ is the probability measure $\mathbb{P}^\upsilon$ on $(S^\infty, \mathcal{A}^\infty)$ such that

$$\mathbb{P}^\upsilon(\xi_{n+1} \in A|\xi_n = x) = Q(x, A) \quad \text{and} \quad \mathbb{P}^\upsilon(\xi_0 \in A) = \upsilon(A).$$

For $\upsilon = \pi$ we denote $\mathbb{P} = \mathbb{P}^\pi$. For $\upsilon = \delta_x$, the Dirac measure, we denote by $\mathbb{P}^x$ and $\mathbb{E}^x$ the probability and conditional expectation for the process started at x. Note that for each x fixed $\mathbb{P}^x(\cdot)$ is a measure on $\mathcal{F}^\infty$, the sigma algebra generated by $\cup_k \mathcal{F}_k$. Also

$$\mathbb{P}(A) = \int \mathbb{P}^x(A)\pi(dx). \tag{2}$$

We mention that any stationary sequence $(Y_k)_{k\in\mathbb{Z}}$ can be viewed as a function of a Markov process $\xi_k = (Y_j; j \leq k)$ with the function $g(\xi_k) = Y_k$. Therefore the theory of stationary processes can be embedded in the theory of Markov chains. So, our results apply to any stationary process with the corresponding interpretation. In the context of a stationary process, a fixed starting point for a corresponding Markov chain means a fixed past trajectory for $k \leq 0$.

All along the paper we shall assume that the Markov chain is ergodic.

Below, we denote by $\Rightarrow$ the convergence in distribution. By $[x]$ we denote the integer part of x.

For a Markov chain, by the quenched CLT (or CLT started at a point) we shall understand the following convergence: there is a positive constant $\sigma \in [0, \infty)$ and a set $S' \subset S$ with $\pi(S') = 1$ such that for $x \in S'$ we have

$$\frac{S_n}{\sqrt{n}} \Rightarrow \sigma N(0, 1) \text{ under } \mathbb{P}^x, \tag{3}$$

and by the quenched functional CLT (which is the same as functional CLT started at a point): there is a set $S' \subset S$ with $\pi(S') = 1$ such that for $x \in S'$

$$\frac{S_{[nt]}}{\sqrt{n}} \Rightarrow \sigma W(t) \text{ under } \mathbb{P}^x, \tag{4}$$

where $W(t)$ denotes the standard Brownian motion and the convergence in distribution is on $D(0, 1)$, the space of functions continuous at the right with limits at the left, endowed with the Skorohod topology.

An important class satisfying quenched functional CLT is the stationary and ergodic martingale differences, as seen in [13,14]. A natural method to prove these types of results for other classes of processes is to use martingale approximations. This method was initiated by Gordin [20].

One of the first results of this type is due to Gordin (published in Ch.4 Section 8 in [4]), who proved the quenched CLT for Markov chains with normal operator ($QQ^* = Q^*Q$), $f \in \mathbb{L}_2^0$, under the condition $f \in (I - Q)\mathbb{L}_2(\pi)$. If the Markov chain is irreducible and aperiodic, then the quenched CLT holds under the condition $\sum_{j=0}^{n} \mathbb{E}_\pi (f Q^j f)$ is convergent (Chen [5]). Without assuming irreducibility conditions, various papers point out rates for convergence to 0 of $\| \sum_{j=0}^{n} Q^j f \|_2 / n$ needed for the quenched results. Among them, we mention papers by Derriennic and Lin [13,14], Wu and Woodroofe [37], Cuny [6], Merlevède et al. [27], Cuny and Peligrad [8], Cuny and Merlevède [7], Cuny and Volný [9], Volný and Woodroofe [36]. Recently, Dedecker et al. [10] showed that the condition $\sum_{j=0}^{\infty} \mathbb{E}_\pi |f Q^j f| < \infty$ leads to the quenched invariance principle.

Our study is motivated by the class considered by Gordin. What can one say about $f \in (I - Q)\mathbb{L}_p(\pi)$ with $1 \le p < 2$? From the paper by Volný and Woodroofe [36] we know that there are examples of functions, $f \in [(I - Q)\mathbb{L}_1(\pi)] \cap \mathbb{L}_2^0(\pi)$ such that $S_n/\sqrt{n}$ satisfies the CLT, but fails to satisfy the quenched CLT.

One of our results shows that for functions of reversible Markov chains one can assume that $f \in [(I - Q)\mathbb{L}_q(\pi)] \cap \mathbb{L}_p^0(\pi)$, with $q \in [1, 2]$, $1/p + 1/q = 1$, for concluding that the quenched functional CLT holds. This result follows from several general preliminary results that have interest in themselves. They specify sufficient conditions for the validity of the quenched CLT and the quenched functional CLT.

