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Two-sided bounds for L_p-norms of combinations of products of independent random variables

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Abstract

We show that for every positive p, the L_p-norm of linear combinations (with scalar or vector coefficients) of products of i.i.d. random variables, whose moduli have a nondegenerate distribution with the p-norm one, is comparable to the l_p-norm of the coefficients and the constants are explicit. As a result the same holds for linear combinations of Riesz products.

We also establish the upper and lower bounds of the L_p-moments of partial sums of perpetuities.

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1. Introduction and main results

Let $X, X_1, X_2, \ldots$ be i.i.d. nondegenerate nonnegative r.v.'s with finite mean. Define

$$R_0 := 1 \quad \text{and} \quad R_i := \prod_{j=1}^{i} X_j \quad \text{for } i = 1, 2, \ldots. \tag{1}$$

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Then obviously for any vectors $v_0, v_1, \ldots, v_n$ in a normed space $(F, \|\ \|)$, $\mathbb{E}\|\sum_{i=0}^n v_i R_i\| \leq \sum_{i=0}^n \|v_i\|\mathbb{E}R_i$. In [17] it was shown that the opposite inequality holds, i.e.

$$\mathbb{E}\left\|\sum_{i=0}^n v_i R_i\right\| \geq c_X \sum_{i=0}^n \|v_i\|\mathbb{E}R_i,$$

where c_X is a constant, which depends only on the distribution of X.

In this paper we present similar estimates for L_p-norms. Our main result is the following.

Theorem 1. *Let $p > 0$ and $X, X_1, X_2, \ldots$ be i.i.d. r.v.'s such that $|X|$ is nondegenerate, $\mathbb{E}|X|^p < \infty$ and let R_i be defined by (1). Then there exist constants $0 < c_{p,X} \leq C_{p,X} < \infty$ which depend only on p and the distribution of X such that for any vectors $v_0, v_1, \ldots, v_n$ in a normed space $(F, \|\ \|)$,*

$$c_{p,X} \sum_{i=0}^n \|v_i\|^p \mathbb{E}|R_i|^p \leq \mathbb{E}\left\|\sum_{i=0}^n v_i R_i\right\|^p \leq C_{p,X} \sum_{i=0}^n \|v_i\|^p \mathbb{E}|R_i|^p.$$

Remark. The assumption that $|X|$ has a nondegenerate distribution is crucial. If $\mathbb{P}(X_i = \pm 1) = 1/2$ then (R_i) are i.i.d. symmetric ± 1 r.v's and by the Khintchine inequality $\mathbb{E}|\sum_{i=1}^n R_i|^p$ is of the order $n^{p/2}$, whereas $\sum_{i=1}^n \mathbb{E}|R_i|^p = n$.

In fact we prove a more general result that does not require the identical distribution assumption. Namely, suppose that

$$X_1, X_2, \ldots \text{ are independent r.v.'s such that } \mathbb{E}|X_i|^p < \infty. \tag{2}$$

Further assumptions depend on whether $p \leq 1$. For $p \in (0, 1]$ we assume that

$$\exists_{\lambda<1} \quad \forall_i\ \mathbb{E}|X_i|^{p/2} \leq \lambda(\mathbb{E}|X_i|^p)^{1/2} \tag{3}$$

and

$$\exists_{\delta>0, A>1} \quad \forall_i\ \mathbb{E}(|X_i|^p - \mathbb{E}|X_i|^p)\mathbb{1}_{\{\mathbb{E}|X_i|^p \leq |X_i|^p \leq A\mathbb{E}|X_i|^p\}} \geq \delta\mathbb{E}|X_i|^p. \tag{4}$$

Theorem 2. *Let $0 < p \leq 1$ and $X_1, X_2, \ldots$ satisfy assumptions (2)–(4). Then for any vectors $v_0, v_1, \ldots, v_n$ in a normed space $(F, \|\ \|)$ we have*

$$c(p, \lambda, \delta, A) \sum_{i=0}^n \|v_i\|^p \mathbb{E}|R_i|^p \leq \mathbb{E}\left\|\sum_{i=0}^n v_i R_i\right\|^p \leq \sum_{i=0}^n \|v_i\|^p \mathbb{E}|R_i|^p,$$

where $c(p, \lambda, \delta, A)$ is a constant which depends only on p, λ, δ and A.

For $p > 1$ to obtain the lower bound we assume that

$$\begin{aligned} &\exists_{\mu>0, A<\infty}\ \forall_i\ \mathbb{E}\|X_i| - \mathbb{E}|X_i\| \geq \mu(\mathbb{E}|X_i|^p)^{1/p} \quad \text{and} \\ &\qquad \mathbb{E}\|X_i| - \mathbb{E}|X_i\|\mathbb{1}_{\{|X_i|>A(\mathbb{E}|X_i|^p)^{1/p}\}} \leq \frac{1}{4}\mu(\mathbb{E}|X_i|^p)^{1/p} \end{aligned} \tag{5}$$

and

$$\exists_{q>\max\{p-1,1\}}\ \exists_{\lambda<1}\ \forall_i\ (\mathbb{E}|X_i|^q)^{1/q} \leq \lambda(\mathbb{E}|X_i|^p)^{1/p}. \tag{6}$$

For the upper bound we need the condition

$$\forall_{k=1,2,\ldots,\lceil p\rceil-1}\ \exists_{\lambda_k<1}\ \forall_i\ (\mathbb{E}|X_i|^{p-k})^{1/(p-k)} \le \lambda_k(\mathbb{E}|X_i|^{p-k+1})^{1/(p-k+1)}. \tag{7}$$

Theorem 3. *Let $p > 1$ and $X_1, X_2, \ldots$ satisfy assumptions (2), (5), (6) and (7). Then for any vectors $v_0, v_1, \ldots, v_n$ in a normed space $(F, \|\ \|)$ we have*

$$c(p,\mu,A,q,\lambda)\sum_{i=0}^{n}\|v_i\|^p\mathbb{E}|R_i|^p \le \mathbb{E}\left\|\sum_{i=0}^{n}v_iR_i\right\|^p \le C(p,\lambda_1,\ldots,\lambda_{\lceil p\rceil-1})\sum_{i=0}^{n}\|v_i\|^p\mathbb{E}|R_i|^p,$$

where $c(p,\mu,A,q,\lambda)$ is a positive constant which depends only on p,μ,A,q and λ and $C(p,\lambda_1,\ldots,\lambda_{\lceil p\rceil-1})$ is a constant which depends only on $p,\lambda_1,\ldots,\lambda_{\lceil p\rceil-1}$.

Remark. Proofs presented below show that Theorem 2 holds with

$$c(p,\lambda,\delta,A)=\frac{\delta^3}{16k},\quad \text{where } k \text{ is an integer such that } k\lambda^{2k-2}\le\frac{\delta^3(1-\lambda)^2}{2^{12}A}.$$

In Theorem 3 we can take

$$C(p,\lambda_1,\ldots,\lambda_{\lceil p\rceil-1})=2^{\frac{p(p+1)}{2}}\prod_{1\le j\le\lceil p\rceil-1}\frac{1}{1-\lambda_j^{p-j}}$$

and

$$c(p,\mu,A,q,\lambda)=\frac{\mu^{3p}}{8k\cdot 2^{10p}\cdot 3^p},$$

where k is an integer such that $k\lambda^{pk}\le\dfrac{(1-\lambda)\mu^{3p}}{8C_0\cdot 2^{10p}\cdot 3^p}$,

$$C_0=(1-\lambda)^{1-p}\left(\frac{2A}{3\lambda}\right)^p\left(\frac{2p}{(q+1-p)\ln 2}\right)^{\frac{p}{q}}48^{\frac{2p^2}{\min\{p-1,1\}}}.$$

Another consequence of Theorem 1 is an estimate for L_p-norms of linear combinations of the Riesz products. Let $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$ be the one dimensional torus and m be the normalized Haar measure on $\mathbb{T}$. The Riesz products are defined on $\mathbb{T}$ by the formula

$$\bar{R}_i(t)=\prod_{j=1}^{i}(1+\cos(n_jt)),\quad i=1,2,\ldots,$$

where $(n_k)_{k\ge1}$ is a lacunary increasing sequence of positive integers.

It is well known that if coefficients n_k grow sufficiently fast then $\|\sum_{i=0}^n a_i\bar{R}_i\|_{L_p(\mathbb{T})}\sim(\mathbb{E}|\sum_{i=0}^n a_iR_i|^p)^{1/p}$ for $p\ge1$, where R_i are products of independent random variables distributed as $\bar{R}_1$. Together with Theorem 1 this gives an estimate for $\|\sum_{i=0}^n a_i\bar{R}_i\|_{L_p(\mathbb{T})}$. Here is the more quantitative result.

Corollary 4. *Suppose that $(n_k)_{k\ge1}$ is an increasing sequence of positive integers such that $n_{k+1}/n_k\ge3$ and $\sum_{k=1}^{\infty}\frac{n_k}{n_{k+1}}<\infty$. Then for any coefficients $a_0,a_1,\ldots,a_n\in\mathbb{R}$ and*

$p \geq 1$,

$$c_p \sum_{i=0}^{n} |a_i|^p \int_{\mathbb{T}} |\bar{R}_i(t)|^p \, \mathrm{d}m(t) \leq \int_{\mathbb{T}} \left| \sum_{i=0}^{n} a_i \bar{R}_i(t) \right|^p \mathrm{d}m(t) \leq C_p \sum_{i=0}^{n} |a_i|^p \int_{\mathbb{T}} |\bar{R}_i(t)|^p \, \mathrm{d}m(t),$$

where $0 < c_p \leq C_p < \infty$ *are constants depending only on* p *and the sequence* (n_k).

Proof. Let $X_1, X_2, \ldots$ be independent random variables distributed as $1 + \cos(Y)$, where Y is uniformly distributed on $[0, 2\pi]$ and R_i be given by (1). By the result of Y. Meyer [18], $\frac{1}{A} \| \sum_{i=0}^{n} a_i \bar{R}_i \|_{L_p} \leq (\mathbb{E} | \sum_{i=0}^{n} a_i R_i|^p)^{1/p} \leq A \| \sum_{i=0}^{n} a_i \bar{R}_i \|_{L_p}$ (in particular also $\frac{1}{A} \| \bar{R}_i \|_{L_p} \leq (\mathbb{E} R_i^p)^{1/p} \leq A \| \bar{R}_i \|_{L_p}$), where A depends only on p and the sequence (n_k). Thus the estimate follows by Theorem 1. □

Theorem 1 has also an immediate application to the stationary $\mathbb{R}^d$-valued solution S of the random difference equation

$$S = XS + B, \tag{8}$$

where the equality is meant in law and (X, B) is a random variable with values in $[0, \infty) \times \mathbb{R}^d$ independent of S such that for some $p > 0$,

$$\mathbb{E} X^p = 1, \quad \mathbb{E} \|B\|^p < \infty \quad \text{and} \quad \mathbb{P}(X = 1) < 1. \tag{GK1}$$

Over the last 40 years Eq. (8) and its various modifications have attracted a lot of attention [1–3,5,8,9,11–16,19,20]. It has a wide spectrum of applications including random walks in random environment, branching processes, fractals, finance and actuarial mathematics, telecommunications, various physical and biological models. In particular, the tail behaviour of S is of interest.

