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Asymptotic mass distribution speed for the one-dimensional heat equation with constant drift and stationary potential

Anja Voß-Böhme

Technische Universität Dresden, Inst. Math. Stoch, Dresden 01062, Germany

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Abstract

We study the long-time behavior of the solution $u(t,x)$ of a Cauchy problem for the one-dimensional heat equation with constant drift and random potential in the quenched setting: $u_t = \frac{1}{2}u_{xx} + hu_x + \xi u$. The initial function is compactly supported. For bounded stationary ergodic potential ξ, we show that u is asymptotically ($t \to \infty$) concentrated in a ball of radius $o(t)$ and center $v_h t$ which is independent of the realization of the random potential. There is a critical drift value h_{cr} where we observe a change from sublinear ($v_h = 0$) to linear ($0 < v_h \leqslant h$) mass propagation.

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1. Introduction and main results

We consider the following one-dimensional Cauchy problem

$$\partial_t u = \tfrac{1}{2}\partial_{xx}u + h\,\partial_x u + \xi(x)u,$$
$$u(0,\cdot) = u_0 \tag{1}$$

on $[0,\infty)\times\mathbb{R}$ for constant drift $h \geqslant 0$ and random functions ξ and u_0 which are defined on some complete probability space $(\hat{\Omega}, \hat{\mathscr{A}}, \hat{P})$. The random process ξ is assumed to be non-negative, $\hat{P}$-a.s. bounded, Hölder-continuous, stationary (in the narrow sense) and ergodic. The initial function u_0 is supposed to be continuous, to be not identically

E-mail address: avoss@math.tu-dresden.de (A. Voß-Böhme).

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equal to zero and to have compact support. Under these assumptions, problem (1) is known to have a unique classical solution.

For fixed realization of the random potential, the solution $u(t,x)$ of (1) can be interpreted as the concentration of particles at point x at time t. These particles spread out spatially with diffusion coefficient $\frac{1}{2}$. They are exposed to some constant external field h and they multiply with rate $\xi(x)$ dependent on the location of the particle.

We are interested in the long-time behavior of $u(t,x)$, in particular in the asymptotic speed of mass propagation. Since the multiplication rate ξ is space dependent, a constant asymptotic velocity will appear only if the multiplication rate is in a sense homogeneous in space. Typical models are the periodic media case and the random space-homogeneous media case. We consider the latter model, i.e., we suppose that ξ is a stationary random process.

It turns out that there are constants $\lambda_h \geqslant 0$ and $v = v_h \geqslant 0$, which depend on the external field h but which are independent of the realization of the random potential and independent of the initial function u_0, that describe the asymptotic mass distribution. If t is large and if the environment is kept fixed, the overall mass grows rather with rate λ_h, and the mass is concentrated approximately in a ball with radius $o(t)$ and with center $-vt$.

One finds that $v < h$ for non-constant potential, which can be interpreted as 'survival of the fittest'. Associate with each particle at point x at time t its 'path of descent' running backwards in time. If the multiplication rate is constant, its ancestor at time 0 is typically located at point $\approx x - ht$. If ξ is non-constant, then there are two competing forces. Particles which stay in regions with large multiplication rates produce a lot of offspring and therefore substantially contribute to the population. On the other hand, all particles are exposed to the external field and are 'punished' for deviating from speed h. It appears that most particles descent from ancestors which are best adapted to the environment, i.e., perform an optimal strategy of sojourn and travel. This results in the 'effective drift' v, i.e., the ancestors at time 0 are typically located at site $\approx x - vt$.

There is another interesting phenomenon. There exists a critical drift value $h_{\mathrm{cr}} \geqslant 0$ where the system undergoes a transition from sublinear to linear propagation. That is, the effective drift equals zero for $0 \leqslant h < h_{\mathrm{cr}}$ and is larger than zero for $h > h_{\mathrm{cr}}$. We will give an example, where this critical value is really larger than zero.

The key in proving our results on the effective drift is a large deviation property for a Brownian motion with drift under exponentially weighted measure. Once we control those probabilities of large deviations we can rely on the Feynman-Kac-representation of $u(t,x)$ to analyze the long-time behavior of u. This approach is easily carried over to a discrete model. Thus, our results are related to those of Greven and den Hollander (1992), who studied the long-time behavior of a branching random walk in a random environment on the one-dimensional lattice. They could ascertain an effective drift v and a phase transition from sublinear to linear propagation as described above. They analyzed variational formulas by means of large deviation techniques but essentially based on the discrete structure of their setting. We handle a larger class of potentials and, moreover, we use a completely different approach. Our study is also related to Sznitman's work. Sznitman (1994,1995) studied probabilities of large deviations for a Brownian motion with drift under exponentially weighted measures. He chooses the

potential to be Poissonian which is a very special example for a stationary process with in-built independence. For his particular kind of potential, he is able to answer questions on the asymptotic mass distribution speed in higher dimensions.

We employ some ideas and techniques from Gärtner (1985), but the object of our study is different from that one. We focus on the conditions for a change from sub-linear to linear propagation for the heat equation with linear reaction term and compactly supported initial function. Besides we rewrite Gärtner's ideas into a generalized eigenfunction equation. Gärtner studied wave front-type propagation in a distinct setting, for reaction terms of Kolmogorov–Petrovskii–Piskunov type and bounded, but non-compactly supported initial function.

Let us now sketch the crucial arguments of our analysis. For convenience, we put $u_0 = \delta_0$, the Delta distribution at point zero. Let $\{X_t^{x,h}\}_{t\geqslant 0}$ be a Brownian motion on $\mathbb{R}$ with constant drift $h \in \mathbb{R}$ starting in $x \in \mathbb{R}$. It is defined by the following Ito stochastic differential equation:

$$\mathrm{d}X_t^{x,h} = \mathrm{d}W_t + h\,\mathrm{d}t, \quad X_0^{x,h} = x, \tag{2}$$

where W_t is a one-dimensional Wiener process defined on a probability space $(\Omega, \mathcal{N}, \mathbf{P})$. Denote by $\mathbf{P}_x^h$ the law of $X_\cdot^{x,h}$. Then, for each realization of the random potential, the solution u of (1) is given by Feynman-Kac's formula,

$$u(t,x) = \mathbf{E}_x^h \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} u_0(X_t), \tag{3}$$

where $(t,x) \in [0,\infty) \times \mathbb{R}$. We define a family of rescaled probability measures $\{\Phi_t^h\}_{t>0}$ on the Borel sets $\mathcal{B}(\mathbb{R})$ via

$$\Phi_t^h(A) := N_t^{-1} \int_A u(t,tx)\,\mathrm{d}x, \quad A \in \mathcal{B}(\mathbb{R}), \tag{4}$$

