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Q1 # Multidimensional Lévy white noise in weighted Besov spaces

Q2 Julien Fageot[a,*], Alireza Fallah[b], Michael Unser[a]

[a] Biomedical Imaging Group, École Polytechnique Fédérale de Lausanne (EPFL), Switzerland
[b] Sharif University of Technology, Theran, Iran

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Abstract

In this paper, we study the Besov regularity of a general d-dimensional Lévy white noise. More precisely, we describe new sample paths properties of a given noise in terms of weighted Besov spaces. In particular, we characterize the smoothness and integrability properties of the noise using the indices introduced by Blumenthal, Getoor, and Pruitt. Our techniques rely on wavelet methods and generalized moments estimates for Lévy noises.

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1. Introduction

This paper is dedicated to the study of the regularity of a general d-dimensional Lévy white noise (also called Lévy noise, or simply noise, throughout the paper) in terms of Besov spaces. It is a continuation of our previous work [13]. A random process is traditionally defined as a collection (X_t) of random variables indexed by $t \in \mathbb{R}$, with some adequate properties. For instance, Lévy processes are described as stochastically continuous random processes with independent and stationary increments [1,36]. However, it is not possible to define the Lévy noise in the traditional framework. In the 1D setting, it is tempting to introduce a Lévy noise as the derivative of a Lévy process, but the well-known issue is that the derivative of a non-trivial Lévy process does not have a pointwise interpretation.

* Corresponding author.
E-mail address: julien.fageot@epfl.ch (J. Fageot).

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An alternative way of introducing random processes is based on the abstract theory of measures on function spaces, as developed by Bogachev [5] among others. In this context, a random process is a random variable that takes values in a function space endowed with the adequate measurable structure. In this spirit, Gelfand [15] and Itô [21] have independently introduced the concept of generalized random processes, defined as random elements in the Schwartz space of generalized functions [41]. This approach was more extensively exposed in [16, Chapter 3] and [22]. The Schwartz space has the advantage of being stable by (weak) differentiation: it therefore includes the d-dimensional Lévy noise, but also all its (partial) derivatives.

Measuring regularity with Besov spaces. Since we are considering processes that have no pointwise interpretation, we should consider function spaces with negative smoothness. When talking about the regularity of random processes, the Sobolev or the Hölder regularities are natural concepts that come into mind. In order to be more general, we will investigate the Besov regularity of a Lévy noise. Besov spaces include both Sobolev and Hölder spaces, and provide a finer measure of the regularity of a function [46,47]. Evaluating the Besov regularity of random processes over $\mathbb{R}^d$ requires the introduction of weights, since they are generally not decreasing towards infinity. Thereafter, we therefore consider weighted Besov spaces or local Besov spaces.

Regularity of Lévy noise and related processes. To the best of our knowledge, the Besov regularity of d-dimensional Lévy noise has never been addressed in full generality. Kusuoka [25] estimated the weighted Sobolev regularity of the Gaussian noise, while Veraar [50] obtained complete results on the local Besov regularity of the Gaussian noise. However, these works are based on intrinsic Gaussian methods and are not easily extended to the non-Gaussian case. In [13], we derived new results on the Besov regularity of the symmetric-α-stable (SαS) noise on the d-dimensional torus. This paper is an extension of [13] in two ways: (1) we consider a noise over $\mathbb{R}^d$ and deduce the local results as corollaries, and (2) we extend the results for a general Lévy noise beyond the SαS setting.

Other important works on the Besov regularity of 1-dimensional Lévy processes must also be mentioned. The pioneer works concern the Brownian motion [3,8,34]; see also [30] for extensions to more general Gaussian processes, including fractional Brownian motion. Stable Lévy processes were studied in [8,33]. Note that Rosenbaum [33] used wavelet techniques similar to ours. The case of general Lévy processes was extensively studied by Schilling, both in the local [37] and weighted cases [38,39]. Herren obtained similar local results in [18]. These authors rely on two indices introduced in [4,31] for the study of non-stable Lévy processes while also providing results for a more general class of Markov processes. Those indices also play a crucial role in the present study. For a comprehensive survey on the Besov regularity of Lévy processes, we refer the reader to [6].

2. Preliminaries

2.1. Generalized processes and the Lévy noise

The stochastic processes of this paper are defined in the framework of generalized random processes [16, Chapter 3]. In particular, this allows us to consider a Lévy noise as a well-defined random process.

The Schwartz space of infinitely smooth and rapidly decaying functions on $\mathbb{R}^d$ is denoted by $\mathcal{S}(\mathbb{R}^d)$. It is endowed with the topology associated with the following notion of convergence: A sequence (φ_n) of functions in $\mathcal{S}(\mathbb{R}^d)$ converges to $\varphi \in \mathcal{S}(\mathbb{R}^d)$ if, for every multiindex $\boldsymbol{\alpha} \in \mathbb{N}^d$ and every $\rho \geq 0$, the functions $\boldsymbol{x} \mapsto |\boldsymbol{x}|^\rho \mathrm{D}^{\boldsymbol{\alpha}}\{\varphi_n\}(\boldsymbol{x})$ converge to $\boldsymbol{x} \mapsto |\boldsymbol{x}|^\rho \mathrm{D}^{\boldsymbol{\alpha}}\{\varphi\}(\boldsymbol{x})$ in $L_2(\mathbb{R}^d)$, where

$|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$. The space $\mathcal{S}(\mathbb{R}^d)$ is a nuclear Fréchet space [43, Section 51]. The topological dual of $\mathcal{S}(\mathbb{R}^d)$ is the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered generalized functions. A *cylindrical set* of $\mathcal{S}'(\mathbb{R}^d)$ is a subset of the form

$$\left\{u \in \mathcal{S}'(\mathbb{R}^d), \ (\langle u, \varphi_1\rangle, \ldots, \langle u, \varphi_n\rangle) \in B\right\}, \tag{1}$$

where $n \geq 1$, $\varphi_1, \ldots, \varphi_n \in \mathcal{S}(\mathbb{R}^d)$, and B is a Borel subset of $\mathbb{R}^n$. We denote by $\mathcal{B}_c\left(\mathcal{S}'(\mathbb{R}^d)\right)$ the *cylindrical σ-field* of $\mathcal{S}'(\mathbb{R}^d)$, defined as the σ-field generated by the cylindrical sets. Then, $\left(\mathcal{S}'(\mathbb{R}^d), \mathcal{B}_c(\mathcal{S}'(\mathbb{R}^d))\right)$ is a measurable space. We fix the probability space $(\Omega, \mathcal{F}, \mathscr{P})$.

Definition 1. A *generalized random process* is a measurable function

$$s : (\Omega, \mathcal{F}) \to \left(\mathcal{S}'(\mathbb{R}^d), \mathcal{B}_c(\mathcal{S}'(\mathbb{R}^d))\right). \tag{2}$$

Its *probability law* is the measure on $\mathcal{S}'(\mathbb{R}^d)$, image of $\mathscr{P}$ by s. For every $B \in \mathcal{B}_c\left(\mathcal{S}'(\mathbb{R}^d)\right)$,

$$\mathscr{P}_s(B) = \mathscr{P}(\{\omega \in \Omega, s(\omega) \in B\}). \tag{3}$$

The *characteristic functional* of s is defined for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$ by

$$\widehat{\mathscr{P}_s}(\varphi) = \int_{\mathcal{S}'(\mathbb{R}^d)} \mathrm{e}^{\mathrm{i}\langle u,\varphi\rangle} \mathrm{d}\mathscr{P}_s(u). \tag{4}$$

A generalized random process is a random element of the space of tempered generalized functions. The characteristic functional is the infinite-dimensional generalization of the characteristic function. It characterizes the law of s in the sense that

$$\mathscr{P}_{s_1} = \mathscr{P}_{s_2} \Leftrightarrow \widehat{\mathscr{P}}_{s_1} = \widehat{\mathscr{P}}_{s_2}, \tag{5}$$

which we denote by $s_1 \overset{(d)}{=} s_2$ (where (d) stands for equality in distribution). Since the space $\mathcal{S}(\mathbb{R}^d)$ is nuclear, the Minlos–Bochner theorem [16,29] gives a complete characterization of admissible characteristic functionals.

Theorem 1 (*Minlos–Bochner Theorem*). *A functional $\widehat{\mathscr{P}}$ on $\mathcal{S}(\mathbb{R}^d)$ is the characteristic functional of a generalized random process s if and only if it is continuous and positive-definite over $\mathcal{S}(\mathbb{R}^d)$ and satisfies $\widehat{\mathscr{P}}(0) = 1$.*

Lévy processes are random processes index by $\mathbb{R}$ with stationary and independent increments. They are deeply related to infinitely divisible random variables [36]. For the same reasons, there is a one-to-one correspondence between infinitely divisible laws and the family of Lévy noises. An infinitely divisible random variable X can be decomposed as $X = X_1 + \cdots + X_N$ for every $N \geq 1$ where the X_n are independent and identically distributed (i.i.d.). The characteristic function of an infinitely divisible random variable can be written as

$$\Phi_X(\xi) = \exp(\Psi(\xi)) \tag{6}$$

with Ψ a suitable continuous function [36, Section 7]. The function Ψ—the continuous log-characteristic function of the infinitely divisible random variable X—is called a *Lévy exponent* (also known as a characteristic exponent). We say moreover that *Ψ satisfies the ϵ-condition* if the moment $\mathbb{E}[|X|^\epsilon]$ of X is finite for some $\epsilon > 0$.

A Lévy exponent Ψ can be uniquely represented by its Lévy triplet (γ, σ^2, ν) as [36, Theorem 8.1]

$$\Psi(\xi) = \mathrm{i}\gamma\xi - \frac{\sigma^2\xi^2}{2} + \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^{\mathrm{i}\xi x} - 1 - \mathrm{i}\xi x \mathbb{1}_{|x|\le 1})\nu(\mathrm{d}x), \tag{7}$$

where $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$, and ν is a Lévy measure, that is, a measure on $\mathbb{R} \setminus \{0\}$ such that $\int_{\mathbb{R}\setminus\{0\}} \inf(1, x^2)\nu(\mathrm{d}x) < \infty$. This is the well-known Lévy–Khintchine decomposition. We remark also that $\Re\Psi \leq 0$.

Let $\boldsymbol{X} = (X_1, \ldots, X_N)$ be a vector of i.i.d. infinitely divisible random variables with common Lévy exponent Ψ. By independence, the characteristic function of $\boldsymbol{X}$ is

$$\Phi_{\boldsymbol{X}}(\boldsymbol{\xi}) = \exp\left(\sum_{n=1}^{N} \Psi(\xi_n)\right) \tag{8}$$

for every $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$. A Lévy noise can be seen as the generalization of this principle in the continuous domain, up to the replacement of the sum in (8) by an integral.

Definition 2. A *Lévy white noise* is a generalized random process w with characteristic functional of the form

$$\widehat{\mathscr{P}}_w(\varphi) = \exp\left(\int_{\mathbb{R}^d} \Psi(\varphi(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right) \tag{9}$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$, where Ψ is a Lévy exponent that satisfies the ϵ-condition.

