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A random flight process associated to a Lorentz gas with variable density in a gravitational field

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Abstract

We investigate a random flight process approximation to a random scatterer Lorentz gas with variable scatterer density in a gravitational field. For power function densities we show how the parameters of the model determine recurrence or transience of the vertical component of the trajectory. Finally, our methods show that, with appropriate scaling of space, time and the density of obstacles, the trajectory of the particle converges to a diffusion with explicitly given parameters.

MSC: 60F17

Keywords: Lorentz model; External field; Invariance principles

1. Introduction

We consider the random flight process that arises as the Boltzmann–Grad limit of a random scatterer model ("Lorentz gas") in a constant gravitational field in dimension three. We also extend our model to other dimensions, where it can be considered as the random walk approximation to the Boltzmann–Grad limit. The Lorentz gas model, which was introduced in

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1905 as a model for the motion of an electron in a metallic body [20], has been studied extensively in the mathematics and physics literature. See [9] for a recent survey. Fundamentally, the model consists of a particle moving in an array of fixed convex scatterers, which are placed either periodically or randomly, and the particle either reflects specularly off of the scatterers (hard core model) or is pushed away via a potential (soft core model). We are motivated by the three dimensional random scatterer hard core model where, in addition to interacting with scatterers, the particle is also pulled down by a constant gravitational field. We generalize the process to arbitrary dimension and investigate whether it is recurrent or transient. We show that dimension three with constant density of scatterers is critical for determining recurrence versus transience with respect to both dimension and the rate at which the density of scatterers increases.

Various aspects of the influence of a gravitational field on a Lorentz gas have previously been investigated, see e.g. [7,28,29,34]. Of this prior work, only [7] has worked directly with the Lorentz gas model. In [7] the authors establish the surprising result that the trajectory of a ball in a two-dimensional, periodic, hard core, Lorentz gas with gravitation is (neighborhood) recurrent [7, Theorem 1]. Heuristically, the pull of gravity is not strong enough to pull the particle to $-\infty$, but rather the scatterers are enough of an obstruction to make the particle bounce back up to some finite energy level infinitely often. In addition to this (neighborhood) recurrence result, a diffusive limit for the particle trajectory is also determined [7, Theorem 2]. We remark that although [7, Theorem 1] is stated with a hypothesis that the particle has a sufficiently high initial speed, as shown in [7, p. 838] all one needs for the particle to return to a fixed finite energy level infinitely often is for the initial speed to be positive (this is a slight oversimplification—in the deterministic setting of [7] the velocity must be uniformly distributed on a particular set specified in [7], but there can be an arbitrarily small upper bound on the initial speed of a particle whose initial velocity is in this set, see [7] for details).

One of the motivations of the present work is to investigate the robustness of these results under perturbations of the model. However, as the authors of [7] mention in their companion paper [6] their approach should extend to the three dimensional case, but the extension currently seems intractable due to the complicated nature of the singularities. Thus we work, as the authors of [28,29,34] do, with the Boltzmann–Grad limit of the random Poisson scatterer Lorentz gas rather than the Lorentz gas itself. Our results suggest that dimension three is the most difficult dimension and that the problem for the periodic Lorentz gas may become tractable again in dimensions four and higher. We determine criteria for the recurrence or transience of the particle trajectory for particular forms of the density of scatterers. Our methods allow us to derive several types of invariance principles in multiple scaling regimes and determine the influence of the density of scatterers on the limiting diffusion. A similar model with constant scatterer density was previously considered in [29], where diffusion limits were obtained but questions of transience and recurrence were not addressed.