where

$$N_t := \int_{\mathbb{R}} u(t,ty)\,\mathrm{d}y$$

is a normalizing constant. Note that Φ_t^h depends on the realization of the random potential, since u does. For fixed environment, Φ_t^h measures the portion of particles that is located within by factor t bloated sets. Using the Feynman-Kac representation (3), we can show that

$$\Phi_t^h(A) = P_t^h(t^{-1}X_t^{0,-h} \in A), \quad A \in \mathcal{B}(\mathbb{R}),$$

where

$$P_t^h(B) := \frac{\mathbf{E}\exp\{\int_0^t \xi(X_s^{0,-h})\,\mathrm{d}s\}\chi_B}{\mathbf{E}\exp\{\int_0^t \xi\left(X_s^{0,-h}\right)\,\mathrm{d}s\}}, \quad B \in \mathcal{N},$$

are probability measures on $(\Omega, \mathcal{N})$ which are obtained from $\mathbf{P}$ by absolutely continuous transformation with exponential weights that depend on the random potential. Based on the results of Gärtner and Freidlin (1985, Section 7.5), we establish large deviation upper estimates for $t^{-1}X_t^{0,-h}$ with respect to P_t^h as $t \to \infty$. In case $h = 0$,

Gärtner and Freidlin proved a large deviation theorem for the hitting times $t^{-1}\tau_0(X_\cdot^{t,0})$ with respect to exponentially weighted measures which are closely related to P_t^0, where τ_y is defined on the set of continuous functions, $\tau_y : C[0,\infty) \to [0,\infty)$,

$$\tau_y(\phi) = \min\{s \geqslant 0:\ \phi(s) = y\}, \quad \phi \in C[0,\infty), \quad y \in \mathbb{R}. \tag{5}$$

They identified the rate function to be the Legendre transform of a numeric function μ which describes the logarithmic asymptotics of a moment generating function,

$$\mu(z) = \lim_{x\to\infty} \frac{1}{x} \ln \mathbf{E}_x^0 \exp\left\{\int_0^{\tau_0(X_\cdot)} (\xi(X_s) + z)\,\mathrm{d}s\right\}, \quad z \in \mathbb{R},$$

and which can be calculated via

$$\mu(z) = \hat{E}\left[\ln \mathbf{E}_1^0 \exp\left\{\int_0^{\tau_0} (\xi(X_s) + z)\,\mathrm{d}s\right\}\right] \quad (\hat{P}\text{-a.s.}).$$

Those results are easily generalized to arbitrary parameter h. One expects that the large deviation rate function for $t^{-1}X_t^{0,-h}$ can be derived from those of $t^{-1}\tau_0(X_\cdot^{x,h})$. Indeed, we define $-\lambda$ to be the inverse of $-\mu$ and show that the Legendre transform of a suitable translate of λ is the rate function which describes probabilities of large deviation for $t^{-1}X_t^0$ with respect to P_t^h. Once we control those small probabilities, it is easily verified that, for almost any realization of the potential ξ, the measures Φ_t^h converge weakly to the Dirac measure δ_{-v} at point $-v$, where $-v$ is the (unique) minimizer of the rate function.

Following the standard procedure in identifying the logarithmic asymptotics of $t^{-1}X_t^{0,h}$ with respect to P_t^h, which was, e.g., elaborated in Freidlin (1985, Section 7.2) in the periodic setting, one were looking for an eigenvalue corresponding to a strictly positive bounded eigenfunction of the operator

$$\tfrac{1}{2}\partial_{xx} - h\,\partial_x + \xi$$

in a suitable subspace of continuous functions on $\mathbb{R}$. But while the space of continuous one-periodic functions is compact, we do not face a compact setting in the random model. Hence standard spectral arguments do not apply. But, re-reading Gärtner's and Freidlin's study, we find that there are a positive stationary process $\{w^h(x)\}_{x\in\mathbb{R}}$ on $(\hat{\Omega}, \hat{\mathscr{A}}, \hat{P})$ and a non-random constant λ_h which is closely related to $\mu^{-1}(-h)$ that satisfy $\hat{P}$-a.s. the second order equation

$$\tfrac{1}{2}(\mathrm{d}^2/\mathrm{d}x^2)w^h - h\,(\mathrm{d}/\mathrm{d}x)w^h + \xi w^h = \lambda_h w^h.$$

The 'generalized eigenfunction' is given by

$$w^h(x) = w_a^h \mathbf{E}_x^{-h} \exp\left\{\int_0^{\tau_a} (\xi(X_s) - \lambda_h)\,\mathrm{d}s\right\},$$

where $a < 0$, $x \geqslant a$ and w_a^h are constants such that $w^h(0) = 1$. Letting $-a$ go to infinity, we see that this function is defined on the whole line. If the drift term h is larger than a critical value h_{cr}, the generalized eigenfunction does neither increase nor decrease exponentially as $|x| \to \infty$, though w^h is not bounded. Using this property, we are able

to prove that, for $h > h_{\mathrm{cr}}, x \in \mathbb{R}, t \geqslant 0$,

$$\mathbf{E}_x^{-h} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} w^h(X_t) = \mathrm{e}^{\lambda_h t} w^h(x). \tag{6}$$

Next we prove that λ_h has the property of a maximal eigenvalue,

$$\lambda_h = \lim_{t\to\infty} \frac{1}{t} \ln \mathbf{E}_0^{-h} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\}. \tag{7}$$

The validity of this property is even extended to $h \leqslant h_{\mathrm{cr}}$ and hence to all $h \geqslant 0$ using elementary estimates and convexity arguments. The last proposition is the basis for the derivation of the large deviation upper bounds and thus enables us to control the measure Φ_t^h.

Let us state our main theorem:

Theorem 1.1. *For $h \geqslant 0$, let u be the solution of the Cauchy problem (1). Then*

(i) $\lim_{t\to\infty} \frac{1}{t} \ln \int_{\mathbb{R}} u(t,x)\,\mathrm{d}x = \lambda_h$ ($\hat{P}$-*a.s.*).

(ii) *There exists a critical value $h_{\mathrm{cr}} \geqslant 0$ such that, for $h \neq h_{\mathrm{cr}}$, the function λ_h is differentiable with respect to h, and Φ_t^h converges weakly to the Dirac measure at point $-v_h$,*

$$\Phi_t^h \xrightarrow{w} \delta_{-v_h} \quad (\hat{P}\text{-a.s.}),$$

where

$$v_h = \begin{cases} 0, & h < h_{\mathrm{cr}}; \\ (\mathrm{d}/\mathrm{d}h)\lambda_h + h, & h > h_{\mathrm{cr}}. \end{cases}$$

The behavior of h_{cr} will be discussed in paragraph 4 with three examples at hand.