I.M. Gelfand and N. Ya. Vilenkin have proved that the functional (9) is a valid characteristic functional on $\mathcal{D}(\mathbb{R}^d)$, the space of compactly supported and infinitely smooth functions, without the ϵ-condition on Ψ [16]. The Schwartz condition is sufficient to extend this result to $\mathcal{S}(\mathbb{R}^d)$ [12, Theorem 3]. Recently, R. Dalang and T. Humeau have shown that this condition is also necessary: A noise with Lévy exponent that does not satisfy the Schwartz condition is almost surely not in $\mathcal{S}'(\mathbb{R}^d)$ [9, Theorem 3.13].

A Lévy noise is stationary, in the sense that $w \overset{(d)}{=} w(\cdot - \boldsymbol{x}_0)$ for every $\boldsymbol{x}_0 \in \mathbb{R}^d$. It is moreover independent at every point, meaning that $\langle w, \varphi\rangle$ and $\langle w, \psi\rangle$ are independent whenever φ and $\psi \in \mathcal{S}(\mathbb{R}^d)$ have disjoint supports. In 1-D, we recover the usual notion of white noise, since w is the derivative in the sense of generalized functions of the Lévy process with the same Lévy exponent. This principle can be extended to any dimension $d \geq 2$: The d-dimensional Lévy noise is the weak derivative $\mathrm{D}_{x_1} \cdots \mathrm{D}_{x_d}\{s\}$ of the d-dimensional Lévy sheet s [9].

2.2. *Weighted Sobolev and Besov spaces*

Our goal is to characterize the smoothness of a Lévy white noise in terms of weighted Besov spaces. All our results related to Besov spaces require the corresponding intermediate result for Sobolev spaces which we introduce in Section 2.2.1.

2.2.1. *Weighted Sobolev spaces*

We set $\langle \boldsymbol{x}\rangle = \sqrt{1 + |\boldsymbol{x}|^2}$. The Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^d)$ is denoted by $\widehat{f}$. For $\tau \in \mathbb{R}$, we define L_τ (the *Bessel operator of order* τ) as the pseudo-differential operator with Fourier multiplier $\langle\cdot\rangle^\tau$. In Fourier domain, we write

$$\widehat{\mathrm{L}_\tau\{\varphi\}}(\boldsymbol{\omega}) := \langle\boldsymbol{\omega}\rangle^\tau\widehat{\varphi}(\boldsymbol{\omega}) \tag{10}$$

for every $\boldsymbol{\omega} \in \mathbb{R}^d$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$. When $\tau > 0$, the operator $\mathrm{I}_\tau = \mathrm{L}_{-\tau}$ is called a Bessel potential [17]. The operator L_τ is self-adjoint, linear, and continuous from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$, since its Fourier multiplier is infinitely smooth and bounded by a polynomial. It can therefore be extended as a linear and continuous operator from $\mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$.

Definition 3. Let $\tau, \rho \in \mathbb{R}$. The *Sobolev space of smoothness* τ is defined by

$$W_2^\tau(\mathbb{R}^d) := \left\{f \in \mathcal{S}'(\mathbb{R}^d),\ \mathrm{L}_\tau\{f\} \in L_2(\mathbb{R}^d)\right\} \tag{11}$$

and the *Sobolev space of smoothness* τ *and decay* ρ is

$$W_2^\tau(\mathbb{R}^d;\rho) := \left\{f \in \mathcal{S}'(\mathbb{R}^d),\ \langle\cdot\rangle^\rho f \in W_2^\tau(\mathbb{R}^d)\right\}. \tag{12}$$

We also set $L_2(\mathbb{R}^d;\rho) := W_2^0(\mathbb{R}^d;\rho)$.

We summarize now the basic properties on weighted Sobolev spaces that are useful for our work, with short proofs for the sake of completeness. More details can be found in [45]; in particular, in Chapter 6, a broader class of weighted spaces with their embedding relations is considered.

Proposition 1. *The following properties hold for weighted Sobolev spaces.*

- *For $\rho, \tau \in \mathbb{R}$, $W_2^\tau(\mathbb{R}^d;\rho)$ is a Hilbert space for the scalar product*

$$\langle f, g\rangle_{W_2^\tau(\mathbb{R}^d;\rho)} := \left\langle \mathrm{L}_\tau\{\langle\cdot\rangle^\rho f\}, \mathrm{L}_\tau\{\langle\cdot\rangle^\rho g\}\right\rangle_{L_2(\mathbb{R}^d)}. \tag{13}$$

  We denote by $\|f\|_{W_2^\tau(\mathbb{R}^d;\rho)} = \langle f, f\rangle^{1/2}_{W_2^\tau(\mathbb{R}^d;\rho)}$ the corresponding norm.
- *For $\rho \in \mathbb{R}$ fixed and for every $\tau_1 \leq \tau_2$, we have the continuous embedding*

$$W_2^{\tau_2}(\mathbb{R}^d;\rho) \subseteq W_2^{\tau_1}(\mathbb{R}^d;\rho). \tag{14}$$

- *For $\tau \in \mathbb{R}$ fixed and for every $\rho_1 \leq \rho_2$, we have the continuous embedding*

$$W_2^{\tau}(\mathbb{R}^d;\rho_2) \subseteq W_2^{\tau}(\mathbb{R}^d;\rho_1). \tag{15}$$

- *For $\rho, \tau \in \mathbb{R}$, the operator $\mathrm{L}_{\tau,\rho} : f \mapsto \langle\cdot\rangle^\rho \mathrm{L}_\tau\{f\}$ is an isometry from $L_2(\mathbb{R}^d)$ to $W_2^{-\tau}(\mathbb{R}^d;-\rho)$.*
- *The dual space of $W_2^\tau(\mathbb{R}^d;\rho)$ is $W_2^{-\tau}(\mathbb{R}^d;-\rho)$ for every $\tau, \rho \in \mathbb{R}$.*
- *We have the countable projective limit*

$$\mathcal{S}(\mathbb{R}^d) = \bigcap_{\tau,\rho\in\mathbb{R}} W_2^\tau(\mathbb{R}^d;\rho) = \bigcap_{n\in\mathbb{N}} W_2^n(\mathbb{R}^d;n). \tag{16}$$

- *We have the countable inductive limit*

$$\mathcal{S}'(\mathbb{R}^d) = \bigcup_{\tau,\rho\in\mathbb{R}} W_2^\tau(\mathbb{R}^d;\rho) = \bigcup_{n\in\mathbb{N}} W_2^{-n}(\mathbb{R}^d;-n). \tag{17}$$

Proof. The space $W_2^\tau(\mathbb{R}^d;\rho)$ inherits the Hilbertian structure of $L_2(\mathbb{R}^d)$. For $\tau_1 \leq \tau_2$ and $\rho_1 \leq \rho_2$, we have moreover the inequalities,

$$\|f\|_{W_2^{\tau_1}(\mathbb{R}^d;\rho)} \leq \|f\|_{W_2^{\tau_2}(\mathbb{R}^d;\rho)}, \tag{18}$$

$$\|f\|_{W_2^{\tau}(\mathbb{R}^d;\rho_1)} \leq \|f\|_{W_2^{\tau}(\mathbb{R}^d;\rho_2)}, \tag{19}$$

from which we deduce (14) and (15). The relation

$$\|\mathrm{L}_{\tau,\rho} f\|_{W_2^{-\tau}(\mathbb{R}^d;-\rho)} = \|\mathrm{L}_{-\tau}\{\langle\cdot\rangle^{-\rho}\mathrm{L}_{\tau,\rho} f\}\|_{L_2(\mathbb{R}^d)} = \|f\|_{L_2(\mathbb{R}^d)} \tag{20}$$

proves that $\mathrm{L}_{\tau,\rho}$ is an isometry. For every $f, g \in L_2(\mathbb{R}^d)$, we have that

$$\langle \mathrm{L}_\tau\{\langle\cdot\rangle^\rho f\}, \mathrm{L}_{-\tau}\{\langle\cdot\rangle^{-\rho} g\}\rangle_{L_2(\mathbb{R}^d)} = \langle f, g\rangle_{L_2(\mathbb{R}^d)}. \tag{21}$$

Since $W_2^\tau(\mathbb{R}^d;\rho) = \{\mathrm{L}_\tau\{\langle\cdot\rangle^\rho f\},\ f \in L_2(\mathbb{R}^d)\}$, we easily deduce the dual of $W_2^\tau(\mathbb{R}^d;\rho)$ from (21). Finally, we can reformulate the topology on $\mathcal{S}(\mathbb{R}^d)$ as (16). This implies directly (17). □

2.2.2. *Weighted Besov spaces*

Following H. Triebel [46], our definitions of weighted Besov spaces are based on wavelets. More traditionally, Besov spaces are introduced through the Fourier transform; see for instance [44]. The use of wavelets is equivalent and is more convenient for our purpose.

Let us first introduce the relevant wavelet bases. We denote by $j \geq 0$ the scaling index and $\boldsymbol{m} \in \mathbb{Z}^d$ the shifting index. Consider ψ_{F} and ψ_{M}, which are the father and mother wavelet of a wavelet basis for $L_2(\mathbb{R})$, respectively. We set $\mathrm{G}^0 = \{M, F\}^d$ and $\mathrm{G}^j = \mathrm{G}^0 \setminus (F, \ldots, F)$ for $j \geq 1$. For a gender $G = (G_1, \ldots, G_d) \in \mathrm{G}^0$ and for every $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we define

$$\psi_G(\boldsymbol{x}) = \prod_{i=1}^d \psi_{G_i}(x_i). \tag{22}$$

Proposition 2 (*Section 1.2.1, [46]*). *For every integer $r_0 \geq 0$, there exist compactly supported wavelets ψ_{F} and ψ_{M} with at least r_0 continuous derivatives such that*

$$\left\{\psi_{j,G,\boldsymbol{m}},\ j \geq 0, G \in \mathrm{G}^j, \boldsymbol{m} \in \mathbb{Z}^d\right\} \tag{23}$$

is an orthonormal basis of $L_2(\mathbb{R}^d)$, where

$$\psi_{j,G,\boldsymbol{m}} := 2^{jd/2}\psi_G(2^j \cdot - \boldsymbol{m}) \tag{24}$$

and ψ_G is defined according to (22).

Concretely, [46] considers separable Daubechies wavelets with the adequate regularity. For $\tau, \rho \in \mathbb{R}$ and $0 < p, q \leq \infty$, the *Besov sequence space* $b_{p,q}^\tau(\rho)$ is the collection of sequences

$$\boldsymbol{\lambda} = \{\lambda_{j,G,\boldsymbol{m}},\ j \geq 0, G \in \mathrm{G}^j, \boldsymbol{m} \in \mathbb{Z}^d\} \tag{25}$$

such that

$$\|\boldsymbol{\lambda}\|_{b_{p,q}^\tau(\rho)} := \left(\sum_{j\geq 0} 2^{jq(\tau - d/p)} \sum_{G \in \mathrm{G}^j} \left(\sum_{\boldsymbol{m}\in\mathbb{Z}^d} \langle 2^{-j}\boldsymbol{m}\rangle^{\rho p} |\lambda_{j,G,\boldsymbol{m}}|^p\right)^{q/p}\right)^{1/q}, \tag{26}$$

with the usual modifications when p and/or $q = \infty$.