It is well known that in law

$$S = \sum_{i=1}^{\infty} R_{i-1} B_i,$$

where $R_{i-1} = X_1 \cdots X_{i-1}$, $R_0 = 1$ and $(X_i, B_i)_{i \geq 1}$ is an i.i.d. sequence of r.v.'s with the same distribution as (X, B). Under the additional assumption that

$$\log X \text{ conditioned on } \{X \neq 0\} \text{ is non lattice and } \mathbb{E} X^p \log^+ X < \infty, \tag{GK2}$$

S has a heavy tail behaviour, i.e. the limit

$$\lim_{t \to \infty} t^p \mathbb{P}(\|S\| > t) = c_\infty(X, B)$$

exists and $c_\infty(X, B)$ is strictly positive provided that $\mathbb{P}(Xv + B = v) < 1$ for every $v \in \mathbb{R}^d$. If $\mathbb{P}(Xv + B = v) = 1$ then $S_n = v - R_{n-1} v \to v = S$. Assumptions (GK1), (GK2) together with $\mathbb{P}(Xv + B = v) < 1$ will be later on referred to as the *Goldie–Kesten conditions*. Let

$$S_n = \sum_{i=1}^{n} R_{i-1} B_i.$$

It turns out that the sequence $\mathbb{E}\|S_n\|^p$ is closely related to $c_\infty(X, B)$. Recently, it has been proved in [6] that under the *Goldie–Kesten conditions* plus a little bit stronger moment assumption $\mathbb{E}(X^{p+\varepsilon} + \|B\|^{p+\varepsilon}) < \infty$ for some $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{np\rho}\mathbb{E}\|S_n\|^p = c_\infty(X, B) > 0,$$

where $\rho := \mathbb{E}X^p \log X$.

Now suppose that X, B are independent. Then Theorem 1 implies that for every n

$$c_{p,X}\mathbb{E}\|B\|^p \le \frac{1}{n}\mathbb{E}\|S_n\|^p \le C_{p,X}\mathbb{E}\|B\|^p, \tag{9}$$

which gives uniform bounds on the Goldie constant $c_\infty(X, B)$ depending only on the law of X and $\mathbb{E}\|B\|^p$ and independent of the dimension. Moreover, in some particular cases when constants $\lambda, \delta, \mu, q, \lambda_k$ in (3)–(7) can be estimated more carefully, (9) may give some information about the size of the Goldie constant which is of some value, especially in the situation when none of the existing formulae for it is satisfactory enough (see [7,10,6,4]).

We can go even further. With a slight modification of the proof we can get rid of independence of X, B and obtain the following theorem.

Theorem 5. *Suppose that F is a separable Banach space. Let $p > 0$ and let an i.i.d. sequence $(X, B), (X_1, B_1), \ldots$ with values in $[0, \infty) \times F$ be such that X is nondegenerate and $\mathbb{E}\|B\|^p, \mathbb{E}X^p < \infty$. Assume additionally that*

$$\mathbb{P}(Xv + B = v) < 1 \quad \text{for every } v \in F. \tag{10}$$

Then there are constants $c_p(X, B) > 0$ which depend on p and the distribution of (X, B) and $C_p(X) < \infty$ which depend on p and the distribution of X such that for every n,

$$c_p(X, B)\mathbb{E}\|B\|^p \sum_{i=1}^n \mathbb{E}R_{i-1}^p \le \mathbb{E}\left\|\sum_{i=1}^n R_{i-1}B_i\right\|^p \le C_p(X)\mathbb{E}\|B\|^p \sum_{i=1}^n \mathbb{E}R_{i-1}^p. \tag{11}$$

Theorem 5 specified to our situation with $\mathbb{E}X^p = 1$ gives

$$c_p(X, B)\mathbb{E}\|B\|^p \le \frac{1}{n}\mathbb{E}\|S_n\|^p \le C_p(X)\mathbb{E}\|B\|^p.$$

This leads to an estimate for the Goldie constant but now with $c_p(X, B), C_p(X)$ depending on the law of (X, B). Again, in particular cases, a careful examination of the constants involved in the proof may give a more satisfactory answer. Also, in view of Theorem 5, it would be worth relaxing the assumptions of [6].

The paper is organized as follows. In Sections 2 and 3 we derive lower bounds in Theorems 2 and 3. Then in Section 4 we establish upper bounds in both theorems. We conclude in Section 5 with a discussion of the proof of Theorem 5.

2. Lower bound for $p > 1$

In this section we will show the lower bound in Theorem 3. Since it is only a matter of normalization we will assume that

$$X_1, X_2, \ldots \text{ are independent r.v.'s such that } \mathbb{E}|X_i|^p = 1. \tag{12}$$

In particular this implies that $\mathbb{E}|R_i|^p = 1$ for all i.

We also set for $k = 1, 2, \ldots$

$$R_{k,k-1} \equiv 1 \quad \text{and} \quad R_{k,i} := \prod_{j=k}^{i} X_i \quad \text{for } i \geq k.$$

Observe that $R_i = R_k R_{k+1,i}$ for $i \geq k \geq 0$.

We begin with several lemmas.

Lemma 6. *Suppose that a r.v. X satisfies $\mathbb{E}\,|\,|X| - \mathbb{E}|X|\,| \geq \mu$ and $\mathbb{E}\,|\,|X| - \mathbb{E}|X|\,|\mathbb{1}_{\{|X|>A\}} \leq \frac{1}{4}\mu$. Then for all $p \geq 1$ and $u, v \in (F, \|\ \|)$ we have*

$$\mathbb{E}\|uX + v\|^p \geq \mathbb{E}\|uX + v\|^p \mathbb{1}_{\{|X|\leq A\}} \geq \frac{\mu^p}{8^p} \min\left\{1, \frac{1}{(\mathbb{E}|X|)^p}\right\} \max\{\|u\|^p, \|v\|^p\}.$$

Proof. Let Y has the same distribution as X conditioned on the set $\{|X| \leq A\}$. Let us define $t := \mathbb{E}Y$. Then $|t| \leq \mathbb{E}|Y| \leq \mathbb{E}|X|$. Clearly, $\mathbb{E}(|X| - \mathbb{E}|X|)_+ = \mathbb{E}(|X| - \mathbb{E}|X|)_- \geq \frac{1}{2}\mu$. Therefore,

$$\begin{aligned}
\mathbb{E}|X - t|\mathbb{1}_{\{|X|\leq A\}} &\geq \mathbb{E}\,|\,|X| - |t|\,|\mathbb{1}_{\{|X|\leq A\}} \geq \mathbb{E}(|X| - |t|)_+ \mathbb{1}_{\{|X|\leq A\}} \\
&\geq \mathbb{E}(|X| - \mathbb{E}|X|)_+ \mathbb{1}_{\{|X|\leq A\}} \\
&= \mathbb{E}(|X| - \mathbb{E}|X|)_+ - \mathbb{E}(|X| - \mathbb{E}|X|)_+ \mathbb{1}_{\{|X|>A\}} \\
&\geq \frac{1}{2}\mu - \mathbb{E}\,|\,|X| - \mathbb{E}|X|\,|\mathbb{1}_{\{|X|>A\}} \geq \frac{1}{2}\mu - \frac{1}{4}\mu = \frac{1}{4}\mu.
\end{aligned}$$

We obtain

$$|t|\mathbb{E}\|uY + v\| = \mathbb{E}\|v(t - Y) + (tu + v)Y\| \geq \|v\|\mathbb{E}|Y - t| - \|tu + v\|\mathbb{E}|Y|.$$

Since $\mathbb{E}\|uY + v\| \geq \|u\mathbb{E}Y + v\| = \|tu + v\|$ and $|t| \leq \mathbb{E}|Y| \leq \mathbb{E}|X|$ we have

$$\begin{aligned}
\mathbb{E}\|uY + v\| &\geq \frac{1}{|t| + \mathbb{E}|Y|}\|v\|\mathbb{E}|Y - t| \geq \frac{\|v\|}{2\mathbb{E}|X|\mathbb{P}(|X| \leq A)}\mathbb{E}|X - t|\mathbb{1}_{\{|X|\leq A\}} \\
&\geq \frac{\mu}{8\mathbb{E}|X|}\frac{\|v\|}{\mathbb{P}(|X| \leq A)}.
\end{aligned}$$

We arrive at

$$\begin{aligned}
\mathbb{E}\|uX + v\|^p \mathbb{1}_{\{|X|\leq A\}} &\geq \left(\mathbb{E}\|uX + v\|\mathbb{1}_{\{|X|\leq A\}}\right)^p = (\mathbb{E}\|uY + v\|\mathbb{P}(|X| \leq A))^p \\
&\geq \frac{\mu^p}{8^p(\mathbb{E}|X|)^p}\|v\|^p.
\end{aligned}$$

We also have

$$\mathbb{E}\|uY + v\| = \mathbb{E}\|u(Y - t) + tu + v\| \geq \|u\|\mathbb{E}|Y - t| - \|tu + v\|.$$

Therefore

$$\mathbb{E}\|uY + v\| \geq \frac{\|u\|}{2}\mathbb{E}|Y - t| \geq \frac{\mu}{8}\frac{\|u\|}{\mathbb{P}(X \leq A)}$$

and as before we get that $\mathbb{E}\|uX + v\|^p \mathbb{1}_{\{X\leq A\}} \geq \frac{\mu^p}{8^p}\|u\|^p$. □

Lemma 7. *Assume that (12) and (5) hold. Then for any $v_0, v_1, \dots, v_n \in (F, \|\ \|)$ we have*

$$\mathbb{E}\left\|\sum_{i=0}^{n} v_i R_i\right\|^p \geq \frac{\mu^{2p}}{64^p} \max_{1\leq i\leq n} \|v_i\|^p \geq \frac{\mu^{2p}}{64^p}\cdot\frac{1}{n}\sum_{i=1}^{n}\|v_i\|^p.$$

Proof. For $1 \leq j \leq n$ we have $\sum_{i=0}^{n} v_i R_i = Y + X_j(v_j R_{j-1} + X_{j+1}Z)$, where Y and Z are independent of X_j and X_{j+1}. Observe that $\mathbb{E}|X_j| \leq 1$ and $\mathbb{E}|X_{j+1}| \leq 1$. Thus, using Lemma 6 twice, we obtain

$$\mathbb{E}\left\|\sum_{i=0}^{n} v_i R_i\right\|^p \geq \frac{\mu^p}{8^p}\mathbb{E}\|v_j R_{j-1} + X_{j+1}Z\|^p \geq \frac{\mu^{2p}}{64^p}\|v_j\|^p\mathbb{E}|R_{j-1}|^p = \frac{\mu^{2p}}{64^p}\|v_j\|^p. \quad \square$$

Lemma 8. *Assume that (12) holds and there exist $q > 1$ and $0 < \lambda < 1$ such that for all i, $(\mathbb{E}|X_i|^q)^{1/q} \leq \lambda$. Then for any $v_0, v_1, \dots, v_n \in (F, \|\ \|)$ and $t > 0$,*

$$\mathbb{P}\left(\left\|\sum_{i=0}^{n} v_i R_i\right\|^p \geq t\sum_{i=0}^{n}\lambda^i\|v_i\|^p\right) \leq (1-\lambda)^{\frac{(1-p)q}{p}} t^{-\frac{q}{p}}.$$

Proof. Using Minkowski's and Hölder's inequalities we obtain

$$\begin{aligned}\left(\mathbb{E}\left\|\sum_{i=0}^{n} v_i R_i\right\|^q\right)^{\frac{1}{q}} &\leq \sum_{i=0}^{n}\left(\mathbb{E}\|v_i R_i\|^q\right)^{\frac{1}{q}} \leq \sum_{i=0}^{n}\|v_i\|\lambda^i = \sum_{i=0}^{n}\|v_i\|\lambda^{\frac{i}{p}}\lambda^{\frac{p-1}{p}i}\\ &\leq \left(\sum_{i=0}^{n}\|v_i\|^p\lambda^i\right)^{\frac{1}{p}}\left(\sum_{i=0}^{n}\lambda^i\right)^{\frac{p-1}{p}}.\end{aligned}$$

Thus,

$$\mathbb{E}\left\|\sum_{i=0}^{n} v_i R_i\right\|^q \leq \left(\sum_{i=0}^{n}\|v_i\|^p\lambda^i\right)^{\frac{q}{p}}(1-\lambda)^{-\frac{(p-1)q}{p}}.$$

By Chebyshev's inequality we get

$$\mathbb{P}\left(\left\|\sum_{i=0}^{n} v_i R_i\right\|^q \geq t^{\frac{q}{p}}\left(\sum_{i=0}^{n}\lambda^i\|v_i\|^p\right)^{\frac{q}{p}}\right) \leq (1-\lambda)^{\frac{(1-p)q}{p}} t^{-\frac{q}{p}}. \quad \square$$

