



# On an approach to boundary crossing by stochastic processes

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## Abstract

In this paper we provide an overview as well as new (definitive) results of an approach to boundary crossing. The first published results in this direction appeared in de la Peña and Giné (1999) book on decoupling. They include order of magnitude bounds for the first hitting time of the norm of continuous Banach-Space valued processes with independent increments. One of our main results is a sharp lower bound for the first hitting time of càdlàg real-valued processes  $X(t)$ , where  $X(0) = 0$  with arbitrary dependence structure:  $ET_r^\gamma \geq \int_0^1 \{a^{-1}(r\alpha)\}^\gamma d\alpha$ , where  $T_r = \inf\{t > 0 : X(t) \geq r\}$ ,  $a(t) = E\{\sup_{0 \leq s \leq t} X(s)\}$  and  $\gamma > 0$ . Under certain extra conditions, we also obtain an upper bound for  $ET_r^\gamma$ . As the main text suggests, although  $T_r$  is defined as the hitting time of  $X(t)$  hitting a level boundary, the bounds developed can be extended to more general processes and boundaries. We shall illustrate applications of the bounds derived for additive processes, Gaussian Processes, Bessel Processes, Bessel bridges among others. By considering the non-random function  $a(t)$ , we can show that in various situations,  $ET_r \approx a^{-1}(r)$ .

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## 1. Introduction

One of the problems of wide interest in the study of stochastic processes involves estimation of  $E[T_r]$ , the expected time at which a process crosses a boundary  $r$ . Several tools have been developed for this purpose including Wald's equations [14], Doob's optional sampling theorem and Burkholder–Davis–Gundy inequality [3]. Our current discussion follows the setting discussed in de la Peña and Giné [7]. To motivate our development, we first present a simplified example where the underlying process is non-random, as is  $f(t) \triangleq \sup_{0 \leq s \leq t} x(s)$ . The time of interest  $t_r$  equals the first time  $t \geq 0$  such that  $x(t)$ , hence,  $f(t)$  reaches or exceeds a fixed horizontal level  $r > 0$ .

Then, it is easy to see that, in the case of continuous functions,

$$f(t_r) = r \quad \text{and} \quad t_r = f^{-1}(r).$$

An immediate question is how to relate the expected hitting time when  $X_t$  is a random stochastic process.

To facilitate the discussion, let  $\{X(t)\}_{t \geq 0}$ ,  $X(0) = 0$ , be a càdlàg process with first passage time across a level boundary  $T_r = \inf\{t > 0 : X(t) \geq r\}$ , and  $T_r = \infty$  when  $X(t) < r$  for all  $t, r > 0$ . In general, there is no explicit closed form solution for  $E[T_r]$  due to the complicated model/dependence structure amongst observations. Let  $a(t) = EM(t)$  where  $M(t) = \sup_{0 \leq s \leq t} X(s)$ . We are interested in obtaining bounds for  $E[T_r]$  as functions of  $a(t)$ . [6] first investigated this problem. Our results extend beyond stopping times for level boundaries. That is because we may embed the boundaries in a transformed process where level boundaries again apply (see Remark 2.2).

To the best of our knowledge, typical methods of obtaining densities or moments of the hitting time assume full knowledge of the distribution or very specific form of bounds to be hit, most of which are level boundaries. In contrast, our approach is based on an approximate knowledge of moments of the maximal process, which are either available in many popular processes for modeling or can be estimated through the empirical observations. In this paper, we obtain a sharp universal lower bound for  $E\{g(T_r)\}$  for any non-decreasing function  $g$ . The main idea inherits the spirit of a natural extension of the concept of boundary crossing by non-random functions to the case of random processes. The maximal process  $a(t)$  can be intuitively interpreted as a natural clock for all processes with the same  $a(t)$ .

The rest of the paper is organized as follows: In Section 2, we obtain our lower bound on  $E\{g(T_r)\}$ . Section 3 elaborates some possible extensions of our methodology that can handle situations in which the expected first hitting time is hard to obtain. Section 4 concludes the paper.

## 2. Main results

For the time being, we assume that  $a(t)$  is continuous and strictly increasing so that  $a^{-1}(\cdot)$  is unambiguously defined. We shall later relax this restriction and show how to extend the result to all càdlàg processes  $X(t)$  with  $X(0) = 0$ .  $a^{-1}(s)$  is unambiguously defined for  $0 < s \leq r$ .

**Theorem 2.1.** (i) Let  $X(t)$  be a càdlàg stochastic process with  $X(0) = 0$ . Denote  $a(t) = E\{\sup_{0 \leq s \leq t} X(s)\}$  and  $T_r = \inf\{t \geq 0 : X(t) \geq r\}$ , where  $r > 0$ . For  $g$  non-decreasing,

$$Eg(T_r) \geq \int_0^1 g\left\{a^{-1}(r\alpha)\right\} d\alpha. \quad (1)$$

In particular,

$$ET_r \geq \int_0^1 a^{-1}(r\alpha) d\alpha. \quad (2)$$

Both of the above two bounds are sharp.