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Definition 4. Let $\tau, \rho \in \mathbb{R}$ and $0 < p, q \leq \infty$. Fix

$$r_0 > \max(\tau, d(1/p - 1)_+ - \tau) \tag{27}$$

and set $(\psi_{j,G,\boldsymbol{m}})$ a wavelet basis of $L_2(\mathbb{R}^d)$ with regularity r_0. The *weighted Besov space* $B^\tau_{p,q}(\mathbb{R}^d; \rho)$ is the collection of generalized function $f \in \mathcal{S}'(\mathbb{R}^d)$ that can be written as

$$f = \sum_{j,G,\boldsymbol{m}} 2^{-jd/2} \lambda_{j,G,\boldsymbol{m}} \psi_{j,G,\boldsymbol{m}} \tag{28}$$

with $\boldsymbol{\lambda} = (\lambda_{j,G,\boldsymbol{m}}) \in b^\tau_{p,q}(\rho)$, where the convergence holds unconditionally in $\mathcal{S}'(\mathbb{R}^d)$.

This definition is usually introduced as a characterization of Besov spaces. When (28) occurs, the representation is unique and we have that [46, Theorem 1.26]

$$\lambda_{j,G,\boldsymbol{m}} = 2^{jd/2} \langle f, \psi_{j,G,\boldsymbol{m}} \rangle. \tag{29}$$

To measure a given Besov regularity (fixed p, q, τ, and ρ), we should select a wavelet with enough regularity for the wavelet coefficients to be well-defined for $f \in B^\tau_{p,q}(\mathbb{R}^d; \rho)$. This is the meaning of (27). Under this condition, and for $f \in B^\tau_{p,q}(\mathbb{R}^d; \rho)$, the quantity

$$\|f\|_{B^\tau_{p,q}(\mathbb{R}^d;\rho)} := \left(\sum_{j \geq 0} 2^{j(\tau - d/p + d/2)q} \sum_{G \in G^j} \left(\sum_{\boldsymbol{m} \in \mathbb{Z}^d} \langle 2^{-j} \boldsymbol{m} \rangle^{\rho p} |\langle f, \psi_{j,G,\boldsymbol{m}} \rangle|^p \right)^{q/p} \right)^{1/q} \tag{30}$$

is finite, with the usual modifications when p and/or $q = \infty$. The quantity (30) is a norm for $p, q \geq 1$, and a quasi-norm otherwise. In any case, the Besov space is complete for its (quasi-)norm, and is therefore a (quasi-)Banach space. We have moreover the equivalence [11, Theorem 4.2.2]

$$f \in B^\tau_{p,q}(\mathbb{R}^d; \rho) \Leftrightarrow \langle \cdot \rangle^\rho f \in B^\tau_{p,q}(\mathbb{R}^d) \tag{31}$$

with $B^\tau_{p,q}(\mathbb{R}^d) := B^\tau_{p,q}(\mathbb{R}^d; 0)$ the classical (non-weighted) Besov space. The family of weighted Besov spaces includes the weighted Sobolev spaces due to the relation [11, Section 2.2.2]

$$B^\tau_{2,2}(\mathbb{R}^d; \rho) = W^\tau_2(\mathbb{R}^d; \rho). \tag{32}$$

Weighted Besov spaces are embedded, as we show in Proposition 3.

Proposition 3. *We fix* $\tau_0, \tau_1, \rho_0, \rho_1 \in \mathbb{R}$ *and* $0 < p_0, q_0, p_1, q_1 \leq \infty$. *We assume that*

$$\tau_0 > \tau_1 \quad \text{and} \quad \rho_0 \geq \rho_1. \tag{33}$$

If, moreover, we have that

$$p_0 \leq p_1 \quad \text{and} \quad \tau_0 - \tau_1 \geq d \left(\frac{1}{p_0} - \frac{1}{p_1} \right) \tag{34}$$

or

$$p_1 \leq p_0 \quad \text{and} \quad \rho_0 - \rho_1 > d \left(\frac{1}{p_1} - \frac{1}{p_0} \right), \tag{35}$$

then we have the continuous embedding

$$B^{\tau_0}_{p_0,q_0}(\mathbb{R}^d; \rho_0) \subseteq B^{\tau_1}_{p_1,q_1}(\mathbb{R}^d; \rho_1). \tag{36}$$

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Proof. A proof of the sufficiency of (34) can be found in [11, Section 4.2.3]. However, we could not find any precise statement of embeddings between Besov spaces for $p_1 \leq p_0$ in the literature, so we provide our own proof for the sufficiency of (35).

First, the parameter q is dominated by parameters τ and p in the sense that, for every $\tau \geq 0$, $\epsilon > 0$, and $0 < p, q, r \leq \infty$, we have the embedding [47, Proposition 2, Section 2.3.2]

$$B^{\tau+\epsilon}_{p,q}(\mathbb{R}^d; \rho) \subseteq B^{\tau}_{p,r}(\mathbb{R}^d; \rho). \tag{37}$$

Note that Triebel considers unweighted spaces in [47], but the extension to the weighted case is obvious. Hence, we restrict ourselves to the case $q_0 = q_1 = q$. Fix $\boldsymbol{\lambda} = \{\lambda_{j,G,\boldsymbol{m}},\ j \geq 0, G \in \mathrm{G}^j, \boldsymbol{m} \in \mathbb{Z}^d\}$. Due to the Hölder inequality, as soon as $1/a + 1/b = 1$, we have, for every $j \geq 0$ and $G \in \mathrm{G}^j$, that

$$\sum_{\boldsymbol{m}\in\mathbb{Z}^d} \langle 2^{-j}\boldsymbol{m}\rangle^{\rho_1 p_1} |\lambda_{j,G,\boldsymbol{m}}|^{p_1}$$
$$\leq \left(\sum_{\boldsymbol{m}\in\mathbb{Z}^d} \langle 2^{-j}\boldsymbol{m}\rangle^{(\rho_1-\rho_0)p_1 b}\right)^{1/b} \left(\sum_{\boldsymbol{m}\in\mathbb{Z}^d} \langle 2^{-j}\boldsymbol{m}\rangle^{\rho_0 p_1 a} |\lambda_{j,G,\boldsymbol{m}}|^{p_1 a}\right)^{1/a}. \tag{38}$$

We choose $a = p_0/p_1 \geq 1$, thus $(\rho_1 - \rho_0)p_1 b = (\rho_1 - \rho_0)/(1/p_1 - 1/p_0) < -d$ by using (35), and $\sum_{\boldsymbol{m}\in\mathbb{Z}^d} \langle 2^{-j}\boldsymbol{m}\rangle^{(\rho_1-\rho_0)p_1 b} < \infty$. Since $ap_1 = p_0$, we rewrite (38) as

$$\left(\sum_{\boldsymbol{m}\in\mathbb{Z}^d} \langle 2^{-j}\boldsymbol{m}\rangle^{\rho_1 p_1} |\lambda_{j,G,\boldsymbol{m}}|^{p_1}\right)^{1/p_1} \leq C \left(\sum_{\boldsymbol{m}\in\mathbb{Z}^d} \langle 2^{-j}\boldsymbol{m}\rangle^{\rho_0 p_0} |\lambda_{j,G,\boldsymbol{m}}|^{p_0}\right)^{1/p_0} \tag{39}$$

with $C > 0$ a finite constant. Using (26), this implies that $\|\boldsymbol{\lambda}\|_{b^{\tau_1}_{p_1,q}(\rho_1)} \leq C' \|\boldsymbol{\lambda}\|_{b^{\tau_0}_{p_0,q}(\rho_0)}$ and consequently the corresponding embedding between Besov sequence spaces. Finally, (36) is a consequence of the isomorphism between Besov sequence spaces and Besov function spaces in Definition 4 (see [46, Theorem 1.26] for more details on the isomorphism). We let the reader adapt the proof when p and/or q are infinite. □

If the only knowledge provided to us is that the generalized function f is in $\mathcal{S}'(\mathbb{R}^d)$, then this is not enough to set the regularity r_0 of the wavelet used to characterize the Besov smoothness of f. However, if we have additional information on f, for instance its inclusion in a Sobolev space, then the situation is different. Proposition 4 gives a wavelet-domain criterion to determine if a generalized function f, known to be in $W_2^{\tau_0}(\mathbb{R}^d; \rho_0)$, is actually in a given Besov space $B^{\tau}_{p,q}(\mathbb{R}^d; \rho)$. Moreover, we also know that any $f \in \mathcal{S}'(\mathbb{R}^d)$ is in some Sobolev space $W_2^{\tau_0}(\mathbb{R}^d; \rho_0)$ because of (17).

Proposition 4. *Let $\tau, \tau_0, \rho, \rho_0 \in \mathbb{R}$ and $0 < p, q \leq \infty$. We set*

$$u > \max(|\tau_0|, |\tau - d(1/p - 1/2)_+|). \tag{40}$$

Then, the generalized function $f \in W_2^{\tau_0}(\mathbb{R}; \rho_0)$ is in $B^{\tau}_{p,q}(\mathbb{R}^d; \rho)$ if and only if

$$\sum_{j\geq 0} 2^{j(\tau-d/p+d/2)q} \sum_{G\in\mathrm{G}^j} \left(\sum_{\boldsymbol{m}\in\mathbb{Z}^d} \langle 2^{-j}\boldsymbol{m}\rangle^{\rho p} |\langle f, \psi_{j,G,\boldsymbol{m}}\rangle|^p\right)^{q/p} < \infty, \tag{41}$$

with $(\psi_{j,G,\boldsymbol{m}})$ a wavelet basis of $L_2(\mathbb{R}^d)$ of regularity u, with the usual modifications when p and/or $q = \infty$.

Proof. Let $\tau_1 < \min(\tau_0, \tau - d(1/p - 1/2)_+)$ and $\rho_1 \leq \min(\rho_0, \rho - d(1/p - 1/2)_+)$. Then, according to Proposition 3, we have the embeddings

$$B^{\tau}_{p,q}(\mathbb{R}^d; \rho) \subseteq W_2^{\tau_1}(\mathbb{R}^d; \rho_1) \quad \text{and} \quad W_2^{\tau_0}(\mathbb{R}^d; \rho_0) \subseteq W_2^{\tau_1}(\mathbb{R}^d; \rho_1).$$

Condition (40) implies that we can apply Definition 4 into the space $W_2^{\tau_1}(\mathbb{R}^d; \rho_1)$. In particular, if $(\psi_{j,G,\boldsymbol{m}})$ is the wavelet basis of Definition 4 with regularity u, and for every function $f \in W_2^{\tau_1}(\mathbb{R}^d; \rho_1)$, then the wavelet coefficients $\langle f, \psi_{j,G,\boldsymbol{m}}\rangle$ are well-defined. Moreover, we have the characterization

$$f \in B^{\tau}_{p,q}(\mathbb{R}^d; \rho) \Leftrightarrow \|f\|_{B^{\tau}_{p,q}(\mathbb{R}^d;\rho)} < \infty$$

for $f \in W_2^{\tau_1}(\mathbb{R}^d; \rho_1)$ and, therefore, for $f \in W_2^{\tau_1}(\mathbb{R}^d; \rho_0)$. $\square$

3. Moment estimates for the Lévy noise

Our goal in this section is to obtain bounds for the pth moments of the random variable $\langle w, \varphi\rangle$, where w is a Lévy noise and $\varphi \in \mathcal{S}(\mathbb{R}^d)$. The bounds are related to the moments of φ. For instance, for a symmetric α-stable noise w_α, it is known [13, Lemma 2] that, for $p > 0$,

$$\mathbb{E}\left[|\langle w_\alpha, \varphi\rangle|^p\right] = C_{p,\alpha}\|\varphi\|_\alpha^p, \tag{42}$$

where $C_{p,\alpha}$ is a finite constant if and only if $\alpha = 2$ (Gaussian case), or $p < \alpha < 2$ (non-Gaussian case).