Denote

$$f_m = \frac{1}{m}(Q + \cdots + Q^m) f \tag{5}$$

and

$$\bar{R}_k^m = \sum_{j=1}^{k} f_m(\xi_j). \tag{6}$$

Theorem 1. *Let* $(X_n)_{n\in\mathbb{Z}}$ *be a stationary sequence of random variables defined by (1) and define* $(\bar{R}_k^m)_{m\geq 1,k\geq 1}$ *by (6). Assume that*

$$\lim_m \limsup_n \mathbb{P}^x\left(\frac{|\bar{R}_n^m|}{\sqrt{n}} > \varepsilon\right) = 0 \quad \pi\text{-a.s.} \tag{7}$$

Then the quenched CLT in (3) holds.

Theorem 2. *Assume that* $(X_n)_{n\in\mathbb{Z}}$ *and* $(\bar{R}_k^m)_{m\geq 1,k\geq 1}$ *are as in Theorem 1 and*

$$\lim_m \limsup_n \mathbb{P}^x\left(\max_{1\leq j\leq n}\frac{|\bar{R}_j^m|}{\sqrt{n}} > \varepsilon\right) = 0 \quad \pi\text{-a.s.} \tag{8}$$

Then the quenched functional CLT in (4) holds.

For $f \in \mathbb{L}_1^0(\pi)$ denote by

$$g_f = \sup_{n\geq 0}\left|\sum_{j=0}^{n} Q^j f\right|. \tag{9}$$

Based on Theorem 2 we shall establish the following theorem:

Theorem 3. *Let* $(X_n)_{n\in\mathbb{Z}}$ *be defined by (1),* f_m *by (5) and* g_f *by (9). Assume the following condition is satisfied:*

$$(f_m g_f)_{m\geq 1} \text{ is uniformly integrable.} \tag{10}$$

Then the quenched functional CLT in (4) holds.

From the proof of Theorem 3 we easily deduce several corollaries. The first corollary is well adapted for strongly mixing sequences:

Corollary 4. *Assume*

$$\lim_{m\to\infty}\sum_{j=1}^{\infty}\mathbb{E}_\pi|(Q^m f)(Q^j f)| = 0. \tag{11}$$

Then the quenched functional CLT in (4) holds.

Remark 5. Condition (11) can be verified in terms of strong mixing coefficients. Practically, we deduce that any strongly mixing sequence satisfying the CLT also satisfies the quenched functional CLT. Therefore our approach also provides a shorter, alternative proof of Corollary 3.5 in [10]. The proof of this remark is postponed to the end of the paper.

Also, as an application to the proof of Theorem 3 we obtain the following:

Corollary 6. *Let* $(X_n)_{n\in\mathbb{Z}}$, f_m, *and* g_f *defined as in Theorem 3. Assume* $f \in \mathbb{L}_p^0(\pi)$ *and* $g_f \in \mathbb{L}_q(\pi)$ *with* $p \in [2,\infty]$, $1/p + 1/q = 1$. *Then the quenched functional CLT holds.*

We say that a Markov chain is reversible if Q is self-adjoint; equivalently (X_0, X_1) and (X_1, X_0) are identically distributed. If the Markov chain is reversible then the following corollary holds.

Corollary 7. *Assume the Markov chain is reversible and*

$$f \in [(I - Q)\mathbb{L}_q(\pi)] \cap \mathbb{L}_p^0(\pi). \tag{12}$$

for $p \in [2, \infty)$, $1/p + 1/q = 1$. *Then the quenched functional CLT holds.*

Let us mention that the class we consider here is of independent interest when compared to the projective condition used in [10], namely $\sum_{j=0}^{\infty} \mathbb{E}|X_0 E(X_j|\mathcal{F}_0)| < \infty$. For instance there are examples which satisfy the conditions of Corollary 6 without satisfying the condition from [10].

Remark 8. There is a stationary and ergodic process of bounded random variables $(X_k)_{k\in Z}$ adapted to a filtration $(\mathcal{F}_k)_{k\in Z}$, such that $\sup_{n\geq 0} |\sum_{j=0}^{n} \mathbb{E}(X_j|\mathcal{F}_0)| \in \mathbb{L}_1$ and $\sum_{j=0}^{\infty} \mathbb{E}|X_0\mathbb{E}(X_j|\mathcal{F}_0)| = \infty$.

We end this section by mentioning two conjectures which deserve further investigation. The results in the paper by Dedecker et al. [10] and the results in this paper suggest the following conjecture, which is a quenched form of the functional CLT in [11].

Conjecture 9. *In the context of Theorem 3 assume*

$$\left| f \sum_{j=0}^{n} Q^j f \right| \text{ is convergent in } \mathbb{L}_1(\pi). \tag{13}$$

Then the quenched functional CLT holds.

For reversible Markov chains we would like to mention the Kipnis and Varadhan [24] conjecture, asking if their functional CLT is quenched. This conjecture is still unsolved.

Conjecture 10. *In the context of Corollary 7 assume*

$$\mathbb{E}_\pi \left(f \sum_{j=0}^{n} Q^j f \right) \text{ is convergent.} \tag{14}$$

Then the quenched functional CLT holds.

Steps towards clarifying this conjecture are contained in the papers by Derriennic and Lin [13] and Cuny and Peligrad [8].