(ii) Suppose that  $\Pr\{T_r \leq \gamma(p)\} = p$ , and that  $\gamma(p)$  is unique (so as to avoid the technical issue of defining the  $p$ th percentile of  $T_r$ ), then

$$\gamma(p) \geq a^{-1}(pr) \quad (3)$$

and the bound is sharp.

**Proof.** (i) Observe that, for  $M(t) = \sup_{0 \leq s \leq t} X(s)$ ,

$$\Pr(T_r \leq t) = \Pr\{M(t) \geq r\} \leq \begin{cases} a(t)/r, & t \leq a^{-1}(r) \\ 1, & t > a^{-1}(r) \end{cases} := F(t), \text{ say.}$$

It follows that  $T_r \stackrel{st}{\geq} Y$ , where  $Y \stackrel{d}{=} F$  and  $A \stackrel{st}{\geq} B$  denotes that  $A$  is stochastically larger than  $B$ . Define  $U$  to be uniformly distributed on  $(0, 1)$  and  $Y = a^{-1}(rU)$ , then

$$\Pr(Y \leq t) = \Pr\{a^{-1}(rU) \leq t\} = \Pr\{U \leq a(t)/r\} = F(t).$$

Next, we define

$$X(t) = \begin{cases} rI(Y \leq t), & t \leq a^{-1}(r) \\ a(t), & t > a^{-1}(r), \end{cases}$$

then

$$EX(t) = EM(t) = \begin{cases} rF(t) = a(t), & t \leq a^{-1}(r) \\ a(t), & t > a^{-1}(r) \end{cases}$$

and hence the  $a(t)$  function for this specially defined process is the given  $a(t)$ . Moreover, for this process,  $T_r = Y$  whose distribution function is  $F$ . Thus, for any process in this class (given  $a(t)$ ),  $T_r \stackrel{st}{\geq} Y$  and the bound is achieved by a member of this class. It follows that,

$$Eg(T_r) \geq Eg(Y) = Eg\{a^{-1}(rU)\} = \int_0^1 g\{a^{-1}(r\alpha)\} d\alpha,$$

and that the bound is sharp.

Proof of (ii) To prove (3), observe that since  $T_r \stackrel{st}{\geq} Y$ , the  $p$ th percentile of  $T_r$  is at least as great as the  $p$ th percentile of  $Y$ , which equals  $a^{-1}(pr)$ .

The above result can be extended to all càdlàg processes  $X(t)$ , where  $X(0) = 0$ ,  $t \geq 0$  as follows. Let  $n$  and  $\tau$  be fixed. For  $t \in \left(\frac{m-1}{2^n}, \frac{m}{2^n}\right]$  with  $0 \leq m \leq 2^n\tau$ , let

$$g^{(n)}(t) = M\left(\frac{m-1}{2^n}\right) + \left\{t - \frac{m-1}{2^n}\right\} \left\{M\left(\frac{m}{2^n}\right) - M\left(\frac{m-1}{2^n}\right)\right\}$$

$$h^{(n)}(t) = g^{(n)}\left(t + \frac{1}{2^n}\right) + \frac{t}{2^n} \quad \text{and}$$

$$M^{(n)}(t) = \min_{1 \leq k \leq n} h^{(k)}(t).$$

By construction,  $M^{(n)}(t)$  is a continuous strictly increasing function of  $t$ ,  $M^{(n)}(t) \geq M(t)$  for all  $t \in [0, \tau]$  and the sequence  $\{M^{(n)}(t)\}$  is non increasing in  $n$  for all  $t$ .

Table 1

Examples of  $a(t)$  and the corresponding sharp lower bounds.

$a(t)$	$a^{-1}(t)$	$\int_0^1 a^{-1}(r\alpha) d\alpha$
$ct^\alpha, c > 0, \alpha > 0$	$(t/c)^{-1/\alpha}$	$\frac{\alpha}{\alpha+1} \left(\frac{r}{c}\right)^{-\alpha}$
$\beta(e^{\lambda t} - 1), \beta > 0, \lambda > 0$	$\lambda^{-1} \log\left(1 + \frac{t}{\beta}\right)$	$\lambda^{-1} \left[\left(1 + \frac{\beta}{r}\right) \log\left(1 + \frac{r}{\beta}\right) - 1\right]$
$\theta \log(1 + ct), \theta > 0, c > 0$	$(e^{t/\theta} - 1)/c$	$c^{-1} \left[\frac{\theta}{r} (e^{r/\theta} - 1) - 1\right]$
$c \tan(\beta t), c > 0, \beta > 0$	$\beta^{-1} \tan^{-1}\left(\frac{t}{c}\right)$	$\beta^{-1} \left[\tan^{-1}\left(\frac{r}{c}\right) - \frac{c}{2r} \log\left(1 + \frac{r^2}{c^2}\right)\right]$