3.1. Indices of a Lévy noise

To generalize (42) for non-stable white noise, we consider the indices introduced in [4,31], which are classical tools to estimate the moments of Lévy processes [10,24,27].

Definition 5. Let Ψ be a Lévy exponent. We consider the two intervals

$$I_0 = \left\{ p \in [0,2], \limsup_{|\xi|\to 0} \frac{\Psi(\xi)}{|\xi|^p} < \infty \right\}, \tag{43}$$

$$I_\infty = \left\{ p \in [0,2], \limsup_{|\xi|\to \infty} \frac{\Psi(\xi)}{|\xi|^p} < \infty \right\}. \tag{44}$$

The *indices* are defined by

$$\beta_0 = \sup I_0, \qquad \beta_\infty = \inf I_\infty. \tag{45}$$

Proposition 5. *Consider a Lévy exponent Ψ with intervals I_0 and I_∞ as in (43) and (44). Then, for $\tilde{\beta}_0 \in I_0$ and $\tilde{\beta}_\infty \in I_\infty$, we have the inequality*

$$\int_{\mathbb{R}^d} |\Psi(\varphi(\boldsymbol{x}))| \mathrm{d}\boldsymbol{x} \leq C\left(\|\varphi\|_{\tilde{\beta}_0}^{\tilde{\beta}_0} + \|\varphi\|_{\tilde{\beta}_\infty}^{\tilde{\beta}_\infty}\right) \tag{46}$$

for all $\varphi \in L_{\tilde{\beta}_0}(\mathbb{R}^d) \bigcap L_{\tilde{\beta}_\infty}(\mathbb{R}^d)$ and some constant $C > 0$.

Proof. The functions $\xi \mapsto |\Psi(\xi)|$ and $\xi \mapsto |\xi|^{\tilde{\beta}_0} + |\xi|^{\tilde{\beta}_\infty}$ are both continuous, the second one being non-vanishing on $\mathbb{R} \setminus \{0\}$ and dominating the first one at zero and at infinity up to some constant. Therefore, there exists a constant $C > 0$ that satisfies

$$|\Psi(\xi)| \leq C \left(|\xi|^{\tilde{\beta}_0} + |\xi|^{\tilde{\beta}_\infty} \right).$$

Integrating the latter equation over $\boldsymbol{x} \in \mathbb{R}^d$ with $\xi = \varphi(\boldsymbol{x})$, we obtain (46). □

3.2. Moment estimates for $\langle w, \varphi \rangle$

We estimate the moments of a random variable by relaying the fractional moments to the characteristic function. Proposition 6 can be found for instance in [10,26,28] with some variations. For the sake of completeness, we recall the proof, similar to the one of [10].

Proposition 6. *For a random variable X with characteristic function Φ_X and $0 < p < 2$, we have the relation*

$$\mathbb{E}[|X|^p] = c_p \int_{\mathbb{R}} \frac{1 - \Re(\Phi_X)(\xi)}{|\xi|^{p+1}} \mathrm{d}\xi \in [0, \infty], \tag{47}$$

for some finite constant $c_p > 0$, where $\Re(z)$ denotes the real part of $z \in \mathbb{C}$.

Proof. For $p \in (0, 2)$, we have, for every $x \in \mathbb{R}$,

$$h(x) = \int_{\mathbb{R}} (1 - \cos(x\xi)) \frac{\mathrm{d}\xi}{|\xi|^{p+1}} = \left(\int_{\mathbb{R}} (1 - \cos(u)) \frac{\mathrm{d}u}{|u|^{p+1}} \right) |x|^p, \tag{48}$$

which is obtained by the change of variable $u = x\xi$. Applying this relation to $x = X$ and denoting $c_p = \left(\int_{\mathbb{R}} (1 - \cos(u)) \frac{\mathrm{d}u}{|u|^{p+1}} \right)^{-1}$, we have by Fubini's theorem that

$$\mathbb{E}[|X|^p] = c_p \mathbb{E}\left[\int_{\mathbb{R}} (1 - \cos(\xi X)) \frac{\mathrm{d}\xi}{|\xi|^{p+1}} \right] \tag{49}$$

$$= c_p \int_{\mathbb{R}} (1 - \Re(\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi X}])) \frac{\mathrm{d}\xi}{|\xi|^{p+1}} \tag{50}$$

$$= c_p \int_{\mathbb{R}} \frac{1 - \Re(\Phi_X)(\xi)}{|\xi|^{p+1}} \mathrm{d}\xi. \quad \square \tag{51}$$

Theorem 2. *Consider a Lévy noise w with indices β_0 and β_∞. Then, for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $0 < p < \beta_0$, we have the inequality*

$$\mathbb{E}\left[|\langle w, \varphi \rangle|^p \right] \leq C \left(\|\varphi\|_{\tilde{\beta}_0}^p + \|\varphi\|_{\tilde{\beta}_\infty}^p \right) \tag{52}$$

for some constant $C > 0$, with $\tilde{\beta}_0 \in I_0$, $\tilde{\beta}_\infty \in I_\infty$, and $p < \tilde{\beta}_0, \tilde{\beta}_\infty$. Moreover, the result is still valid for $p = \tilde{\beta}_0 = \tilde{\beta}_\infty = 2$ if $\beta_0 = 2 \in I_0$ (finite-variance case).

Proof. We start with a preliminary property: There exists a constant $C > 0$ such that, for every $z \in \mathbb{C}$ with $\Re(z) \leq 0$, we have that

$$|1 - \mathrm{e}^z| \leq C \left(1 - \mathrm{e}^{-|z|} \right). \tag{53}$$

Indeed, the function $h(z) = \frac{|1-\mathrm{e}^{z}|}{1-\mathrm{e}^{-|z|}}$ is easily shown to be bounded for $\Re(z) \leq 0$ by a continuity argument.

Defining $X = \langle w, \varphi \rangle$, the characteristic function of X is

$$\Phi_X(\xi) = \exp\left(\int_{\mathbb{R}^d} \Psi(\xi \varphi(\boldsymbol{x})) \mathrm{d}\boldsymbol{x} \right). \tag{54}$$

Moreover, from Proposition 5, we have that

$$\int_{\mathbb{R}^d} |\Psi(\xi\varphi)| \leq C \left(\|\varphi\|_{\tilde{\beta}_0}^{\tilde{\beta}_0} |\xi|^{\tilde{\beta}_0} + \|\varphi\|_{\tilde{\beta}_\infty}^{\tilde{\beta}_\infty} |\xi|^{\tilde{\beta}_\infty} \right). \tag{55}$$

We therefore have that

$$\begin{aligned} 1 - \Re(\Phi_X)(\xi) &\leq |1 - \Phi_X(\xi)| \\ &\overset{(i)}{\leq} C \left(1 - \exp\left(- \left| \int \Psi(\xi\varphi) \right| \right) \right) \\ &\overset{(ii)}{\leq} C \left(1 - \exp\left(- \int |\Psi(\xi\varphi)| \right) \right) \\ &\overset{(iii)}{\leq} C' \left(1 - \exp(-\|\varphi\|_{\tilde{\beta}_0}^{\tilde{\beta}_0} |\xi|^{\tilde{\beta}_0}) \exp(-\|\varphi\|_{\tilde{\beta}_\infty}^{\tilde{\beta}_\infty} |\xi|^{\tilde{\beta}_\infty}) \right) \\ &\overset{(iv)}{\leq} C' \left(\left(1 - \exp(-\|\varphi\|_{\tilde{\beta}_0}^{\tilde{\beta}_0} |\xi|^{\tilde{\beta}_0}) \right) + \left(1 - \exp(-\|\varphi\|_{\tilde{\beta}_\infty}^{\tilde{\beta}_\infty} |\xi|^{\tilde{\beta}_\infty}) \right) \right), \end{aligned} \tag{56}$$

where (i) comes from (53), (ii) and (iii) from the fact that $x \mapsto 1 - \mathrm{e}^{-x}$ is increasing, (iii) from (55), and (iv) from the remark that $(1 - xy) \leq (1 - x) + (1 - y)$. Finally, by a simple change of variable that for $\alpha \in (0, 2)$ and $p < \alpha$, there exists a constant $c_{p,\alpha}$ such that

$$\int_{\mathbb{R}} \frac{1 - \mathrm{e}^{-|x\xi|^\alpha}}{|\xi|^{p+1}} \mathrm{d}\xi = c_{p,\alpha} |x|^p. \tag{57}$$

Applying this result with $x = \|\varphi\|_{\tilde{\beta}_0}$, $\alpha = \tilde{\beta}_0$ and $x = \|\varphi\|_{\tilde{\beta}_\infty}$, $\alpha = \tilde{\beta}_\infty$, respectively, we deduce using (47) that

$$\mathbb{E}[|X|^p] = c_p \int_{\mathbb{R}} \frac{1 - \Re(\Phi_X)(\xi)}{|\xi|^{p+1}} \mathrm{d}\xi \leq C'' \left(\|\varphi\|_{\tilde{\beta}_0}^p + \|\varphi\|_{\tilde{\beta}_\infty}^p \right), \tag{58}$$

ending the proof.

The finite-variance case (for which $\beta_0 = 2 \in I_0$) cannot be deduced with the same arguments, since (47) is not valid any more. However, we know in this case that

$$\mathbb{E}[\langle w, \varphi \rangle^2] = \sigma^2 \|\varphi\|_2^2 + \gamma^2 \left(\int_{\mathbb{R}^d} \varphi \right)^2 \leq \sigma^2 \|\varphi\|_2^2 + \gamma^2 \|\varphi\|_1^2, \tag{59}$$

where σ^2 and γ are the variance and the mean of the infinitely divisible random variable with the same Lévy exponent as w [49, Proposition 4.15], respectively. Hence, the result is still valid. □

We take advantage of Theorem 2 in a slightly less general form and apply it to wavelets, which are rescaled versions of an initial function at resolution $j = 0$. Specifically, for $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $j \geq 0$, and $\boldsymbol{m} \in \mathbb{Z}^d$, we set $\varphi_{j,\boldsymbol{m}} = 2^{jd/2} \varphi(2^j \cdot - \boldsymbol{m})$.

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Corollary 1. *Let w be a Lévy noise with indices β_0 and β_∞. We assume either that $\beta_\infty < \beta_0$, or that $\beta_\infty = \beta_0 \in I_\infty \bigcap I_0$. We fix $p < \beta \in I_0 \bigcap I_\infty$. Then, there exists a constant C such that, for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $j \geq 0$, and $\boldsymbol{m} \in \mathbb{Z}^d$,*

$$\mathbb{E}\left[|\langle w, \varphi_{j,\boldsymbol{m}}\rangle|^p\right] \leq C 2^{jdp(1/2-1/\beta)} \|\varphi\|_\beta^p. \tag{60}$$

Moreover, this result is still valid if $p = \beta = 2 \in I_0$.