The Boltzmann–Grad limit is a low density limit in which the number of scatterers in a fixed box goes to infinity while, at the same time, the size of each scatterer goes to zero in such a way that the distribution of the distance between scattering events for the tracer particle has a non-degenerate limit. When the centers of scatterers are placed according to a Poisson process and the rates are chosen appropriately, the asymptotic behavior of the moving particle is described by a Markovian random flight process [10,30,31]. The Markovian nature of the Boltzmann–Grad limit is due to the following two observations: (i) re-collisions with scatterers become unlikely as the size of each scatterer goes to zero, and (ii) the Poisson nature of the scatterer locations means that knowing the location of one scatterer does not give information about the locations of the other scatterers. Since analyzing the random Lorentz gas directly is beyond the capability of current techniques, this random flight model is commonly studied in both the mathematics literature [2,4,29,34] and the physics literature [1,8,23,32] to gain insight into the behavior of random Lorentz gas models. Random flight processes also arise in settings other than Lorentz gas models. For example, the random flight process we study here also appears as a model for a particle percolating through a porous medium, see [35] and the references therein.

1.1. The model

Let us now introduce our model carefully. We will use the notation $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$. We will denote the $(d-1)$-dimensional sphere in $\mathbb{R}^d$ by $\mathbf{S}^{d-1} := \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| = 1\}$ and we will typically reserve the following notation for its elements, $\mathbf{u} = (u_1, u_2, \ldots, u_d) \in \mathbf{S}^{d-1}$. We will denote components of other vectors in a similar way. The notation $d\mathbf{x}$ will refer to d-dimensional Lebesgue measure.

We are primarily interested in the process in dimension three and we start by explaining the Boltzmann–Grad limit. Fix $g > 0$ and $h : \mathbb{R} \to \mathbb{R}$. The constant g will serve as the strength of the gravitational field, which will be directed towards $-\infty$ in the last coordinate and will not act on the other coordinates, and the density of scatterers will be determined by h. We will assume that the density of scatterers depends only on the distance from the plane $\mathbb{R}^2 \times \{0\}$. This does not effect taking the Boltzmann–Grad limit, but makes our analysis of the limiting process tractable. In the Boltzmann–Grad limit, we let the size of the scatterers tend to 0 as the number of scatterers tends to ∞. In particular, assume spherical scatterers with radius $1/R$ are placed so their centers are the points of a Poisson process with intensity $R^2 h(x_3) d\mathbf{x}$. Since, typically, the trajectory of a particle in a gravitational field does not intersect itself, the arguments of [30,31] can easily be adapted to include the gravitational field and produce the following result: if the initial position and velocity of the particle is absolutely continuous with respect to Lebesgue measure on the constant energy surface then the distribution of the position and velocity process of the particle converges as $R \to \infty$, in the sense of convergence of finite dimensional distributions, to a Markovian random flight process $(\mathbf{X}(t), \mathbf{V}(t))_{t \geq 0}$ with generator

$$\begin{aligned} \widehat{D} f(\mathbf{x}, \mathbf{v}) &= \mathbf{v} \cdot \nabla_{\mathbf{x}} f(\mathbf{x}, \mathbf{v}) - g \frac{\partial}{\partial v_3} f(\mathbf{x}, \mathbf{v}) \\ &\quad + h(x_3) \|\mathbf{v}\| \int_{\mathbf{S}^2} (f(\mathbf{x}, \|\mathbf{v}\| \mathbf{u}) - f(\mathbf{x}, \mathbf{v})) \sigma(d\mathbf{u}), \end{aligned} \tag{1.1}$$

where σ is the normalized surface measure on the unit sphere $\mathbf{S}^2$, see [31]. More generally, in any dimension d we will consider the process $(\mathbf{X}(t), \mathbf{V}(t))_{t \geq 0}$ with generator

$$\begin{aligned} \widehat{D} f(\mathbf{x}, \mathbf{v}) &= \mathbf{v} \cdot \nabla_{\mathbf{x}} f(\mathbf{x}, \mathbf{v}) - g \frac{\partial}{\partial v_d} f(\mathbf{x}, \mathbf{v}) \\ &\quad + h(x_d) \|\mathbf{v}\| \int_{\mathbf{S}^{d-1}} (f(\mathbf{x}, \|\mathbf{v}\| \mathbf{u}) - f(\mathbf{x}, \mathbf{v})) \sigma(d\mathbf{u}), \end{aligned} \tag{1.2}$$

where σ is the normalized surface measure on the unit sphere $\mathbf{S}^{d-1}$. In dimensions other than 3 the Boltzmann–Grad limit of the Lorentz gas has a similar generator, but instead of the integral being against the normalized surface measure it is against a kernel that depends on $\mathbf{v}$, see Appendix. For physical reasons we are most interested in dimension 3, so we take advantage of the simplification provided by taking a uniform scattering direction. We note that even in other dimensions we expect the process we consider to have similar qualitative behavior to the