Lemma 9. *Let Y, Z be random vectors with values in a normed space F and let $p \geq 1$. Suppose that $\mathbb{E}\|Y\|^{p-1}\|Z\| \leq \gamma\mathbb{E}\|Z\|^p$. Then*

$$\mathbb{E}\|Y+Z\|^p \geq \mathbb{E}\|Y\|^p + \left(\frac{1}{3^p} - 2p\gamma\right)\mathbb{E}\|Z\|^p.$$

Proof. For any real numbers a, b we have $|a+b|^p \geq |a|^p - p|a|^{p-1}|b|$. If, additionally, $|a| \leq \frac{1}{3}|b|$, then $|a+b|^p \geq |a|^p + \frac{1}{3^p}|b|^p$. Taking $a = \|Y\|$, $b = -\|Z\|$ and using the inequality

$\|Y+Z\| \geq |\|Y\| - \|Z\||$ we obtain

$$\begin{aligned}
\mathbb{E}\|Y+Z\|^p &= \mathbb{E}\|Y+Z\|^p \mathbb{1}_{\left\{\|Y\| \leq \frac{1}{3}\|Z\|\right\}} + \mathbb{E}\|Y+Z\|^p \mathbb{1}_{\left\{\|Y\| > \frac{1}{3}\|Z\|\right\}} \\
&\geq \mathbb{E}\|Y\|^p \mathbb{1}_{\left\{\|Y\| \leq \frac{1}{3}\|Z\|\right\}} + \frac{1}{3^p}\mathbb{E}\|Z\|^p \mathbb{1}_{\left\{\|Y\| \leq \frac{1}{3}\|Z\|\right\}} \\
&\quad + \mathbb{E}\|Y\|^p \mathbb{1}_{\left\{\|Y\| > \frac{1}{3}\|Z\|\right\}} - p\mathbb{E}\|Y\|^{p-1}\|Z\| \mathbb{1}_{\left\{\|Y\| > \frac{1}{3}\|Z\|\right\}} \\
&= \mathbb{E}\|Y\|^p + \frac{1}{3^p}\mathbb{E}\|Z\|^p \left(1 - \mathbb{1}_{\left\{\|Y\| > \frac{1}{3}\|Z\|\right\}}\right) \\
&\quad - p\mathbb{E}\|Y\|^{p-1}\|Z\| \mathbb{1}_{\left\{\|Y\| > \frac{1}{3}\|Z\|\right\}}.
\end{aligned}$$

Note that

$$\mathbb{E}\left(\frac{1}{3^p}\|Z\|^p + p\|Y\|^{p-1}\|Z\|\right) \mathbb{1}_{\left\{\|Y\| > \frac{1}{3}\|Z\|\right\}} \leq \left(\frac{1}{3} + p\right) \mathbb{E}\|Y\|^{p-1}\|Z\| \leq 2p\gamma \mathbb{E}\|Z\|^p.$$

Therefore,

$$\mathbb{E}\|Y+Z\|^p \geq \mathbb{E}\|Y\|^p + \frac{1}{3^p}\mathbb{E}\|Z\|^p - 2p\gamma\mathbb{E}\|Z\|^p. \quad \square$$

We are now able to state the key proposition which will easily yield the lower bound in Theorem 3.

Proposition 10. *Let $p > 1$ and suppose that r.v.'s $X_1, X_2, \ldots$ satisfy assumptions (12), (5) and (6). Then there exist constants $\varepsilon_0, \varepsilon_1, C_0 > 0$ depending only on p, μ, A, q and λ such that for any vectors $v_0, v_1, \ldots, v_n$ in a normed space $(F, \|\ \|)$ and $k \geq 1$ we have*

$$\mathbb{E}\left\|\sum_{i=0}^n v_i R_i\right\|^p \geq \varepsilon_0\|v_0\|^p + \sum_{i=1}^n \left(\frac{\varepsilon_1}{k} - c_i\right)\|v_i\|^p, \tag{13}$$

where

$$c_i = 0 \quad \text{for } 1 \leq i \leq k-1, \qquad c_i = \Phi \sum_{j=k}^i \lambda^j \quad \text{for } i \geq k \text{ and } \Phi = C_0\lambda^{(p-1)k}.$$

Proof. Define

$$\varepsilon_0 := \min\left\{\frac{1}{4\cdot 3^p}, \frac{\mu^p}{8\cdot 24^p}\right\}, \qquad \varepsilon_1 := \min\left\{\frac{\mu^p}{8^p}, \frac{\mu^{2p}}{2^{p-1}64^p}\right\}\varepsilon_0,$$

where the value of C_0 will be chosen later. In the proof by ε_2, C_2, C_3 we denote finite nonnegative constants that depend only on parameters $p, \mu.A, q$ and λ.

We fix $k \geq 1$ and prove (13) by induction on n. From Lemmas 6 and 7 we obtain

$$\mathbb{E}\left\|\sum_{i=0}^n v_i R_i\right\|^p \geq 2\varepsilon_0\|v_0\|^p, \qquad \mathbb{E}\left\|\sum_{i=0}^n v_i R_i\right\|^p \geq \frac{2\varepsilon_1}{n}\sum_{i=1}^n \|v_i\|^p.$$

Therefore for $n \leq k$ we have

$$\mathbb{E}\left\|\sum_{i=0}^n v_i R_i\right\|^p \geq \varepsilon_0\|v_0\|^p + \frac{\varepsilon_1}{k}\sum_{i=1}^n \|v_i\|^p.$$

Suppose that the induction assertion holds for $n \geq k$. We show it for $n+1$. To this end we consider two cases.

Case 1. $\varepsilon_0\|v_0\|^p \leq \Phi \sum_{i=k}^{n+1} \lambda^i \|v_i\|^p$.

Applying the induction assumption conditionally on X_1 we obtain

$$\begin{aligned}
\mathbb{E}\left\|\sum_{i=0}^{n+1} v_i R_i\right\|^p &\geq \varepsilon_0 \mathbb{E}\|v_0 + v_1 X_1\|^p + \sum_{i=2}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-1}\right)\mathbb{E}\|X_1 v_i\|^p \\
&\geq \frac{\varepsilon_1}{k}\|v_1\|^p + \sum_{i=2}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-1}\right)\|v_i\|^p \\
&\geq \varepsilon_0\|v_0\|^p - \Phi\sum_{i=k}^{n+1}\lambda^i\|v_i\|^p + \frac{\varepsilon_1}{k}\|v_1\|^p + \sum_{i=2}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-1}\right)\|v_i\|^p \\
&= \varepsilon_0\|v_0\|^p + \sum_{i=1}^{n+1}\left(\frac{\varepsilon_1}{k} - c_i\right)\|v_i\|^p,
\end{aligned}$$

where the second inequality follows from Lemma 6.

Case 2. $\varepsilon_0\|v_0\|^p > \Phi \sum_{i=k}^{n+1} \lambda^i \|v_i\|^p$.

Define the event $A_k \in \sigma(X_1, \ldots, X_k)$ by

$$A_k := \left\{|X_1| \leq A, |R_{2,k}| \leq 2^{\frac{1}{q}}\lambda^{k-1}\right\}.$$

By the induction assumption used conditionally on $X_1, \ldots, X_k$ we have

$$\begin{aligned}
\mathbb{E}\left\|\sum_{i=0}^{n+1} v_i R_i\right\|^p \mathbb{1}_{\Omega\setminus A_k} &\geq \varepsilon_0 \mathbb{E}\left\|\sum_{i=0}^{k} v_i R_i\right\|^p \mathbb{1}_{\Omega\setminus A_k} \\
&\quad + \sum_{i=k+1}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-k}\right)\mathbb{E}\|v_i R_k\|^p \mathbb{1}_{\Omega\setminus A_k}. \qquad (14)
\end{aligned}$$

We have by Chebyshev's inequality and (6),

$$\mathbb{P}\left(|R_{2,k}| \geq 2^{\frac{1}{q}}\lambda^{k-1}\right) \leq \frac{\mathbb{E}|R_{2,k}|^q}{2\lambda^{(k-1)q}} \leq \frac{1}{2}. \qquad (15)$$

Together with (5) it implies $\mathbb{P}(A_k) > 0$. Let (Y, Y', Z) have the same distribution as the random vector $(\sum_{i=k}^{n+1} v_i R_i, \sum_{i=k}^{n+1} v_i R_{k+1,i}, \sum_{i=0}^{k-1} v_i R_i)$ conditioned on the event A_k. Note that

$$\mathbb{E}\left\|\sum_{i=0}^{n+1} v_i R_i\right\|^p \mathbb{1}_{A_k} = \mathbb{P}(A_k)\mathbb{E}\|Y + Z\|^p.$$

Applying Lemma 6 conditionally we obtain

$$\begin{aligned}
\mathbb{E}\|Z\|^p &= \frac{1}{\mathbb{P}(A_k)}\mathbb{E}\left\|\sum_{i=0}^{k-1} v_i R_i\right\|^p \mathbb{1}_{\{|X_1|\leq A\}}\mathbb{1}_{\left\{|R_{2,k}|\leq 2^{\frac{1}{q}}\lambda^{k-1}\right\}} \\
&\geq \frac{\mu^p}{8^p}\|v_0\|^p \frac{\mathbb{P}\left(|R_{2,k}| \leq 2^{\frac{1}{q}}\lambda^{k-1}\right)}{\mathbb{P}(A_k)} = \frac{\mu^p}{8^p}\|v_0\|^p \frac{1}{\mathbb{P}(|X_1| \leq A)} \geq \frac{\mu^p}{8^p}\|v_0\|^p. \qquad (16)
\end{aligned}$$

Note that Y' has the same distribution as $\sum_{i=k}^{n+1} v_i R_{k+1,i}$ and is independent of Z. We have for $t > 0$,

$$\begin{aligned}\mathbb{P}(\|Y\|^p \geq t\mathbb{E}\|Z\|^p) &\leq \mathbb{P}\left(A^p\lambda^{p(k-1)}2^{\frac{p}{q}}\|Y'\|^p \geq t\frac{\mu^p}{8^p}\|v_0\|^p\right)\\ &\leq \mathbb{P}\left(A^p\lambda^{p(k-1)}2^{\frac{p}{q}}\|Y'\|^p \geq t\frac{\mu^p}{8^p}\frac{\Phi}{\varepsilon_0}\sum_{i=k}^{n+1}\lambda^i\|v_i\|^p\right)\\ &= \mathbb{P}\left(\|Y'\|^p \geq tC_0\varepsilon_2\sum_{i=k}^{n+1}\lambda^{i-k}\|v_i\|^p\right) \leq C_1(tC_0)^{-\frac{q}{p}}, \end{aligned} \tag{17}$$

where the last inequality follows by Lemma 8 (recall that ε_2 and C_1 denote constants depending on p, μ, A, q and λ).

