Stochastic Processes and their Applications 48 (1993) 123-137
North-Holland

Pricing options on securities with discontinuous returns

Indrajit Bardhan*
Goldman Sachs & Co., New York, USA

Xiulu Chao**
Division of Industrial and Management Engineering, New Jersey Institute of Technology, Newark, NJ, USA

Received 6 April 1992
Revised 24 September 1992

We consider a financial market where the asset prices are driven by a multidimensional Brownian motion process and a multidimensional point process of random jumps admitting stochastic intensity. Using the equivalent martingale measure approach, we construct hedging portfolios for European and American contingent claims. We also present a valuation equation that must be satisfied by any derivative security and can be solved numerically to obtain option prices.

point processes * stochastic intensity * equivalent martingale measure * European and American options * valuation equation

1. Introduction

In this paper, we consider a financial market in which securities are allowed to have discontinuous returns. There is one bond and m risky stocks being traded continuously over a finite time horizon. The security prices are driven by a d-dimensional Brownian motion process and an $(m-d)$-dimensional point process. The Brownian motion represents the continuous flow of information into the market, while the point process represents sudden shocks. The point processes are quite general, non-Markovian but admit stochastic intensity. The jump sizes are allowed to be random. The stocks are assumed to pay out a stream of dividends as well.

Using a boundedness condition on the risk-aversion premium for jumps, we identify an equivalent risk-neutral probability measure, under which the total return on investment in any stock is equal to the riskless return on the bond. This risk-neutral measure is then applied to the construction of hedging portfolios for European and American contingent claims. We use the generalized Ito formula and the martingale representation theorem (e.g., Protter, 1990), as well as concepts from optimal stopping theory. These hedging portfolios enable us to characterize the evolution of the price of a contingent claim over its lifetime.

Correspondence to: Dr. Indrajit Bardhan, Goldman Sachs & Co., New York, NY 10004, USA.

* This work was done when the author was at the Department of Industrial Engineering and Operations Research, Columbia University, New York, U.S.A.

** This research is partially supported by SBR, NJIT.

Finally, we focus on derivative securities, i.e., contingent claims whose payoffs depend on the prices of the risky stocks. Here we derive a differential-difference equation that is satisfied by the fair price of any derivative security. This valuation equation can be solved numerically to price all sorts of options. Due to the presence of jump uncertainty, the equation is not entirely free of investor preferences. For a call option under deterministic coefficients, we obtain a closed-form solution similar to Merton's 1976 result.

The equivalent martingale measure approach adopted in this paper was motivated by the article of Karatzas (1989). First introduced by Harrison and Kreps (1979), it has been extensively used in the pricing of derivative securities in models without jumps. Cox and Ross (1975) and Merton (1976) were the first to introduce jump-diffusions as models of stock price behaviour. There is now a substantial amount of literature on the subject of option pricing in the presence of jumps (e.g., Jarrow and Rudd, 1983; Jones, 1984; Bates, 1988). Most of the work, however, has been focussed on Poisson processes. Pontier and Picque (1990) derive a valuation equation for European options in the presence of non-homogeneous Poisson jumps. Xue (1991) considers a model with jumps coming from a compensated Poisson process. Our model allows for jump processes that are considerably more general, requiring only the existence of a stochastic intensity. A related single-dimensional model was studied by Aase (1988).

The paper is organized as follows. In Sections 1 and 2, we describe the financial market and the equivalent risk-neutral measure that can be constructed from security returns. Section 4 deals with the hedging and pricing of European contingent claims while Section 5 does the same for American contingent claims. Finally in Section 6, we present the valuation equation for derivative securities and simplifications for special cases.

2. The financial market

Consider a financial market subject to both diffusive uncertainty as well as jump uncertainty. Uncertainty enters through the components of an $\mathbb{R}^d$-valued Brownian motion $W(t)=(W_1(t),\ldots,W_d(t))^{\mathrm{T}}$, and the components of a $(m-d)$-dimensional multivariate jump process $N(t)=(N_1(t),\ldots,N_{m-d}(t))^{\mathrm{T}}$. $W(t)$ is defined on a probability space $(\Omega^W, \mathscr{F}^W, P^W)$, and $N(t)$ is defined on a probability space $(\Omega^N, \mathscr{F}^N, P^N)$.

Let $(\Omega, \mathscr{F}, P)$ be the product probability space, i.e., $\Omega=\Omega^W\times\Omega^N$, $\mathscr{F}=\mathscr{F}^W\otimes\mathscr{F}^N$, and $P=P^W\otimes P^N$. Together, there are m sources of uncertainty present. The time horizon is from 0 to time T.

Each jump process is a *point process* $\{t_n^{(k)};\ n\geq 1\}$, where $t_n^{(k)}$ is the time of the nth jump. We denote

$$N_k(t)=\sup\{n:\ t_n^{(k)}\leq t\}. \tag{2.1}$$

as the number of type k random jumps to the market by time t. N_k will represent both the kth point process, as well as the term in (2.1). The kth jump process is assumed to admit a $(P, \mathscr{F}_t)$-*stochastic intensity* $\lambda^{(k)}(t)$. Put simply, $\lambda^{(k)}(t)$ is the rate of the jump process at time t. The process $\lambda^{(k)}$ is $\{\mathscr{F}_t\}$-predictable, positive and uniformly bounded over $[0, T]$. For further details on point processes, see Bremaud (1981).