Define correspondingly the hitting time  $T_r^{(n)} = \inf\{t \geq 0 : M^{(n)}(t) \geq r\}$  and  $a^{(n)}(t) = E\{M^{(n)}(t)\}$ . Since  $M^{(n)}(t)$  is a non-increasing sequence in  $n$ , it follows that  $T_r^{(n)} \leq T_r^{(n+1)} \leq \dots \leq T_r^{(\infty)}$ . We can, therefore, write

$$\begin{aligned} \Pr(T_r \leq t) &\leq \Pr(T_r^{(n)} \leq t) = \Pr(M^{(n)}(t) \geq r) \\ &\leq \Pr\left\{M^{(n)}(t) \wedge r \geq r\right\} \\ &\leq E\left\{\frac{M^{(n)}(t) \wedge r}{r}\right\} \rightarrow E\left\{\frac{M(t) \wedge r}{r}\right\} \quad (\text{by DCT}) \\ &\leq \left\{\frac{EM(t)}{r} \wedge 1\right\} = \left\{\frac{a(t)}{r} \wedge 1\right\} = F(t), \quad \text{as } n \rightarrow \infty. \end{aligned}$$

The proof proceeds as in the case treated before.  $\square$

**Remark 2.1.** Theorem 2.1 can be extended to any càdlàg process  $Y(t)$  on  $[0, \tau]$ . To see this simply let  $X(t) = Y(t) - Y(0)$ , in which case  $X(0) = 0$ .

Closed form lower bounds for various  $a(t)$  can be obtained. In particular, we present Table 1 that illustrates common forms of  $a(t)$  and the corresponding sharp lower bounds.

**Remark 2.2** (See [9]). The versatility of the sharp lower bound developed is manifested via the fact that the bound at which the process of interest hits can be of a very general form in lieu of some restrictive level/linear bounds.

Suppose that we are interested in the first hitting time of real-valued process  $X(t)$  with respect to asymmetric boundaries  $a$  and  $b$  such that  $a < 0 < b$ . We may define  $\zeta_{a,b}(x) = x/aI(x < 0) + x/bI(x \geq 0)$  and it follows that  $T_{a,b} := \inf\{t > 0 : X(t) \notin [a, b]\} = \inf\{t > 0 : \zeta_{a,b}(X_t) \geq 1\}$  with  $a(t) = E\left[\sup_{0 \leq s \leq t} \zeta_{a,b}\{X(s)\}\right] = E\left[\sup_{0 \leq s \leq t} I\{X(s) < 0\}a^{-1}X(s) + I\{X(s) \geq 0\} \times b^{-1}X(s)\right]$ .

The above formulation can be further generalized into cases where the first hitting time is defined as  $T_r = \inf\{t \geq 0 : \zeta\{X(t)\} > r\}$ . For processes  $X(t), t > 0$  taking values in measurable space  $\mathcal{S}$  and  $\zeta : \mathcal{S} \rightarrow \mathbb{R}^+$ . It can be used to handle cases in which the volume/surface area of a multidimensional process reaching a general deterministic threshold which bears a physical meaning. For example,  $\zeta(x) = x^+, x \in \mathbb{R}, \zeta(x, y) = |x - y|, x, y \in \mathbb{R}^d, \zeta(x, y) = \rho(x - y)$  for a metric  $\rho$ . When the boundary  $\eta(\cdot)$  is a stochastic process, define  $Y(t) = X(t)/\eta(t)$  and  $T_1 = \inf\{t : Y(t) > 1\}$ , where  $X$  is similarly defined as above. Then, the hitting time of the process  $X(t)$  reaching  $\eta(t)$  is given by  $T_r$  with  $a(t) = E\{\sup_{0 \leq s \leq t} X(s)/\eta(s)\}$ .

**Remark 2.3.** Since  $ET_r \geq \int_0^1 a^{-1}(r\alpha)d\alpha$ , whenever the latter integral is infinite, we have  $ET_r = \infty$ . To illustrate this phenomenon, we offer the following example.

**Example 2.1.** Let

$$a(t) = \begin{cases} t/2, & 0 \leq t \leq 1 \\ 1 - 1/(2t), & t > 1. \end{cases}$$

Then,

$$a^{-1}(s) = \begin{cases} 2s, & 0 \leq s \leq 1/2 \\ \frac{1}{2}(1-s)^{-1}, & 1/2 < s < 1. \end{cases}$$

As a result, if  $r$  is chosen to be 1, then by [Theorem 2.1](#)

$$\begin{aligned} ET_r &\geq \int_0^1 a^{-1}(r\alpha)d\alpha = \int_0^1 a^{-1}(\alpha)d\alpha = \int_0^{1/2} 2\alpha d\alpha \\ &\quad + \frac{1}{2} \int_{1/2}^1 (1-\alpha)^{-1} d\alpha = \frac{1}{4} - \log(1-\alpha) \Big|_{1/2}^1 = \infty. \end{aligned}$$

Hence,  $ET_1 = \infty$ . We can, therefore, determine if an expected hitting time is infinite by considering its  $a^{-1}(\cdot)$ .