2. Preliminary considerations

The method we shall use in our proofs is based on a martingale approximation depending on a certain parameter which is fixed at the beginning and after that we let it grow to ∞. To deal with this parameter, we start by pointing out several preliminary considerations for convergence in distribution. From Theorem 3.2 in [3], it is well-known the following result:

Lemma 11. *Assume that the elements* $(X_{n,m}, X_n)$ *are defined on the same probability space with values in* $S \times S$*, where* S *is a metric space. Assume that*

$$X_{n,m} \Rightarrow_n Y_m \Rightarrow_m X$$

and

$$\lim_m \limsup_n \mathbb{P}(d(X_{n,m}, X_n) \geq \varepsilon) = 0. \tag{15}$$

Then

$$X_n \Rightarrow X.$$

If the metric space is separable and complete, then one does not have to assume $Y_m \Rightarrow X$. This result (see for instance Theorem 2 in [12]) is given in the following lemma where the variables are denoted as in Lemma 11.

Lemma 12. *Assume the metric space S is separable and complete. Assume that for every m*

$$X_{n,m} \Rightarrow Y_m \quad \text{as } n \to \infty$$

and condition (15) is satisfied. Then there is a S-valued random variable X such that

$$Y_n \Rightarrow X \quad \text{and} \quad X_n \Rightarrow X \quad \text{as } n \to \infty.$$

These considerations suggest that the conditions of Lemma 11 are too strong. Indeed, we can formulate the following lemma.

Lemma 13. *In Lemma 11 condition (15) can be replaced by*

$$\liminf_m \limsup_n \mathbb{P}(d(X_{n,m}, X_n) \geq \varepsilon) = 0. \tag{16}$$

Proof of Lemma 13. Let F be a closed set. Define $F_\varepsilon = \{x : d(x, F) \leq \varepsilon\}$. Then, by Portmanteau theorem (Theorem 2.1 in [3]),

$$\limsup_n \mathbb{P}(X_{n,m} \in F_\varepsilon) \leq \mathbb{P}(Y_m \in F_\varepsilon).$$

Since

$$\mathbb{P}(X_n \in F) \leq \mathbb{P}(X_{n,m} \in F_\varepsilon) + \mathbb{P}(d(X_{n,m}, X_n) \geq \varepsilon),$$

by combining these results, we deduce that

$$\begin{aligned}\limsup_n \mathbb{P}(X_n \in F) &\leq \limsup_n \mathbb{P}(X_{n,m} \in F_\varepsilon) + \limsup_n \mathbb{P}(d(X_{n,m}, X_n) \geq \varepsilon)\\ &\leq \mathbb{P}(Y_m \in F_\varepsilon) + \limsup_n \mathbb{P}(d(X_{n,m}, X_n) \geq \varepsilon).\end{aligned}$$

Therefore taking the limit inferior when $m \to \infty$ we obtain by (16) and Portmanteau Theorem that

$$\begin{aligned}\limsup_n \mathbb{P}(X_n \in F) &\leq \liminf_m [\mathbb{P}(Y_m \in F_\varepsilon) + \limsup_n \mathbb{P}(d(X_{n,m}, X_n) \geq \varepsilon)]\\ &\leq \limsup_m \mathbb{P}(Y_m \in F_\varepsilon) + \liminf_m \limsup_n \mathbb{P}(d(X_{n,m}, X_n) \geq \varepsilon)\\ &\leq \mathbb{P}(X \in F_\varepsilon).\end{aligned}$$

Now we take a sequence $F_\varepsilon \downarrow F$ as $\varepsilon \downarrow 0$, the result follows by applying again the Portmanteau theorem. □

One of the difficulties in proving quenched results is the fact that, under $\mathbb{P}^x$, the Markov chain is no longer strictly stationary. Since we are interested in proving quenched results which are almost sure results, and also the quenched functional form of the CLT, we need to use maximal inequalities. There are not too many maximal inequalities available in the nonstationary context.

A useful maximal inequality is an easy consequence of inequality (3.9) given in the book by Rio [32] (see also [11]).

Lemma 14. *Assume that* (X_k) *is a sequence of real valued centered random variables in* $\mathbb{L}_2(\Omega, \mathcal{K}, \mathbb{P})$*, adapted to an increasing filtration of sub-sigma fields of* $\mathcal{K}$*,* $(\mathcal{F}_n)$*. Then*

$$\mathbb{E}(\max_{1\leq k\leq n} S_k^2) \leq 8\sum_{k=1}^{n}\mathbb{E}(X_k^2) + 16\sum_{k=1}^{n}\mathbb{E}|X_k\mathbb{E}(S_n - S_k|\mathcal{F}_k)|.$$

One of the basic results used in our proofs is the functional CLT for martingales in the following form:

Theorem 15. *Assume that* (D_n) *is a sequence of martingale differences on a probability space* $(\Omega, \mathcal{K}, \mathbb{P})$ *adapted to an increasing filtration of sub-sigma fields of* $\mathcal{K}$*,* $(\mathcal{F}_n)$*. Assume that the following two conditions hold*

$$\left(\frac{1}{\sqrt{n}}\max_{0\leq k\leq n}|D_k|\right)_{n\geq 1} \text{ is uniformly integrable} \tag{17}$$

and for each t*,* $0 \leq t \leq 1$

$$\frac{1}{n}\sum_{k=0}^{[nt]} D_k^2 \to t\sigma^2 \text{ in probability.} \tag{18}$$