There are $m+1$ securities being traded continuously. One of these is a risk-free asset, with price $P_0(t)$ given by

$$\mathrm{d}P_0(t) = P_0(t) r(t)\,\mathrm{d}t, \qquad P_0(0) = 1. \tag{2.2}$$

The other m securities are risky assets, called *stocks*, subject to the uncertainty in the market. The price of the ith stock $P_i(t)$ is governed by a linear stochastic differential equation

$$\mathrm{d}P_i(t) = P_i(t-)\left(b_i(t)\,\mathrm{d}t + \sum_{j=1}^{d} \sigma_{ij}(t)\,\mathrm{d}W_j(t) + \sum_{k=1}^{m-d} \rho_{ik}(t)\,\mathrm{d}Q_k(t)\right), \tag{2.3}$$

where

$$Q_k(t) = N_k(t) - \int_0^t \lambda^{(k)}(s)\,\mathrm{d}s \tag{2.4}$$

represents the contribution of the jumps to the security returns. $r(t)$ is the instantaneous *rate of interest*. $b(t) = (b_1(t), \ldots, b_m(t))^{\mathrm{T}}$ is the vector of the instantaneous *appreciation rates* on the stocks. $\tilde{\sigma}(t) \triangleq [\sigma(t), \rho(t)]$ is the $m \times m$ *volatility matrix* process. All these processes are assumed to be predictable with respect to $\{\mathscr{F}_t\}$, and are bounded uniformly in $(t, \omega) \in [0, T] \times \Omega$. In addition, $\rho_{ik}(t) > -1$ for all i, k and $t \in [0, T]$, to ensure limited liability of the stock. Finally, the *covariance matrix* process $a(t) \triangleq \tilde{\sigma}(t)\tilde{\sigma}^{\mathrm{T}}(t)$ is assumed to be strongly nondegenerate.

Note that the sizes of the jumps in the security returns are random, with the randomness coming from the process ρ. However, the effect of a jump is *predictable*, given that ρ itself is predictable. This means that, at time $t-$, the effect of a possible jump at t is known.

In addition, each stock pays out a continuous stream of dividends determined by a *dividend rate* process $\delta_i(t)$, $0 \leq t \leq T$, i.e., the dividend paid out for each dollar invested in the stock. The δ process is assumed to be predictable and bounded, like the b and r processes.

From point process theory, the Q_k processes of (2.3) are actually P-martingales. Since W is a P-martingale too, the price process of stock i, P_i, is a semimartingale with drift rate $b_i(t)$, while the instantaneous expected return from investment in stock i is $b_i(t) + \delta_i(t)$.

Define the *discount factor* as

$$\beta(t) \triangleq \frac{1}{P_0(t)} = \exp\left\{-\int_0^t r(s)\,\mathrm{d}s\right\}. \tag{2.5}$$

It represents the riskless appreciation rate in the market. The next section introduces a new risk-neutral measure on the sample space. Under this measure, the discounted value of investment in any stock is a martingale, i.e., the expected total instantaneous return on investment in stock i—including dividend return as well as price appreciation—is equal to the riskless interest rate r.

3. Risk-neutral measure and admissible policies

The $\mathbb{R}^m$-valued process of *relative risk* is defined as

$$\theta(t) \triangleq (\tilde{\sigma}(t))^{-1}[b(t)+\delta(t)-r(t)] = \begin{bmatrix} \theta_W(t) \\ \theta_Q(t) \end{bmatrix}, \tag{3.1}$$

where $\theta_W(t)$ is an $\mathbb{R}^d$-valued process and $\theta_Q(t)$ is an $\mathbb{R}^{m-d}$-valued process. The process θ is bounded, measurable and predictable w.r.t. $\{\mathscr{F}_t\}$, by the assumptions on b, δ, r and $\tilde{\sigma}$. It represents the relative risk-premium as implied by stock returns and stock volatilities. Define the following processes:

$$\tilde{W}(t) \triangleq W(t) + \int_0^t \theta_W(s)\,\mathrm{d}s, \qquad \tilde{Q}(t) \triangleq Q(t) + \int_0^t \theta_Q(s)\,\mathrm{d}s. \tag{3.2}$$

Then (2.3) can be written as

$$\mathrm{d}P_i(s) = P_i(s)[r(s)\,\mathrm{d}s - \delta_i(s)\,\mathrm{d}s + \sum_{j=1}^{d} \sigma_{ij}(s)\,\mathrm{d}\tilde{W}_j(s) + \sum_{k=1}^{m-d} \rho_{ik}(s)\,\mathrm{d}\tilde{Q}_k(s)]. \tag{3.3}$$

We now introduce the boundedness condition on the relative-risk of the jumps. For the kth jump process, we require the following inequality:

$$\theta_Q^{(k)}(t) < \lambda^{(k)}(t), \tag{3.4}$$

where $\theta_Q^{(k)}$ is the kth element of θ_Q. The relative risk-premium process is bounded from above. This condition will be required to construct the risk-neutral measure. Intuitively speaking, the risk-neutral measure changes the drift in the stock prices to $r-\delta_i$. If the risk-premium is positive, then in a sense, the drift b_i is greater than $r-\delta_i$ and must be brought down by the new measure. In this case, if the risk-premium is higher than the rate at which jumps are contributing to the upward drift, we cannot get a measure to bring the stock price drift down to the level we want.

Consider the processes

$$Z_W(t) \triangleq \exp\left\{ -\int_0^t \theta_W^{\mathrm{T}}(s)\,\mathrm{d}W(s) - \frac{1}{2}\int_0^t \|\theta_W(s)\|^2\,\mathrm{d}s \right\}; \tag{3.5}$$

$$Z_Q(t) \triangleq \prod_{1 \le k \le m-d} \left(\prod_{n \ge 1} ((\mu^{(k)}(t_n^{(k)})+1)1_{\{t_n^{(k)} \le t\}} + 1_{\{t_n^{(k)} > t\}}) \times \exp\left\{ -\int_0^t \mu^{(k)}(s)\lambda^{(k)}(s)\,\mathrm{d}s \right\} \right); \tag{3.6}$$

where

$$\mu^{(k)}(t) \triangleq -\theta_Q^{(k)}(t)/\lambda^{(k)}(t). \tag{3.7}$$

Note that $\mu^{(k)}$ is well defined since the denominator of (3.7) is bounded away from 0.

The next lemma describes the risk-neutral measure. The claims in the lemma have been proved in Section 4 of Bardhan and Chao (1991).