Boltzmann–Grad limit of the Lorentz gas. Our process essentially corresponds to the random walk in gravity in a field of scatterers and random walks have long been, and continue to be, a source of intuition for the behavior of Lorentz gases, see e.g. [9,24,27,36] for a non-exhaustive list. Moreover, and more rigorously, there are many universality results in the literature on transport processes, of which our process is a special case, that show the reflection directions average out on a very short time scale [11,25,26]. This is due to the diffusive heuristic that reflections happen on a shorter time scale than the particles movement, see e.g. Theorems 1.2 and 1.3 for the exact form in this setting. Thus the reflection direction will mix before a far away observer will see the particle move, and to an observer watching from far away it will seem that the particle has reflected in a uniform direction essentially because if the particle approaches a scatterer in a uniformly random direction, the direction will still be uniformly random after the scattering event has occurred. See [26,25] for detailed arguments making this rigorous in settings similar to ours and [14] for a classical version of the argument from the physics literature in a similar setting. Our expectation is further supported by the rapid decay of auto-correlations of the velocity of periodic Lorentz gas models, which goes back to [5] and continues to play a role in more recent work such as [7], which is very close to our present setting and, when taken together with our results, confirms the qualitative similarity when $d = 2$ and $h = 1$.

The process $(\mathbf{X}(t), \mathbf{V}(t))_{t\geq 0}$ can be constructed iteratively by interspersing periods of deterministic motion in a gravitation field (flights) with random perturbations to the velocity at discrete times, which explains the name "random flight process". We now give the details of this construction. Let $(\boldsymbol{\Lambda}(\mathbf{x}, \mathbf{v}, t))_{t\geq 0}$ with $\mathbf{x}, \mathbf{v} \in \mathbb{R}^d$ and $\|\mathbf{v}\|^2/2 + gx_d = E$ denote the solution to the initial value problem

$$\begin{cases} \boldsymbol{\Lambda}'' & \equiv -g\mathbf{e}_d, \\ \boldsymbol{\Lambda}(0) & = \mathbf{x}, \\ \boldsymbol{\Lambda}'(0) & = \mathbf{v}, \end{cases} \tag{1.3}$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_d$ are the standard basis vectors of $\mathbb{R}^d$. We construct our process $((\mathbf{X}(t), \mathbf{V}(t)),\ t \geq 0)$ recursively as follows. Set $(\mathbf{X}(0), \mathbf{V}(0)) = (\mathbf{x}, \mathbf{v})$ and let $T_0 = 0$. For $k \geq 1$, assuming we have defined $((\mathbf{X}(t), \mathbf{V}(t)))_{0\leq t\leq T_{k-1}}$, we let $\mathbf{U}_{k-1}$ be independent of this part of the path and uniformly distributed on $\mathbf{S}^{d-1}$ and let T_k satisfy

$$\begin{aligned} &\mathbb{P}\left(T_k - T_{k-1} > t \mid \mathbf{U}_{k-1}, ((\mathbf{X}_t, \mathbf{V}_t))_{0\leq t\leq T_{k-1}}\right) \\ &\quad = \exp\Bigg(-\int_0^t h\left(\boldsymbol{\Lambda}\Big(\mathbf{X}(T_{k-1}), \|\mathbf{V}(T_{k-1})\|\mathbf{U}_{k-1}, s\Big)\right) \\ &\qquad \times \left\|\boldsymbol{\Lambda}'\Big(\mathbf{X}(T_{k-1}), \|\mathbf{V}(T_{k-1})\|\mathbf{U}_{k-1}, s\Big)\right\| ds\Bigg). \end{aligned} \tag{1.4}$$