Then

$$\frac{\sum_{k=0}^{[nt]} D_k}{\sqrt{n}} \Rightarrow |\sigma|W(t).$$

This theorem follows from Theorem 2.3 in [19] combined with the commentaries on pages 316–317 of this paper. Indeed, according to the sequence of implications on page 316 of this book, the conditions (A_a) and $(R_{a,t})$ of their Theorem 2.3 are verified under (18) and

$$\frac{1}{\sqrt{n}}\max_{0\leq k\leq n}|D_k| \to 0 \quad \text{in } \mathbb{L}_1. \tag{19}$$

Then, by arguments on page 317 both conditions (17) and (18) imply condition (19).

3. Proofs

Proof of Theorems 1 and 2. We start with a martingale construction. The construction of the martingale decomposition is inspired by works of Gordin [20], Heyde [23], Gordin–Lifshitz [21]; see also Theorem 8.1 in [4,24,25]. The form we use here was initiated by Wu and Woodroofe [37], and further exploited by Zhao and Woodroofe [38], Peligrad [29], Gordin and Peligrad [22] among others. We briefly give it here for completeness.

We introduce a parameter, an integer $m \geq 1$ (kept fixed for the moment), and introduce the functions

$$v_k = (I + Q + \cdots + Q^{k-1})f. \tag{20}$$

Define the stationary sequence of random variables:

$$\theta_0^m = \frac{1}{m}\sum_{k=1}^{m} v_k(\xi_0), \qquad \theta_k^m = \theta_0^m \circ T^k.$$

Denote by

$$D_k^m = D_k^m(\xi_k, \xi_{k+1}) = \theta_{k+1}^m - \mathbb{E}_k(\theta_{k+1}^m); \qquad M_n^m = \sum_{k=1}^{n} D_k^m. \tag{21}$$

Then, $(D_k^m)_{k\in\mathbb{Z}}$ is a martingale difference sequence which is stationary and ergodic and $(M_n^m)_{n\geq 0}$ is a martingale. So we have

$$X_k = D_k^m + \theta_k^m - \theta_{k+1}^m + f_m(\xi_k),$$

with f_m defined by (5). Therefore

$$S_k = M_k^m + \theta_1^m - \theta_{k+1}^m + \overline{R}_k^m, \tag{22}$$

where we implemented the notation

$$\bar{R}_k^m = \sum_{j=1}^{k} f_m(\xi_j).$$

With the notation

$$R_k^m = \theta_1^m - \theta_{k+1}^m + \bar{R}_k^m, \tag{23}$$

we have the following martingale decomposition

$$S_k = M_k^m + R_k^m. \tag{24}$$

We shall prove now the quenched functional CLT for the martingale M_n^m. We shall verify the conditions of the functional CLT given in Theorem 15.

We start by noticing that $(M_n^m)_n$ is also a martingale under $\mathbb{P}^x$ (since $\mathbb{E}^x(D_k^m|\mathcal{F}_{k-1}) = \mathbb{E}(D_k^m|\mathcal{F}_{k-1})$ by the fact that the Markov chain has the same transitions under $\mathbb{P}$ and $\mathbb{P}^x$). We verify first condition (18). Since M_n^m is a martingale with stationary and ergodic increments, by Birkhoff's ergodic theorem, for every $0 \leq t \leq 1$,

$$\frac{1}{n}\sum_{k=1}^{[nt]}(D_k^m)^2 \to t\mathbb{E}(D_0^m)^2 \quad \mathbb{P}\text{-a.s.}$$

and therefore for every $0 \leq t \leq 1$ and π-almost all x

$$\frac{1}{n}\sum_{k=1}^{[nt]}(D_k^m)^2 \to t\mathbb{E}(D_0^m)^2 \quad \mathbb{P}^x\text{-a.s.} \tag{25}$$

In order to verify (17), for proving uniform integrability it is enough to show that for π-almost all x, for some constant C_x we have

$$\sup_n \frac{1}{n}\mathbb{E}^x(\max_{1\leq k\leq n}(D_k^m)^2) \leq C_x. \tag{26}$$

Clearly

$$\frac{1}{n}\mathbb{E}^x(\max_{1\le k\le n}(D_k^m)^2) \le \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}^x(D_k^m)^2.$$

Note that $D_0^m = D_0^m(\xi_1, \xi_0)$ and then, denoting by $h(y) = E((D_0^m(\xi_1, \xi_0))^2|\xi_0 = y)$, by the Markov property it follows that $\mathbb{E}^x(D_k^m)^2 = Q^k h(x)$. By Hopf's ergodic theorem for Markov operators (see Theorem 11.4 in [18]) we obtain

$$\limsup_n \frac{1}{n}\mathbb{E}^x(\max_{1\le k\le n}(D_k^m)^2) \le \limsup_n \frac{1}{n}\sum_{k=1}^{n} Q^k h(x) = \mathbb{E}(D_0^m)^2 \quad \pi\text{-a.s.}$$

and (26) follows.