Lemma 3.1. *The process Z defined by*

$$Z(t) \triangleq Z_W(t) Z_Q(t), \tag{3.8}$$

is a P-*martingale with $E[Z(T)]=1$. Define an auxiliary probability measure on $(\Omega, \mathscr{F}_T)$ as*

$$\tilde{P}(A) \triangleq E[Z(T) 1_A], \quad A \in \mathscr{F}_T. \tag{3.9}$$

Then $\tilde{W}$ and $\tilde{Q}$ are martingales under $\tilde{P}$. In particular, the jump process N_k admits $(\tilde{\mathrm{P}}, \mathscr{F}_t)$-stochastic intensity $\tilde{\lambda}^{(k)}(t) = (\mu^{(k)}(t)+1)\lambda^{(k)}(t)$. $\square$

We obtain the following explicit expression for the discounted stock prices:

$$\begin{aligned}\beta(t) P_i(t) = {} & \exp\left\{-\int_0^t \delta_i(s)\,\mathrm{d}s\right\} P_i(0) \\ & \prod_{k=1}^{m-d} \left(\prod_{n=1}^{N_k(t)} \left(1+\rho_{ik}(t_n^{(k)})\right) \exp\left\{-\int_0^t \rho_{ik}(s)\tilde{\lambda}^{(k)}(s)\,\mathrm{d}s\right\}\right) \\ & \times \exp\left\{\int_0^t \sigma_i^T(s)\,\mathrm{d}\tilde{W}(s) - \tfrac{1}{2}\int_0^t \|\sigma_i(s)\|^2\,\mathrm{d}s\right\},\end{aligned} \tag{3.10}$$

where σ_i denotes the ith row of σ. The expected appreciation rate is exactly $r-\delta_i$ for the ith stock. Thus, under $\tilde{P}$, the total expected return from investment in any stock is exactly equal to the interest rate r.

Consider now a small investor, with initial capital $x>0$, who uses his wealth for consumption and investment in the financial market. His investment policy is described by a *portfolio process* $\pi(t)$, $0 \le t \le T$, an $\mathbb{R}^d$-valued process that represents the dollar investment that the investor maintains in the d stocks. It is assumed to be $\mathscr{F}_t$-predictable and $\int_0^T \|\pi(t)\|^2\,\mathrm{d}t < \infty$ a.s. On the other hand, $C(t)$, $0 \le t \le T$, is a non-negative *consumption process*, assumed to be non-decreasing and predictable w.r.t. $\{\mathscr{F}_t\}$. with $C(0)=0$ and $C(T)<\infty$ a.s. The investor's *wealth process* X satisfies

$$\begin{aligned}\mathrm{d}X(t) = {} & r(t)X(t)\,\mathrm{d}t - \mathrm{d}C(t) + \pi^{\mathrm{T}}(t)[b(t)+\delta(t)-r(t)1]\,\mathrm{d}t \\ & + \pi^{\mathrm{T}}(t)\sigma(t)\,\mathrm{d}W(t) + \pi^{\mathrm{T}}(t)\rho(t)\,\mathrm{d}Q(t), \\ = {} & r(t)X(t)\,\mathrm{d}t - \mathrm{d}C(t) + \pi^{\mathrm{T}}(t)\sigma(t)\,\mathrm{d}\tilde{W}(t) + \pi^{\mathrm{T}}(t)\rho(t)\,\mathrm{d}\tilde{Q}(t).\end{aligned} \tag{3.11}$$

A solution to this differential equation that satisfies $X(0)=x\geqslant 0$ is

$$\beta(t)X(t)=x-\int_0^t \beta(s)\,\mathrm{d}C(s)+\int_0^t \beta(s)\pi^{\mathrm{T}}(s)\sigma(s)\,\mathrm{d}\tilde{W}(s)$$
$$+\int_0^t \beta(s)\pi^{\mathrm{T}}(s)\rho(s)\,\mathrm{d}\tilde{Q}(s). \tag{3.12}$$

So

$$M(t)\triangleq\beta(t)X(t)+\int_0^t \beta(s)\,\mathrm{d}C(s), \tag{3.13}$$

is a local martingale under $\tilde{P}$, by Lemma 3.1. This is because $\int_0^T \|\pi(t)\|^2\,\mathrm{d}t<\infty$ a.s., and the processes β and $\tilde{\sigma}$ are bounded. For later use, define the process

$$\zeta(t)\triangleq\beta(t)Z(t). \tag{3.14}$$

A portfolio and consumption processes pair (π, C) is considered *admissible* for initial capital $x\geqslant 0$ if the associated wealth process X satisfies $X(T)\geqslant 0$ and $X(t)\geqslant -K$ $\forall 0\leqslant t\leqslant T$ a.s., for some non-negative and P-integrable random variable $K=K(\pi, C)$. Denote the class of admissible pairs as $\mathscr{A}(x)$.

For any $(\pi, C)\in\mathscr{A}(x)$, the local martingale M of (3.13) is bounded from below, and is hence a super-martingale. We then get the inequality

$$\tilde{E}\left[\beta(\tau)X(\tau)+\int_0^\tau \beta(s)\,\mathrm{d}C(s)\right]\leqslant x, \tag{3.15}$$

for all $\{\mathscr{F}_t\}$-stopping times τ which are less than T. Here, $\tilde{E}$ denotes expectation under the measure $\tilde{P}$.

4. Valuation of European contingent claims

In this section, the risk-neutral measure is used to value *European contingent claims.* A European contingent claim is a financial instrument that has a dividend payoff of $\nu(t)$, $t\in[0, T]$, and a liquidation value of S. Here, ν is non-negative bounded and progressively measurable w.r.t. $\{\mathscr{F}_t\}$, while S is a nonnegative $\mathscr{F}_T$-measurable random variable. In addition, ν and S are assumed to satisfy

$$\tilde{E}\left[\beta(T)S+\int_0^T \beta(s)\nu(s)\,\mathrm{d}s\right]<\infty. \tag{4.1}$$

We denote the arbitrage value or fair value of the claim as $e(t)$. To determine the fair value of the claim, we will find a portfolio-consumption pair (π, C) such that the consumption process is equal a.s. to the dividend of the claim, and the terminal wealth of the pair is equal a.s. to the liquidation value of the claim. This pair can then be used to hedge the claim's payoff. The claim must be valued at the minimum initial capital required to finance such a hedging portfolio-consumption strategy.