2. Generalized eigenvalue and eigenfunction

Let $\mathbf{P}_x^h$ denote the law of a Brownian motion with constant drift $h \in \mathbb{R}$ starting in $x \in \mathbb{R}$, as defined in (2). Let us denote by $U_\delta[y]$, $\delta > 0$, the δ-neighborhood of a point $y \in \mathbb{R}$.

We suppose that the process $\{\xi(x)\}_{x\in\mathbb{R}}$ on $(\hat{\Omega}, \hat{\mathscr{A}}, \hat{P})$ is stationary in the narrow sense and ergodic. Moreover, ξ is assumed to be $\hat{P}$-a.s. non-negative, Hölder-continuous and bounded, i.e., there is a $c > 0$ such that

$$0 \leqslant \xi \leqslant c \quad (\hat{P}\text{-a.s.}).$$

We denote by $\tau_a = \tau_a(X.)$ the hitting time of a point $a \in \mathbb{R}$, as defined in (5).

By Freidlin (1985), the limit

$$\mu(z) = \lim_{t\to\infty} \frac{1}{t} \ln \mathbf{E}_t^0 \exp\left\{\int_0^{\tau_0(X.)} (\xi(X_s) + z)\,\mathrm{d}s\right\} \quad (\hat{P}\text{-a.s.}), \tag{8}$$

exists for all $z \in \mathbb{R}$ and is non-random. Following Freidlin (1985, Section 7.5), the function $\mu(z)$ has the following properties ($\hat{P}$-a.s.):

(A) There exists a non-random number $\bar{g} \in [-c, 0]$,

$$\bar{g} = -\lim_{t\to\infty} \frac{1}{t} \ln \inf_{x\in U_\delta[0]} \mathbf{E}_x^0 \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} \chi_{X_t \in U_\delta[0]} \quad (\hat{P}\text{-a.s.}),$$

$\delta > 0$ arbitrary, such that $\mu(z) \leqslant 0$ for $z \leqslant \bar{g}$ and $\mu(z) = \infty$ for $z > \bar{g}$.

(B) The function $\mu(z)$ is continuously differentiable, the derivative $\mu'(z)$ is positive and monotonously non-decreasing for $z < \bar{g}$. Moreover, $\mu'(z) \to 0$ as $z \to -\infty$.

(C) $-\sqrt{2|z|} \leqslant \mu(z) \leqslant 0$ for $z \leqslant \bar{g}$.

Additionally, it is easy to derive that

(D) $\mu(z) \leqslant -\sqrt{2|z+c|}$ for $z \leqslant -c$.

We define the inverse function

$$\lambda(z) = \begin{cases} -\mu^{-1}(-z) & \text{if } z \geqslant h_{\mathrm{cr}}, \\ -\bar{g} & \text{if } 0 \leqslant z < h_{\mathrm{cr}}, \end{cases} \tag{9}$$

where

$$h_{\mathrm{cr}} := -\mu(\bar{g}). \tag{10}$$

For $z < 0$, we define λ to be the inverse of some related function $\bar{\mu}$,

$$\begin{aligned} \bar{\mu}(z) &:= \lim_{t\to\infty} \frac{1}{t} \ln \mathbf{E}_{-t}^0 \exp\left\{\int_0^{\tau_0} (\xi(X_s) + z)\,\mathrm{d}s\right\} \\ &= \lim_{t\to\infty} \frac{1}{t} \ln \mathbf{E}_t^0 \exp\left\{\int_0^{\tau_0} (\xi(-X_s) + z)\,\mathrm{d}s\right\} \quad (\hat{P}\text{-a.s.}). \end{aligned}$$

It is obvious that $\bar{\mu}(z)$ satisfies (A)–(D) with the same point of discontinuity $\bar{g}$. Denote $\bar{h}_{\mathrm{cr}} := -\bar{\mu}(\bar{g})$. We set

$$\lambda(z) = \begin{cases} -\bar{\mu}^{-1}(z) & \text{if } z \leqslant -\bar{h}_{\mathrm{cr}}, \\ -\bar{g} & \text{if } -\bar{h}_{\mathrm{cr}} < z < 0. \end{cases} \tag{11}$$

The function $\lambda|\mathbb{R} \to [0, \infty)$ is convex, continuous and continuously differentiable on $\mathbb{R} \setminus \{h_{\mathrm{cr}}, -\bar{h}_{\mathrm{cr}}\}$.

For any $h \in \mathbb{R}$, we define now constants λ_h by

$$\lambda_h := \lambda(h) - \tfrac{1}{2}h^2. \tag{12}$$

Obviously, we have

$$0 \leqslant \lambda_h \leqslant c. \tag{13}$$

Let

$$w^h(x) := w_a^h \mathrm{e}^{(x-a)h} \mathbf{E}_x^0 \exp\left\{\int_0^{\tau_a} [\xi(X_s) - \lambda(h)]\,\mathrm{d}s\right\} \quad (\hat{P}\text{-a.s.})$$

for any $h \geqslant h_{\mathrm{cr}}, a < 0, x \geqslant a$, where w_a^h is a constant such that $w^h(0) = 1$. By the strong Markov property, the (random) function w^h does not depend on the choice of a and therefore w^h is defined for any $x \in \mathbb{R}$.

It is easy to see that w^h is strictly positive. Moreover, it follows from (8) and (9) that w^h is finite and does neither increase nor decrease exponentially,

$$\lim_{x \to \pm\infty} \tfrac{1}{x} \ln w^h(x) = 0 \quad (\hat{P}\text{-a.s.}). \tag{14}$$

The function $w^h(x)$ is continuously differentiable in x, the derivative $(\mathrm{d}/\mathrm{d}x)w^h$ is absolutely continuous and the following equation holds a.e. with respect to the Lebesgue measure,

$$\tfrac{1}{2}(\mathrm{d}^2/\mathrm{d}x^2)w^h - h\,(\mathrm{d}/\mathrm{d}x)w^h + \xi w^h = \lambda_h w^h. \tag{15}$$

We are going to show that this generalized eigenvalue equation can be translated into a generalized eigenvalue equation for corresponding operators, which is summarized in the following lemma.

Lemma 2.1. *For any $t > 0, x \in \mathbb{R}$, $h \geqslant h_{\mathrm{cr}}$, we have*

$$\mathbf{E}_x^{-h} \exp\left\{ \int_0^t \xi(X_s)\,\mathrm{d}s \right\} w^h(X_t) = \mathrm{e}^{t\lambda_h} w^h(x) \quad (\hat{P}\text{-a.s.}). \tag{16}$$

If w^h was bounded, (16) would follow directly from Feynman-Kac's formula. We prove the proposition directly by using (14) and Girsanov's formula.