Proof. Remark first that the assumptions on β_0 and β_∞ imply that $I_0 \bigcap I_\infty \neq \emptyset$. We apply Theorem 2 with $\tilde{\beta}_\infty = \tilde{\beta}_0 = \beta$. In particular, we have that $\mathbb{E}\left[|\langle w, \varphi_{j,\boldsymbol{m}}\rangle|^p\right] \leq C\|\varphi_{j,\boldsymbol{m}}\|_\beta^p$. The result follows from the relation

$$\|\varphi_{j,\boldsymbol{m}}\|_\beta^p = 2^{jdp/2}\left(\int_{\mathbb{R}^d} |\varphi(2^j\boldsymbol{x} - \boldsymbol{m})|^\beta \mathrm{d}\boldsymbol{x}\right)^{p/\beta} = 2^{jdp(1/2-1/\beta)}\|\varphi\|_\beta^p, \tag{61}$$

the last equality being obtained by the change of variable $\boldsymbol{y} = 2^j\boldsymbol{x} - \boldsymbol{m}$. The result is still valid for $p = \beta = 2$ for which we can still apply Theorem 2. □

3.3. Application of moment estimates to the extension of $\langle w, \varphi\rangle$ for non-smooth functions

A generalized random process s is a random variable from Ω to $\mathcal{S}'(\mathbb{R}^d)$. Alternatively, it can be seen as a linear and continuous map[1] from $\mathcal{S}(\mathbb{R}^d)$ to the space $L_0(\Omega)$ of real random variables, that associates to $\varphi \in \mathcal{S}(\mathbb{R}^d)$ the random variable $\langle s, \varphi\rangle$. The space $L_0(\Omega)$ is a Fréchet space associated with the convergence in probability. We also define the spaces $L_p(\Omega)$ for $0 < p < \infty$ associated for $p \geq 1$ ($p < 1$, respectively) with the norm (the quasi-norm, respectively) $\|X\|_{L_p(\Omega)} = (\mathbb{E}[|X|^p])^{1/p}$. See [22, Section 2.2] for more details.

To measure the Besov regularity of a Lévy noise, we shall consider random variables $\langle w, \varphi\rangle$ for test functions φ not in $\mathcal{S}(\mathbb{R}^d)$. We handle this by extending the domain of test functions through which one can observe a generalized random process.

Lemma 1. *Let $0 < p, \beta < \infty$. Consider a generalized random process s. We assume that, for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$,*

$$\mathbb{E}[|\langle s, \varphi\rangle|^p] \leq C\|\varphi\|_\beta^p \tag{62}$$

for some constant $C > 0$. Then, we can extend s as a linear and continuous map from $L_\beta(\mathbb{R}^d)$ to $L_0(\Omega)$. Moreover, (62) remains valid for $\varphi \in L_\beta(\mathbb{R}^d)$.

Before proving this result, we remark that it immediately implies Corollary 2.

Corollary 2. *Under the conditions of Corollary 1, we can extend $\langle w, \varphi\rangle$ for $\varphi \in L_\beta(\mathbb{R}^d)$. Moreover, (60) remains valid for any $\varphi \in L_\beta(\mathbb{R}^d)$.*

Proof of Lemma 1. The result is deduced by applying a standard density argument. Specifically, since $\mathcal{S}(\mathbb{R}^d)$ is dense in $L_\beta(\mathbb{R}^d)$ (well-known for $\beta \geq 1$ and easily extended for $\beta < 1$), we can approximate a function $\varphi \in L_\beta(\mathbb{R}^d)$ by a sequence (φ_n) of functions in $\mathcal{S}(\mathbb{R}^d)$. Then, $(\langle s, \varphi_n\rangle)$

[1] This is not as obvious as it might seem in infinite-dimension, and is again due to the nuclear structure of $\mathcal{S}(\mathbb{R}^d)$. For the links between E'-valued random variables and linear functionals from E to $L_0(\Omega)$ (with E' the dual of E), see [22, Section 2.3], in particular Theorems 2.3.1 and 2.3.2.

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is a Cauchy sequence in $L_p(\Omega)$ and therefore converges to some random variable $\langle s, \varphi \rangle$, due to the relation

$$\mathbb{E}[|\langle s, \varphi_n \rangle - \langle s, \varphi_m \rangle|^p] = \mathbb{E}[|\langle s, \varphi_n - \varphi_m \rangle|^p] \leq C \|\varphi_n - \varphi_m\|_\beta^p. \tag{63}$$

We easily show that the limit does not depend on the sequence (φ_n) so that $\langle s, \varphi \rangle$ is uniquely defined. Finally, (62) is still valid for $\varphi \in L_\beta(\mathbb{R}^d)$ by continuity of s from $L_\beta(\mathbb{R}^d)$ to $L_p(\Omega)$. □

4. Measurability of weighted Besov spaces

A generalized random process is a measurable function from Ω to $\mathcal{S}'(\mathbb{R}^d)$, endowed with the cylindrical σ-field $\mathcal{B}_c(\mathcal{S}'(\mathbb{R}^d))$. In the next sections, we shall investigate in which Besov space (local or weighted) is a given Lévy noise. Here, we first show that this question is meaningful in the sense that any Besov space $B^\tau_{p,q}(\mathbb{R}^d; \rho)$ is measurable in $\mathcal{S}'(\mathbb{R}^d)$.

Proposition 7. *For every $0 < p, q \leq \infty$ and $\tau, \rho \in \mathbb{R}$, we have that*

$$B^\tau_{p,q}(\mathbb{R}^d; \rho) \in \mathcal{B}_c\left(\mathcal{S}'(\mathbb{R}^d)\right), \tag{64}$$

with $\mathcal{B}_c\left(\mathcal{S}'(\mathbb{R}^d)\right)$ the cylindrical σ-field on $\mathcal{S}'(\mathbb{R}^d)$.

The proof of this result is very similar to the one of [13, Theorem 4], except we work now over $\mathbb{R}^d$ and deal with weights. In particular, we shall rely on [13, Lemma 1].

Proof. We obtain the desired result in three steps. We treat the case $p, q < \infty$ and let the reader adapt the proof for p and/or $q = \infty$.

- First, we show that $W^\tau_2(\mathbb{R}^d; \rho) \in \mathcal{B}_c\left(\mathcal{S}'(\mathbb{R}^d)\right)$ for every $\tau, \rho \in \mathbb{R}$. Let $(h_n)_{n \in \mathbb{N}}$ be an orthonormal basis of $L_2(\mathbb{R}^d)$, with $h_n \in \mathcal{S}(\mathbb{R}^d)$ for all $n \geq 0$. (We can for instance consider the Hermite functions, based on Hermite polynomials, see [42, Section 2] or [22, Section 1.3] for the definitions.) The interest of having basis functions in $\mathcal{S}(\mathbb{R}^d)$ is that we have the characterization

$$L_2(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d), \sum_{n \in \mathbb{N}} |\langle f, h_n \rangle|^2 < \infty \right\}. \tag{65}$$

  More generally, with the notations of Section 2.2.1, $f \in W^\tau_2(\mathbb{R}^d; \rho)$ if and only if $\mathrm{L}_\tau\{\langle \cdot \rangle^\rho f\} \in L_2(\mathbb{R}^d)$, from which we deduce that

$$W^\tau_2(\mathbb{R}^d; \rho) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d), \sum_{n \in \mathbb{N}} |\langle f, \langle \cdot \rangle^\rho \mathrm{L}_\tau\{h_n\} \rangle|^2 < \infty \right\}. \tag{66}$$

  We can therefore apply [13, Lemma 1] with $\alpha = 2$, $S = \mathbb{N}$, and $\varphi_n = \langle \cdot \rangle^\rho \mathrm{L}_\tau\{h_n\}$, to deduce that $W^\tau_2(\mathbb{R}^d; \rho) \in \mathcal{B}_c(\mathcal{S}'(\mathbb{R}^d))$.
- For any $\tau, \rho \in \mathbb{R}$, the cylindrical σ-field of $W^\tau_2(\mathbb{R}^d; \rho)$ is the σ-field $\mathcal{B}_c(W^\tau_2(\mathbb{R}^d; \rho))$ generated by the sets

$$\left\{ u \in W^\tau_2(\mathbb{R}^d; \rho), \ \ (\langle u, \varphi_1 \rangle, \ldots, \langle u, \varphi_n \rangle) \in B \right\}, \tag{67}$$

  where $N \geq 1$, $\varphi_1, \ldots, \varphi_N \in W^{-\tau}_2(\mathbb{R}^d; -\rho)$, and B is a Borelian subset of $\mathbb{R}^N$. Then, $W^\tau_2(\mathbb{R}^d; -\rho) \in \mathcal{B}_c(\mathcal{S}'(\mathbb{R}^d))$ implies that

$$\mathcal{B}_c(W^\tau_2(\mathbb{R}^d; \rho)) \subset \mathcal{B}_c(\mathcal{S}'(\mathbb{R}^d)). \tag{68}$$

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- Finally, we show that $B^{\tau}_{p,q}(\mathbb{R}^d;\rho) \in \mathcal{B}_c(W_2^{\tau_1}(\mathbb{R}^d;\rho_1))$ for some $\tau_1, \rho_1 \in \mathbb{R}$. Coupled with (68), we deduce (64).

  Fix $\tau_1 \leq \tau + d\,(1/2 - 1/p)$ and $\rho_1 < \rho + d\,(1/p - 1/2)$. According to Proposition 3, we have the embedding $B^{\tau}_{p,q}(\mathbb{R}^d;\rho) \subseteq W_2^{\tau_1}(\mathbb{R}^d;\rho_1)$. Now, we can rewrite Proposition 4 (with $\tau_0 = \tau_1$ and $\rho_0 = \rho_1$) as

$$B^{\tau}_{p,q}(\mathbb{R}^d;\rho) = \Bigg\{ f \in W_2^{\tau_1}(\mathbb{R}^d;\rho_1), \\ \sum_{j,G} \left(\sum_{\boldsymbol{m}} |\langle f, 2^{j(\tau - d/p + d/2)} \langle 2^{-j}\boldsymbol{m}\rangle^{\rho} \psi_{j,G,\boldsymbol{m}} \rangle|^p \right)^{q/p} < \infty \Bigg\}. \quad (69)$$

Q4 Again, we apply [13, Lemma 1] with $S = \{(j,G), j \in \mathbb{Z}, G \in \mathrm{G}_j\}$, $n = (j,G)$, $T_n = T_{(j,G)} = \mathbb{Z}^d$, $\varphi_{n,m} = \varphi_{j,G,\boldsymbol{m}} = \langle 2^{j(\tau-d/p+d/2)} \langle 2^{-j}\boldsymbol{m}\rangle^{\rho} \psi_{j,G,\boldsymbol{m}} \rangle$, $\alpha = p$, and $\beta = q/p$, to deduce that $B^{\tau}_{p,q}(\mathbb{R}^d;\rho) \in \mathcal{B}_c(W_2^{\tau_1}(\mathbb{R}^d;\rho_1))$. Remark that, strictly speaking, Lemma 1 in [13] is stated for T_n finite, but the proof is easily adapted to T_n countable. $\square$

Proposition 7 suggests that the framework of generalized random processes is particularly well-suited to addressing regularity issues. By comparison, we recall that the space $\mathcal{C}(\mathbb{R}^d)$ of continuous functions is not measurable with respect to the topological σ-field on the space of (pointwise) functions from $\mathbb{R}^d$ to $\mathbb{R}$, while we have that

$$\mathcal{C}(\mathbb{R}^d) \in \mathcal{B}_c(\mathcal{D}'(\mathbb{R}^d)), \quad (70)$$

which is the cylindrical σ-field of the space of generalized functions (not necessarily tempered) [14, Proposition III.3.3]. See [7] for a discussion on the measurability of function spaces and the advantages of generalized random processes.