By Theorem 15 it follows that for π-almost all x we have

$$\frac{M_{[nt]}^m}{\sqrt{n}} \Rightarrow |\sigma_m| W(t) \text{ under } \mathbb{P}^x, \tag{27}$$

where $W(t)$ is the standard Brownian motion and

$$\sigma_m^2 = \mathbb{E}(D_0^m)^2. \tag{28}$$

By stationarity, by the fact that θ_0^m is in $\mathbb{L}_2$ we have

$$\frac{\max_{1\le k\le n}|\theta_k^m|}{\sqrt{n}} \to 0 \quad \mathbb{P}\text{-a.s.}$$

To see it, just start from $\sum_n \mathbb{P}(|\theta_0^m|^2 > \varepsilon n) < \infty$ and apply the Borel–Cantelli lemma (see also page 171 in [4]).

Therefore, for π-almost all x

$$\frac{\max_{1\le k\le n}|\theta_k^m|}{\sqrt{n}} \to 0 \quad \mathbb{P}^x\text{-a.s.} \tag{29}$$

If we assume (7) then clearly by (29) we obtain

$$\lim_m \limsup_n \mathbb{P}^x\left(\frac{|S_n - M_n^m|}{\sqrt{n}} > \varepsilon\right) = 0 \quad \pi\text{-a.s.}$$

Clearly (27) implies that for each $m \ge 1$

$$\frac{M_n^m}{\sqrt{n}} \Rightarrow |\sigma_m| Z \text{ under } \mathbb{P}^x$$

where Z has a standard normal distribution. By applying Lemma 12 we obtain that $|\sigma_m|Z$ converges in distribution to a random variable Y, which is also the limiting distribution of $S_n/\sqrt{n}$ under $\mathbb{P}^x$. Clearly Y has a normal distribution with variance $\sigma^2 = \lim_m \sigma_m^2$, where $\sigma \in [0, \infty)$.

Now, by taking into account (8), we have

$$\lim_m \limsup_n \mathbb{P}^x\left(\max_{1\le j\le n}\frac{|S_n - M_n^m|}{\sqrt{n}} > \varepsilon\right) = 0 \quad \pi\text{-a.s.}$$

and, by Lemma 12, as explained before, we get both that $\mathbb{E}(D_0^m)^2 \to \sigma^2$ and that the quenched functional CLT holds with the limit $\sigma W(t)$. $\square$

Remark 16. We point out that the proofs of Theorems 1 and 2 also indicate how to identify the constant σ^2 which appears in the limit as

$$\sigma^2 = \lim_{m\to\infty}\lim_{n\to\infty}\mathbb{E}(D_n^m)^2,$$

where D_n^m was defined in (21).

Remark 17. By Lemma 13, Theorems 1 and 2 also hold if we replace in conditions (7) and (8) the limit when $m \to \infty$ by $\liminf_{m\to\infty}$ and we add the condition

$$\mathbb{E}(D_0^m)^2 \text{ is convergent as } m \to \infty. \tag{30}$$

Condition (30) is verified in many situations including classes of normal and reversible Markov chains, as shown by Gordin and Lifshitz [21] and Kipnis and Varadhan [24] among others.

We shall establish next a maximal inequality needed to verify condition (8).

Proposition 18. *For any $h \in \mathbb{L}_2^0(\pi)$ such that $\mathbb{E}_\pi(|hg_h|) < \infty$, we have the following maximal inequality*

$$\limsup_n \frac{\mathbb{E}^x(\max_{1\le k\le n} S_k^2(h))}{n} \le 24\mathbb{E}_\pi(|hg_h|) \quad \pi\text{-a.s.} \tag{31}$$

Proof. We start by applying Rio's maximal inequality given in Lemma 14 which implies that

$$\mathbb{E}^x(\max_{1\le k\le n} S_k^2(h)) \le 8\sum_{u=1}^{n}\mathbb{E}^x(h^2(\xi_u)) + 16\sum_{u=1}^{n-1}\mathbb{E}^x\left|h(\xi_u)\sum_{k=1}^{n-u}Q^k h(\xi_u)\right|.$$

So

$$\mathbb{E}^x(\max_{1\le k\le n} S_k^2(h)) \le 24\sum_{j=1}^{n}Q^j\left[\sup_{k\ge 0}\left|\sum_{u=0}^{k}hQ^u h\right|\right](x).$$

By the Hopf ergodic theorem for Markov operators

$$\frac{1}{n}\sum_{j=1}^{n-1}Q^j\left[\sup_{k\ge 0}\left|\sum_{u=0}^{k}hQ^u h\right|\right](x) \to \mathbb{E}_\pi \sup_{n\ge 0}\left|\sum_{u=0}^{n}hQ^u h\right| \quad \pi\text{-a.s.}$$

which leads by the previous considerations to (31) by the definition of g_h. □

Proof of Theorem 3. The proof consists in verifying condition (8) of Theorem 2.