To this end, let $H_e(x)$ denote the class of $(\pi, C)\in\mathscr{A}(x)$ such that $C(t)\geqslant\int_0^t \nu(s)\,\mathrm{d}s$ a.s. and $X(T)=S$ a.s. Clearly, $e(0)=\inf\{x\colon \exists(\pi, C)\in H_e(x)\}$.

Theorem 4.1. *The fair price of the European contingent claim is given by*

$$e(0)=\tilde{E}\left[\beta(T)S+\int_0^T \beta(s)\nu(s)\,\mathrm{d}s\right]. \tag{4.2}$$

There exists a unique (up to equivalence) portfolio-consumption pair $(\pi, C)\in H_e(e(0))$ that hedges the claim's payoff. The evolution of the claim's price process is given by

$$e(t)=\frac{1}{\beta(t)}\,\tilde{E}\left[\beta(T)S+\int_t^T \beta(s)\nu(s)\,\mathrm{d}s\,|\,\mathscr{F}_t\right]. \tag{4.3}$$

Proof. Let $\hat{x}=\tilde{E}[\beta(T)S+\int_0^T\beta(s)\nu(s)\,\mathrm{d}s]$ and $C(t)=\int_0^t\nu(s)\,\mathrm{d}s$. First, we show that there exists a unique (up to equivalence) portfolio process π such that $(\pi, C)\in\mathscr{A}(\hat{x})$ and the corresponding wealth process satisfies $X(T)=S$ a.s. Since by (3.15) any other hedging portfolio is bound to cost at least as much as $\hat{x}$, we have $e(0)=\hat{x}$. The associated wealth process provides the evolution of the claim's arbitrage value.

To this end, consider the $\mathscr{F}_T$-measurable r.v. $D\triangleq\beta(T)S+\int_0^T\beta(s)\nu(s)\,\mathrm{d}s$, and the $(P, \mathscr{F}_t)$-martingale

$$u(t)\triangleq\tilde{E}(D\,|\,\mathscr{F}_t)-\tilde{E}D,\quad 0\leqslant t\leqslant T. \tag{4.4}$$

u can be represented as a stochastic integral w.r.t. (W, Q) because the family of martingales $\{W_j;\,1\leqslant j\leqslant d\}$ and $\{Q_k;\,1\leqslant k\leqslant m-d\}$ has the predictable representation property on the product space (e.g. Protter, 1990; and Bremaud, 1981). We can then write u as a stochastic integral w.r.t. $(\tilde{W}, \tilde{Q})$ (Bardhan and Chao, 1991). Thus

$$u(t)=\int_0^t \eta_W(s)\,\mathrm{d}\tilde{W}(s)+\int_0^t\eta_Q(s)\,\mathrm{d}\tilde{Q}(s), \tag{4.5}$$

for some $\{\mathscr{F}_t\}$-predictable $\mathbb{R}^d$-valued process η_W and some $\{\mathscr{F}_t\}$-predictable $\mathbb{R}^{m-d}$-valued process η_Q, with $\int_0^T(\|\eta_W(s)\|^2+\|\eta_Q(s)\|^2)\,\mathrm{d}s<\infty$ a.s. Define the portfolio process to be

$$\pi(t)=P_0(t)(\tilde{\sigma}^{\mathrm{T}}(t))^{-1}\begin{bmatrix}\eta_W(t)\\ \eta_Q(t)\end{bmatrix}. \tag{4.6}$$

This process is $\{\mathscr{F}_t\}$-predictable, and satisfies $\int_0^T\|\pi(t)\|^2\,\mathrm{d}t<\infty$ a.s. The corresponding wealth process X is

$$\beta(t)X(t)\triangleq\hat{x}-\int_0^t\beta(s)\,\mathrm{d}C(s)+u(t)=\tilde{E}\left[\beta(T)S+\int_t^T\beta(s)\,\mathrm{d}C(s)\,|\,\mathscr{F}_t\right]. \tag{4.7}$$

So $X(t)\geqslant 0$, $\forall t\in[0, T]$ and $X(T)=S$ a.s. Thus (π, C) is an admissible pair, and is the desired hedging strategy, proving (4.2) and (4.3).

As for uniqueness, let π_1, π_2 be two portfolios such that (π_1, C) and (π_2, C) are both in $\mathscr{A}(\hat{x})$, let X_1, X_2 be the corresponding wealth processes and M_1, M_2 the corresponding martingales from (3.13). Then

$$(M_1 - M_2)(t) = \beta(t)(X_1(t) - X_2(t)) = \int_0^t \beta(s)(\pi_1(s) - \pi_2(s))^{\mathrm{T}}\sigma(s)\,\mathrm{d}\tilde{W}(s) + \int_0^t \beta(s)(\pi_1(s) - \pi_2(s))^{\mathrm{T}}\rho(s)\,\mathrm{d}\tilde{Q}(s), \tag{4.8}$$

is a martingale also. Moreover, since $M_1(T) = M_2(T) = D$, this martingale must be identically zero, whereby its predictable quadratic variations process must be zero. Using Protter (1990, p. 64),

$$\langle M_1 - M_2\rangle(t) = \int_0^t \beta^2(s)\|(\pi_1(s) - \pi_2(s))^{\mathrm{T}}\sigma(s))\|^2\,\mathrm{d}s + \int_0^t \beta^2(s)(\pi_1(s) - \pi_2(s))^{\mathrm{T}}\rho(s)\,\mathrm{diag}(\tilde{\lambda}(s))\rho^{\mathrm{T}}(s)(\pi_1(s) - \pi_2(s))\,\mathrm{d}s = 0,$$

$$0 \leqslant t \leqslant T, \tag{4.9}$$

where $\mathrm{diag}(\tilde{\lambda}(s))$ is an $(m-d)$-dimensional diagonal matrix with $\tilde{\lambda}^{(k)}(s)$ as the diagonal elements. Thus π_1, π_2 are equivalent. □