Proof. Fix $x \in \mathbb{R}$, $t > 0$. Let $a < \min\{x, 0\}$. Fix $\hat{\omega} \in \hat{\Omega}$. Using Girsanov's formula we have

$$\begin{aligned}
&\mathrm{e}^{-t\lambda_h} \mathbf{E}_x^{-h} \exp\left\{ \int_0^t \xi(X_s)\,\mathrm{d}s \right\} w^h(X_t) \\
&\quad = \mathrm{e}^{-t\lambda_h} \mathbf{E}_x^{0} \exp\left\{ \int_0^t \xi(X_s)\,\mathrm{d}s - \tfrac{1}{2}h^2 t + h(x - X_t) \right\} w^h(X_t) \\
&\quad \geqslant \mathbf{E}_x^{0} \exp\left\{ \int_0^t [\xi(X_s) - \lambda(h)]\,\mathrm{d}s + h(x - X_t) \right\} w^h(X_t) \chi_{\tau_a(X_\cdot) > t} \\
&\quad = \mathbf{E}_x^{0} \exp\left\{ \int_0^t [\xi(X_s) - \lambda(h)]\,\mathrm{d}s + h(x - X_t) \right\} \chi_{\tau_a > t} \\
&\qquad \times w_a^h \exp\{(X_t - a)h\} \mathbf{E}_{X_t}^{0} \exp\left\{ \int_0^{\tau_a} [\xi(X_s) - \lambda(h)]\,\mathrm{d}s \right\}.
\end{aligned}$$

By the Markov property, we get

$$\mathrm{e}^{-t\lambda_h} \mathbf{E}_x^{-h} \exp\left\{ \int_0^t \xi(X_s)\,\mathrm{d}s \right\} w^h(X_t)$$

$$\geqslant w_a^h \mathrm{e}^{(x-a)h} \mathbf{E}_x^0 \exp\left\{\int_0^{\tau_a} [\xi(X_s) - \lambda(h)]\,\mathrm{d}s\right\} \chi_{\tau_a > t}$$

$$= w^h(x) - w_a^h \mathrm{e}^{(x-a)h} \mathbf{E}_x^0 \exp\left\{\int_0^{\tau_a} [\xi(X_s) - \lambda(h)]\,\mathrm{d}s\right\} \chi_{\tau_a \leqslant t}. \tag{17}$$

The last term on the right-hand side is estimated by Hölder's inequality,

$$w_a^h \mathrm{e}^{(x-a)h} \mathbf{E}_x^0 \exp\left\{\int_0^{\tau_a} [\xi(X_s) - \lambda(h)]\,\mathrm{d}s\right\} \chi_{\tau_a \leqslant t}$$

$$\leqslant w_a^h \mathrm{e}^{(x-a)h} \mathbf{E}_x^0 \exp\left\{\int_0^{t} \xi(X_s)\,\mathrm{d}s\right\} \chi_{\tau_a \leqslant t}$$

$$\leqslant \left[\mathbf{E}_x^0 \exp\left\{2\int_0^t \xi(X_s)\,\mathrm{d}s\right\}\right]^{1/2} w_a^h \mathrm{e}^{(x-a)h} [\mathbf{P}_x^0(\tau_a \leqslant t)]^{1/2}.$$

The first factor does not depend on a. The second term vanishes for $a \to -\infty$. Indeed, since $w_a^h = w^h(a)$, we get from (14) that $w_a^h \leqslant \mathrm{e}^{-ha}$ for any a sufficiently close to $-\infty$. Hence

$$\limsup_{a\to-\infty} w_a^h \mathrm{e}^{(x-a)h} [\mathbf{P}_x^0(\tau_a \leqslant t)]^{1/2} \leqslant \lim_{a\to-\infty} \exp\left\{-2ha + xh - \frac{(x-a)^2}{2t}\right\} = 0.$$

Since the value $a < \min\{x, 0\}$ in (17) is arbitrary, we conclude

$$\mathrm{e}^{-t\lambda_h} \mathbf{E}_x^{-h} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} w^h(X_t) \geqslant w^h(x). \tag{18}$$

For the upper estimate, we get analogously

$$\mathrm{e}^{-t\lambda_h} \mathbf{E}_x^{-h} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} w^h(X_t)$$

$$= w_a^h \mathrm{e}^{(x-a)h} \mathbf{E}_x^0 \exp\left\{\int_0^t [\xi(X_s) - \lambda(h)]\,\mathrm{d}s\right\} \chi_{\tau_a(X_\cdot) > t}$$

$$\times \mathbf{E}_{X_t}^0 \exp\left\{\int_0^{\tau_a} [\xi(X_s) - \lambda(h)]\,\mathrm{d}s\right\}$$

$$+ \mathbf{E}_x^0 \exp\left\{\int_0^t [\xi(X_s) - \lambda(h)]\,\mathrm{d}s\right\} \mathrm{e}^{(x-X_t)h} w^h(X_t) \chi_{\tau_a(X_\cdot) \leqslant t}$$

$$\leqslant w_a^h \mathrm{e}^{(x-a)h} \mathbf{E}_x^0 \exp\left\{\int_0^{\tau_a} [\xi(X_s) - \lambda(h)]\,\mathrm{d}s\right\}$$

$$+ \mathbf{E}_x^0 \exp\left\{\int_0^t [\xi(X_s) - \lambda(h)]\,\mathrm{d}s\right\} \mathrm{e}^{(x-X_t)h} w^h(X_t) \chi_{\tau_a \leqslant t}$$

$$= w^h(x) + \mathbf{E}_x^0 \exp\left\{\int_0^t [\xi(X_s) - \lambda(h)]\,\mathrm{d}s\right\} \mathrm{e}^{(x-X_t)h} w^h(X_t) \chi_{\tau_a \leqslant t}.$$

Again the last term is estimated by Hölder's inequality,

$$\mathbf{E}_x^0 \exp\left\{\int_0^t [\xi(X_s)-\lambda(h)]\,\mathrm{d}s\right\} \mathrm{e}^{(x-X_t)h} w^h(X_t)\chi_{\tau_a\leqslant t}$$

$$\leqslant \left[\mathbf{E}_x^0 \exp\left\{2\int_0^t [\xi(X_s)-\lambda(h)]\,\mathrm{d}s\right\} \mathrm{e}^{2(x-X_t)h}\{w^h(X_t)\}^2\right]^{1/2} [\mathbf{P}_x^0(\tau_a\leqslant t)]^{1/2}.$$

Both the factor

$$\exp\left\{2\int_0^t [\xi(X_s)-\lambda(h)]\,\mathrm{d}s\right\}$$

and

$$\mathrm{e}^{2(x-X_t)h}\left\{w^h(X_t)\right\}^2 \leqslant K^2\mathrm{e}^{2xh+2(h+1)|X_t|}$$

are square-integrable with respect to $\mathbf{P}_x^0$. This follows from the fact that ξ is bounded and that, by (14) and the continuity of w^h, there is a constant $K>0$ such that $w^h(y)<Ke^{|y|}$ for any $y\in\mathbb{R}$. Thus, the first expectation above is finite and does not depend on a, whereas the last probability vanishes as $a\to-\infty$. We get

$$\mathrm{e}^{-t\lambda_h}\mathbf{E}_x^{-h}\exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} w^h(X_t)\leqslant w^h(x). \qquad \square$$

To analyze the asymptotic mass propagation in our original Cauchy problem, which amounts to an analysis of large deviations for a Brownian motion with drift under exponentially weighted measures, we need the following theorem.