For $t \in [T_{k-1}, T_k]$ we then define

$$\begin{aligned} \mathbf{X}(t) &:= \boldsymbol{\Lambda}(\mathbf{X}(T_{k-1}), \|\mathbf{V}(T_{k-1})\|\mathbf{U}_{k-1}, t - T_{k-1}), \\ \mathbf{V}(t) &:= \boldsymbol{\Lambda}'(\mathbf{X}(T_{k-1}), \|\mathbf{V}(T_{k-1})\|\mathbf{U}_{k-1}, t - T_{k-1}). \end{aligned} \tag{1.5}$$

We note that, under very mild assumptions, $T_k \to \infty$ a.s., and thus this defines the path of the particle for all times. Intuitively, T_k defines the kth reflection of our particle by a scatterer.

At this point we make a simple but important observation. By conservation of energy,

$$\|\mathbf{V}(t)\| = \sqrt{2(E - gX_d(t))},$$

so that, if we define

$$v(\mathbf{x}) = \sqrt{2(E - g x_d)}, \tag{1.6}$$

then

$$\mathbf{X}(t) = \boldsymbol{\Lambda}\left(\mathbf{X}(T_{k-1}), v(\mathbf{X}(T_{k-1}))\mathbf{U}_{k-1}, t - T_{k-1}\right) \quad \text{for } t \in [T_{k-1}, T_k]. \tag{1.7}$$

Since $\mathbf{U}_{k-1}$ is independent of $((\mathbf{X}(t), \mathbf{V}(t)))_{0 \le t \le T_{k-1}}$, this implies that if we define $\mathbf{X}_k = \mathbf{X}(T_k)$, then $(\mathbf{X}_k)_{k \ge 1}$ is a Markov chain. That the index in this chain starts at 1 is an artifact of our deterministic choice of $\mathbf{V}(0)$. If instead of choosing $\mathbf{V}(0) = \mathbf{v}_0$ in the construction above we take $\mathbf{V}(0) = v(\mathbf{X}(0))\mathbf{U}$, with $\mathbf{U}$ uniformly distributed on $\mathbf{S}^{d-1}$, then $(\mathbf{X}_k)_{k \ge 0}$ is a Markov chain and its transition operator is

$$\widehat{P} f(\mathbf{x}) = \mathbb{E}\left[f\left(\boldsymbol{\Lambda}\left(\mathbf{x}, v(\mathbf{x})\mathbf{U}, N(\mathbf{x}, \mathbf{U})\right)\right)\right], \tag{1.8}$$

where $N(\mathbf{x}, \mathbf{u})$ is a random variable with distribution

$$\mathbb{P}\left(N(\mathbf{x}, \mathbf{u}) > t\right) = \exp\left(-\int_0^t h[\boldsymbol{\Lambda}(\mathbf{x}, v(\mathbf{x})\mathbf{u}, s)] v[\boldsymbol{\Lambda}(\mathbf{x}, v(\mathbf{x})\mathbf{u}, s)] ds\right), \tag{1.9}$$

and conditional on $\mathbf{U} = \mathbf{u}$, $N(\mathbf{x}, \mathbf{U})$ is distributed like $N(\mathbf{x}, \mathbf{u})$.

To simplify matters, we will assume that the particle has zero total energy, i.e., $E = 0$, so that our particle's motion remains in the half-space $\mathbb{R}^{d-1} \times (-\infty, 0]$. This is purely a normalization assumption and has no substantive impact on our results: conservation of energy shows that for $E > 0$, the d'th coordinate of the trajectory is bounded by E/g and our results hold verbatim upon translating the $x_d = E/g$ affine plane to the plane $x_d = 0$. With the assumption $E = 0$, between reflections the particle travels along the gravitational parabola

$$\left\{\boldsymbol{\Lambda}(\mathbf{x}, \mathbf{u}, t) := \sum_{i=1}^{d-1} \left(x_i + u_i \sqrt{2g|x_d|}\, t\right) \mathbf{e}_i + \left(x_d + u_d \sqrt{2g|x_d|}\, t - \frac{g}{2} t^2\right) \mathbf{e}_d, \ t \ge 0\right\}. \tag{1.10}$$