**Example 2.2.** Suppose that  $EM(t) \leq \kappa(t)$  for all  $t$ , and that is the only information that we have. Then, by [Theorem 2.1](#)

$$Eg(T_r) \geq \int_0^1 g\{\kappa^{-1}(r\alpha)d\alpha\}$$

and

$$\gamma_p \geq \kappa^{-1}(pr)$$

and these bounds are sharp. To see this, given  $\kappa$ , we construct our process as above, substituting  $\kappa$  for  $a$ . For the constructed process  $T_r \stackrel{d}{=} \kappa^{-1}(rU)$  and  $EM(t) = \kappa(t)$ . For any process in the collection  $\{a(t) \leq \kappa(t)\}$  for all  $t$ ,

$$T_r \stackrel{st}{\geq} a^{-1}(rU) \geq \kappa^{-1}(rU).$$

**Example 2.3.** If  $a(t)$  is concave, then

$$ET_r \geq a^{-1}\left(\frac{r}{2}\right).$$

To see this, note that  $\int_0^1 a^{-1}(r\alpha)d\alpha = Ea^{-1}(rU) \geq a^{-1}\{E(rU)\} = a^{-1}(r/2)$ , which is the bound discussed in [\[2\]](#). However, in general  $\int_0^1 a^{-1}(r\alpha)d\alpha$  will exceed  $a^{-1}(r/2)$ . For instance, if  $a(t) = ct^{1/2}$ , for some  $c > 0$ . It follows that  $a^{-1}(s) = (s/c)^2$  and that  $a^{-1}(r/2) = r^2/(4c^2)$  and  $\int_0^1 (r\alpha/c)^2 d\alpha = r^2/(3c^2) > r^2/(4c^2)$ .

The following example, adapted from [\[2\]](#), shows that in general, it is not possible to obtain an upper bound without further restriction on  $a(t)$ .

**Example 2.4.** Suppose  $X(t) = tY$ , with  $Y$  a non-negative random variable. Then

$$a(t) = tEY \quad \text{and} \quad ET_r = rE(Y^{-1}).$$

Suppose  $Y$  is exponentially distributed with mean 1. Then  $EY = 1$  while  $E(Y^{-1}) = \infty$ . Therefore,  $ET_r = \infty$ .

As a result, additional restriction(s) is/are required for us to obtain an upper bound for the expected hitting time. The following condition is an example which ensures the existence of an upper bound for  $ET_r$  which will be used in [Example 3.1](#) to get the order of magnitude for Gaussian process.

**Theorem 2.2.** Suppose that a function  $g(r)$ ,  $r > 0$  satisfies

- (i)  $g(r) > 0, r > 0$
- (ii)  $\Pr\{\sup_{0 \leq s \leq t} X(s) \leq r\} \leq C\{g(r)/t\}^\theta$ .

Then, for any  $\gamma \in (0, \theta)$

$$ET_r^\gamma \leq C'\{g(r)\}^\gamma, \quad r > 0,$$

where  $C'$  depends only on  $C$ ,  $\gamma$  and  $\theta$ .

**Proof.** First note that  $g(r) > 0$ . We have for  $0 < \gamma < \theta$ ,

$$\begin{aligned} ET_r^\gamma &= \gamma \int_0^\infty t^{\gamma-1} \Pr(T_r > t) dt \\ &\leq \gamma \int_0^\infty t^{\gamma-1} \Pr\{M(t) \leq r\} dt \\ &= \int_0^{g(r)} t^{\gamma-1} \Pr\{M(t) \leq r\} dt + \int_{g(r)}^\infty t^{\gamma-1} \Pr\{M(t) \leq r\} dt \\ &\leq \gamma \int_0^{g(r)} t^{\gamma-1} dt + C\gamma\{g(r)\}^\theta \int_{g(r)}^\infty t^{-\theta+\gamma-1} dt \\ &= \{g(r)\}^\gamma + C\{g(r)\}^\theta \frac{\gamma}{\theta - \gamma} \frac{1}{\{g(r)\}^{\theta-\gamma}} \\ &= \left(1 + C \frac{\gamma}{\theta - \gamma}\right) \{g(r)\}^\gamma. \quad \square \end{aligned}$$

**Remark 2.4.** Let  $X(i) \geq c > 0$  be arbitrary dependent random variables with  $EX_i = EX < \infty$ . Let  $T_r = \inf\{n : \sum_{i=1}^n X(i) > r\}$ ,  $r > 0$ . Then  $T_r \leq \lceil r/c \rceil$ , which implies that  $ET_r^\gamma \leq \lceil r/c \rceil^\gamma$ , for  $\gamma > 0$ . Using the result of [Theorem 2.1](#), we can also obtain  $ET_r^\gamma \geq (r/EX)^\gamma/2$ .