Theorem 2.2.

(i) *For any $h\in\mathbb{R}$ and each $x\in\mathbb{R}$, there exists the limit*

$$\lim_{t\to\infty}\frac{1}{t}\ln \mathbf{E}_x^{-h}\exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\}=\lambda_h \quad (\hat{P}\text{-a.s.}).$$

(ii) *The function λ_h, $h\in\mathbb{R}$, is non-random and finite. For $0\leqslant h\neq h_{\mathrm{cr}}$, the function λ_h is continuously differentiable. The function $\lambda_h+\frac{1}{2}h^2$ is monotonously non-decreasing and convex.*

The second part of this theorem is a recapitulation of the most important properties of the function λ_h which were derived above.

The proof of Theorem 2.2(i) is split into two parts. First, we restrict ourselves to the case $h\geqslant h_{\mathrm{cr}}$ and rely on the generalized eigenvalue equation (16). This is done in the next lemma. Second, we use elementary estimates and convexity arguments to carry forward the results to $h<h_{\mathrm{cr}}$. This is done in Lemma 2.4.

Lemma 2.3. *Suppose $h\geqslant h_{\mathrm{cr}}$ and let be $x\in\mathbb{R}$. Then*

$$\lim_{t\to\infty}\frac{1}{t}\ln \mathbf{E}_x^{-h}\exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\}=\lambda_h \quad (\hat{P}\text{-a.s.}).$$

Proof. Fix $h \geqslant h_{\mathrm{cr}}$, $x \in \mathbb{R}$ and $\hat{\omega} \in \hat{\Omega}$. Choose $D > \max\{4h, \sqrt{8(c-\lambda_h)}\}$, where c is the upper bound of ξ. Then we have $D > 4h$ for sufficiently large $t \geqslant 0$ and $c - D^2/8 < \lambda_h$. We estimate

$$\begin{aligned}
\mathbf{E}_x^{-h} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} \chi_{|X_t|>Dt} &\leqslant \mathrm{e}^{ct}\mathbf{P}_x^{-h}(|X_t| > Dt) \\
&= \mathrm{e}^{ct}\mathbf{P}(|W_t - ht + x| > Dt) \\
&\leqslant 2\mathrm{e}^{ct}\mathbf{P}(W_t \geqslant Dt/2) \\
&\leqslant \exp\{ct - D^2 t/8\}. \qquad (19)
\end{aligned}$$

Hence

$$\limsup_{t\to\infty} \frac{1}{t} \ln \mathbf{E}_x^{-h} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} \chi_{|X_t|>Dt} < c - D^2/8 < \lambda_h. \qquad (20)$$

Using (14) we find that

$$\mathrm{e}^{-\alpha t} \sup_{|x|\leqslant Dt} w^h(x) \leqslant 1, \quad \mathrm{e}^{\alpha t} \inf_{|x|\leqslant Dt} w^h(x) \geqslant 1 \qquad (21)$$

for any $\alpha > 0$, $D > 0$ and sufficiently large t. Thus,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t} \ln \mathbf{E}_x^{-h} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} \chi_{|X_t|\leqslant Dt} \\
&\quad\leqslant \limsup_{t\to\infty} \frac{1}{t} \ln \mathbf{E}_x^{-h} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} \mathrm{e}^{\alpha t} w^h(X_t) \\
&\quad\leqslant \alpha + \lambda_h + \limsup_{t\to\infty} \frac{1}{t} \ln w^h(x) \\
&\quad= \alpha + \lambda_h.
\end{aligned}$$

The last inequality follows from Lemma 2.1. Since α can be chosen arbitrarily small, we have together with (20),

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t} \ln \mathbf{E}_x^{-h} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} \\
&\quad\leqslant \limsup_{t\to\infty} \frac{1}{t} \ln \left[\mathbf{E}_x^{-h} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} \chi_{|X_t|>Dt}\right. \\
&\qquad \left. + \mathbf{E}_x^{-h} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} \chi_{|X_t|\leqslant Dt}\right] \\
&\quad\leqslant \lambda_h. \qquad (22)
\end{aligned}$$

For the lower estimate, we observe that, because of (14),

$$\mathbf{E}_x^{-h} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} \chi_{|X_t|>Dt} w^h(X_t)$$

$$\leqslant \mathbf{E}_x^{-h} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s + \alpha|X_t|\right\} \chi_{|X_t|>Dt},$$

for any $0<\alpha<1, D>0$ and sufficiently large t. By transformation of drift we get

$$\mathbf{E}_x^{-h} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s + \alpha|X_t|\right\} \chi_{|X_t|>Dt}$$

$$\leqslant \mathbf{E}_x^{-h+\alpha} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s - \alpha h t + \frac{1}{2}\alpha^2 t + \alpha x\right\} \chi_{X_t>Dt}$$

$$+ \mathbf{E}_x^{-h-\alpha} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s + \alpha h t + \frac{1}{2}\alpha^2 t - \alpha x\right\} \chi_{X_t<-Dt}.$$

Following the same argument as in (20), we get, for suitable $D>0$,

$$\limsup_{t\to\infty} \frac{1}{t} \ln \mathbf{E}_x^{-h} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s + \alpha|X_t|\right\} \chi_{|X_t|>Dt} < \lambda_h - 1 \tag{23}$$

and hence

$$\limsup_{t\to\infty} \frac{1}{t} \ln \mathbf{E}_x^{-h} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} w^h(X_t)\chi_{|X_t|>Dt} < \lambda_h - 1.$$