We investigate questions of transience and recurrence for the d'th coordinate of the random flight process (1.2) when h is of the form $h(\mathbf{x}) = h(x_d) = c|x_d|^\lambda$ for some $\lambda \ge 0$. Interest in variable densities of scatterers for random Lorentz gasses goes back at least to [30] since one wants to understand how the qualitative features of the process depend on the model choices. Having a variable density of scatterers also makes sense in a number of physical situations: for example one can think of our particle as a light particle in Earth's atmosphere with the 0 energy barrier being the top of the atmosphere. The increasing density of scatterers then corresponds to the fact that Earth's atmosphere gets denser. Alternatively, if one is thinking of our particle as diffusing through a porous medium as in, e.g., [35], this corresponds to the medium becoming less porous as the particle travels further into the medium. In our setting where we have surprising recurrence properties for a particle being pulled to infinity by a gravitational force in a constant density of scatterers, such as those in [7,29], it is particularly interesting to investigate whether or not changes in the density of scatterers can impact recurrence and transience. Other densities are also likely to be of interest, but the ones we have chosen give a natural class for investigating the effect of an increasing density because of the easily tunable parameter λ that exhibits a phase transition depending simply on the relationship between λ and the ambient dimension.

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Since our force acts only in the d'th coordinate, under this assumption on h the evolution of $((X_d(t), V_d(t)),\ t \geq 0)$ becomes a Markov process with generator

$$Df(y,v) = v\frac{\partial}{\partial y}f(y,v) - g\frac{\partial}{\partial v}f(y,v) + h(y)\sqrt{2g|y|}\int_{\mathbf{S}^{d-1}}\left(f\left(y, \sqrt{2g|y|}\mathbf{u}\right) - f(y,v)\right)\sigma(d\mathbf{u}), \tag{1.11}$$

and if we observe the process only at reflection times, $(X_{k,d})_{k\geq 0}$ is a Markov chain with transition operator

$$Pf(y) = \mathbb{E}\left[f\left(\Lambda_d\left(y\mathbf{e}_d, \sqrt{2g|y|}\mathbf{U}, N(y\mathbf{e}_d, \mathbf{U})\right)\right)\right]. \tag{1.12}$$

For ease of notation, we set $N(y,\mathbf{u}) = N(y\mathbf{e}_d,\mathbf{u})$. Since our force acts only in the d'th coordinate, determining transience versus recurrence for the d'th coordinate is equivalent to determining transience versus recurrence of the particle's kinetic energy. Our approach to transience versus recurrence naturally leads to some invariance principles, which we explore as well. Interestingly, the scaling is non-Brownian for most values of λ. The methods we use can also be used to establish invariance principles for more general h, and we sketch how this is done. In subsequent work of the second author and other coauthors this approach was extended to study these types of random flight processes in a general force and scattering density [12].

Suppose that $(\mathbf{X}(t), \mathbf{V}(t))_{t\geq 0}$ has the generator (1.2) and let $(\mathbf{X}(t))_{t\geq 0} = \{(X_1(t), \ldots, X_d(t))\}_{t\geq 0}$. The processes $(\mathbf{X}(t))_{t\geq 0}$ and $(X_d(t),\ t \geq 0)$ are not Markov. The concepts of recurrence and transience are typically applied to Markov processes so we need the following definition. Let $\mathbf{0} = (0, \ldots, 0)$ and assume that $(\mathbf{X}(0), \mathbf{V}(0)) = (\mathbf{0}, \mathbf{0})$. We say that $(X_d(t),\ t \geq 0)$ is neighborhood recurrent if for every $y < 0$, the process $(\mathbf{X}(t), \mathbf{V}(t))_{t\geq 0}$ hits $\mathbb{R}^{d-1} \times [y, 0] \times \mathbb{R}^d$ infinitely often, a.s. We say that $(X_d(t),\ t \geq 0)$ is recurrent if for every $y \leq 0$, the process $(\mathbf{X}(t), \mathbf{V}(t))_{t\geq 0}$ hits $\mathbb{R}^{d-1} \times \{y\} \times \mathbb{R}^d$ infinitely often, a.s.