In [Section 3](#) we construct the function  $a(t)$  for several processes.

### 3. Examples and applications

In this section, we shall introduce a number of examples that illustrate how our bounds developed in the previous section will be useful in various applications. Readers may refer to [\[8\]](#) for more examples as well as discussion of relevant results of [Example 3.1](#).

**Example 3.1 (Real-Valued Separable Centered Gaussian Processes).** Let  $\{X(t), t \geq 0\}$  be Gaussian, separable, centered and  $\mathbb{R}$ -valued. Also, assume that  $\sigma^2(t) = EX^2(t), t \geq 0$  is

continuous. We define  $\sigma^*(t) = \sup_{0 \leq s \leq t} \sigma(s)$ ,  $t > 0$ . Note that since  $\sigma(t)$  is continuous,  $\sigma^*(t) = \max_{0 \leq s \leq t} \sigma(s) = \sigma(s^*)$  for some  $s^* \in [0, t]$ ;  $\sigma^*(t)$  is continuous and non-decreasing. We consider  $\zeta(t) = X^+(t)$  and  $\zeta(t) = |X(t)|$ . For  $\zeta(t) = |X(t)|$ , it is trivial to have a crude decomposition:

$$\begin{aligned} \Pr \left\{ \sup_{0 \leq s \leq t} \zeta(s) \leq r \right\} &\leq \Pr\{|X(s^*)| \leq r\} \\ &= \sqrt{\frac{2}{\pi}} \int_0^{r/\sigma(s^*)} e^{-x^2/2} dx < \sqrt{\frac{2}{\pi}} \frac{r}{\sigma(s^*)} = \sqrt{\frac{2}{\pi}} \frac{r}{\sigma^*(t)}. \end{aligned} \quad (4)$$

Hence, if  $\sigma^*(t) \geq ct^p$  ( $p > 0$ ), then  $\Pr\{\sup_{0 \leq s \leq t} \zeta(s) \leq r\} \leq C(r^{1/p}/t)^p$ . It follows that for any  $0 < \gamma < p$ ,  $ET_r^\gamma \leq C'r^{\gamma/p}$ ,  $r > 0$ . For example, if  $X$  is a Brownian motion ( $X(0) = 0$ ) then  $p = 1/2$ . Using the fact that  $X_t$  has independent increments, one obtains

$$\Pr\{\zeta^*(t) \leq r\} \leq \prod_{l=1}^k \Pr \left\{ \sup_{0 \leq s \leq t} \zeta_l(s) \leq 2r \right\} \leq C_k \left( \frac{r^2}{t} \right)^{k/2},$$

where  $\zeta_l^*(t) = \sup_{(l-1)\frac{t}{k} \leq s \leq l\frac{t}{k}} |X(s) - X\{(l-1)\frac{t}{k}\}|$  and  $k$  is a positive integer. It follows that for any  $\gamma > 0$ ,  $ET_r^\gamma \leq C'r^{2\gamma}$ ,  $r > 0$ . For  $\zeta_t = X_t^+$  and  $M(t) = E\{\sup_{0 \leq s \leq t} X(s)\}$ , no simple conclusion can be drawn from  $\Pr\{M(t) \leq r\}$  except that for some trivial cases such as when  $X_t$  is a Brownian motion.

In this example, our focus is on the lower bound of  $ET_r^\gamma$ . We consider  $\zeta_t = |X_t|$ . By symmetry,

$$\Pr \left( \sup_{0 \leq s \leq t} |X_s| \geq r \right) \leq 2 \Pr \{M(t) \geq r\}.$$

Hence, similar to the previous case, for  $T_r \triangleq \inf\{t > 0 : |X_t| = r\}$ ,  $r > 0$  and any  $\gamma > 0$ ,  $\epsilon \in (0, 1)$ ,

$$ET_r^\gamma \geq \int_0^1 \{(r\alpha)/C'\}^{2\gamma} d\alpha = \frac{r^{2\gamma}}{C'}.$$

By Theorem 2.2 and the fact that  $\Pr\{M(t) \leq r\} \leq \sqrt{2/\pi}r/t$  due to (4), we can see that

$$ET_r^\gamma \leq \sqrt{2/\pi}r^{2\gamma}, \quad r > 0.$$

**Example 3.2 (Additive Processes).** Let  $X(t)$  be a process in  $\mathbb{R}^d$  with independent increments, right-continuous with left limits paths.  $X(t)$  is called additive if  $X(t)$  is continuous in probability and  $X(0) = 0$ . The class of additive processes represents a wide family of non-homogeneous processes including Lévy processes.