We start by applying Proposition 18 to $S_k(f_m)$, where f_m is defined by (5). Note that $\bar{R}_k^m$ defined by (6) is equal to $S_k(f_m)$. For all m fixed

$$\limsup_n \frac{\mathbb{E}^x(\max_{1\le k\le n}(\bar{R}_k^m)^2)}{n} \le 24\mathbb{E}_\pi\left[\sup_{k\ge 0}\left|\sum_{j=0}^{k}f_m Q^j f_m\right|\right] \quad \pi\text{-a.s.} \tag{32}$$

Then, we have

$$\left|\sum_{j=0}^{n} Q^j f_m\right| = \frac{1}{m}\left|\sum_{j=0}^{n}\sum_{k=1}^{m} Q^{j+k} f\right| \le \frac{1}{m}\sum_{k=1}^{m}\left|\sum_{j=k}^{n+k} Q^j f\right|$$
$$\le 2\sup_n\left|\sum_{j=0}^{n} Q^j f\right| \le 2g_f,$$

which, combined with (32), leads to

$$\limsup_n \frac{\mathbb{E}^x(\max_{1\le k\le n}(\bar{R}_k^m)^2)}{n} \le 48\mathbb{E}_\pi(|f_m g_f|) \quad \pi\text{-a.s.}$$

Clearly, by using this last inequality, in order to prove (8), it remains to show

$$\mathbb{E}_\pi(|f_m g_f|) \to 0 \quad \text{as } m \to \infty. \tag{33}$$

By Hopf's ergodic theorem for Markov operators (Theorem 11.4 in [18])

$$f_m \to 0\ \pi\text{-a.s.} \quad \text{so } f_m g_f \to 0\ \pi\text{-a.s.}$$

and also, because by condition (10), $(f_m g_f)_{m\ge1}$ is uniformly integrable, it follows that

$$f_m g_f \to 0 \quad \text{in } \mathbb{L}_1(\pi). \quad \square$$

Proof of Corollary 4. Note that, by the triangle inequality, (11) implies (33) and the proof of Theorem 3 applies.

Proof of Corollary 6. We start from (33) and apply Hölder's inequality, so

$$\mathbb{E}_\pi(|f_m g_f|) \le \mathbb{E}_\pi^{1/p}(|f_m|^p)\mathbb{E}_\pi^{1/q}(|g_f|^q).$$

By the mean ergodic theorem for the Dunford–Schwartz operators on a Banach space (see Theorem 8.18 in [18]) $\mathbb{E}_\pi(|f_m|^p) \to 0$ as $m \to \infty$, and the result follows. Also note that we can take $p = \infty$ and $q = 1$. $\square$

Proof of Corollary 7. We shall verify the condition of Corollary 6. If $f \in (I - Q)\mathbb{L}_q(\pi)$ there is $h \in \mathbb{L}_q(\pi)$ such that $f = (I - Q)h$.

Then, by Hölder's inequality

$$\mathbb{E}_\pi(\sup_n |(I + Q + \cdots + Q^{n-1})f|^q) = \mathbb{E}_\pi(\sup_n |(I - Q^n)h|^q)$$
$$\le 2^{q-1}[\mathbb{E}_\pi(|h|^q) + \mathbb{E}_\pi(\sup_n(|Q^n h|^q))].$$

By the Stein Theorem (see [34]), $\sup_n |Q^n h|$ is in $\mathbb{L}_q(\pi)$ and there is a constant K such that $\mathbb{E}_\pi \sup_n(|Q^n h|^q) \le K\mathbb{E}_\pi(|h|^q)$. Therefore g_f is in $\mathbb{L}_q(\pi)$ and we can apply Corollary 6 to obtain the result. $\square$

Proof of the Remark 8. It is convenient to specify this example in terms of a stationary process defined by a dynamical system. The proof of this remark follows by analyzing the example given in [17,16].

We consider an ergodic dynamical system $(\Omega, \mathcal{A}, \mu, T)$, with μ nonatomic and strictly positive entropy. Let $\mathcal{B}$ and $\mathcal{C}$ be two independent sub-sigma algebras of $\mathcal{A}$ such that $T^{-1}\mathcal{C} = \mathcal{C}$.

Let $(e_i)_{i\in\mathbb{Z}}$ be a sequence of independent identically distributed Rademacher random variables with parameter $1/2$, measurable with respect to $\mathcal{B}$ and denote by $\mathcal{F}_0$ the σ-algebra generated by $\mathcal{C}$ and $(e_i)_{i\le 0}$. We consider an increasing sequence of integers (N_k), and mutually disjoint sets $(A_k)_{k\in\mathbb{Z}}$, $A_k\in\mathcal{C}$ such that

(1) $\frac{2}{3}\rho_k \le \mu(A_k) \le \rho_k$for all $k\in\mathbb{N}^*$ where $\rho_k = a^k$ for $0 < a \le 1/4$.
and
(2) for all $k\in\mathbb{N}$ and all $i, j\in\{0,\dots,N_k\}$, $\mu(T^{-i}A_k\Delta T^{-j}A_k)\le\varepsilon_k$ where (ε_k) will be selected later.

The existence of the sequence $(A_k)_{k\in\mathbb{Z}}$ with the above properties was explained in Lemma 2 of Durieu and Volný [17].