5. The Lévy noise on weighted Sobolev spaces

In order to characterize the Besov smoothness of a Lévy noise, we first obtain information on their Sobolev smoothness.

Proposition 8. *A Lévy noise w with indices $\beta_0 > 0$ and β_∞ is in the weighted Sobolev space $W_2^{-\tau}(\mathbb{R}^d;-\rho)$ if*

$$\rho > \frac{d}{\beta_0} \quad \text{and} \quad \tau > \frac{d}{2}. \quad (71)$$

Proof. As we have seen in Proposition 1, we have the countable projective limit

$$\mathcal{S}(\mathbb{R}^d) = \bigcap_{\tau,\rho \in \mathbb{R}} W_2^{\tau}(\mathbb{R}^d;\rho) = \bigcap_{n \in \mathbb{N}} W_2^{n}(\mathbb{R}^d;n).$$

We are in the context of [19, Theorem A.2]. It implies in particular that, if, for some $\rho_0 \in \mathbb{R}$,

- the characteristic functional $\widehat{\mathscr{P}}_w$ of w is continuous over $L_2(\mathbb{R}^d;\rho_0)$, and
- the identity operator I is Hilbert–Schmidt from $W_2^{\tau}(\mathbb{R}^d;\rho)$ to $L_2(\mathbb{R}^d;\rho_0)$,

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then $w \in W_2^{-\tau}(\mathbb{R}^d, -\rho) = \left(W_2^{\tau}(\mathbb{R}^d; \rho)\right)'$ almost surely. For the rest of the proof, we therefore fix $\rho > \frac{d}{\beta_0}$ and $\tau > \frac{d}{2}$. We also set ρ_0 such that

$$d\left(\frac{1}{\beta_0} - \frac{1}{2}\right) < \rho_0 < \rho - \frac{d}{2}. \tag{72}$$

The lower bound on ρ_0 will imply the continuity of $\widehat{\mathscr{P}}_w$ on $L_2(\mathbb{R}^d; \rho_0)$ while the upper bound will be sufficient to ensure that the identity is Hilbert–Schmidt.

Continuity of $\widehat{\mathscr{P}}_w$. Fix $\epsilon > 0$ small enough such that

$$\beta_0 - \epsilon > 0 \quad \text{and} \quad \rho_0 > d\left(\frac{1}{\beta_0 - \epsilon} - \frac{1}{2}\right). \tag{73}$$

Applying Proposition 5 with $\tilde{\beta}_\infty = 2$ and $\tilde{\beta}_0 = \beta_0 - \epsilon$, we deduce that

$$\int_{\mathbb{R}^d} |\Psi(\varphi(\boldsymbol{x}))| \mathrm{d}\boldsymbol{x} \leq C\left(\|\varphi\|_{\tilde{\beta}_0}^{\tilde{\beta}_0} + \|\varphi\|_2^2\right). \tag{74}$$

Since $\rho_0 > d\left(\frac{1}{\beta_0} - \frac{1}{2}\right) \geq 0$, we have that $\|\varphi\|_{L_2(\mathbb{R}^d)}^2 \leq \|\varphi\|_{L_2(\mathbb{R}^d;\rho_0)}^2$. Moreover, using the Hölder inequality, we get

$$\|\varphi\|_{\tilde{\beta}_0}^{\tilde{\beta}_0} = \int_{\mathbb{R}^d} |\varphi(\boldsymbol{x})|^{\tilde{\beta}_0} \mathrm{d}\boldsymbol{x} \leq \int_{\mathbb{R}^d} (|\varphi(\boldsymbol{x})|^{\tilde{\beta}_0} \langle \boldsymbol{x}\rangle^{\rho_0 \tilde{\beta}_0})^p \mathrm{d}\boldsymbol{x} \int_{\mathbb{R}^d} \langle \boldsymbol{x}\rangle^{-\rho_0 \tilde{\beta}_0 q} \mathrm{d}\boldsymbol{x} \tag{75}$$

for any $1 \leq p, q \leq \infty$ such that $1/p + 1/q = 1$. Setting $p = 2/\tilde{\beta}_0 \geq 1$, we have $q = \frac{2}{2 - \tilde{\beta}_0}$. Therefore,

$$\|\varphi\|_{\tilde{\beta}_0}^{\tilde{\beta}_0} \leq \int_{\mathbb{R}^d} \left(|\varphi(\boldsymbol{x})| \langle \boldsymbol{x}\rangle^{\rho_0}\right)^2 \mathrm{d}\boldsymbol{x} \int_{\mathbb{R}^d} \langle \boldsymbol{x}\rangle^{-\frac{2\tilde{\beta}_0 \rho_0}{2 - \tilde{\beta}_0}} \mathrm{d}\boldsymbol{x}, \tag{76}$$

the last integral being finite due to (73), which implies that $\frac{2\tilde{\beta}_0 \rho_0}{2 - \tilde{\beta}_0} > d$. Finally, injecting (76) into (74), we obtain the inequalities Q5

$$|\log \widehat{\mathscr{P}}_w(\varphi)| \leq \int_{\mathbb{R}^d} |\Psi(\varphi(\boldsymbol{x}))| \mathrm{d}\boldsymbol{x} \leq C' \|\varphi\|_{L_2(\mathbb{R}^d;\rho_0)}^2. \tag{77}$$

This implies that $\widehat{\mathscr{P}}_w$ is well-defined over $L_2(\mathbb{R}^d; \rho_0)$ and continuous at $\varphi = 0$. Since $\widehat{\mathscr{P}}_w$ is positive-definite, it is therefore continuous over $L_2(\mathbb{R}^d; \rho_0)$ [20].

Hilbert–Schmidt condition. The identity is actually a compact operator, and therefore a Hilbert–Schmidt one, from $W_2^{\tau}(\mathbb{R}^d; \rho)$ to $L_2(\mathbb{R}^d; \rho_0)$, under the conditions that $\tau > d/2$ and $\rho - \rho_0 > d/2$. This is a special case of a general result on compactness in weighted Triebel–Lizorkin spaces [11, Section 4.2.3]. □

6. The Lévy noise on weighted Besov spaces

We investigate here the Besov smoothness of a Lévy noise over the complete domain $\mathbb{R}^d$. The paths of a nontrivial white noise w are never included in $B_{p,q}^{\tau}(\mathbb{R}^d)$, since their is no decay at infinity. For this reason, and as for Sobolev spaces, we consider the weighted Besov spaces, introduced in Section 2.2. The main course of this section is to prove Theorem 3.

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Theorem 3. *Consider a Lévy noise w with indices $\beta_0 > 0$ and β_∞. Let $0 < p, q \leq \infty$, $\tau, \rho \in \mathbb{R}$. If*

$$\rho > \frac{d}{\min(p, \beta_0)} \quad \text{and} \quad \tau > d\left(1 - \frac{1}{\max(p, \beta_\infty)}\right), \tag{78}$$

then $w \in B^{-\tau}_{p,q}(\mathbb{R}^d; -\rho)$ a.s.

Proof. We start with some preliminary remarks.

- First of all, it is sufficient to prove (78) for $p = q$, the other cases being deduced by the embedding relations $B^{-\tau+\epsilon}_{p,p}(\mathbb{R}^d; -\rho) \subseteq B^{-\tau}_{p,q}(\mathbb{R}^d; -\rho)$ already seen in (37). Therefore, a different parameter q can always be absorbed at the cost of an arbitrarily small smoothness, which is still possible in our case since the condition on τ in (78) is a strict inequality. For the same reason, it is admissible to consider that $p < \infty$.
- Second, we know from Proposition 8 that, for a fixed $\epsilon > 0$ and with probability 1,

$$w \in W_2^{-d/2-\epsilon}\left(\mathbb{R}^d; -\frac{d}{\beta_0} - \epsilon\right). \tag{79}$$

From now on, we fix $p = q, \tau, \rho$. We can apply Proposition 4 with $\tau_0 = -d/2 - \epsilon$ and $\rho_0 = d(\frac{1}{\beta_0} - \frac{1}{2}) + \epsilon$. We set u according to (40) and consider $(\psi_{j,G,\boldsymbol{m}})$ a wavelet basis with regularity r_0 thereafter.

First case: $\beta_\infty < \beta_0$ **or** $\beta_\infty = \beta_0 \in I_0 \bigcap I_\infty$. We fix $p < \beta \in I_\infty \bigcap I_0$. We are by assumption in the conditions of Corollary 1. In particular, Corollary 2 applies: The random variable $\langle w, \varphi \rangle$ is well-defined for any $\varphi \in L_\beta(\mathbb{R}^d)$. In particular, since $\beta \in (0, 2]$, the Daubechies wavelets $\psi_{j,G,\boldsymbol{m}}$, which are compactly supported and in $L_2(\mathbb{R}^d)$, are in $L_\beta(\mathbb{R}^d)$, so that the random variables $\langle w, \psi_{j,G,\boldsymbol{m}} \rangle$ are well-defined and (60) is applicable to them. We shall show that $w \in B^{-\tau}_{p,p}(\mathbb{R}^d; -\rho)$ a.e. if

$$\rho > d/p \quad \text{and} \quad \tau > d(1 - 1/\beta). \tag{80}$$

To show that $w \in B^{-\tau}_{p,p}(\mathbb{R}^d; -\rho)$ with probability 1, it is sufficient to show that

$$\mathbb{E}\left[\|w\|^p_{B^{-\tau}_{p,p}(\mathbb{R}^d;-\rho)}\right] = \sum_{j \geq 0}\left(2^{j(-\tau p - d + dp/2)} \sum_{G \in \mathrm{G}^j, \boldsymbol{m} \in \mathbb{Z}^d} \frac{\mathbb{E}[|\langle w, \psi_{j,G,\boldsymbol{m}} \rangle|^p]}{\langle 2^{-j}\boldsymbol{m} \rangle^{\rho p}}\right) < \infty. \tag{81}$$

The noise w being stationary, $\mathbb{E}[|\langle w, \psi_{j,G,\boldsymbol{m}} \rangle|^p]$ does not depend on the shift index $\boldsymbol{m}$. Moreover, using (60) with $a = 2^{-j}$, we have that

$$\mathbb{E}[|\langle w, \psi_{j,G,\boldsymbol{m}} \rangle|^p] \leq C 2^{jpd(1/2-1/\beta)} \|\psi_G\|^p_\beta. \tag{82}$$