As specified in [15], there are two measures and two kernels for an additive process: (the jump measure)  $\mu = \sum_{t \geq 0} I(\Delta X_t \neq 0) \delta_{(t, \Delta X_t)}$  on  $\mathbb{R}_+ \times \mathbb{R}^d$ , where  $\delta_a$  is the Dirac point mass at  $a \in \mathbb{R}_+ \times \mathbb{R}^d$  and (the intensity measure)  $\nu(B) = E\mu(B)$ ,  $B \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$ ;  $\mu_t(A) = \nu([0, t] \times A) = E\mu_t(A)$ . Here,  $\nu_t$  denote a Lévy measure for fixed  $t$ . The characteristic function for  $X(t)$  can be expressed via the form of  $E \exp(i\langle \lambda, X(t) \rangle) = \exp \Psi_t(\lambda)$ ,  $\lambda \in \mathbb{R}^d$ , where  $\Phi_t(\lambda) = i\langle B_t, \lambda \rangle - \frac{1}{2}\langle \lambda, Q_t \lambda \rangle + \int \{\exp(i\langle \lambda, x \rangle) - 1 - i\langle \lambda, x \rangle I(|x| \leq 1)\} \nu_t(dx)$ , where

$B_t = (B_t^{(1)}, B_t^{(2)}, \dots, B_t^{(d)}) \in \mathbb{R}^d$  is continuous with  $B_0 = 0$ ;  $Q_t = \{q_{ij}(t)\}_{d \times d}$  is a non-negative definite symmetric  $d \times d$  matrix which defines centered Gaussian process. For fixed  $\lambda$ ,  $\langle \lambda, Q_t \lambda \rangle$  is a non-decreasing continuous functions with  $q_{ii}(0) = 0$ ,  $i \neq j$ .

The characteristics of the  $i$ th component  $X_t^{(i)}$  of  $X_t$  are  $B_t^{(i)}$ ,  $C_t^{(i)} = q_{ii}(t)$ ,  $v_t^{(i)}(B) = v_t(\{x \in \mathbb{R}^d : x_i \in B\})$ ,  $B \in \mathcal{B}(\mathbb{R})$  respectively. Hence  $\Psi_t(\lambda) = i\lambda B_t - \frac{1}{2}\lambda^2 C_t + \int \{\exp(i\lambda x) - 1 - i\lambda x I(|x| \leq 1)\} v_t(dx)$ . Denote, for  $r > 0$  and  $t \geq 0$  the following terms:

$$\begin{aligned} G_t(r) &= \int_{|x| > r} v_t(dx), \\ K_t(r) &= r^{-2} \left\{ C_t + \int_{|x| \leq r} x^2 v_t(dx) \right\}, \\ M_t(r) &= r^{-1} \left| B_t + \int_{1 \leq |x| \vee 1} x v_t(dx) - \int_{r \wedge 1 < |x| \leq 1} x v_t(dx) \right|, \\ M_t^*(r) &= \max_{0 \leq s \leq t} M_s(r), \\ y_t(r) &= G_t(r) + K_t(r) + M_t^*(r). \end{aligned} \quad (5)$$

For any process  $X_t \in \mathbb{R}^d$  with additive components,  $y_t(r) = \sum_{i=1}^d y_t^{(i)}(r)$ , where  $y_t^{(i)}(r)$  are given by (5) for their respective components  $X_t^{(i)}$  of  $X_t$ . In Lemma 2.1 of [15], it is proved that, for all  $r > 0$ ,  $t \geq 0$ ,

$$\Pr \left\{ \max_{0 \leq s \leq t} X(s) \geq r \right\} \leq \pi_d y_t(r),$$

where  $\pi_d = aK(d)$ ,  $a = 2^{-1}(3 + \sqrt{5})$ ,  $K(d) = 3d^2$ ,  $d > 1$ ,  $K(1) = 1$ . Observe that

$$a(t) = E \sup_{0 \leq s \leq t} X(s) = \int_0^\infty \Pr \left\{ \sup_{0 \leq s \leq t} X(s) > \xi \right\} d\xi \leq \int_0^\infty \pi_d y_t(\xi) d\xi := A(t),$$

we have  $a^{-1}(t) \geq A^{-1}(t)$ . It follows that

$$Eg(T_r) \geq \int_0^1 g\{a^{-1}(r\alpha)\} d\alpha \geq \int_0^1 g\{A^{-1}(r\alpha)\} d\alpha,$$

which gives a general lower bound for an additive process. Note that in many settings, the term  $y_t(r)$  can be explicitly specified. Again, in general, this bound is larger than that obtained in Lemma 5.1 of [15]. It should also be noted that additive processes, especially Markov additive processes (MAPs) are commonly used for modeling risk processes in which premium rates, claim arrivals and claim sizes will affect the amount of reserve for an insurance company for example; see [12]. Our result provides a lower bound for the hitting time of such MAPs which may represent a default for instance.