Note that D can be chosen independent of α, since α is small. Thus, we have from (21) and Lemma 2.1

$$\liminf_{t\to\infty} \frac{1}{t} \ln \mathbf{E}_x^{-h} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} \chi_{|X_t|\leqslant Dt}$$

$$\geqslant \liminf_{t\to\infty} \frac{1}{t} \ln \left[\mathbf{E}_x^{-h} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} \mathrm{e}^{-\alpha t} w^h(X_t)\right.$$

$$\left. - \mathbf{E}_x^{-h} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} \mathrm{e}^{-\alpha t} w^h(X_t)\chi_{|X_t|>Dt}\right]$$

$$= \liminf_{t\to\infty} \frac{1}{t} \ln \left[\mathrm{e}^{(\lambda_h-\alpha)t} w^h(x)\right.$$

$$\left. - \mathbf{E}_x^{-h} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} \mathrm{e}^{-\alpha t} w^h(X_t)\chi_{|X_t|>Dt}\right]$$

$$= \lambda_h - \alpha$$

for any $0 < \alpha < 1$ and suitable $D > 0$. Hence

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{t} \ln \mathbf{E}_x^{-h} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} \\
&\quad \geqslant \liminf_{t\to\infty} \frac{1}{t} \ln \mathbf{E}_x^{-h} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} \chi_{|X_t|\leqslant Dt} \\
&\quad \geqslant \lambda_h.
\end{aligned}$$

Together with (22) the claim is proven. □

By symmetry, it is easy to see that the assertion of the above lemma is valid for $h \leqslant -\bar{h}_{\mathrm{cr}}$ as well.

A convex function that is larger or equal than $-\bar{g}$ on the interval $[-\bar{h}_{\mathrm{cr}}, h_{\mathrm{cr}}]$ and equals $-\bar{g}$ at the end points of the interval is equal to $-\bar{g}$ on the whole interval. This is the idea of the proof of the next lemma.

Lemma 2.4. *Suppose that* $-\bar{h}_{\mathrm{cr}} \leqslant h \leqslant h_{\mathrm{cr}}$ *and let be* $x \in \mathbb{R}$. *Then*

$$\lim_{t\to\infty} \frac{1}{t} \ln \mathbf{E}_x^{-h} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} = \lambda_h \quad (\hat{P}\text{-a.s.}).$$

Proof. Fix $\hat{\omega} \in \hat{\Omega}$ and $x \in \mathbb{R}$. By Girsanov's formula, it is sufficient to show that, for $z \in [-\bar{h}_{\mathrm{cr}}, h_{\mathrm{cr}}]$,

$$\lim_{t\to\infty} \frac{1}{t} \ln \mathbf{E}_x^0 \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s - zX_t\right\} = -\bar{g}.$$

Let be $z \in \mathbb{R}$, $\delta > 0$ and $x \in \mathbb{R}$. Then by property (A),

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{t} \ln \mathbf{E}_x^0 \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s - zX_t\right\} \\
&\quad \geqslant \liminf_{t\to\infty} \frac{1}{t} \ln \mathbf{E}_x^0 \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s - zX_t\right\} \chi_{X_t\in U_\delta(0)} \\
&\quad \geqslant \liminf_{t\to\infty} \frac{1}{t} \ln \inf_{x\in U_\delta(0)} \mathbf{E}_x^0 \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s - |z|\delta\right\} \chi_{X_t\in U_\delta(0)} \\
&\quad \geqslant \liminf_{t\to\infty} \left(-\frac{|z|\delta}{t}\right) + \liminf_{t\to\infty} \frac{1}{t} \ln \inf_{x\in U_\delta(0)} \mathbf{E}_x^0 \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} \chi_{X_t\in U_\delta(0)} \\
&\quad = -\bar{g}. \qquad (24)
\end{aligned}$$

The function

$$\tilde{\lambda}(z) := \limsup_{t\to\infty} \frac{1}{t} \ln \mathbf{E}_x^0 \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s - zX_t\right\}$$

is convex as can easily be seen by Hölder's inequality. Since $\tilde{\lambda}(h_{\mathrm{cr}}) = \lambda(h_{\mathrm{cr}}) = -\bar{g}$ and by the remark following Lemma 2.3, $\tilde{\lambda}(-\bar{h}_{\mathrm{cr}}) = \lambda(-\bar{h}_{\mathrm{cr}}) = -\bar{g}$, we find

$$\limsup_{t\to\infty} \frac{1}{t} \ln \mathbf{E}_x^0 \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s - zX_t\right\} \leqslant -\bar{g}$$

for any $z \in (-\bar{h}_{\mathrm{cr}}, h_{\mathrm{cr}})$. This proves the claim. □

3. Asymptotic speed of mass propagation

We shall now consider the asymptotic mass distribution of the solution u of our originally Cauchy problem (1). To this end, we study the asymptotic behavior of the measure Φ_t^h, which was defined in (4). After some preliminary lemmata, we will prove our main Theorem 1.

First, we prove a simple lemma which helps us to handle the measure Φ_t^h.

Lemma 3.1. *For any* $A \in \mathscr{B}(\mathbb{R}), t > 0, h \in \mathbb{R}$,

$$\int_A u(t,x)\,\mathrm{d}x = \mathbf{E}_0^{-h} \chi_A(X_t) \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\}.$$

Proof. Fix $h \in \mathbb{R}$, $\hat{\omega} \in \hat{\Omega}$ and $t > 0$. The function

$$P^h(t,x,\Gamma) := \mathbf{E}_x^h \chi_\Gamma(X_t) \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\},$$

$\Gamma \in \mathscr{B}(\mathbb{R})$, $x \in \mathbb{R}$, is a transition function with density $p^h(t,x,y)$ satisfying Kolmogorov's backward differential equation

$$\frac{\partial p^h(t,x,y)}{\partial t} = \frac{1}{2}\frac{\partial^2 p^h(t,x,y)}{\partial x^2} + h\frac{\partial p^h(t,x,y)}{\partial x} + \xi(x)p^h(t,x,y),$$

as well as Kolmogorov's forward differential equation

$$\frac{\partial p^h(t,x,y)}{\partial t} = \frac{1}{2}\frac{\partial^2 p^h(t,x,y)}{\partial y^2} - h\frac{\partial p^h(t,x,y)}{\partial y} + \xi(y)p^h(t,x,y),$$

(cp. Dynkin, 1965). Thus, it is easy to see that

$$p^h(t,x,y) = p^{-h}(t,y,x).$$

By the Feynman-Kac formula and by using that $u_0 = \delta_0$, we get

$$\begin{aligned}\int_{\mathbb{R}} u(t,x)\chi_A(x)\,\mathrm{d}x &= \int_{\mathbb{R}}\int_{\mathbb{R}} p^h(t,x,y)\,u_0(y)\,\chi_A(x)\,\mathrm{d}y\,\mathrm{d}x \\ &= \mathbf{E}_0^{-h}\chi_A(X_t)\,\exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\},\end{aligned}$$

for any $A \in \mathscr{B}(\mathbb{R})$. □

Now we consider a Wiener process with drift under exponentially weighted measures which depend on the environment. We derive large deviation estimates using

Gärtner-Ellis' theorem. The logarithmic asymptotics of the suitable moment generating function are determined by means of the generalized eigenvalue equation which has been studied in paragraph 2. Recall that $U_\varepsilon[y], \varepsilon > 0$ denotes the ε-neighborhood of a point $y \in \mathbb{R}$ and $U_\varepsilon^c[y]$ is it's complement.