Hence, we deduce that

$$\mathbb{E}\left[\|w\|^p_{B^{-\tau}_{p,p}(\mathbb{R}^d;-\rho)}\right] \leq C' \sum_{j \geq 0} 2^{j(-\tau p - d + dp - dp/\beta)} \sum_{\boldsymbol{m} \in \mathbb{Z}^d} \langle 2^{-j}\boldsymbol{m} \rangle^{-\rho p}, \tag{83}$$

where $C' = C \sum_{G \in \mathrm{G}^0} \|\psi_G\|^p_\beta$ is a finite constant. The sum $\sum_{\boldsymbol{m} \in \mathbb{Z}^d} \langle 2^{-j}\boldsymbol{m} \rangle^{-\rho p}$ is finite if and only if $\rho > d/p$, in which case there exists a constant $C_0 > 0$ such that

$$\sum_{\boldsymbol{m} \in \mathbb{Z}^d} \langle 2^{-j}\boldsymbol{m} \rangle^{-\rho p} \underset{j \to \infty}{\sim} C_0 2^{jd}. \tag{84}$$

Indeed, we have the convergence of the Riemann sums

$$\frac{1}{n^d}\sum_{\boldsymbol{m}\in\mathbb{Z}^d}\left\langle\frac{\boldsymbol{m}}{n}\right\rangle^{-\rho p}\underset{n\to\infty}{\longrightarrow}\int_{\mathbb{R}^d}\langle\boldsymbol{x}\rangle^{-\rho p}\mathrm{d}\boldsymbol{x}<\infty \tag{85}$$

and (84) is proved setting $2^j=n$, for $C_0=\int_{\mathbb{R}^d}\langle\boldsymbol{x}\rangle^{-\rho p}\mathrm{d}\boldsymbol{x}$. Finally, for $\rho>d/p$, the quantity $\mathbb{E}\left[\|w\|^p_{B^{-\tau}_{p,p}(\mathbb{R}^d;-\rho)}\right]$ is finite if

$$\sum_{j\geq 0}2^{j(-\tau p+dp-dp/\beta)}<\infty, \tag{86}$$

which happens when

$$\tau-d+d/\beta>0. \tag{87}$$

We have shown that $w\in B^{-\tau}_{p,p}(\mathbb{R}^d;-\rho)$ under the conditions of (80), as expected. We now split the domain of p:

- if $p\leq\beta_\infty$, by choosing β close enough to β_∞ (or equal if $\beta_0=\beta_\infty$), we obtain that $w\in B^{-\tau}_{p,p}(\mathbb{R}^d;-\rho)$ if $\rho>d/p$ and $\tau>d-d/\beta_\infty$;
- if $\beta_\infty<p<\beta_0$, by choosing β close enough to p, we obtain that $w\in B^{-\tau}_{p,p}(\mathbb{R}^d;-\rho)$ if $\rho>d/p$ and $\tau>d-d/p$.

We summarize the situation by $w\in B^{-\tau}_{p,p}(\mathbb{R}^d;-\rho)$ if $\rho>d/p$ and $\tau>d-d/\max(p,\beta_\infty)$, which corresponds to (78) for $p<\beta_0$. Finally, the case $p\geq\beta_0$ is deduced from the result for $p<\beta_0$ (by considering values of p arbitrarily close to β_0) and the embedding (34).

Second case: general (β_0,β_∞). A Lévy noise w can be decomposed as

$$w=w_1+w_2, \tag{88}$$

where w_1 and w_2 are independent, w_1 is a compound-Poisson noise, and w_2 is finite-variance. To see that, we invoke the Lévy–Itô decomposition, see for instance [36, Chapter 4]. It means in particular that $\beta_0(w_1)=\beta_0>0$ and $\beta_\infty(w_1)=0$. Therefore, w_1 is covered by the first case. Moreover, $\beta_\infty(w_2)=\beta_\infty$ and $\beta_0(w_2)=2\in I_0(w_2)$. Again, w_2 is covered by the first case. Indeed, it is obvious if $\beta_\infty<2$. But if $\beta_\infty=2$, we have that $\beta_\infty(w_2)=\beta_0(w_2)=2\in I_0(w_2)\bigcap I_\infty(w_2)$. Hence, w is the sum of two processes w_1 and w_2 that are in $B^{-\tau}_{p,q}(\mathbb{R}^d;-\rho)$ under the conditions (78). Besov spaces being linear spaces, these conditions are also sufficient for w. $\square$

7. The Lévy noise on local Besov spaces

The space of infinitely smooth and compactly supported functions is denoted by $\mathcal{D}(\mathbb{R}^d)$. Its topological dual is $\mathcal{D}'(\mathbb{R}^d)$, the space of generalized functions, not necessarily tempered. In the same way that we defined generalized random processes over $\mathcal{S}'(\mathbb{R}^d)$, we can also define generalized random processes over $\mathcal{D}'(\mathbb{R}^d)$. This is actually the original approach of Gelfand and Vilenkin in [16]. As we briefly saw in Section 2.1, the class of Lévy white noises over $\mathcal{D}'(\mathbb{R}^d)$ is strictly larger than the one over $\mathcal{S}'(\mathbb{R}^d)$. A Lévy noise over $\mathcal{D}'(\mathbb{R}^d)$ is also in $\mathcal{S}'(\mathbb{R}^d)$ if and only if its Lévy exponent satisfies the Schwartz condition [9] or, equivalently, if and only if its index β_0 is not 0. Until now, we have only considered a Lévy white noise for which $\beta_0\neq 0$. Since we shall now focus on the local Besov smoothness of a given noise, the two equivalent conditions are now superfluous.

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Definition 6. Let $\tau \in \mathbb{R}$ and $0 < p, q \leq \infty$. The *local Besov space* $B^{\tau,\mathrm{loc}}_{p,q}(\mathbb{R}^d)$ is the collection of functions $f \in \mathcal{D}'(\mathbb{R}^d)$ such that $f \times \varphi \in B^{\tau}_{p,q}(\mathbb{R}^d)$ for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

The weighted and local Besov regularities are linked according to Proposition 9.

Proposition 9. *Let $\tau, \rho \in \mathbb{R}$, $0 < p, q \leq \infty$. We have the continuous embedding*

$$B^{\tau}_{p,q}(\mathbb{R}^d; \rho) \subseteq B^{\tau,\mathrm{loc}}_{p,q}(\mathbb{R}^d). \tag{89}$$

The local regularity of a Lévy noise is directly obtained from the previous results, essentially up to the case of compound-Poisson noise with $\beta_0 = 0$. Before stating the main result of this section, we therefore have to analyze the compound-Poisson case.

Definition 7. A *compound-Poisson noise* is a Lévy noise with a Lévy exponent of the form

$$\Psi(\xi) = \exp\left(\lambda(\widehat{\mathcal{P}}_{\mathrm{jump}}(\xi) - 1)\right), \tag{90}$$

where $\lambda > 0$ is called the *Poisson parameter* and $\mathcal{P}_{\mathrm{jump}}$ is a probability law on $\mathbb{R} \setminus \{0\}$ called the *law of jumps*.

Compound-Poisson random variables are infinitely divisible [36], so that (90) defines a valid Lévy exponent.

Lemma 2. *Let $\tau \in \mathbb{R}$ and $0 < p, q \leq \infty$. Consider a compound-Poisson noise w. If*

$$\tau > d\left(1 - \frac{1}{p}\right), \tag{91}$$

then $w \in B^{-\tau,\mathrm{loc}}_{p,q}(\mathbb{R}^d)$.

Proof. Let λ and $\mathcal{P}_{\mathrm{jump}}$ be the Poisson parameter and the law of jumps of w, respectively. The compound-Poisson noise w can be written as

$$w = \sum_{n \in \mathbb{N}} a_k \delta(\cdot - \boldsymbol{x}_k), \tag{92}$$

where (a_k) are i.i.d. with law $\mathcal{P}_{\mathrm{jump}}$ and $(\boldsymbol{x}_k)$ are such that the number of $\boldsymbol{x}_k$ in any Borelian $B \in \mathbb{R}^d$ is a Poisson random variable with parameter $\lambda\mu(B)$, μ denoting the Lebesgue measure on $\mathbb{R}^d$. This result can be seen as a consequence of the Lévy–Itô decomposition where (92) is the form of the Poisson random measure part of the decomposition. We refer to [48, Theorem 1] for a proof of the equivalence between (90) and (92) in the framework of generalized random processes. For $\varphi \in \mathcal{D}(\mathbb{R}^d)$, the function φ being compactly supported, the generalized random process $w \times \varphi = \sum_{n \in \mathbb{N}} a_n \varphi(\boldsymbol{x}_n)\delta(\cdot - \boldsymbol{x}_n)$ is almost surely a finite sum of shifted Dirac functions. Hence, it has the Besov regularity of a single Dirac function, which is precisely (91); see [40, p. 164]. $\square$

Corollary 3. *Let $0 < p, q \leq \infty$, $\tau \in \mathbb{R}$. Consider a Lévy noise w with indices β_0, β_∞. If*

$$\tau > d\left(1 - \frac{1}{\max(p, \beta_\infty)}\right), \tag{93}$$

then $w \in B^{-\tau,\mathrm{loc}}_{p,q}(\mathbb{R}^d)$ a.s.

Proof. The case $\beta_0 > 0$ is a direct consequence of Theorem 3 and Proposition 9. Let us assume now that $\beta_0 = 0$. Again, we can split w as $w_1 + w_2$, where w_1 is compound-Poisson and w_2 is finite-variance. For w_2, we can still apply Theorem 3. We can therefore restrict our attention to the case of compound-Poisson noises with $\beta_0 = 0$. But we have seen that the compound-Poisson case – regardless of β_0 – was covered in Lemma 2. Since $\beta_\infty = 0$ for compound-Poisson noises, Lemma 2 is consistent with (93), finishing the proof. □

8. Discussion and examples

8.1. Discussion and comparison with known results

Sobolev regularity of a Lévy noise. It is noteworthy to observe that the results in Sections 5 and 6, while based on very different techniques, yield exactly the same estimates when applied to Sobolev spaces. Indeed, by applying Theorem 3 with $p = q = 2$, we recover exactly (71) due to the relations $\min(2, \beta_0) = \beta_0$ and $\max(2, \beta_\infty) = 2$. Theorem 3 is therefore the generalization of Proposition 8, from Sobolev to Besov spaces.

Interestingly, the Sobolev smoothness parameter τ of a Lévy noise does not depend on the noise: The universal sufficient condition is $\tau > d/2$. Moreover, we conjecture that this condition is also necessary, in the sense that $w \notin W_2^\tau(\mathbb{R}^d; \rho)$ with probability 1 for $\tau \geq d/2$ for any ρ and any noise w. The situation is different when considering Besov smoothness for $p \neq 2$.

Hölder regularity of a Lévy noise. We obtain the Hölder regularity of the Lévy noise by setting $p = q = \infty$ in Theorem 3. Because $\min(\infty, \beta_0) = \beta_0$ and $\max(\infty, \beta_\infty) = \infty$, we deduce Corollary 4.