The function f is then defined as

$$f = \sum_{k\ge 1} e_{-N_k}\mathbf{1}_{A_k}. \tag{34}$$

The function f defined in (34) is centered, $\mathcal{F}_0$-measurable and bounded.

For any $i\in\mathbb{Z}$, let now $X_i = f\circ T^i$. This sequence is adapted to the stationary and nondecreasing sequence of σ-algebras $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ where $\mathcal{F}_i = T^{-i}(\mathcal{F}_0)$. Note that the sequence $(e_i)_{i\in\mathbb{Z}}$ is adapted to $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ and $\mathbb{E}(e_i|\mathcal{F}_0) = e_i\mathbf{1}_{i\le 0}$ almost surely. Also, for all k and i, $\mathbf{1}_{A_k}\circ T^i$ is $\mathcal{F}_0$-measurable and the e_i's and the $\mathbf{1}_{A_k}$'s are independent. Clearly, for any $i\in\mathbb{N}$,

$$\begin{aligned}\mathbb{E}(X_i|\mathcal{F}_0) &= \sum_{k\ge 1} e_{-N_k+i}\mathbf{1}_{i\le N_k}\mathbf{1}_{T^{-i}(A_k)}\\ &= \sum_{k\ge 1} e_{-N_k+i}\mathbf{1}_{i\le N_k}\mathbf{1}_{A_k} + \sum_{k\ge 1} e_{-N_k+i}\mathbf{1}_{i\le N_k}(\mathbf{1}_{T^{-i}(A_k)\setminus A_k} - \mathbf{1}_{A_k\setminus T^{-i}(A_k)}).\end{aligned} \tag{35}$$

So, by using the fact that the e_j's and f are bounded by one, and selecting N_k, ε_k such that $\sum_{k\ge 1} N_k\varepsilon_k < \infty$, we obtain

$$\begin{aligned}\sum_{i\ge 1}\mathbb{E}\left|\sum_{k\ge 1} e_{-N_k+i}\mathbf{1}_{i\le N_k}(\mathbf{1}_{T^{-i}(A_k)\setminus A_k} - \mathbf{1}_{A_k\setminus T^{-i}(A_k)})\right| &\le \sum_{i\ge 1}\sum_{k\ge 1}\mathbf{1}_{i\le N_k}[\mu(T^{-i}(A_k)\Delta A_k)]\\ &\le \sum_{k\ge 1} N_k\varepsilon_k < \infty.\end{aligned} \tag{36}$$

Therefore, since f is bounded, by (35) and (36), in order to show that $\sum_{i\ge 0}\mathbb{E}|f\mathbb{E}(X_i|\mathcal{F}_0)| = \infty$ holds, it is enough to show that

$$\sum_{i\ge 1}\mathbb{E}\left|f\sum_{k\ge 1} e_{-N_k+i}\mathbf{1}_{i\le N_k}\mathbf{1}_{A_k}\right| = \infty. \tag{37}$$

By the fact that (A_k) are disjoint

$$\begin{aligned}\sum_{i\ge 1}\mathbb{E}\left|f\sum_{k\ge 1} e_{-N_k+i}\mathbf{1}_{i\le N_k}\mathbf{1}_{A_k}\right| &= \sum_{i\ge 1}\mathbb{E}\left|\sum_{u\ge 1} e_{-N_u}\mathbf{1}_{A_u}\sum_{k\ge 1} e_{-N_k+i}\mathbf{1}_{i\le N_k}\mathbf{1}_{A_k}\right|\\ &= \sum_{i\ge 1}\mathbb{E}\left|\sum_{k\ge 1} e_{-N_k}e_{-N_k+i}\mathbf{1}_{i\le N_k}\mathbf{1}_{A_k}\right|\\ &= \sum_{i\ge 1}\sum_{k\ge 1}\mathbb{E}|e_{-N_k}e_{-N_k+i}\mathbf{1}_{i\le N_k}\mathbf{1}_{A_k}|\end{aligned}$$

$$= \sum_{i\ge 1}\sum_{k\ge 1} \mathbf{1}_{i\le N_k}\mu(A_k) \ge \frac{2}{3}\sum_{i\ge 1}\sum_{k\ge 1}\mathbf{1}_{i\le N_k}\rho_k$$
$$= \frac{2}{3}\sum_{k\ge 1} N_k\rho_k .$$

On the another hand, by (35) and (36),

$$\mathbb{E}\sup_n \left|\sum_{i=1}^{n}\mathbb{E}(X_i|\mathcal{F}_0)\right| \le \sum_{k\ge 1}\mathbb{E}\sup_n\left|\sum_{i=1}^{n\wedge N_k} e_{-N_k+i}\mathbf{1}_{A_k}\right| + \sum_{k\ge 1} N_k\varepsilon_k . \tag{38}$$

By the fact that (e_i)'s and (A_k)'s are independent and by Doob's maximal inequality we obtain

$$\sum_{k\ge 1}\mathbb{E}\sup_n\left|\sum_{i=1}^{n\wedge N_k} e_{-N_k+i}\right|\mathbf{1}_{A_k} = \sum_{k\ge 1}\mathbb{E}\max_{1\le j\le N_k}\left|\sum_{i=1}^{j} e_{-N_k+i}\right|\mu(A_k)$$
$$\le \sum_{k\ge 1}\mathbb{E}\max_{1\le j\le N_k}\left|\sum_{i=1}^{j} e_{-N_k+i}\right|\rho_k \le \sum_{k\ge 1}\sqrt{N_k}\rho_k .$$

To finish the proof of this remark we have to select sequences such that $\sum_{k\ge 1} N_k\varepsilon_k < \infty$, $\sum_{k\ge 1} N_k\rho_k = \infty$ and $\sum_{k\ge 1}\sqrt{N_k}\rho_k < \infty$.