In order to use Lemma 9 we would like to estimate $\mathbb{E}\|Y\|^{p-1}\|Z\|$. To this end take $\delta > 0$ and observe first that

$$\begin{aligned}\mathbb{E}\|Y\|^{p-1}\|Z\| \leq\ & \mathbb{E}\|Y\|^{p-1}\|Z\|\mathbb{1}_{\{\|Y\|^p\leq\delta\mathbb{E}\|Z\|^p\}} + \mathbb{E}\|Y\|^{p-1}\|Z\|\mathbb{1}_{\{\|Z\|^p\leq\delta\mathbb{E}\|Z\|^p\}}\\ &+ \mathbb{E}\|Y\|^{p-1}\|Z\|\mathbb{1}_{\{\|Y\|^p>\delta\mathbb{E}\|Z\|^p\}}\mathbb{1}_{\{\|Z\|^p>\delta\mathbb{E}\|Z\|^p\}}. \end{aligned} \tag{18}$$

Clearly,

$$\mathbb{E}\|Y\|^{p-1}\|Z\|\mathbb{1}_{\{\|Y\|^p\leq\delta\mathbb{E}\|Z\|^p\}} \leq \delta^{\frac{p-1}{p}}(\mathbb{E}\|Z\|^p)^{\frac{p-1}{p}}\mathbb{E}\|Z\| \leq \delta^{\frac{p-1}{p}}\mathbb{E}\|Z\|^p. \tag{19}$$

To estimate the next term in (18) note that

$$\mathbb{E}\|Y\|^{p-1}\|Z\|\mathbb{1}_{\{\|Z\|^p\leq\delta\mathbb{E}\|Z\|^p\}} \leq \delta^{1/p}(\mathbb{E}\|Z\|^p)^{1/p}\mathbb{E}\|Y\|^{p-1}.$$

Using estimate (17) we obtain

$$\begin{aligned}\mathbb{E}\|Y\|^{p-1} &= (\mathbb{E}\|Z\|^p)^{\frac{p-1}{p}}\int_0^\infty \mathbb{P}\left(\|Y\|^p \geq s^{\frac{p}{p-1}}\mathbb{E}\|Z\|^p\right)\,\mathrm{d}s\\ &\leq (\mathbb{E}\|Z\|^p)^{\frac{p-1}{p}}\int_0^\infty \min\left\{1, C_1C_0^{-\frac{q}{p}}s^{-\frac{q}{p-1}}\right\}\,\mathrm{d}s\\ &\leq (\mathbb{E}\|Z\|^p)^{\frac{p-1}{p}}\left(1 + C_2C_0^{-\frac{q}{p}}\right), \end{aligned}$$

where the last inequality follows since $q > p - 1$. Thus,

$$\mathbb{E}\|Y\|^{p-1}\|Z\|\mathbb{1}_{\{\|Z\|^p\leq\delta\mathbb{E}\|Z\|^p\}} \leq \delta^{1/p}\left(1 + C_2C_0^{-\frac{q}{p}}\right)\mathbb{E}\|Z\|^p. \tag{20}$$

We are left with estimating the last term in (18). We have

$$\begin{aligned}&\mathbb{E}\|Y\|^{p-1}\|Z\|\mathbb{1}_{\{\|Y\|^p>\delta\mathbb{E}\|Z\|^p\}}\mathbb{1}_{\{\|Z\|^p>\delta\mathbb{E}\|Z\|^p\}}\\ &\quad= \sum_{m=0}^\infty \mathbb{E}\|Y\|^{p-1}\|Z\|\mathbb{1}_{\{2^m\delta\mathbb{E}\|Z\|^p<\|Y\|^p\leq 2^{m+1}\delta\mathbb{E}\|Z\|^p\}}\mathbb{1}_{\{\|Z\|^p>\delta\mathbb{E}\|Z\|^p\}}\\ &\quad\leq \sum_{m=0}^\infty 2^{(m+1)\frac{p-1}{p}}\delta^{\frac{p-1}{p}}\mathbb{E}(\mathbb{E}\|Z\|^p)^{\frac{p-1}{p}}\|Z\|\mathbb{1}_{\{2^m\delta\mathbb{E}\|Z\|^p<\|Y\|^p\}}\mathbb{1}_{\{\|Z\|^p>\delta\mathbb{E}\|Z\|^p\}} \end{aligned}$$

$$\leq \delta^{\frac{p-1}{p}} \sum_{m=0}^{\infty} 2^{(m+1)\frac{p-1}{p}} \mathbb{E}\left(\frac{\|Z\|^p}{\delta}\right)^{\frac{p-1}{p}} \|Z\| \mathbb{1}_{\{2^m \delta \mathbb{E}\|Z\|^p < \|Y\|^p\}}$$
$$= \sum_{m=0}^{\infty} 2^{(m+1)\frac{p-1}{p}} \mathbb{E}\|Z\|^p \mathbb{1}_{\{2^m \delta \mathbb{E}\|Z\|^p < \|Y\|^p\}}.$$

Recall that Z and Y' are independent. Therefore as in (17) we get

$$\begin{aligned}\mathbb{E}\|Z\|^p \mathbb{1}_{\{2^m \delta \mathbb{E}\|Z\|^p < \|Y\|^p\}} &\leq \mathbb{E}\|Z\|^p \mathbb{1}_{\left\{\|Y'\|^p \geq 2^m \delta C_0 \varepsilon_2 \sum_{i=k}^{n+1} \lambda^{i-k} \|v_i\|^p\right\}} \\ &= \mathbb{E}\|Z\|^p \mathbb{P}\left(\|Y'\|^p \geq 2^m \delta C_0 \varepsilon_2 \sum_{i=k}^{n+1} \lambda^{i-k} \|v_i\|^p\right) \\ &\leq \mathbb{E}\|Z\|^p C_1 (2^m \delta C_0)^{-\frac{q}{p}}.\end{aligned}$$

We arrive at

$$\begin{aligned}\mathbb{E}\|Y\|^{p-1}\|Z\| \mathbb{1}_{\{\|Y\|^p > \delta \mathbb{E}\|Z\|^p\}} \mathbb{1}_{\{\|Z\|^p > \delta \mathbb{E}\|Z\|^p\}} &\leq \mathbb{E}\|Z\|^p C_1 (\delta C_0)^{-\frac{q}{p}} \sum_{m=0}^{\infty} 2^{(m+1)\frac{p-1}{p}} 2^{-\frac{mq}{p}} \\ &\leq \mathbb{E}\|Z\|^p C_3 (\delta C_0)^{-\frac{q}{p}}, \end{aligned} \tag{21}$$

where we have used the fact that $q > p - 1$.

Estimates (18)–(21) imply

$$\mathbb{E}\|Y\|^{p-1}\|Z\| \leq \mathbb{E}\|Z\|^p \left(\delta^{\frac{p-1}{p}} + \delta^{1/p}\left(1 + C_2 C_0^{-\frac{q}{p}}\right) + C_3 (\delta C_0)^{-\frac{q}{p}}\right).$$

Now we choose $\delta = \delta(p)$ sufficiently small and then $C_0 = C_0(p, A, \mu.q, \lambda)$ sufficiently large to obtain

$$\mathbb{E}\|Y\|^{p-1}\|Z\| \leq \frac{1}{4p3^p} \mathbb{E}\|Z\|^p. \tag{22}$$

From Lemma 9 we deduce

$$\mathbb{E}\|Y + Z\|^p \geq \mathbb{E}\|Y\|^p + \frac{1}{2 \cdot 3^p} \mathbb{E}\|Z\|^p.$$

Hence

$$\mathbb{E}\left\|\sum_{i=0}^{n+1} v_i R_i\right\|^p \mathbb{1}_{A_k} \geq \frac{1}{2 \cdot 3^p} \mathbb{E}\left\|\sum_{i=0}^{k-1} v_i R_i\right\|^p \mathbb{1}_{A_k} + \mathbb{E}\left\|\sum_{i=k}^{n+1} v_i R_i\right\|^p \mathbb{1}_{A_k}. \tag{23}$$

Lemma 6 and (15) yield

$$\mathbb{E}\left\|\sum_{i=0}^{k-1} v_i R_i\right\|^p \mathbb{1}_{A_k} \geq \frac{\mu^p}{8^p} \|v_0\|^p \mathbb{P}\left(|R_{2,k}| \leq 2^{\frac{1}{q}} \lambda^{k-1}\right) \geq \frac{1}{2} \cdot \frac{\mu^p}{8^p} \|v_0\|^p.$$

It follows that

$$\frac{1}{2 \cdot 3^p} \mathbb{E}\left\|\sum_{i=0}^{k-1} v_i R_i\right\|^p \mathbb{1}_{A_k} \geq \varepsilon_0 \|v_0\|^p + \varepsilon_0 \mathbb{E}\left\|\sum_{i=0}^{k-1} v_i R_i\right\|^p \mathbb{1}_{A_k}. \tag{24}$$

By the induction assumption we obtain

$$\mathbb{E}\left\|\sum_{i=k}^{n+1} v_i R_i\right\|^p \mathbb{1}_{A_k} \geq \varepsilon_0 \mathbb{E}\|v_k R_k\|^p \mathbb{1}_{A_k} + \sum_{i=k+1}^{n+1} \left(\frac{\varepsilon_1}{k} - c_{i-k}\right) \mathbb{E}\|v_i R_k\|^p \mathbb{1}_{A_k}. \tag{25}$$

Combining (23)–(25) we get

$$\begin{aligned}
\mathbb{E}\left\|\sum_{i=0}^{n+1} v_i R_i\right\|^p \mathbb{1}_{A_k} &\geq \varepsilon_0 \|v_0\|^p + \varepsilon_0 \mathbb{E}\left\|\sum_{i=0}^{k-1} v_i R_i\right\|^p \mathbb{1}_{A_k} + \varepsilon_0 \mathbb{E}\|v_k R_k\|^p \mathbb{1}_{A_k} \\
&\quad + \sum_{i=k+1}^{n+1} \left(\frac{\varepsilon_1}{k} - c_{i-k}\right) \mathbb{E}\|v_i R_k\|^p \mathbb{1}_{A_k} \\
&\geq \varepsilon_0 \|v_0\|^p + \frac{\varepsilon_0}{2^{p-1}} \mathbb{E}\left\|\sum_{i=0}^{k} v_i R_i\right\|^p \mathbb{1}_{A_k} \\
&\quad + \sum_{i=k+1}^{n+1} \left(\frac{\varepsilon_1}{k} - c_{i-k}\right) \mathbb{E}\|v_i R_k\|^p \mathbb{1}_{A_k}.
\end{aligned}$$

This inequality together with (14) and Lemma 7 yields

$$\begin{aligned}
\mathbb{E}\left\|\sum_{i=0}^{n+1} v_i R_i\right\|^p &\geq \varepsilon_0 \|v_0\|^p + \frac{\varepsilon_0}{2^{p-1}} \mathbb{E}\left\|\sum_{i=0}^{k} v_i R_i\right\|^p + \sum_{i=k+1}^{n+1} \left(\frac{\varepsilon_1}{k} - c_{i-k}\right) \mathbb{E}\|v_i R_k\|^p \\
&\geq \varepsilon_0 \|v_0\|^p + \frac{\varepsilon_1}{k} \sum_{i=1}^{k} \|v_i\|^p + \sum_{i=k+1}^{n+1} \left(\frac{\varepsilon_1}{k} - c_{i-k}\right) \|v_i\|^p \\
&\geq \varepsilon_0 \|v_0\|^p + \sum_{i=1}^{n+1} \left(\frac{\varepsilon_1}{k} - c_i\right) \|v_i\|^p. \quad \square
\end{aligned}$$

We are ready to prove the lower L_p-estimate for $p > 1$.

Proof of the lower bound in Theorem 3. For sufficiently large k we have for all i,

$$c_i \leq \frac{\Phi \lambda^k}{1-\lambda} = \frac{C_0 \lambda^{pk}}{1-\lambda} \leq \frac{\varepsilon_1}{2k}.$$

Thus, Proposition 10 yields

$$\mathbb{E}\left\|\sum_{i=0}^{n} v_i R_i\right\|^p \geq \varepsilon_0 \|v_0\|^p + \frac{\varepsilon_1}{2k} \sum_{i=1}^{n} \|v_i\|^p \geq \varepsilon \sum_{i=0}^{n} \|v_i\|^p,$$

where $\varepsilon := \min\{\varepsilon_0, \frac{\varepsilon_1}{2k}\}$. $\square$

Remark. Observe that $\mu \leq \mathbb{E}\,||X_i| - \mathbb{E}|X_i|\,| \leq 2\mathbb{E}|X_i| \leq 2(\mathbb{E}|X_i|^p)^{1/p} = 2$. This shows that

$$\varepsilon_0 = \frac{\mu^p}{8 \cdot 24^p}, \qquad \varepsilon_1 = \frac{\mu^{2p}}{2^{p-1} 64^p} \cdot \varepsilon_0 \quad \text{and} \quad \min\left\{\varepsilon_0, \frac{\varepsilon_1}{2k}\right\} = \frac{\mu^{3p}}{8k \cdot 2^{10p} \cdot 3^p}.$$

Other constants used in the proof of Proposition 10 may be estimated as follows

$$\varepsilon_2 = \frac{\mu^p \lambda^p}{8^p A^p \varepsilon_0} 2^{-\frac{p}{q}} \geq \left(\frac{3\lambda}{2A}\right)^p, \qquad C_1 = (1-\lambda)^{\frac{(1-p)q}{p}} \varepsilon_2^{-\frac{q}{p}} \leq (1-\lambda)^{\frac{(1-p)q}{p}} \left(\frac{2A}{3\lambda}\right)^q,$$

$$C_2 \leq \frac{p-1}{q+1-p} C_1 \quad \text{and} \quad C_3 = \frac{2^{\frac{q}{p}}}{2^{\frac{q+1-p}{p}} - 1} C_1 \leq \frac{2p}{(q+1-p)\ln 2} C_1.$$

Hence we can for example take

$$\delta := 48^{-\frac{p^2}{\min\{p-1,1\}}} \quad \text{and} \quad C_0 := (1-\lambda)^{1-p} \left(\frac{2A}{3\lambda}\right)^p \left(\frac{2p}{(q+1-p)\ln 2}\right)^{\frac{p}{q}} 48^{\frac{2p^2}{\min\{p-1,1\}}},$$

then each term $\delta^{(p-1)/p}$, $\delta^{1/p}$, $\delta^{1/p} C_2 C_0^{-q/p}$ and $C_3 (\delta C_0)^{-q/p}$ is not greater than 48^{-p}, hence then $(16p3^p)^{-1}$ and (22) holds.