Lemma 3.2. *Let $v = (\mathrm{d}/\mathrm{d}h)\lambda_h + h$. For any $\varepsilon > 0$, the following inequality holds $\hat{P}$-a.s.,*

$$\limsup_{t\to\infty} \frac{1}{t} \ln \mathbf{E}_0^{-h} \chi_{U_\varepsilon^c[-v]}(t^{-1}X_t) \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} < \lambda_h. \tag{25}$$

Proof. Fix $\hat{\omega} \in \hat{\Omega}$. Let $X_\cdot^{x,h}$ denote a Brownian motion on $(\Omega, \mathcal{N}, \mathbf{P})$ with constant drift $h \in \mathbb{R}$ starting in $x \in \mathbb{R}$, as defined in (2). Denote by $\mathbf{P}_x^h$ the law of $X_\cdot^{x,h}$. The equality

$$P_t^h(B) := \frac{\mathbf{E}\exp\{\int_0^t \xi(X_s^{0,-h})\,\mathrm{d}s\}\chi_B}{\mathbf{E}\exp\{\int_0^t \xi(X_s^{0,-h})\,\mathrm{d}s\}}, \quad B \in \mathcal{N},$$

defines a family of weighted probability measures P_t^h, $t > 0$, in the measure space $(\Omega, \mathcal{N})$. We will use large deviation estimates for the family of random vectors

$$\eta_t = \frac{1}{t} X_t^{0,-h}$$

on the probability spaces $(\Omega, \mathcal{N}, \mathbf{P}_0^{h,t})$, $t > 0$. Consider the function $G^{h,t} : \mathbb{R} \to [0, \infty)$, $h \in \mathbb{R}$, $t > 0$,

$$\begin{aligned} G^{h,t}(z) &:= E_t^h \exp\{tz\eta_t\} \\ &= E_t^h \exp\{zX_t^{0,-h}\}. \end{aligned}$$

Girsanov's theorem yields

$$\begin{aligned} &\mathbf{E}_0^{-h+z} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} \\ &\quad = \mathbf{E}_0^{-h} \exp\left\{zX_t - \frac{1}{2}t[(h-z)^2 - h^2]\right\} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\}. \end{aligned}$$

Hence

$$\begin{aligned} G^{h,t}(z) &= E_t^h \exp\{zX_t^{0,-h}\} \\ &= \frac{\mathbf{E}_0^{-h+z} \exp\{\int_0^t \xi(X_s)\,\mathrm{d}s\} \exp\{\frac{1}{2}t[(h-z)^2 - h^2]\}}{\mathbf{E}_0^{-h} \exp\{\int_0^t \xi(X_s)\,\mathrm{d}s\}}, \end{aligned}$$

and thus, by Theorem 2.2

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t} \ln G^{h,t}(z) &= \lambda_{h-z} - \lambda_h + \tfrac{1}{2}(h-z)^2 - \tfrac{1}{2}h^2 \\ &= \lambda(h-z) - \lambda(h) \\ &=: H(z). \end{aligned} \tag{26}$$

It follows from the properties of λ, which were summarized in Theorem 2.2, that the function $H(z)$, $z \in \mathbb{R}$, is finite, convex, and continuous differentiable for any z different

from the critical values $h_{\mathrm{cr}}, \bar{h}_{\mathrm{cr}}$, which were defined in (10). The Fenchel-Legendre transform S of H, defined by the equality

$$S(y) := \sup_{z\in\mathbb{R}}[yz - H(z)], \qquad y \in \mathbb{R},$$

is therefore convex and lower semicontinuous. Its level sets

$$\Lambda(s) := \{y \in \mathbb{R} : S(y) \leqslant s\}$$

are compact.

By Gärtner–Ellis' theorem, the function S is the rate function for the large deviation upper estimate of (η_t, P_t^h). Hence

$$\limsup_{t\to\infty} \frac{1}{t} \ln P_t^h(\eta_t \in F) \leqslant -\inf_{y\in F} S(y) \tag{27}$$

for any closed set $F \subset \mathbb{R}$. Therefore

$$\limsup_{t\to\infty} \frac{1}{t} \ln \frac{\mathbf{E}_0^{-h} \exp\{\int_0^t \xi(X_s)\,\mathrm{d}s\}\chi_{U_\varepsilon^c[-v]}(t^{-1}X_t)}{\mathbf{E}_0^{-h} \exp\{\int_0^t \xi(X_s)\,\mathrm{d}s\}} \leqslant -\inf_{y\in U_\varepsilon^c[-v]} S(y)$$

for any $\varepsilon > 0$ and $v = (\mathrm{d}/\mathrm{d}h)\lambda(h)$. Since

$$\lim_{t\to\infty} t^{-1} \ln \mathbf{E}_0^{-h} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} = \lambda_h$$

by Theorem 2.2, and since $U_\varepsilon^c[-v]$ is closed, we get the claim if we can show that $-v$ is the only zero of S and consequently $\inf_{y\in U_\varepsilon^c[-v]} S(y) > 0$. Indeed, if $h \notin \{-\bar{h}_{\mathrm{cr}}, h_{\mathrm{cr}}\}$, then $H(\gamma)$ is differentiable at $\gamma = 0$ and hence it is easily verified that $S(-v) = 0$. Suppose now that $S(x) = 0$ for some $x \in \mathbb{R}$. Then $H(\gamma) \geqslant x\gamma$ for any $\gamma \in \mathbb{R}^1$ and hence

$$H'(0) = \lim_{\gamma\downarrow 0} \frac{H(\gamma)}{\gamma} \geqslant x \geqslant \lim_{\gamma\uparrow 0} \frac{H(\gamma)}{\gamma} = H'(0).$$

Thus, $x = -v$. □

Now we have got all means to prove the above Theorem 1 on the asymptotic speed of mass propagation. Recall definition (4) of the measure Φ_t^h and note that Φ_t^h is well defined. Indeed, by Lemma 3.1 we find that the denominator is strictly positive and finite. We show that Φ_t^h converges weakly to the Dirac measure δ_{-v} at point $-v$ ($\hat{P}$-a.s.).

Proof of Theorem 1. By Lemma 3.1, we have

$$\int_{\mathbb{R}} u(t,x)\,\mathrm{d}x = \mathbf{E}_0^{-h} \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\}. \tag{28}$$

Hence part (i) of Theorem 1 follows directly from Lemma 2.2.