5. Valuation of American contingent claims

This section discusses *American contingent claims*. An American contingent claim is a financial instrument that allows the holder the choice of an exercise time $\tau \in \mathscr{S}_{0,T}$, where T is the expiration date of the claim and $\mathscr{S}_{u,v}$ is the set of $\{\mathscr{F}_t\}$-stopping times that take values in the interval $[u, v]$. The claim guarantees the investor a dividend of $\nu(t)$, $t \in [0, \tau]$ and a payment of $f(\tau)$ upon exercise. Here, ν is a dividend process as in the last section, and $\{f(t);\ t \in [0, T]\}$ is a right-continuous, non-negative $\{\mathscr{F}_t\}$-adapted process satisfying

$$E\left(\sup_{t\in[0,T]}\left(f(t) + \int_0^t \nu(s)\,\mathrm{d}s\right)\right)^a < \infty \quad \text{for some } a > 1. \tag{5.1}$$

As in the previous section, we wish to compute $a(0)$, the price of such a claim at time 0, and also $a(t)$, the evolution of this price over the life of the claim. Once again, this is achieved by finding a portfolio-consumption pair (π, C) that hedges the claim's payoff.

The hedging strategy (π, C) should satisfy

(i) $C(t) \geqslant \int_0^t \nu(s)\,\mathrm{d}s$,

(ii) $X(t) \geqslant f(t)$, $t \in [0, T]$; $X(T) = f(T)$,

almost surely, where X is the wealth process associated with (π, C).

Let $H_a(x)$ be the class of hedging strategies that are financible with initial capital x. Then the fair price (current price) of the claim is given by

$$a(0) = \inf\{x\colon \exists(\pi, C) \in H_a(x)\}. \tag{5.2}$$

Theorem 5.1. *The fair price of the American contingent claim is given by*

$$a(0)=\sup_{\tau\in\mathcal{S}_{0,T}}\tilde{E}\left[\beta(\tau)f(\tau)+\int_0^\tau\beta(s)\nu(s)\,\mathrm{d}s\right]. \tag{5.3}$$

There exists a hedging strategy $(\pi, C)\in H_a(a(0))$, *and the evolution of the claim's price at any time is*

$$a(t)=\frac{1}{\beta(t)}\operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{t,T}}\tilde{E}\left[\beta(\tau)f(\tau)+\int_t^\tau\beta(s)\nu(s)\,\mathrm{d}s\,|\,\mathcal{F}_t\right]\quad a.s.\quad \forall t\in[0,T]. \tag{5.4}$$

The optimal time to exercise the claim is given by

$$\tau^*=\inf\{t\in[0,T]: a(t)=f(t)\}. \tag{5.5}$$

Proof. Define

$$Q(t)\triangleq\beta(t)f(t)+\int_0^t\beta(s)\nu(s)\,\mathrm{d}s, \tag{5.6}$$

and

$$\hat{x}\triangleq\sup_{\tau\in\mathcal{S}_{0,T}}\tilde{E}Q(\tau). \tag{5.7}$$

From the definition of hedging strategies and (3.15), any hedging strategy has to start out with a level of initial capital that is at least as great as $\hat{x}$. We show that it is possible to hedge using initial capital of $\hat{x}$, thereby implying that the fair value of the claim is indeed $\hat{x}$.

From the theory of optimal stopping (for references, see Karatzas, 1989), there exists a nonnegative, right-continuous with left-hand limits, $\tilde{P}$-supermartingale Y which satisfies

$$Y(t)=\operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{t,T}}\tilde{E}\left[\beta(\tau)f(\tau)+\int_0^\tau\beta(s)\nu(s)\,\mathrm{d}s\,|\,\mathcal{F}_t\right]\quad \text{a.s.} \tag{5.8}$$

The process Y admits the unique Doob–Meyer decomposition

$$Y(t)=Y(0)+M(t)-A(t),\quad 0\leq t\leq T, \tag{5.9}$$

where $\{M(t),\mathcal{F}_t\}$ is a $\tilde{P}$-martingale and A is a non-decreasing process of finite variation, with $M(0)=A(0)=0$. Of course, from (5.6),

$$Y(0)=\sup_{\tau\in\mathcal{S}_{0,T}}\tilde{E}\left[\beta(\tau)f(\tau)+\int_0^\tau\beta(s)\nu(s)\,\mathrm{d}s\right]=\hat{x}. \tag{5.10}$$

As in Theorem 4.1, M can be represented as

$$M(t)=\int_0^t\beta(s)\pi^{\mathrm{T}}(s)\sigma(s)\,\mathrm{d}\tilde{W}(s)+\int_0^t\beta(s)\pi^{\mathrm{T}}(s)\rho(s)\,\mathrm{d}\tilde{Q}(s),\quad 0\leq t\leq T, \tag{5.11}$$

where π is an $\{\mathscr{F}_t\}$-predictable process with $\int_0^T \|\pi(t)\|^2 \, dt < \infty$ a.s. Now defining

$$X(t) \triangleq \frac{1}{\beta(t)} \left[Y(t) - \int_0^t \beta(s)\nu(s) \, ds \right], \tag{5.12}$$

$$C(t) \triangleq \int_0^t \nu(s) \, ds + \int_0^t \frac{1}{\beta(s)} \, dA(s), \tag{5.13}$$

it is possible to verify that X is the wealth process associated with (π, C). Also, $C(t) \geq \int_0^t \nu(s) \, ds$ and $X(t) \geq f(t)$ for all $t \in [0, T]$, with $X(T) = f(T)$, all of these almost surely. Thus (π, C) is a hedging strategy financible with $\hat{x}$, which proves the claim. Since (π, C) provides the minimum cost hedging strategy for the American contingent claim, the wealth process X gives the arbitrage value $a(t)$ of the claim at any time t before it is exercised. That the optimal exercise time is given by (5.5), is a simple consequence of the theory of optimal stopping. □

Clearly, if the process Q is a $\tilde{P}$-submartingale, the claim will not be exercised before expiration. Assume that a specific stock i does not pay any dividends. An American stock option written on this stock has the payoff

$$f(t) = (P_i(t) - K)^+, \tag{5.14}$$

where K is the exercise price. Since the stock pays no dividend, (3.10) says that the discounted stock price process βP_i is a martingale. By Jensen's inequality for convex functions, the process $\beta(t)f(t)$ is a submartingale. Since the dividend stream is non-negative, the process Q of (5.6) is a submartingale, too. This is Merton's result that an American option on a stock without dividends should not be exercised before maturity. This result holds true even though the other stocks do pay dividends.