3. Lower bound for $p \leq 1$

In this section we prove the lower bound in Theorem 2. We will also assume normalization (12) and use similar notation as for $p > 1$.

We begin with a result similar to Lemma 6.

Lemma 11. *Let X be a random variable such that $\mathbb{E}|X|^p = 1$. Then for every $A > 1$ and u, v in a normed space $(F, \|\ \|)$ we have*

$$\mathbb{E}\|uX + v\|^p \geq \mathbb{E}\|uX + v\|^p \mathbb{1}_{\{|X|^p \leq A\}} \geq \delta \max\{\|u\|^p, \|v\|^p\},$$

where

$$\delta := \mathbb{E}(|X|^p - 1)\mathbb{1}_{\{1 \leq |X|^p \leq A\}}.$$

Proof. Since $\mathbb{E}|X|^p = 1$ we have

$$\delta \leq \mathbb{E}(|X|^p - 1)\mathbb{1}_{\{1 \leq |X|^p\}} = \mathbb{E}(1 - |X|^p)\mathbb{1}_{\{|X|^p \leq 1\}} \leq \mathbb{P}(|X|^p \leq 1) \leq \mathbb{P}(|X|^p \leq A). \quad (26)$$

The triangle inequality yields $\|uX + v\| \geq \big|\|u\||X| - \|v\|\big|$. Thus, it suffices to prove

$$\mathbb{E}\big|\|u\||X| - \|v\|\big|^p \mathbb{1}_{\{|X|^p \leq A\}} \geq \delta \max\{\|u\|^p, \|v\|^p\}. \quad (27)$$

If $u = 0$ then this inequality is satisfied due to (26). In the case $u \neq 0$ divide both sides of (27) by $\|u\|^p$ to see that it is enough to show

$$\mathbb{E}|\,|X| - t|^p \mathbb{1}_{\{|X|^p \leq A\}} \geq \delta \max\{t^p, 1\} \quad \text{for } t \geq 0.$$

To prove this inequality let us consider two cases. First assume that $t \in [0, 1]$. Then we have

$$\begin{aligned}\mathbb{E}||X| - t|^p \mathbb{1}_{\{|X|^p \leq A\}} &\geq \mathbb{E}||X| - t|^p \mathbb{1}_{\{1 \leq |X|^p \leq A\}} \geq \mathbb{E}(|X|^p - t^p)\mathbb{1}_{\{1 \leq |X|^p \leq A\}} \\ &\geq \mathbb{E}(|X|^p - 1)\mathbb{1}_{\{1 \leq |X|^p \leq A\}} = \delta = \delta \max\{t^p, 1\}.\end{aligned}$$

In the case $t > 1$ it suffices to note that

$$\begin{aligned}\mathbb{E}||X| - t|^p \mathbb{1}_{\{|X|^p \le A\}} &\ge \mathbb{E}||X| - t|^p \mathbb{1}_{\{|X|^p \le 1\}} \ge \mathbb{E}(t^p - |X|^p)\mathbb{1}_{\{|X|^p \le 1\}} \\ &\ge t^p \mathbb{E}(1 - |X|^p)\mathbb{1}_{\{|X|^p \le 1\}} \ge \delta t^p = \delta \max\{t^p, 1\},\end{aligned}$$

where the last inequality follows from (26). □

As a consequence, in the same way as in Lemma 7, we derive the following estimate.

Lemma 12. *Let r.v.'s $X_1, X_2, \ldots$ satisfy (12) and (4). Then for any vectors $v_0, v_1, \ldots, v_n \in F$ we get*

$$\mathbb{E}\left\|\sum_{i=0}^n v_i R_i\right\|^p \ge \delta^2 \max_{1\le i\le n} \|v_i\|^p \ge \frac{\delta^2}{n}\sum_{i=1}^n \|v_i\|^p.$$

Lemma 13. *Suppose that random variables $X_1, X_2, \ldots$ satisfy assumptions (12) and (3). Then for all vectors $v_1, v_2, \ldots$ in $(F, \|\ \|)$ we have*

$$\mathbb{P}\left(\left\|\sum_{i=0}^n v_i R_i\right\|^p \ge \frac{t}{1-\lambda}\sum_{i=0}^n \lambda^i \|v_i\|^p\right) \le \frac{1}{\sqrt{t}} \quad \text{for } t > 0.$$

Proof. Note that

$$\mathbb{E}\left\|\sum_{i=0}^n v_i R_i\right\|^{p/2} \le \sum_{i=0}^n \|v_i\|^{p/2}\mathbb{E}|R_i|^{p/2} \le \sum_{i=0}^n \lambda^i \|v_i\|^{p/2}.$$

By the Cauchy–Schwarz inequality we get

$$\left(\sum_{i=0}^n \lambda^i \|v_i\|^{p/2}\right)^2 \le \sum_{i=0}^n \lambda^i \sum_{i=0}^n \lambda^i \|v_i\|^p \le \frac{1}{1-\lambda}\sum_{i=0}^n \lambda^i \|v_i\|^p.$$

Thus, using Chebyshev's inequality we arrive at

$$\begin{aligned}\mathbb{P}\left(\left\|\sum_{i=0}^n v_i R_i\right\|^p \ge \frac{t}{1-\lambda}\sum_{i=0}^n \lambda^i \|v_i\|^p\right) &\le \mathbb{P}\left(\left\|\sum_{i=0}^n v_i R_i\right\|^{p/2} \ge \sqrt{t}\sum_{i=0}^n \lambda^i \|v_i\|^{p/2}\right) \\ \le \left(\sqrt{t}\sum_{i=0}^n \lambda^i \|v_i\|^{p/2}\right)^{-1} &\mathbb{E}\left\|\sum_{i=0}^n v_i R_i\right\|^{p/2} \le \frac{1}{\sqrt{t}}. \quad \square\end{aligned}$$

Our next lemma is in the spirit of Lemma 9, but it has a simpler proof.

Lemma 14. *Let Y, Z be random vectors with values in a normed space $(F, \|\ \|)$ such that*

$$\mathbb{E}\|Z\|^p \mathbb{1}_{\left\{\|Y\|^p \ge \frac{1}{8}\mathbb{E}\|Z\|^p\right\}} \le \frac{1}{8}\mathbb{E}\|Z\|^p.$$

Then

$$\mathbb{E}\|Y + Z\|^p \ge \mathbb{E}\|Y\|^p + \frac{1}{2}\mathbb{E}\|Z\|^p.$$

Proof. For any $u, v \in F$ we have $\|u+v\|^p \geq \big|\|u\| - \|v\|\big|^p \geq \|u\|^p - \|v\|^p$, therefore

$$\begin{aligned}
\mathbb{E}\|Y+Z\|^p &\geq \mathbb{E}(\|Y\|^p + \|Z\|^p - 2\|Z\|^p)\mathbb{1}_{\{\|Y\|^p \geq \frac{1}{8}\mathbb{E}\|Z\|^p\}} \\
&\quad + \mathbb{E}(\|Y\|^p + \|Z\|^p - 2\|Y\|^p)\mathbb{1}_{\{\|Y\|^p < \frac{1}{8}\mathbb{E}\|Z\|^p\}} \\
&\geq \mathbb{E}\|Y\|^p + \mathbb{E}\|Z\|^p - 2\mathbb{E}\|Z\|^p \mathbb{1}_{\{\|Y\|^p \geq \frac{1}{8}\mathbb{E}\|Z\|^p\}} - 2\mathbb{E}\|Y\|^p \mathbb{1}_{\{\|Y\|^p < \frac{1}{8}\mathbb{E}\|Z\|^p\}} \\
&\geq \mathbb{E}\|Y\|^p + \mathbb{E}\|Z\|^p - 2\cdot\frac{1}{8}\mathbb{E}\|Z\|^p - 2\cdot\frac{1}{8}\mathbb{E}\|Z\|^p \\
&= \mathbb{E}\|Y\|^p + \frac{1}{2}\mathbb{E}\|Z\|^p. \quad \square
\end{aligned}$$

The proof of the lower bound for $p \leq 1$ is similar to the proof for $p > 1$ and it relies on a proposition similar to Proposition 10.

Proposition 15. *Let $0 < p \leq 1$ and suppose that r.v.'s $X_1, X_2, \ldots$ satisfy assumptions (12), (3) and (4). Then for any vectors $v_0, v_1, \ldots, v_n$ in a normed space $(F, \|\ \|)$ and any integer $k \geq 1$ we have*

$$\mathbb{E}\left\|\sum_{i=0}^n v_i R_i\right\|^p \geq \varepsilon_0\|v_0\|^p + \sum_{i=1}^n \left(\frac{\varepsilon_1}{k} - c_i\right)\|v_i\|^p,$$

where $\varepsilon_0 = \delta/8$, $\varepsilon_1 = \delta^3/8$ and

$$c_i = 0 \quad \text{for } 1 \leq i \leq k-1, \qquad c_i = \Phi\sum_{j=k}^i \lambda^j \quad \text{for } i \geq k \text{ and } \Phi = \frac{2^8 A}{1-\lambda}\lambda^{k-2}.$$

Proof. For $n \leq k$ the assertion follows by Lemmas 11 and 12, since $\varepsilon_0 \leq \delta/2$ and $\varepsilon_1/k \leq \varepsilon_1/n \leq \delta^2/(2n)$. For $n \geq k$ we proceed by induction on n.

Case 1. $\varepsilon_0\|v_0\|^p \leq \Phi\sum_{i=k}^{n+1}\lambda^i\|v_i\|^p$.

In this case the induction step is the same as in the proof of Proposition 10.

Case 2. $\varepsilon_0\|v_0\|^p > \Phi\sum_{i=k}^{n+1}\lambda^i\|v_i\|^p$.