**Example 3.3 (Bessel Processes).** In this example, we study the potential applications of Bessel processes which are frequently applied in problems in mathematical finance, optimal control and neuroscience. For example, in finance, Cox–Ingersoll–Ross (CIR) processes are used to model interest rates; in neuroscience, the firing time of a neuron is usually modeled as the hitting time of a stochastic process that captures the membrane/cellular behavior. Recall that for the  $\delta$ -dimensional Bessel process starting from  $y$ , the solution of the following stochastic differential



equation is:

$$Z^{\delta,y}(t) = Z^{\delta,y}(0) + \frac{\delta-1}{2} \int_0^t \{Z^{\delta,y}(s)\}^{-1} ds + B(t),$$

where  $Z^{\delta,y}(0) = y \geq 0$ ,  $\{B(t)\}_{t \geq 0}$  is a one-dimensional Brownian motion. The index and the dimension of the process are denoted by  $\nu$  and  $\delta$  respectively, where  $\nu = \delta/2 - 1$ . The densities of the hitting times of the  $\delta$ -dimensional Bessel process have been carefully studied in [5] for specific choices of boundaries in which cases explicit closed-formed densities can be obtained. In particular, for  $\tau_l = \inf\{t > 0 : Z^{\delta,x}(t) = l\}$ , we get

$$E_x\{\exp(-\lambda\tau_l)\} = \frac{x^{-\nu} I_\nu(x\sqrt{2\lambda})}{l^{-\nu} I_\nu(l\sqrt{2\lambda})}, \quad x > 0 \quad \text{and}$$

$$E_0\{\exp(-\lambda\tau_l)\} = \frac{(l\sqrt{2\lambda})^\nu}{2^\nu \Gamma(\nu+1)} \frac{1}{I_\nu(l\sqrt{2\lambda})},$$

where  $I_\nu(z) = \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^{\nu+2n} \frac{1}{n! \Gamma(\nu+n+1)}$  denotes the Bessel function. Although, the expectations  $E_x[\tau_l]$  and  $E_0[\tau_l]$  can be obtained via differentiation, the general form of  $E_x[g(\tau_l)]$  is difficult to obtain. As an application, we consider the hitting time of the CIR process, which is the solution of the following stochastic differential equation:

$$dX^\delta(t) = (a + bX^\delta(t))dt + c\sqrt{X^\delta(t)}dB(t),$$

with  $X(0)^\delta = x(0) \geq 0$ ,  $a \geq 0$ ,  $b \in \mathbb{R}$ ,  $c > 0$  and  $\{B(t)\}_{t \geq 0}$  is a standard Brownian motion. Let  $\delta = 4a^2/c$ . It can be proved that the CIR process has the same distribution as  $\{\tilde{X}(t), t > 0\}$  where

$$\tilde{X}(t) = e^{bt} Y^\delta \left\{ \frac{c^2}{4b} (1 - e^{-bt}) \right\}$$

with  $\tilde{X}(0) = Y^\delta(0)$ , where  $Y^\delta$  is the square of a Bessel process in dimension  $\delta = 4a/c^2$ . Denote  $T_l = \inf\{t \geq 0 : X^\delta(t) = l\}$  the hitting time of a given level  $l$  for the CIR process, it can be shown that  $T_l$  has the same distribution as  $-b^{-1} \log(1 - 4\tau_\psi b/c^2)$ , where  $\tau_\psi = \inf\{t \geq 0 : Y^\delta(t) = l(1 - 4b/c^2)t\}$ . In [5], the level boundary  $l$  leads to the following inequality

$$\Pr(T_l > t) \geq 1 - \Pr\left\{\Xi_{N^\epsilon} \geq \frac{c^2}{4b} (1 - e^{-bt})\right\},$$

which leads to the bound for  $ET_l \geq \int_0^\infty \left[1 - \Pr\left\{\Xi_{N^\epsilon} \geq \frac{c^2}{4b} (1 - e^{-bt})\right\}\right] dt$ , where  $\Xi_{N^\epsilon}$  can be obtained via simulations as proposed in [5]. By [10], the distribution, and hence the expectation, of the maximum of the Bessel process can be specified, i.e.  $a(t)$  can be obtained. Using Theorem 2.1, we can obtain a lower bound for  $g(T_l)$ . In fact, for a more general boundary to be hit by the CIR process, the method proposed [5] may not provide a (closed-form) bounds for  $\Pr(T_l > t)$ ; This adds usefulness to our simple, yet general, approach, which works even for hitting times of Bessel processes with non-integral dimensions.