Sometimes the American feature of the claim is restricted to a set of stopping times $\mathscr{S}_{0,T}^{\text{res}} \subset \mathscr{S}_{0,T}$. For example, a deferred American option does not allow the bearer to exercise the option before a stipulated date. Many warrants and convertible bonds issued by companies have this feature as well. As long as $\mathscr{S}_{u,t}^{\text{res}}$ is stable under the supremum operator, the results of this section are still valid with the only change of replacing $\mathscr{S}_{u,t}$ by $\mathscr{S}_{u,t}^{\text{res}}$ everywhere.

6. The valuation equation for derivative securities

In this section, we use hedging arguments to derive a valuation equation that must be satisfied by option prices, and indicate how this equation can be used numerically to solve for option prices in the presence of jump uncertainty. We consider *derivative securities*, viz., contingent claims whose payoffs depend on the prices of the securities $(P_1, \ldots, P_m)$. In these cases, the price of the claim can be expressed as a function of time t and the vector of security prices $P \equiv (P_1, \ldots, P_m)$.

Consider an European option which offers a dividend ν and a terminal payoff G. The dividend stream and the terminal payoff both depend on the values of the securities being traded, i.e., ν is a measurable function $[0, T] \times \mathbb{R}^d \to \mathbb{R}$, and G is a measurable function $\mathbb{R}^d \to \mathbb{R}$. Note that ν and G depend only on the current prices on the stock and not the path of the price process (e.g., look-back options).

The results of Section 4 indicate that there exists a replicating portfolio for the derivative security. We would like to relate the price of the derivative and the composition of replicating portfolio to the price of the underlying securities.

Assume that all market coefficients are deterministic. Then the price of the derivative security is a function of time and the security prices, i.e., there exists a function V:[0, $T] \times \mathbb{R}^d \to \mathbb{R}$, $C^{1,2}$ on $[0, T)$, such that

$$X(t) = V(t, P(t)), \quad t \in [0, T]. \tag{6.1}$$

Theorem 6.1. *The price of any derivative security must satisfy the following differential-difference equation*:

$$\begin{aligned}
&\Bigg[\frac{\partial V}{\partial t}(t, P(t)) + \sum_{i=1}^{m} \frac{\partial V}{\partial p_i}(t, P(t))P_i(t)[r(t) - \delta_i(t)] \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 V}{\partial p_i \partial p_j}(t, P(t))P_i(t)P_j(t)\sigma_i^{\mathrm{T}}(t)\sigma_j(t) - r(t)V(t, P(t))\Bigg] \\
&\quad - \sum_{i=1}^{m}\sum_{k=1}^{m-d} \frac{\partial V}{\partial p_i}(t, P(t))P_i(t)\tilde{\lambda}^{(k)}(t) \\
&\quad + \sum_{k=1}^{m-d} \tilde{\lambda}^{(k)}(t)[(V(t, P(t)(1+\rho^{(k)}(t))) - V(t, P(t)))] \\
&\quad = -\nu(t, P(t)), \quad t \in [0, T),
\end{aligned} \tag{6.2}$$

subject to the boundary condition $V(T, P(T)) = G(P(T))$. *The replicating portfolio is given by*

$$\pi(t, P(t)) = \tilde{\sigma}^{-1}(t)\begin{bmatrix} \psi^{w}(t, P(t)) \\ \psi^{q}(t, P(t)) \end{bmatrix}, \tag{6.3}$$

where

$$\psi_j^{w}(t, P) \triangleq \sum_{i=1}^{m} \frac{\partial V}{\partial p_i}(t, P)P_i\sigma_{ij}(t), \quad 1 \leqslant j \leqslant d,$$

and

$$\psi_k^{q}(t, P) \triangleq (V(t, P(1+\rho^{(k)}(t))) - V(t, P)), \quad 1 \leqslant k \leqslant m-d.$$

Proof. The wealth process of the hedging strategy is given as

$$\begin{aligned}
X(t) - X(0) = &\int_0^t r(s)X(s)\,\mathrm{d}s + \int_0^t \pi^{\mathrm{T}}(s)\sigma(s)\,\mathrm{d}\tilde{W}(s) \\
&\int_0^t \pi^{\mathrm{T}}(s)\rho(s)\,\mathrm{d}\tilde{Q}(s) \\
&- \int_0^t \nu(s, P(s))\,\mathrm{d}s.
\end{aligned} \tag{6.4}$$

On the other hand, applying Itô's lemma to the function $V(t, P(t))$ gives us

$$V(t, P(t)) - V(0, P(0))$$

$$= \int_0^t \frac{\partial V}{\partial s}(s, P(s))\, \mathrm{d}s + \sum_{i=1}^m \int_0^t \frac{\partial V}{\partial p_i}(s, P(s)) P_i(s) r(s)\, \mathrm{d}s - \delta_i(s)\, \mathrm{d}s$$

$$+ \sigma_i^T(s)\, \mathrm{d}\tilde{W}(s)$$

$$+ \frac{1}{2} \sum_{i,j=1}^m \int_0^t \frac{\partial^2 V}{\partial p_i \partial p_j}(s, P(s)) P_i(s) P_j(s) \sigma_i^{\mathrm{T}}(s) \sigma_j(s)\, \mathrm{d}s$$

$$- \sum_{i=1}^m \sum_{k=1}^{m-d} \int_0^t \frac{\partial V}{\partial p_i}(s, P(s)) P_i(s) \tilde{\lambda}^{(k)}(s)\, \mathrm{d}s$$

$$+ \sum_{k=1}^{m-d} \int_0^t (V(s-, P(s-)(1+\rho^{(k)}(s))) - V(s-, P(s-)))\, \mathrm{d}N_k(s). \quad (6.5)$$