Let us define the set

$$A_k := \{|X_1|^p \leq A, |R_{2,k}|^p \leq 4\lambda^{2k-2}\}.$$

By the induction hypothesis we have

$$\begin{aligned}
&\mathbb{E}\left\|\sum_{i=0}^{n+1} v_i R_i\right\|^p \mathbb{1}_{\Omega\setminus A_k} \\
&\quad \geq \varepsilon_0\mathbb{E}\left\|\sum_{i=0}^{k} v_i R_i\right\|^p \mathbb{1}_{\Omega\setminus A_k} + \sum_{i=k+1}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-k}\right)\mathbb{E}\|v_i R_k\|^p \mathbb{1}_{\Omega\setminus A_k}. \qquad (28)
\end{aligned}$$

By Chebyshev's inequality and (3) we get

$$\mathbb{P}(|R_{2,k}|^p > 4\lambda^{2k-2}) \leq \frac{\mathbb{E}|R_{2,k}|^{p/2}}{2\lambda^{k-1}} \leq \frac{1}{2}, \qquad (29)$$

in particular $\mathbb{P}(A_k) > 0$. Let Y, Y', Z be defined as in the proof of Proposition 10. As in (16) we show that Lemma 11 yields $\mathbb{E}\|Z\|^p \geq \delta\|v_0\|^p$. We have $\|Y\|^p \leq 4A\lambda^{2k-2}\|Y'\|^p$, variables Y'

and Z are independent and Y' has the same distribution as $\sum_{i=k}^{n+1} v_i R_{k+1,i}$. Thus,

$$\begin{aligned}
\mathbb{E}\|Z\|^p \mathbb{1}_{\{\|Y\|^p \geq \frac{1}{8}\mathbb{E}\|Z\|^p\}} &\leq \mathbb{E}\|Z\|^p \mathbb{1}_{\{4A\lambda^{2k-2}\|Y'\|^p \geq \frac{\delta}{8}\|v_0\|^p\}} \\
&= \mathbb{E}\|Z\|^p \mathbb{P}\left(\|Y'\|^p \geq \frac{1}{4A\lambda^{2k-2}}\varepsilon_0\|v_0\|^p\right) \\
&\leq \mathbb{E}\|Z\|^p \mathbb{P}\left(\left\|\sum_{i=k}^{n+1} v_i R_{k+1,i}\right\|^p \geq \frac{2^6}{1-\lambda}\sum_{i=k}^{n+1}\lambda^{i-k}\|v_i\|^p\right) \\
&\leq \frac{1}{8}\mathbb{E}\|Z\|^p,
\end{aligned}$$

where the second inequality follows by the assumptions of Case 2 and the definition of Φ and the last one by Lemma 13. Hence, Lemma 14 yields

$$\mathbb{E}\|Y+Z\|^p \geq \mathbb{E}\|Y\|^p + \frac{1}{2}\mathbb{E}\|Z\|^p.$$

Thus,

$$\mathbb{E}\left\|\sum_{i=0}^{n+1} v_i R_i\right\|^p \mathbb{1}_{A_k} \geq \frac{1}{2}\mathbb{E}\left\|\sum_{i=0}^{k-1} v_i R_i\right\|^p \mathbb{1}_{A_k} + \mathbb{E}\left\|\sum_{i=k}^{n+1} v_i R_i\right\|^p \mathbb{1}_{A_k}. \tag{30}$$

Using Lemma 11 and (29) we obtain

$$\mathbb{E}\left\|\sum_{i=0}^{k-1} v_i R_i\right\|^p \mathbb{1}_{A_k} \geq \delta\|v_0\|^p \mathbb{P}(|R_{2,k}| \leq 4\lambda^{2k-2}) \geq \frac{\delta}{2}\|v_0\|^p.$$

Since $\varepsilon_0 \leq \frac{1}{4}$ and $\varepsilon_0 \leq \delta/8$, it follows that

$$\frac{1}{2}\mathbb{E}\left\|\sum_{i=0}^{k-1} v_i R_i\right\|^p \mathbb{1}_{A_k} \geq \varepsilon_0\|v_0\|^p + \varepsilon_0\mathbb{E}\left\|\sum_{i=0}^{k-1} v_i R_i\right\|^p \mathbb{1}_{A_k}. \tag{31}$$

By the induction assumption we obtain

$$\mathbb{E}\left\|\sum_{i=k}^{n+1} v_i R_i\right\|^p \mathbb{1}_{A_k} \geq \varepsilon_0\mathbb{E}\|v_k R_k\|^p \mathbb{1}_{A_k} + \sum_{i=k+1}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-k}\right)\mathbb{E}\|v_i R_k\|^p \mathbb{1}_{A_k}. \tag{32}$$

Combining (30)–(32) we arrive at

$$\begin{aligned}
\mathbb{E}\left\|\sum_{i=0}^{n+1} v_i R_i\right\|^p \mathbb{1}_{A_k} &\geq \varepsilon_0\|v_0\|^p + \varepsilon_0\mathbb{E}\left\|\sum_{i=0}^{k-1} v_i R_i\right\|^p \mathbb{1}_{A_k} + \varepsilon_0\mathbb{E}\|v_k R_k\|^p \mathbb{1}_{A_k} \\
&\quad + \sum_{i=k+1}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-k}\right)\mathbb{E}\|v_i R_k\|^p \mathbb{1}_{A_k} \\
&\geq \varepsilon_0\|v_0\|^p + \varepsilon_0\mathbb{E}\left\|\sum_{i=0}^{k} v_i R_i\right\|^p \mathbb{1}_{A_k} \\
&\quad + \sum_{i=k+1}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-k}\right)\mathbb{E}\|v_i R_k\|^p \mathbb{1}_{A_k}.
\end{aligned}$$

Combining this inequality with (28) yields

$$\begin{aligned}\mathbb{E}\left\|\sum_{i=0}^{n+1} v_i R_i\right\|^p &\geq \varepsilon_0\|v_0\|^p + \varepsilon_0\mathbb{E}\left\|\sum_{i=0}^{k} v_i R_i\right\|^p + \sum_{i=k+1}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-k}\right)\mathbb{E}\|v_i R_k\|^p \\ &\geq \varepsilon_0\|v_0\|^p + \frac{\varepsilon_1}{k}\sum_{i=1}^{k}\|v_i\|^p + \sum_{i=k+1}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-k}\right)\|v_i\|^p \\ &\geq \varepsilon_0\|v_0\|^p + \sum_{i=1}^{n+1}\left(\frac{\varepsilon_1}{k} - c_i\right)\|v_i\|^p,\end{aligned}$$

where in the second inequality we used Lemma 12. □

We are now ready to establish the lower L_p-bound for $p \leq 1$.

Proof of the lower bound in Theorem 2. To show the lower bound let us choose k such that

$$k\lambda^{2k-2} \leq \frac{\delta^3(1-\lambda)^2}{2^{12}A}.$$

Then

$$c_i \leq \Phi\frac{\lambda^k}{1-\lambda} = \frac{2^8 A\lambda^{2k-2}}{(1-\lambda)^2} \leq \frac{\varepsilon_1}{2k}.$$

Therefore, Proposition 15 implies

$$\mathbb{E}\left\|\sum_{i=0}^{n} v_i R_i\right\|^p \geq \frac{\delta}{8}\|v_0\|^p + \frac{\delta^3}{16k}\sum_{i=1}^{n}\|v_i\|^p \geq \frac{\delta^3}{16k}\sum_{i=0}^{n}\|v_i\|^p. \quad \square$$

4. Upper bounds

The upper bound in Theorem 2 immediately follows by the inequality $(a+b)^p \leq a^p + b^p$, $a, b \geq 0$, $p \in (0, 1]$. To get the upper bound in Theorem 3 we prove the following result.

Proposition 16. *Let $p > 0$ and $X_1, X_2, \ldots$ be independent r.v.'s such that $\mathbb{E}|X_i|^p < \infty$ for all i and*

$$\forall_{1\leq k<\lceil p\rceil}\ \exists_{\lambda_k<1}\ \forall_i\ (\mathbb{E}|X_i|^{p-k})^{1/(p-k)} \leq \lambda_k(\mathbb{E}|X_i|^{p-k+1})^{1/(p-k+1)}. \tag{33}$$

Then for any vectors $v_0, v_1, \ldots, v_n$ in a normed space $(F, \|\ \|)$ we have

$$\mathbb{E}\left\|\sum_{i=0}^{n} v_i R_i\right\|^p \leq C(p)\sum_{i=0}^{n}\|v_i\|^p\mathbb{E}|R_i|^p, \tag{34}$$

where $C(p) = 1$ for $p \leq 1$ and for $p > 1$,

$$C(p) = 2^p\left(1 + C(p-1)\frac{\lambda_1^{p-1}}{1-\lambda_1^{p-1}}\right) \leq 2^p\frac{C(p-1)}{1-\lambda_1^{p-1}}.$$

Proof. We have $\|\sum_{i=0}^n v_i R_i\| \leq \sum_{i=0}^n \|v_i\||R_i|$ and $|R_i| = \prod_{j=1}^i |X_j|$, so it is enough to consider the case when $F = \mathbb{R}$, $v_k \geq 0$ and variables X_j are nonnegative. Since it is only a matter of normalization we may also assume that $\mathbb{E}X_i^p = 1$ for all i.

We proceed by induction on $m := \lceil p \rceil$. If $m = 1$, i.e. $0 < p \leq 1$ then the assertion easily follows, since $(x+y)^p \leq x^p + y^p,\ x, y \geq 0$.

Suppose that $m > 1$ and (34) holds in the case $p \leq m$. Take p such that $m < p \leq m+1$. Observe that

$$(x+y)^p \leq x^p + 2^p(yx^{p-1} + y^p) \quad \text{for } x, y \geq 0. \tag{35}$$

Indeed, either $x \leq y$ and then $(x+y)^p \leq 2^p y^p$, or $0 \leq y < x$ and then by the convexity of x^p, $((x+y)^p - x^p)/y \leq ((2x)^p - x^p)/x = (2^p - 1)x^{p-1}$.

We have by (35)

$$\mathbb{E}\left|\sum_{i=0}^n v_i R_i\right|^p \leq \mathbb{E}\left|\sum_{i=1}^n v_i R_i\right|^p + 2^p\left(v_0 \mathbb{E}\left|\sum_{i=1}^n v_i R_i\right|^{p-1} + v_0^p\right).$$

Iterating this inequality we get

$$\mathbb{E}\left|\sum_{i=0}^n v_i R_i\right|^p \leq v_n^p \mathbb{E} R_n^p + 2^p\left(\sum_{k=0}^{n-1} v_k \mathbb{E} R_k \left(\sum_{i=k+1}^n v_i R_i\right)^{p-1} + \sum_{i=0}^{n-1} v_i^p \mathbb{E} R_i^p\right).$$

However, $\mathbb{E}R_k(\sum_{i=k+1}^n v_i R_i)^{p-1} = \mathbb{E}R_k^p \mathbb{E}(\sum_{i=k+1}^n v_i R_{k+1,i})^{p-1}$ and $\mathbb{E}R_k^p = \prod_{j=1}^k \mathbb{E}X_j^p = 1$. Hence

$$\mathbb{E}\left|\sum_{i=0}^n v_i R_i\right|^p \leq 2^p \sum_{i=0}^n v_i^p + 2^p \sum_{k=0}^{n-1} v_k \mathbb{E}\left(\sum_{i=k+1}^n v_i R_{k+1,i}\right)^{p-1}.$$

The induction assumption yields

$$\begin{aligned}
\mathbb{E}\left(\sum_{i=k+1}^n v_i R_{k+1,i}\right)^{p-1} &\leq C(p-1) \sum_{i=k+1}^n v_i^{p-1} \mathbb{E} R_{k+1,i}^{p-1} \\
&= C(p-1) \sum_{i=k+1}^n v_i^{p-1} \prod_{j=k+1}^i \mathbb{E} X_j^{p-1} \\
&\leq C(p-1) \sum_{i=k+1}^n v_i^{p-1} \lambda_1^{(p-1)(i-k)},
\end{aligned}$$

where the last inequality follows by (33). To finish the proof we observe that

$$\begin{aligned}
\sum_{k=0}^{n-1} v_k \sum_{i=k+1}^n v_i^{p-1} \lambda_1^{(p-1)(i-k)} &\leq \sum_{0 \leq k < i \leq n} \left(\frac{1}{p} v_k^p + \frac{p-1}{p} v_i^p\right) \lambda_1^{(p-1)(i-k)} \\
&\leq \sum_{i=0}^n v_i^p \sum_{j=1}^\infty \lambda_1^{(p-1)j} = \frac{\lambda_1^{p-1}}{1 - \lambda_1^{p-1}} \sum_{i=0}^n v_i^p. \quad \square
\end{aligned}$$

Remark. It is not hard to show by induction on $\lceil p \rceil$ that

$$C(p) \leq 2^{\frac{p(p+1)}{2}} \prod_{1 \leq j \leq \lceil p \rceil - 1} \frac{1}{1 - \lambda_j^{p-j}}.$$