**Example 3.4 (Bessel Bridge).** The process  $\{X(t)\}_{t \in [0,T]}$  denotes the  $\delta$ -Bessel bridge with  $X(0) = a \in \mathbb{R}$  and  $X(T) = c \in \mathbb{R}$ .  $X$  can be thought of the process of the Bessel process conditioned to take the value  $c$  at a certain fixed time point  $T$ . Let  $P$  denote probability measure on  $\Omega$  that defines the Bessel process as described in Example 3.2, for  $u \in \mathbb{R}$  and

measurable subsets  $A \subset \Omega$ , there exists a probability kernel  $u \times A \mapsto \eta_u(A)$  such that  $P(A) = \int_{\mathbb{R}} \eta_n(A) \mu(du)$ , where  $\mu$  is the distribution of the  $\delta$ -dimensional Bessel process. It is known, say see [11], that the Bessel bridge process satisfies the following SDE with  $t \in [0, \tau_0] = [0, \inf\{s : X(s) = 0\}]$ :

$$dX(t) = \left\{ \frac{\delta - 1}{2X(t)} - \frac{X(t)}{T - t} \right\} dt + dW(t), \quad X(0) = a > 0.$$

For hitting of a level boundary, [11] shows that the distribution of the first time a  $\delta$ -Bessel bridge is given by  $\delta \in \{1, 3\}$  and such that  $X(0) = a > 0$  and  $X(T) = 0$ . If  $0 < b < a$ , for  $\tau := \inf\{s > 0 : X(s) = b\}$  then,

$$Q(\tau \in dt) = \frac{h(t, b)}{h(0, a)} \left( \frac{b}{a} \right)^{v+|v|} \frac{a-b}{\sqrt{2\pi t^3}} e^{-\frac{(a-b)^2}{2t}}, \quad t \leq T,$$

where  $h(t, x) := \frac{T}{(T-t)^{\delta/2}} \exp\{-\frac{x^2}{2(T-t)}\}$ ,  $v = \delta/2 - 1$  and  $Q(\cdot) = \frac{h\{t, Y(t)\}}{h(0, a)} P(\cdot)$  denotes another probability measure that connects the Bessel bridge dynamics with the underlying Bessel process; see [11] for the details. Note that for a non-level/non-linear bound, the explicit form of the distribution of the hitting time of a  $\delta$ -dimensional Bessel bridge process with  $\delta \notin \{1, 3\}$  has, to our best knowledge, not yet been obtained.

Recall that in [13], the moments of the maximum of the  $\delta$ -dimensional Bessel bridge process are given by

$$E\left[\sup_{0 \leq s < 1} X(s)^\kappa\right] = \frac{2^{1-\kappa/2}}{2^{(\delta-2)/2} \Gamma(\delta/2)} \sum_{n=1}^{\infty} \frac{j^{\kappa-1-v}}{J_{v+1}(j_{v,n})},$$

where the definitions of  $J_v$  and  $j_{v,n}$  can be found in Appendix of [13]. Hence, in this case,  $a(t)$  can be obtained by using this result. It follows, from Theorem 2.1, that the lower bound for the expectation of the hitting time of a Bessel bridge process is given by  $\int_0^1 g\{a^{-1}(r\alpha)\} d\alpha$ .

As discussed in [1], the short rate model specifies the zero-coupon bond price as follows:

$$p(t, s) = E\left[\exp\left(-\int_t^s r(u)du\right) \middle| \mathcal{F}_t\right].$$

Examples including Vasicek or Ornstein–Uhlenbeck processes quantify the short rate  $r$  via the dynamics shown below:

$$dr(t) = \{b - cr(t)\}dt + \sigma dW(t) + dX(t), \quad t > 0,$$

where  $X(t)$ , a  $\delta$ -dimensional Bessel bridge, is regarded as the counter-party credit-index process which represents the time of default as described in [4]. It follows that

$$r(t) = e^{-ct}r(0) + \int_0^t e^{-c(t-u)}bdu + \int_0^t \sigma e^{-c(t-u)}dW(u) + \int_0^t e^{-c(t-u)}dX(u),$$

and that the zero coupon bond price can be expressed as

$$\begin{aligned} p(0, t) &= E\left[\exp\left\{-ce^{-ct} - \int_0^t ce^{-c(t-u)}X(u)du\right\}\right] \\ &\geq \exp\left[-ce^{-ct} - cE\left\{\int_0^t e^{-c(t-u)}X(u)du\right\}\right]. \end{aligned}$$

The lower bound introduced for the Bessel bridge process  $X(t)$  can be used to provide a lower bound for the bond price that takes counter-party risk into account.

#### 4. Conclusion

In this paper, we introduce an approach to boundary crossing problems. By considering the non-random function  $a(t) = E \sup_{0 \leq s \leq t} X(s)$ , where  $X(t), t > 0$  denotes a stochastic process with  $X(0) = 0$ , we can show that in various situations,  $ET_r \approx a^{-1}(r)$ , where  $T_r = \inf\{t : X(t) \geq r\}$ . In particular, bounds for the expectation of the stopping time  $T_r$  of any càdlàg stochastic process are derived; see [Remarks 2.1](#) and [2.2](#) where more general processes and boundaries are treated. In situations where (an upper bound on) the moment of the maximal process is available, the results shown can be helpful for the estimation of  $ET_r$ . As we can see in the examples discussed earlier, for example the sum of discrete non-negative random variables and the Gaussian processes in particular, it is evident that our approach (discovered considering decoupling) can provide a generic framework for studying the boundary crossing problem for a wide class of processes.

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