This selection is possible. For instance, we can take $\rho_k = 4^{-k}$, $N_k = 4^k$ and $\varepsilon_k = 8^{-k}$.

Proof of the Remark 5 (*Application to Strong Mixing Sequences*)**.** We shall apply now Corollary 4 to strong mixing sequences.

For the random variable X, define the "upper tail" quantile function q by

$$q(u) = \inf\{t \ge 0 : \mathbb{P}(|X_0| > t) \le u\} .$$

Relevant to this application is the following lemma.

Lemma 19. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\mathcal{M}$ be a sub σ-algebra of $\mathcal{A}$. Let X and Y be two square integrable identically distributed random variables. Denote by q their common quantile function. Then*

$$\mathbb{E}|X\mathbb{E}(Y|\mathcal{M})| \le 3\int_0^{\bar\alpha} q^2 du,$$

where

$$\bar\alpha = \bar\alpha(Y,\mathcal{M}) = \sup_{t\in\mathbb{R}}\mathbb{E}|\mathbb{P}(Y\le t|\mathcal{M}) - \mathbb{P}(Y\le t)|.$$

Inspired by the proof of Lemma 2 in [28], this lemma can be obtained directly, by truncation arguments. It can also be obtained by using Lemma 4 in [26], combined with Rio's covariance inequality (Theorem 1.1 in [32]). The proof is left to the reader.

Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary sequence of real valued random variables. We shall interpret it as a function of a Markov chain $\xi_k = (X_j, j \le k)$, $f(\xi_k) = X_k$, and define the σ-algebra $\mathcal{F}_0 = \sigma(X_i, i \le 0)$. For any $k \in \mathbb{N}$ also define

$$\bar\alpha_k = \bar\alpha(X_k, \mathcal{F}_0).$$

Recall that the strong mixing coefficient of Rosenblatt [33], defined by

$$\alpha_k = \sup_{A \in \sigma(Y_k), B \in \mathcal{F}_0} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|,$$

is such that $\bar{\alpha}_k \le 2\alpha_k$. (See page 8 in [32].)

By using our Corollary 4 we shall establish the following result (see also Corollary 3.5 in [10]).

For any nonnegative random variable Z, we define the quantile function q_Z of Z by $q(u) = \inf\{t \ge 0 : \mathbb{P}(|Z| > t) \le u\}$.

Proposition 20. *Assume $(X_i)_{i \in \mathbb{Z}}$ is a stationary and ergodic sequence of random variables and $|X_0|$ has quantile function q. Also assume*

$$\sum_{j \ge 1} \int_0^{\bar{\alpha}_j} q^2 du < \infty. \tag{39}$$

Then the quenched functional CLT holds.

Proof. Note that $\mathbb{E}_\pi |(Q^m f)(Q^j f)| \le \min(\mathbb{E}|f(\xi_m)(Q^j f)(\xi_0)|,\ \mathbb{E}|f(\xi_j)(Q^m f)(\xi_0)|)$. So, by Lemma 19 we obtain

$$\sum_{j \ge 1} \mathbb{E}|(Q^m f)(Q^j f)| \le 3 \sum_{j \ge 1} \min\left(\int_0^{\bar{\alpha}_j} q^2 du, \int_0^{\bar{\alpha}_m} q^2 du\right). \tag{40}$$

If we impose condition (39), this condition implies $\bar{\alpha}_m \to 0$ and also allows us to apply the discrete Lebesgue dominated theorem in (40). So condition (11) is satisfied and the result follows. $\square$

We easily recognize condition (39) as being the usual condition, optimal in some sense, used in the context of invariance principles for strongly mixing sequences (see [15]).

Note that X_0 is distributed as $q(U)$ where U is a uniform random variable. Therefore we can give sufficient conditions for the validity of (39) in terms of moments of X_0 and mixing rates.

For instance if X_0 is almost surely bounded by a constant, condition (39) is satisfied as soon as $\sum_{j \ge 1} \bar{\alpha}_j < \infty$. If for a $\delta > 0$ we have $\mathbb{E}(|X_0|^{2+\delta}) < \infty$, then condition (39) is satisfied provided $\sum_{j \ge 1} j^{2/\delta} \bar{\alpha}_j < \infty$ (see [15]).

Acknowledgments

The authors are indebted to Florence Merlevède for helpful discussions. Many thanks are going to the referee for carefully reading the manuscript and for numerous suggestions which improved the presentation of the paper. This research was supported in part by a Charles Phelps Taft Memorial Fund grant and the NSF grant DMS-1512936.

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