Let us prove part (ii). Lemma 3.2 yields that

$$\lim_{t\to\infty} \frac{1}{t} \ln \int_{U_\varepsilon^c[-v]} u(t,tx)\,\mathrm{d}x$$

$$= \lim_{t\to\infty} \frac{1}{t} \ln \left[t^{-1} \mathbf{E}_0^{-h} \chi_{U_\varepsilon^c[-v]}(t^{-1}X_t) \exp\left\{ \int_0^t \xi(X_s)\,\mathrm{d}s \right\} \right]$$

$$< \lambda_h.$$

According to Theorem 2.2(i),

$$\lim_{t\to\infty} \frac{1}{t} \ln \int_{\mathbb{R}} u(t, ty)\,\mathrm{d}y = \lambda_h.$$

Therefore, we have

$$\lim_{t\to\infty} \Phi_t^h(U_\varepsilon^c[-v]) = 0, \quad \lim_{t\to\infty} \Phi_t^h(U_\varepsilon[-v]) = 1 \tag{29}$$

for any $\varepsilon > 0$.

Let $A \in \mathscr{B}(\mathbb{R})$ be a δ_{-v}-continuity set, i.e., we require that $-v$ is not contained in the boundary ∂A of the set A. Then $-v$ is contained in either the interior $\mathrm{int}(A)$ of A or the interior $\mathrm{int}(A^c)$ of the complement A^c. In the first case, we conclude

$$1 \geqslant \lim_{t\to\infty} \Phi_t^h(A) \geqslant \lim_{t\to\infty} \Phi_t^h(U_\varepsilon[-v]) = 1 = \delta_{-v}(A)$$

for some $\varepsilon > 0$. In the second case, we find

$$0 \leqslant \lim_{t\to\infty} \Phi_t^h(A) \leqslant \lim_{t\to\infty} \Phi_t^h(U_\varepsilon^c[-v]) = 0 = \delta_{-v}(A)$$

for some $\varepsilon > 0$. Hence Φ_t^h converges weakly to the Dirac measure δ_{-v} at point $-v$ ($\hat{P}$-a.s.). □

4. Examples

We schedule three examples. The first one illustrates that there is $h_{\mathrm{cr}} > 0$ in a really random situation. In the second example, we see that our results for random potentials agree with those in the periodic case. Here we get $h_{\mathrm{cr}} = 0$ but $v < h$ at least for small $h > 0$. The last example connects our setting with that one of Greven and den Hollander who considered a branching random walk in random environment on the one-dimensional lattice. In the first and in the third example, the potential ξ is not continuous but piecewise continuous. This difficulty can be overcome by imposing gluing conditions at the points of discontinuity.

1. Let the random potential ξ be a stationary Markov process with two states a and b ($0 \leqslant a < b$). The transition densities ν (transition from a to b) and γ (transition from b to a) are to be positive. Then it is known from Gärtner (1985) that

$$\mu(z) = \begin{cases} -\displaystyle\int_{\beta(z)}^{\alpha(z)} x\,[p^z(x) + q^z(x)]\,\mathrm{d}x & \text{if } z \leqslant -b, \\ +\infty & \text{if } z > -b, \end{cases}$$

where

$$\alpha(z) = \sqrt{-2(z+a)}, \quad \beta(z) = \sqrt{-2(z+b)},$$

and

$$p^z(x) = \frac{D(z)}{\alpha^2(z) - x^2} H^z(x),$$

$$q^z(x) = \frac{D(z)}{x^2 - \beta^2(z)} H^z(x),$$

with

$$H^z(x) = \left(\frac{\alpha(z) - x}{\alpha(z) + x}\right)^{\nu/2\alpha(z)} \left(\frac{x - \beta(z)}{x + \beta(z)}\right)^{\gamma/2\beta(z)}.$$

Here $D(z)$ is a constant such that

$$\int_{\beta(z)}^{\alpha(z)} [p^z(x) + q^z(x)]\,\mathrm{d}x = 1. \tag{30}$$

Set $\alpha := \alpha(b) > 0$ and consider, for $0 < x < \alpha$,

$$p^b(x) + q^b(x) = \lim_{z \uparrow -b} (p^z(x) + q^z(x))$$

$$= D\left(\frac{\alpha - x}{\alpha + x}\right)^{\nu/2\alpha} \mathrm{e}^{-\gamma/x} \left[\frac{1}{\alpha^2 - x^2} + \frac{1}{x^2}\right],$$

where $D := D(b)$ is the normalizing constant of (30). Obviously, $x(p^b(x) + q^b(x)) > 0$ for $0 < x < \alpha$. Since the function $x(p^z(x) + q^z(x))$ is continuous in (x, z) for $z \leqslant -b, \beta(z) \leqslant x \leqslant \alpha(z)$, we can exchange the integral with the limit and get

$$h_{\mathrm{cr}} = -\mu(b) = \int_0^\alpha x\,[p^b(x) + q^b(x)]\,\mathrm{d}x > 0.$$

2. Let now $\xi(x, \hat{\omega}) = c(x + \zeta(\hat{\omega}))$, where $c(x)$ is a periodic continuous positive function of period 1 and $\zeta(\hat{\omega})$ is a random variable uniformly distributed in the unit interval $[0, 1]$. In this case the function $\lambda(z)$, $z \in \mathbb{R}$, is the eigenvalue of the differential operator

$$L^z = \tfrac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}x^2} - z \frac{\mathrm{d}}{\mathrm{d}x} + \left[c(x) + \frac{z^2}{2}\right],$$

corresponding to a positive and periodic eigenfunction of period 1 (cp. Freidlin, 1985, Section 7.6). Thus, $\mu(-\lambda(z)) = -z$ for all $z \geqslant 0$, and therefore $h_{\mathrm{cr}} = -\mu(\bar{g}) = \lim_{h \downarrow 0}(-\mu(-\lambda(h))) = 0$.

Ellerich has shown in Ellerich (1997, Chapter 2.1), that $(\mathrm{d}/\mathrm{d}h)\lambda(h) < h$ at least for small $h > 0$. Hence we have $v_h < h$ in our example.

3. Let $\{\xi_k\}_{k \in \mathbf{Z}}$ be a sequence of independent, identically distributed random variables on $(\hat{\Omega}, \hat{\mathscr{A}}, \hat{P})$ with values in some compact interval. We put

$$\xi(x) = \sum_{k \in \mathbf{Z}} \xi_k \chi_{[k, k+1)}(x), \quad x \in \mathbb{R}.$$

This potential is bounded, stationary and strongly mixing. It satisfies all assumptions on the potential that are made in this study, except that on continuity. But as mentioned above, this can be overcome by imposing gluing conditions.

Since this special potential is constant on intervals of the form $[k, k+1)$, $k \in \mathbf{Z}$, it best corresponds to the setting on the lattice which was chosen by Greven and den Hollander (1992).

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