Corollary 4. *The Lévy noise w with indices $\beta_0 > 0$ and β_∞ is in the weighted Hölder space $H^{-\tau}(\mathbb{R}^d; -\rho)$ if*

$$\rho > d/\beta_0, \qquad \tau > d. \tag{94}$$

Similar to the Sobolev regularity, the Hölder regularity of a Lévy noise that we obtained is independent of the noise type. However, the Gaussian noise has a local Hölder regularity of $(-\tau)$ for every $\tau > \frac{d}{2}$ [50]. It means that our bounds for the regularity are suboptimal for the Gaussian case. By contrast, we conjecture that the condition $\tau > d$ is optimal for non-Gaussian Lévy noises.

The regularity of a Lévy noise for general p. Fixing the parameters $p = q > 0$, we define

$$\tau_p(w) = \min\{\tau \in \mathbb{R}, w \in B_{p,p}^{-\tau,\mathrm{loc}}(\mathbb{R}^d) \text{ a.s.}\}. \tag{95}$$

The quantity $\tau_p(w)$ measures the regularity of the Lévy noise w for the L_p-(quasi-)norm. In Corollary 3, we have seen that $\tau_p(w) \leq d\left(\frac{1}{\max(p,\beta_\infty)} - 1\right)$, a quantity that does not depend on β_0. We conjecture that

$$\tau_p(w) = d\left(\frac{1}{\max(p, \beta_\infty)} - 1\right) \tag{96}$$

for a non-Gaussian noise. If this is true, then the quantity $\max(p, \beta_\infty)$ is a measure of the regularity of a Lévy noise for the L_p-(quasi-)norm.

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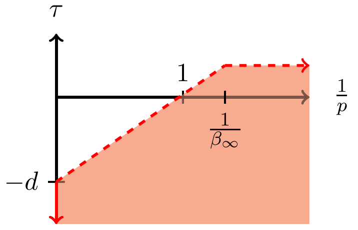
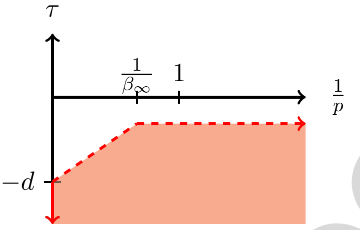


(a) $0 < \beta_\infty < 1$.



(b) $1 < \beta_\infty < 2$.

Fig. 1. Besov localization of a general Lévy white noise. A noise process is almost surely in a given local Besov space $B^{\tau,\mathrm{loc}}_{p,q}(\mathbb{R}^d)$ if $(1/p, \tau)$ is located in the shaded region.

We summarize the local results of Corollary 3 with the diagram of Fig. 1. We use the classical $(1/p, \tau)$-representation, which is most convenient for visualization. We indeed see in (96) that the parameters $1/p$ and τ are linked with a linear relation for $p \leq \beta_\infty$.

Our results can be compared with previous ones for Lévy processes. Since for $d = 1$, a Lévy process is the integrated version of the corresponding Lévy noise, its regularity can be obtained simply by adding 1 to the one of the noise. This allows us to recover the local regularity results obtained by several authors for Lévy processes or subfamilies. In particular, our results are in agreement with [18, Theorem 3.2] (that is restricted to the case $p \geq 1$) and [37, Theorem 1.1]. To summarize, a Lévy process is in $B^{\tau,\mathrm{loc}}_{p,p}(\mathbb{R})$ almost surely if

$$\tau < \frac{1}{\max(p, \beta_\infty)}. \tag{97}$$

Weights and Lévy noises. As for the regularity, we can define for $p = q > 0$ the optimal weight

$$\rho_p(w) = \min\{\rho \in \mathbb{R},\ \exists \tau \in \mathbb{R},\ w \in B^{-\tau}_{p,p}(\mathbb{R}^d; \rho) \text{ a.s.}\}. \tag{98}$$

According to Theorem 3, we have that $\rho_p(w) \leq \frac{d}{\min(p,\beta_0)}$. We conjecture that

$$\rho_p(w) = \frac{d}{\min(p, \beta_0)} \tag{99}$$

for every noise w with infinite variance (typically if $\beta_0 < 2$).

If this conjecture is true, then $\rho_\infty(w) = d/\beta_0$. When β_0 goes to 0, we need stronger and stronger weights to include the Lévy noise into the corresponding Hölder space. The limit case is $\beta_0 = 0$ for which we only have local results. Indeed, polynomial weights are not increasing fast enough to compensate the erratic behavior of the noise. This is consistent with the fact that a Lévy noise with $\beta_0 = 0$ is not tempered [9].

8.2. Besov regularity of some specific noises

Let us now apply our results to important subfamilies of Lévy noises. We start by recalling the indices of the considered white noise. We give in Table 1 the Lévy exponent and the probability density of the underlying infinitely divisible law, when they can be expressed in a closed form. All the considered distributions are known to be infinitely divisible. For Gaussian, SαS, or compound-Poisson noises, this can be easily seen from the definition. For the others, it is a non-trivial fact, and we refer to [36] for more details and references to literature.

Table 1
Blumenthal–Getoor indices of Lévy exponent.

Lévy noise	Parameter	$\Psi(\xi)$	$p_{\mathrm{id}}(x)$	β_0	β_∞	*cf.*
Gaussian	$\sigma^2 > 0$	$-\sigma^2\xi^2/2$	$\frac{\mathrm{e}^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}$	2	2	[50]
Pure drift	$\gamma \in \mathbb{R}$	$\mathrm{i}\gamma\xi$	$\delta(\cdot - \gamma)$	1	1	
SαS	$\alpha \in (0, 2)$	$-\lvert\xi\rvert^\alpha$	–	α	α	[35]
Sum of SαS	$\alpha, \beta \in (0, 2)$	$-\lvert\xi\rvert^\alpha - \lvert\xi\rvert^\beta$	–	$\min(\alpha, \beta)$	$\max(\alpha, \beta)$	[35]
Laplace	–	$-\log(1+\xi^2)$	$\frac{1}{2}\mathrm{e}^{-\lvert x\rvert}$	2	0	[23]
Sym-gamma	$c > 0$	$-c\log(1+\xi^2)$	–	2	0	[23]
Poisson	$\lambda > 0$	$\lambda(\mathrm{e}^{\mathrm{i}\xi} - 1)$	–	2	0	
Compound-Poisson	$\lambda > 0, \mathbb{P}_J$	$\lambda(\widehat{\mathbb{P}}_J(\xi) - 1)$	–	variable	0	[49]
Inverse Gaussian	–	–	$\frac{\mathrm{e}^{-x}}{\sqrt{2\pi x^{3/2}}}\mathbb{1}_{x\geq 0}$	2	1/2	[2]

Q6 We moreover remark that any combination of β_0 and β_∞ is possible, as stated in Proposition 10 (see Fig. 2).

Proposition 10. *For every $\beta_0, \beta_\infty \in [0, 2]$, there exists a Lévy noise with indices β_0 and β_∞.*

Proof. We shall define the Lévy exponent Ψ of w, and therefore w itself, by its Lévy triplet according to (7). When $\gamma = \sigma^2 = 0$ and ν is symmetric, the Lévy white noise with Lévy triplet $(0, 0, \nu)$ has indices given by [10, Section 3.1]

$$\begin{aligned}\beta_\infty &= \inf_{p\in[0,2]}\left\{\int_{|x|\leq 1}|x|^p\nu(\mathrm{d}x) < \infty\right\},\\ \beta_0 &= \sup_{p\in[0,2]}\left\{\int_{|x|>1}|x|^p\nu(\mathrm{d}x) < \infty\right\}.\end{aligned} \tag{100}$$

For $0 < \beta_0 \leq 2$ and $0 \leq \beta_\infty < 2$, we set

$$\nu_{\beta_0}(x) = |x|^{-(\beta_0+1)}\mathbb{1}_{|x|>1}, \qquad \nu^{\beta_\infty}(x) = |x|^{-(\beta_\infty+1)}\mathbb{1}_{|x|\leq 1}.$$

Moreover, for $\beta_0 = 0$ and $\beta_\infty = 2$, we set

$$\nu_0(x) = (1 + |\log x|)^{-2}|x|^{-1}\mathbb{1}_{|x|>1}, \qquad \nu^2(x) = (1 + |\log x|)^{-2}|x|^{-3}\mathbb{1}_{|x|\leq 1}.$$

For $0 \leq \beta_0, \beta_\infty \leq 2$ and defining $\nu_{\beta_0}^{\beta_\infty} = \nu_{\beta_0} + \nu^{\beta_\infty}$, we see easily that

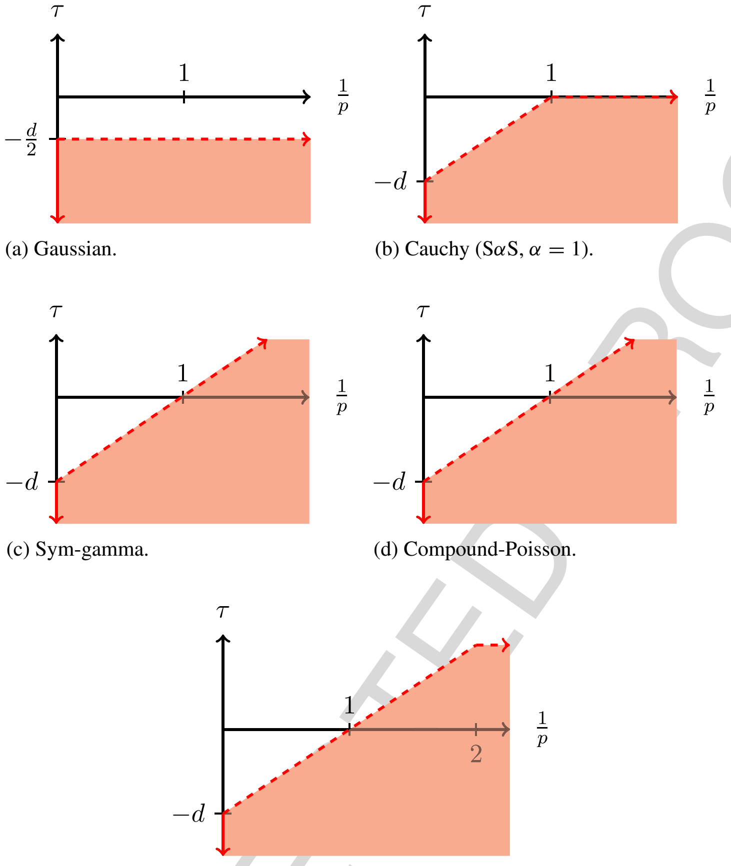
$$\int_{\mathbb{R}\setminus\{0\}} \inf(1, x^2)\nu_{\beta_0}^{\beta_\infty}(\mathrm{d}x) < \infty,$$

so that $\nu_{\beta_0}^{\beta_\infty}$ is a Lévy measure. Based on (100), we also see that the associated indices are β_0 and β_∞. □

Uncited references

Q7

[32].



(a) Gaussian.
(b) Cauchy (SαS, $\alpha = 1$).
(c) Sym-gamma.
(d) Compound-Poisson.
(e) Inverse Gaussian.

Fig. 2. Besov localization of specific Lévy white noises. A noise process is almost surely in a given Besov space $B_{p,q}^{\tau,\mathrm{loc}}(\mathbb{R}^d)$ if $(1/p, \tau)$ is located in the shaded region.

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