5. Stochastic recursions

The proof of Theorem 5 is only a slight modification of the proof of Theorem 1. Normalizing we may always assume $\mathbb{E}X^p = 1$. The upper bound follows as in the proof of Proposition 16 (see more details below). To show the lower bound we consider two cases:

$$\text{There are } w, u \in F \text{ such that } w + B + Xu = 0 \text{ a.e.} \tag{C1}$$

or

$$\mathbb{P}(w + B + Xu = 0) < 1 \quad \text{for every } w, u \in F. \tag{C2}$$

In case (C1) we get

$$\begin{aligned}\sum_{i=1}^{n} R_{i-1}B_i &= \sum_{i=1}^{n} R_{i-1}(-w - X_i u) = -\sum_{i=1}^{n} R_{i-1}w - \sum_{i=1}^{n} R_i u \\ &= -\sum_{i=1}^{n} R_{i-1}(w+u) + u - R_n u.\end{aligned}$$

Notice that

$$\mathbb{E}\left\|\sum_{i=1}^{n} R_{i-1}(w+u)\right\|^p = \|w+u\|^p \mathbb{E}\left|\sum_{i=1}^{n} R_{i-1}\right|^p \geq c_{p,X} n \|w+u\|^p,$$

where the last inequality follows by Theorem 1 with $F = \mathbb{R}$ and $v_i = 1$. Assumption (10) implies $w + u \neq 0$. Moreover,

$$\mathbb{E}\|u - R_n u\|^p \leq 2^p \|u\|^p (1 + \mathbb{E}R_n^p) = 2^{p+1}\|u\|^p.$$

Hence for $n \geq n_0 = n_0(X, B)$ and $c = c(p, X, B) = \frac{1}{2^{p+1}} c_{p,X} \|w+u\|^p$,

$$\mathbb{E}\left\|\sum_{i=1}^{n} R_{i-1}B_i\right\|^p = \mathbb{E}\left\|\sum_{i=1}^{n} R_{i-1}(w+u) - (u - R_n u)\right\|^p \geq cn.$$

To get the lower bound in (11) for $1 \leq n < n_0$ we observe that

$$\begin{aligned}cnn_0 &\leq \mathbb{E}\left\|\sum_{i=1}^{nn_0} R_{i-1}B_i\right\|^p = \mathbb{E}\left\|\sum_{k=0}^{n_0-1}\sum_{i=kn+1}^{(k+1)n} R_{i-1}B_i\right\|^p \\ &\leq n_0^p \sum_{k=0}^{n_0-1} \mathbb{E}\left\|\sum_{i=kn+1}^{(k+1)n} R_{i-1}B_i\right\|^p = n_0^p \sum_{k=0}^{n_0-1} \mathbb{E}R_{kn}^p \mathbb{E}\left\|\sum_{i=kn+1}^{(k+1)n} R_{kn+1,i-1}B_i\right\|^p \\ &= n_0^p n_0 \mathbb{E}\left\|\sum_{i=1}^{n} R_{i-1}B_i\right\|^p,\end{aligned}$$

where the last equality follows since $\sum_{i=kn+1}^{(k+1)n} R_{kn+1,i-1}B_i$ has the same distribution as $\sum_{i=1}^{n} R_{i-1}B_i$.

It is worth mentioning that the estimate $\mathbb{E}(\sum_{i=1}^{n} R_{i-1})^p \geq cn$ was first observed in [4] under the *Goldie–Kesten conditions*. In fact, a stronger statement was proved there: $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}(\sum_{i=1}^{n} R_{i-1})^p$ exists and it is strictly positive. Note also that if $u = -w$, i.e. assumption (10)

is not satisfied, then

$$\mathbb{E}\left\|\sum_{i=1}^{n} R_{i-1}B_i\right\|^p = \mathbb{E}\|u - R_n u\|^p \le 2^{p+1}\|u\|^p$$

and the lower bound in (11) cannot hold for large n.

In the sequel, to derive the lower bound it is enough to consider case (C2). The following lemma is then a counterpart of Lemmas 6 and 11.

Lemma 17. *Suppose that X is a nonnegative, nondegenerate r.v., B is a random vector with values in a separable Banach space F, $\mathbb{E}X^p, \mathbb{E}\|B\|^p < \infty$ and for any $u, w \in F$, $\mathbb{P}(B + Xu = w) < 1$. Then there exist constants $A < \infty$ and $\delta > 0$, depending only on the distribution of (B, X) and p, such that*

$$\mathbb{E}\|w + B + Xu\|^p \mathbb{1}_{\{X \le A\}} \ge \delta \max\{\|w\|^p, \|u\|^p, \mathbb{E}\|B\|^p\}.$$

Proof. By δ_1 and δ_2 we will denote in the sequel positive constants depending only on the distribution of (B, X) and p. Lemmas 6 and 11 yield

$$\mathbb{E}\|w + Xu\|^p \ge \delta_1 \max\{\|w\|^p, \|u\|^p\} \quad \text{for any } w, u \in F.$$

Since $\|u_1 + u_2\|^p \le 2^p(\|u_1\|^p + \|u_2\|^p)$ for any $u_1, u_2 \in F$, we get

$$\mathbb{E}\|w + B + Xu\|^p \ge 2^{-p}\mathbb{E}\|w + Xu\|^p - \mathbb{E}\|B\|^p \ge 2^{-p-1}\delta_1 \max\{\|w\|^p, \|u\|^p, \mathbb{E}\|B\|^p\},$$

provided that $\max\{\|w\|^p, \|u\|^p\} \ge M := 2^{p+1}\max\{1, \delta_1^{-1}\}\mathbb{E}\|B\|^p$. Let

$$\alpha := \inf\left\{\mathbb{E}\|w + B + Xu\|^p\colon \max\{\|w\|^p, \|u\|^p\} \le M\right\}.$$

First we observe that $\alpha > 0$. Indeed, assume that $\alpha = 0$. Then there exist sequences (u_n), (w_n) in F such that $\|u_n\|^p \le M$, $\|w_n\|^p \le M$ and $\mathbb{E}\|w_n + B + Xu_n\|^p \to 0$. We have

$$\begin{aligned}
&\mathbb{E}\|w_n + B + Xu_n\|^p + \mathbb{E}\|w_m + B + Xu_m\|^p \\
&\quad \ge 2^{-p}\mathbb{E}\|(w_n + B + Xu_n) - (w_m + B + Xu_m)\|^p \\
&\quad \ge 2^{-p}\delta_1 \max\{\|w_n - w_m\|^p, \|u_n - u_m\|^p\}.
\end{aligned}$$

Thus both sequences (u_n) and (w_n) satisfy the Cauchy condition, hence they are convergent, respectively to u and w. But then $\mathbb{E}\|w + B + Xu\|^p = \lim_n \mathbb{E}\|w_n + B + Xu_n\|^p = 0$, which contradicts our assumptions.

Therefore $\alpha > 0$ and for $\max\{\|w\|^p, \|u\|^p\} \le M$ we get

$$\mathbb{E}\|w + B + Xu\|^p \ge \alpha \ge \alpha \max\left\{\frac{1}{M}\|w\|^p, \frac{1}{M}\|u\|^p, \frac{1}{\mathbb{E}\|B\|^p}\mathbb{E}\|B\|^p\right\}.$$

This way we showed that

$$\mathbb{E}\|w + B + Xu\|^p \ge \delta_2 \max\{\|w\|^p, \|u\|^p, \mathbb{E}\|B\|^p\} \quad \text{for any } w, u \in F.$$

To finish the proof it is enough to note that

$$\begin{aligned}
\mathbb{E}\|w + B + Xu\|^p \mathbb{1}_{\{X > A\}} &\le 3^p \mathbb{E}(\|w\|^p + \|B\|^p + \|u\|^p)\mathbb{1}_{\{X > A\}} \\
&\le \frac{\delta_2}{2}\max\{\|w\|^p, \|u\|^p, \mathbb{E}\|B\|^p\},
\end{aligned}$$

provided that A is large enough. □

For the rest of the proof of the lower bound in (11) we do not need to assume that (X_i, B_i) are i.i.d., but we need uniformity in Lemma 17, i.e. the condition

$$\exists_{\delta>0, A<\infty}\ \forall_i\ \forall_{w,u\in F}\ \mathbb{E}\|w + B_i + X_i u\|^p \mathbb{1}_{\{X_i^p \le A\mathbb{E}X_i^p\}} \ge \delta \max\{\mathbb{E}\|B_i\|^p, \|w\|^p, \|u\|^p \mathbb{E}X_i^p\}. \tag{36}$$

More precisely, the following theorems hold.

Theorem 18. *Let $0 < p \le 1$ and let $(X_1, B_1), (X_2, B_2) \ldots \in \mathbb{R}^+ \times F$ be a sequence of independent random variables such that $\mathbb{E}\|B_i\|^p, \mathbb{E}X_i^p < \infty$. Suppose that conditions (3) and (36) are satisfied. Then there is a constant $c(p, \lambda, \delta, A)$ such that for every n,*

$$c(p, \lambda, \delta, A) \sum_{i=1}^n (\mathbb{E}R_{i-1}^p)\mathbb{E}\|B_i\|^p \le \mathbb{E}\left\|\sum_{i=1}^n R_{i-1}B_i\right\|^p \le \sum_{i=1}^n (\mathbb{E}R_{i-1}^p)\mathbb{E}\|B_i\|^p. \tag{37}$$

Theorem 19. *Let $p > 1$ and let $(X_1, B_1), (X_2, B_2) \ldots \in \mathbb{R}^+ \times F$ be a sequence of independent random variables such that $\mathbb{E}\|B_i\|^p, \mathbb{E}X_i^p < \infty$. Suppose that conditions (6), (7) and (36) are satisfied. Then there are constants $c = c(p, q, \lambda, \delta, A), C(p, \lambda_1, \ldots \lambda_{\lceil p\rceil - 1})$ such that for every n,*

$$\begin{aligned} c(p, \lambda, \delta, A) \sum_{i=1}^n (\mathbb{E}R_{i-1}^p)\mathbb{E}\|B_i\|^p &\le \mathbb{E}\left\|\sum_{i=1}^n R_{i-1}B_i\right\|^p \\ &\le C(p, \lambda_1, \ldots, \lambda_{\lceil p\rceil - 1}) \sum_{i=1}^n (\mathbb{E}R_{i-1}^p)\mathbb{E}\|B_i\|^p. \end{aligned} \tag{38}$$

Since it is only a matter of normalization we may and will assume that $\mathbb{E}X_i^p = 1$.

First we prove the upper bound in (38). Proceeding by induction, as in the proof of Proposition 16, we get

$$\begin{aligned} \mathbb{E}\left\|\sum_{i=1}^n R_{i-1}B_i\right\|^p &\le \mathbb{E}\left\|\sum_{i=2}^n R_{i-1}B_i\right\|^p + 2^p\left(\mathbb{E}\|B_1\|\left(\sum_{i=2}^n R_{i-1}\|B_i\|\right)^{p-1} + \mathbb{E}\|B_1\|^p\right) \\ &= \mathbb{E}\left\|\sum_{i=2}^n R_{i-1}B_i\right\|^p + 2^p\left(\mathbb{E}\|B_1\|X_1^{p-1}\left(\sum_{i=2}^n R_{2,i-1}\|B_i\|\right)^{p-1} + \mathbb{E}\|B_1\|^p\right). \end{aligned}$$

Iterating this inequality we obtain

$$\begin{aligned} &\mathbb{E}\left\|\sum_{i=1}^n R_{i-1}B_i\right\|^p \\ &\le \mathbb{E}\|B_n\|^p + 2^p\sum_{k=1}^{n-1}\mathbb{E}\|B_k\|R_{k-1}R_k^{p-1}\left(\sum_{i=k+1}^n R_{k+1,i-1}\|B_i\|\right)^{p-1} + 2^p\sum_{i=1}^{n-1}\mathbb{E}\|B_i\|^p \\ &\le 2^p\sum_{i=1}^n \mathbb{E}\|B_i\|^p + 2^p\sum_{k=1}^{n-1}\mathbb{E}\|B_k\|X_k^{p-1}\mathbb{E}\left(\sum_{i=k+1}^n R_{k+1,i-1}\|B_i\|\right)^{p-1}. \end{aligned}$$

By the induction assumption

$$\mathbb{E}\left(\sum_{i=k+1}^{n} R_{k+1,i-1}\|B_i\|\right)^{p-1} \le C(p-1)\sum_{i=k+1}^{n}(\mathbb{E}R_{k+1,i-1}^{p-1})\mathbb{E}\|B_i\|^{p-1} \le C(p-1)\sum_{i=k+1}^{n}\lambda_1^{(i-1-k)(p-1)}\mathbb{E}\|B_i\|^{p-1}.$$

Hence,

$$\mathbb{E}\left\|\sum_{i=1}^{n} R_{i-1}B_i\right\|^p \le 2^p\sum_{i=1}^{n}\mathbb{E}\|B_i\|^p + 2^pC(p-1)\sum_{k=1}^{n-1}\sum_{i=k+1}^{n}\lambda_1^{(i-1-k)(p-1)}\mathbb{E}\|B_k\|X_k^{p-1}\|B_i\|^{p-1}.$$

To finish the proof of the upper bound we observe that for $k < i$,

$$\mathbb{E}\|B_k\|X_k^{p-1}\|B_i\|^{p-1} \le \frac{1}{p}\mathbb{E}(\|B_k\|^p + (p-1)X_k^p\|B_i\|^p) = \frac{1}{p}(\mathbb{E}\|B_k\|^p + (p-1)\mathbb{E}\|B_i\|^p).$$

Therefore,

$$\sum_{k=1}^{n-1}\sum_{i=k+1}^{n}\lambda_1^{(i-1-k)(p-1)}\mathbb{E}\|B_k\|X_k^{p-1}\|B_i\|^{p-1} \le \sum_{k=1}^{n-1}\sum_{i=k+1}^{n}\lambda_1^{(i-1-k)(p-1)}\left(\frac{1}{p}\mathbb{E}\|B_k\|^p + \frac{p-1}{p}\mathbb{E}\|B_i\|^p\right) \le \frac{1}{1-\lambda_1^{p-1}}\sum_{i=1}^{n}\mathbb{E}\|B_i\|^p$$

and the conclusion follows.