Rewrite the last term on the right in terms of integrals of $\tilde{Q}$ and $\tilde{\lambda}$ and compare the coefficients of (6.4) and (6.5) to get

$$\frac{\partial V}{\partial s}(s, P(s)) + \sum_{i=1}^m \frac{\partial V}{\partial p_i}(s, P(s)) P_i(s)[r(s) - \delta_i(s)]$$

$$+ \frac{1}{2} \sum_{i,j=1}^m \frac{\partial^2 V}{\partial p_i \partial p_j}(s, P(s)) P_i(s) P_j(s) \sigma_i^{\mathrm{T}}(s) \sigma_j(s)$$

$$- \sum_{i=1}^m \sum_{k=1}^{m-d} \frac{\partial V}{\partial p_i}(s, P(s)) P_i(s) \tilde{\lambda}^{(k)}(s)$$

$$+ \sum_{k=1}^{m-d} (V(s, P(s)(1+\rho^{(k)}(s))) - V(s, P(s))) \tilde{\lambda}^{(k)}(s)$$

$$= r(s) V(s, P(s)) - \nu(s, P(s)). \quad (6.6)$$

Rearranging terms gives (6.2). The boundary condition is merely a restatement of the liquidation value of the derivative security. Comparing coefficients also gives us the form of the replicating portfolio π in (6.3). □

The coefficients r, $\tilde{\sigma}$ and $\tilde{\lambda}^{(k)}$ can actually be arbitrary functions of the current stock prices. This equation is not free of investor preferences because evaluating $\tilde{\lambda}$ involves the parameter μ, which is derived from the relative risk premium θ.

Define the total jump intensity as

$$\tilde{\lambda}(t) \triangleq \sum_{k=1}^{m-d} \tilde{\lambda}^{(k)}(t), \quad (6.7)$$

and the probability mass distribution $\tilde{\phi}(t, z) \triangleq \tilde{P}\{\Delta P(t)/P(t-) = z | \Delta P(t) > 0\}$ as

$$\tilde{\phi}(t, \rho_{(t)}^{(k)}) \triangleq \tilde{\lambda}^{(k)}(t)/\tilde{\lambda}(t), \quad 1 \leqslant k \leqslant m-d. \quad (6.8)$$

This defines a *marked point process* with total stochastic intensity $\tilde{\lambda}$. The size of a jump at time t can take on any one of $(m-d)$ values with the probability distribution $\tilde{\phi}(t,\cdot)$. In terms of these parameters, (6.2) becomes

$$\begin{aligned}
&\Bigg[\frac{\partial V}{\partial t}(t,P(t))+\sum_{i=1}^{m}\frac{\partial V}{\partial p_i}(t,P(t))P_i(t)[r(t)-\delta_i(t)] \\
&\quad+\frac{1}{2}\sum_{i,j=1}^{m}\frac{\partial^2 V}{\partial p_i\,\partial p_j}(t,P(t))P_i(t)P_j(t)\sigma_i^T(t)\sigma_j(t)-r(t)V(t,P(t))\Bigg] \\
&\quad-\sum_{i=1}^{m}\frac{\partial V}{\partial p_i}(t,P(t))P_i(t)\tilde{\lambda}(t) \\
&\quad+\tilde{\lambda}(t)E_{\tilde{\phi}(t)}[(V(t,P(t)(1+z))-V(t,P(t)))]=-\nu(t,P(t)).
\end{aligned} \tag{6.9}$$

This valuation equation is the jump-diffusion analogue of the PDE derived by Black-Scholes (1971) and by Merton (1973) for the pure diffusion case. This equation has been derived by Pontier and Picque (1990) for non-homogeneous Poisson jumps. A similar equation has been obtained by Aase (1988) for the one-dimensional case. Bates (1988) has derived a similar PDE using general equilibrium arguments. He restricts his attention to the case of Poisson jumps, but does not require the boundedness of the jump sizes. Note that it is the risk-neutral stochastic intensity that enters the equation, not the original one.

A crucial fact is that under the assumption of jump-diffusion, a complete system, i.e., as set of spanning securities, cannot be provided by only one stock and one bond. Yet, for derivative securities that depend only on one stock, say the 1st stock, some simplifications can be made. Consider the situation when there is only one Brownian motion and $m-1$ jump sources. Furthermore, let the final payoff function depend only on $P_1(T)$, e.g., a call option on stock 1. Since neither the payoff function g nor the dividends ν depend on the prices of the other $m-1$ stocks, the valuation equation reduces to

$$\begin{aligned}
&\Bigg[\frac{\partial V}{\partial t}(t,P_1(t))+\frac{\partial V}{\partial p_i}(t,P_1(t))P_1(t)[r(t)-\delta_1(t)] \\
&\quad+\frac{1}{2}\frac{\partial^2 V}{\partial p_1^2}(t,P_1(t))(P_1(t)\sigma_{11}(t))^2-r(t)V(t,P_1(t))\Bigg] \\
&\quad-\frac{\partial V}{\partial p_1}(t,P_1(t))P_1(t)\tilde{\lambda}(t)+\tilde{\lambda}(t)E_{\tilde{\phi}(t)}[(V(t,P_1(t)(1+z))-V(t,P_1(t)))] \\
&=-\nu(t,P_1(t)),
\end{aligned} \tag{6.10}$$

subject to the final condition that $V(T,P_1)=g(P_1)$. Though the valuation equation seems to be independent of the other stocks, we must remember that $\tilde{\lambda}$ and $\tilde{\phi}$ both depend on μ, which is computed from the actual returns on *both the stocks*. Thus computing the coefficients of the jump terms accounts for the presence of another stock.