To prove the lower bounds in (37) and (38) we follow closely arguments of Sections 2 and 3, making use of (36) whenever Lemma 6 or Lemma 11 are used. For instance, to obtain the estimate

$$\mathbb{E}\left\|w + \sum_{i=1}^{n} R_{i-1}B_i\right\|^p \ge \delta \max_{1\le j\le n}\mathbb{E}\|B_j\|^p \ge \frac{\delta}{n}\sum_{j=1}^{n}\mathbb{E}\|B_j\|^p \tag{39}$$

we proceed as follows. For $1 \le j \le n$ we have

$$\mathbb{E}\left\|w + \sum_{i=1}^{n} R_{i-1}B_i\right\|^p \ge \mathbb{E}\left\|w + \sum_{i=1}^{n} R_{i-1}B_i\right\|^p \mathbb{1}_{\{R_{j-1}>0\}} = \mathbb{E}R_{j-1}^p\|Y_j + B_j + X_jZ_j\|^p,$$

where

$$Y_j := \left(w + \sum_{i=1}^{j-1} R_{i-1}B_i\right)\frac{1}{R_{j-1}}\mathbb{1}_{\{R_{j-1}>0\}} \quad \text{and} \quad Z_j := \sum_{i=j+1}^{n} R_{j+1,i-1}B_i.$$

Since variables R_{j-1}, Y_j and Z_j are independent of (X_j, B_j), condition (36) yields

$$\mathbb{E}\left\|w+\sum_{i=1}^{n} R_{i-1}B_i\right\|^p \geq \delta\mathbb{E}R_{j-1}^p\mathbb{E}\|B_j\|^p = \delta\mathbb{E}\|B_j\|^p.$$

Similar argument used for $j=1$ yields

$$\mathbb{E}\left\|w+\sum_{i=1}^{n} R_{i-1}B_i\right\|^p \geq \delta\|w\|^p. \tag{40}$$

For the rest of this section let us concentrate on the case $p \leq 1$, presenting only the parts of the argument that are specific for the setting of Theorem 18. If $p>1$ the argument is completely analogous. In this situation Lemma 13 holds with the same proof.

Lemma 20. *Suppose the assumptions of Theorem 18 are satisfied. Then for $t>0$,*

$$\mathbb{P}\left(\left\|\sum_{i=1}^{n} X_1\cdots X_{i-1}B_i\right\|^p \geq \frac{t}{1-\lambda}\sum_{i=1}^{n}\lambda^{i-1}\mathbb{E}\|B_i\|^p\right) \leq t^{-1/2}.$$

The main proposition (analogous to Proposition 15) can be formulated as follows.

Proposition 21. *Suppose that the assumptions of Theorem 18 are satisfied and $\mathbb{E}X_i^p = 1$ for all i. Then for any $w\in F$ and $k=1,2,\ldots$ we have*

$$\mathbb{E}\left\|w+\sum_{i=1}^{n} R_{i-1}B_i\right\|^p \geq \varepsilon_0\|w\|^p + \sum_{i=1}^{n}\left(\frac{\varepsilon_1}{k}-c_i\right)\mathbb{E}\|B_i\|^p,$$

where $\varepsilon_0=\delta/8$, $\varepsilon_1=\delta\varepsilon_0$,

$$c_i = 0 \quad \text{for } 1\leq i\leq k-1, \qquad c_i = \Phi\sum_{j=k}^{i}\lambda^{j-1}, \quad i\geq k \text{ and } \Phi = \frac{2^8A}{(1-\lambda)}\lambda^{k-2}.$$

Proof. For $n\leq k$ the assertion follows by (39) and (40). For $n\geq k$ we proceed by induction. To simplify the notation let for $k=1,2,\ldots$ and $w\in F$,

$$S_{k,n}(w) := w+\sum_{i=k}^{n} R_{k,i-1}B_i \quad \text{and} \quad S_n(w) := S_{1,n}(w) = w+\sum_{i=1}^{n} R_{i-1}B_i.$$

Observe that the random variable $S_{k,n}(w)$ is independent of $(X_i, B_i)_{i\leq k-1}$.

As in the proof of Proposition 15 we consider two cases. First assume that

$$\varepsilon_0\|w\|^p \leq \Phi\sum_{i=k}^{n+1}\lambda^{i-1}\mathbb{E}\|B_i\|^p. \tag{41}$$

We have

$$\mathbb{E}\|S_{n+1}(w)\|^p = \mathbb{E}X_1^p\|S_{2,n+1}(w')\|^p\mathbb{1}_{\{X_1>0\}} + \mathbb{E}\|w+B_1\|^p\mathbb{1}_{\{X_1=0\}},$$

where $w' = X_1^{-1}(w+B_1)\mathbb{1}_{\{X_1>0\}}$. Hence by the induction assumption (used conditionally on (X_1, B_1)) we get

$$\begin{aligned}
\mathbb{E}\|S_{n+1}(w)\|^p &\geq \mathbb{E}X_1^p \left(\varepsilon_0\|w'\|^p + \sum_{i=2}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-1}\right)\mathbb{E}\|B_i\|^p\right)\mathbb{1}_{\{X_1>0\}} \\
&\quad + \mathbb{E}\|w + B_1\|^p \mathbb{1}_{\{X_1=0\}} \\
&= \varepsilon_0\mathbb{E}\|w + B_1\|^p \mathbb{1}_{\{X_1>0\}} + \sum_{i=2}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-1}\right)\mathbb{E}X_1^p\|B_i\|^p\mathbb{1}_{\{X_1>0\}} \\
&\quad + \mathbb{E}\|w + B_1\|^p \mathbb{1}_{\{X_1=0\}} \\
&\geq \varepsilon_0\mathbb{E}\|w + B_1\|^p + \sum_{i=2}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-1}\right)\mathbb{E}\|B_i\|^p \\
&\geq \varepsilon_1\mathbb{E}\|B_1\|^p + \sum_{i=2}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-1}\right)\mathbb{E}\|B_i\|^p + \varepsilon_0\|w\|^p - \Phi\sum_{i=k}^{n+1}\lambda^{i-1}\mathbb{E}\|B_i\|^p \\
&= \varepsilon_0\|w\|^p + \varepsilon_1\mathbb{E}\|B_1\|^p + \sum_{i=2}^{n+1}\left(\frac{\varepsilon_1}{k} - c_i\right)\mathbb{E}\|B_i\|^p,
\end{aligned}$$

where we used independence of X_1 and B_i for $i \geq 2$, normalization $\mathbb{E}X_1^p\mathbb{1}_{\{X_1>0\}} = \mathbb{E}X_1^p = 1$ and inequalities (39) and (41).

Now suppose that

$$\varepsilon_0\|w\|^p > \Phi\sum_{i=k}^{n+1}\lambda^{i-1}\mathbb{E}\|B_i\|^p$$

and let

$$U_k := \{X_1^p \leq A,\, R_{2,k}^p \leq 4\lambda^{2k-2}\}.$$

We have

$$\begin{aligned}
\mathbb{E}\|S_{n+1}(w)\|^p\mathbb{1}_{\Omega\setminus U_k} &= \mathbb{E}R_k^p\|S_{k+1,n+1}(w')\|^p\mathbb{1}_{\Omega\setminus U_k}\mathbb{1}_{\{R_k>0\}} \\
&\quad + \mathbb{E}\left\|w\sum_{i=1}^{k}R_{i-1}B_i\right\|^p\mathbb{1}_{\Omega\setminus U_k}\mathbb{1}_{\{R_k=0\}},
\end{aligned}$$

where $w' = (R_k)^{-1}(w + \sum_{i=1}^k R_{i-1}B_i)\mathbb{1}_{\{R_k>0\}}$. Hence by the induction assumption

$$\begin{aligned}
\mathbb{E}\|S_{n+1}(w)\|^p\mathbb{1}_{\Omega\setminus U_k} &\geq \mathbb{E}R_k^p\left(\varepsilon_0\|w'\|^p + \sum_{i=k+1}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-k}\right)\mathbb{E}\|B_i\|^p\right)\mathbb{1}_{\Omega\setminus U_k}\mathbb{1}_{\{R_k>0\}} \\
&\quad + \mathbb{E}\left\|w + \sum_{i=1}^{k}R_{i-1}B_i\right\|^p\mathbb{1}_{\Omega\setminus U_k}\mathbb{1}_{\{R_k=0\}} \\
&= \varepsilon_0\mathbb{E}\left\|w + \sum_{i=1}^{k}R_{i-1}B_i\right\|^p\mathbb{1}_{\Omega\setminus U_k}\mathbb{1}_{\{R_k>0\}} \\
&\quad + \mathbb{E}R_k^p\sum_{i=k+1}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-k}\right)\mathbb{E}\|B_i\|^p\mathbb{1}_{\Omega\setminus U_k}\mathbb{1}_{\{R_k>0\}}
\end{aligned}$$

$$
\begin{aligned}
&+\mathbb{E}\left\|w+\sum_{i=1}^{k} R_{i-1}B_i\right\|^p \mathbb{1}_{\Omega\setminus U_k}\mathbb{1}_{\{R_k=0\}}\\
&\geq \varepsilon_0\mathbb{E}\left\|w+\sum_{i=1}^{k} R_{i-1}B_i\right\|^p \mathbb{1}_{\Omega\setminus U_k}+\mathbb{E}R_k^p\sum_{i=k+1}^{n+1}\left(\frac{\varepsilon_1}{k}-c_{i-k}\right)\mathbb{E}\|B_i\|^p\mathbb{1}_{\Omega\setminus U_k}\\
&=\varepsilon_0\mathbb{E}\left\|w+\sum_{i=1}^{k} R_{i-1}B_i\right\|^p \mathbb{1}_{\Omega\setminus U_k}+\sum_{i=k+1}^{n+1}\left(\frac{\varepsilon_1}{k}-c_{i-k}\right)\mathbb{E}\|R_kB_i\|^p\mathbb{1}_{\Omega\setminus U_k}.
\end{aligned}
$$

To finish the proof we define (Z, Y, Y') as the random variable

$$\left(w+\sum_{i=1}^{k} X_k\cdots X_{i-1}B_i,\ \sum_{i=k+1}^{n+1} X_1\cdots X_{i-1}B_i,\ \sum_{i=k+1}^{n+1} X_{k+1}\cdots X_{i-1}B_i\right)$$

conditioned on U_k and we proceed as in the proof of Proposition 15. □

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