For deterministic r, σ, $\tilde{\lambda}$ and $\tilde{\phi}$, the jumps processes are non-homogeneous compound Poisson processes under the new measure. One can then obtain explicit results in some cases. For example, if the distribution of jumps sizes is time-invariant, in terms of the risk-neutral coefficients, we get the following expression for the value of a European call option on stock 1, with exercise price K and maturity τ:

$$C(\tau, P_1) = \sum_0^\infty \frac{\exp[-\int_0^\tau \tilde{\lambda}(s)\,\mathrm{d}s](\int_0^\tau \tilde{\lambda}(s)\,\mathrm{d}s)^n}{n!}$$
$$\times E_n\left[\mathcal{BS}\left(\tau, P_1 X_n \exp\left[-\int_0^\tau \tilde{\lambda}(s)\,\mathrm{d}s\right], K, \sigma_{11}, r, \delta_1\right)\right], \qquad (6.11)$$

where X_n is a random variable having the distribution of the product of n i.i.d. random variables, each of which is $1+z$ and z is distributed according to $\tilde{\phi}$. E_n is expectation under the distribution of X_n. BS($\cdot$) is a generalized version of the celebrated Black–Scholes call option formula:

$$\mathrm{BS}(t, X, K, \sigma, r, d) = X \exp\left[-\int_0^t d(s)\,\mathrm{d}s\right]\Phi(n_1)$$
$$- K \exp\left[-\int_0^t r(s)\,\mathrm{d}s\right]\Phi(n_2), \qquad (6.12)$$

with

$$n_1 = \frac{\ln(X/K) + \int_0^t (r(s) - d(s) + \frac{1}{2}\sigma^2(s))\,\mathrm{d}s}{(\int_0^t \sigma^2(s)\,\mathrm{d}s)^{1/2}},$$
$$n_2 = n_1 - \left(\int_0^t \sigma^2(s)\,\mathrm{d}s\right)^{1/2}, \qquad (6.13)$$

and $\Phi(\cdot)$ is the standard normal distribution function. Simple substitution determines that (6.11) is indeed a solution to (6.10) with deterministic coefficients (along the lines of Merton, 1976). Using the risk-neutral parameters $\tilde{\lambda}$ and $\tilde{\phi}$, we can apply Merton's analysis, which assumes that investors are risk-neutral to jump risk.

In (6.11), the term μ is the only quantity that depends on investor preferences, entering through $\tilde{\lambda}$ and $\tilde{\phi}$. One can find implied values of μ from the price of one option and use it to price another option on the same stock. For Poisson jumps, μ is a constant and it is relatively simple to find its implied value.

One can similarly price a European put option, using (6.11) and the *put-call parity* relationship (e.g., Merton, 1973). For a derivative security with any arbitrary payoff $G(P)$, (6.11) still holds with BS replaced with the appropriate integral of G under a normal distribution with mean $\int_0^\tau r(s)\,\mathrm{d}s$ and variance $\int_0^\tau \sigma^2(s)\,\mathrm{d}s$. If the derivative security pays out an absolute dividend $\nu(t)$, then one would add the term $\int_0^\tau \exp(-\int_0^s r(u)\,\mathrm{d}u)\nu(s)\,\mathrm{d}s$ to the term in (6.11). On the other hand, if the security gives out a dividend yield $\psi(t)$ then one should multiply the term in (6.11) by $\exp(\int_0^\tau \psi(s)\,\mathrm{d}s$. If r, σ or ν are functions of P as well, then explicit calculations are not possible.

The valuation equation (6.2) is satisfied by all derivative securities, European as well as American. This integro-differential equation can be solved numerically using trapezoidal methods to value prices of simple European options (e.g., Bates, 1988). For more exotic options such as capped options or knockout options, one merely has to include appropriate boundary conditions during the numerical procedure (e.g., Bardhan, 1991). In pricing American options, one has to check for boundary conditions of optimal exercise. This is a free-boundary problem, with the free-boundary being the boundary of optimal exercise. By recursively solving backwards, and comparing the holding value of the option against the payoff value—$a(t)=f(t)$ condition from the last section—one can compute not only the current price of the option but also the optimal exercise boundary.

Acknowledgement

We would like to thank an anonymous referee for helpful comments and for having brought to our attention the paper by Aase (1988).

References

K.K. Aase, Contingent claims valuation when the security price is a combination of an Ito process and a random point process, Stochastic Process. Appl. 28 (1988) 185-220.

I. Bardhan, Option pricing when volatility and interest rates are random—A finite difference method approach, internal paper (Goldman, Sachs & Co., 1991).

I. Bardhan and X. Chao, Point processes and risk-neutral measures for securities with discontinuous returns, Tech. Rep. Ser. No. 9108, Div. of Indus. and Manag. Eng., New Jersey Inst. of Technol. (Newark, NJ, 1991).

D.S. Bates, Crashes, options, and international asset substitutability, PhD dissertation, Dept. of Economics, Princeton Univ. (Princeton, NJ, 1988).

F. Black and M. Scholes, The pricing of options and corporate liabilities, J. Polit. Econom. 8 (1971) 637-659.

P. Bremaud, Point Processes and Queues: Martingale Dynamics (Springer, Berlin, 1981).

J.C. Cox and S.A. Ross, The valuation of options for alternative stochastic processes, J. Financ. Econom. 3 (1985) 145-166.

J.M. Harrison and D. Kreps, Martingales and multiperiod securities markets, J. Econom. Theory 20 (1979) 381-408.

R.A. Jarrow and A. Rudd, Options Pricing (Irwin, Homewood, IL, 1983).

E.P. Jones, Option arbitrage and strategy with large price changes, J. Finan. Econom. 13 (1984) 91-113.

I. Karatzas, Some optimization problems in the theory of continuous trading, SIAM J. Control Optim. 27 (1989) 1221-1259.

R.C. Merton, The theory of rational option pricing, Bell J. Econom. Manag. Sci. 4 (1973) 141-183.

R.C. Merton, Option pricing when underlying stock returns are discontinuous, J. Financ. Econom. 3 (1976) 125-144.

M. Picque and M. Pontier, Optimal portfolio for a small investor in a market with discontinuous prices, Appl. Math. Optimization 22 (1990) 287-310.

P. Protter, Stochastic Integration and Differential Equations (Springer, New York, 1990).

X.X. Xue, Martingale representation theory for a class of Levy processes and its applications, PhD. dissertation, Dept. of Statist., Columbia Univ. (New York, 1991).