Our main result on transience versus recurrence in the case $h(\mathbf{x}) = h(x_d) = c|x_d|^\lambda$ is the following theorem.

Theorem 1.1. *Let $(\mathbf{X}(t), \mathbf{V}(t))_{t\geq 0}$ be the Markov process with generator (1.2) started from $(\mathbf{0}, \mathbf{0})$ with gravitation g and scatterer density $h(\mathbf{x}) = h(x_d) = c|x_d|^\lambda$, with $c > 0$ and $\lambda \geq 0$. Let $(\mathbf{X}(t))_{t\geq 0} = \{(X_1(t), \ldots, X_d(t))\}_{t\geq 0}$.*

1. *If $d = 1$ then $(X_d(t),\ t \geq 0)$ is recurrent.*
2. *If $d \in \{2, 3\}$ then $(X_d(t),\ t \geq 0)$ is neighborhood recurrent but not recurrent.*
3. *If $d \geq 4$ then $(X_d(t),\ t \geq 0)$ is transient if $\lambda < (d-3)/2$ and neighborhood recurrent (but not recurrent) if $\lambda > (d-3)/2$.*

Note that recurrence trivially fails when $d \geq 2$ because conservation of energy implies that X_d can visit 0 only if one of the reflections puts all of the velocity in the positive x_d-direction, an event that happens with probability 0. However, in the regimes where $X_d(t)$ is neighborhood recurrent and $d \geq 2$ we will show that 0 is the only number in $(-\infty, 0]$ that $X_d(t)$ does not visit infinitely often. Both recurrence and neighborhood recurrence imply the interesting result that gravity is not sufficiently strong to pull the particle through the field of scatterers to $-\infty$.

We note that already for the case $d = 3$ we have to do careful calculations to show that the process is neighborhood recurrent when $\lambda = 0$, which is the "critical" case in dimension 3. In this we are aided by the fact that h is constant in this case. Even more delicate calculations are likely to be needed to determine whether the process is transient or recurrent when $d \geq 4$ and $\lambda = (d-3)/2$ so we leave this case open.

Our approach to proving Theorem 1.1 leads naturally to two invariance principles, the first for the process observed at reflection times and the second for the process on its natural time scale.

Theorem 1.2. *Let* $(\mathbf{X}_k)_{k\geq 0} = \{(X_{1,k}, \ldots, X_{d,k})\}_{k\geq 0}$ *be the Markov chain with transition operator (1.8) with gravitation* g *and scatterer density* $h(\mathbf{x}) = h(x_d) = c|x_d|^\lambda$, *with* $c > 0$ *and* $\lambda \geq 0$. *Let*

$$d' = \frac{d+1+2\lambda}{2+2\lambda}.$$

Under these conditions, regardless of the distribution of X_0,

$$\left(\frac{1}{n^{\frac{1}{2+2\lambda}}} X_{d,[nt]},\ t \geq 0\right) \to_d \left(-\rho_{d'}\left(\frac{2}{dc^2}(1+\lambda)^2 t\right)^{1/(1+\lambda)},\ t \geq 0\right),$$

where the convergence is in distribution on the Skorokhod space $D(\mathbb{R}_+, \mathbb{R})$ *and* $(\rho_{d'}(t),\ t \geq 0)$ *is a* d'*-dimensional Bessel process started at* 0.

The standard classification of recurrence versus transience for Bessel processes shows that the limiting process is recurrent at 0 if $\lambda > (d-3)/2$, transient if $\lambda < (d-3)/2$, and neighborhood recurrent at 0 if $\lambda = (d-3)/2$. This suggests that for the process in Theorem 1.1 neighborhood recurrence is observable on the diffusive scale, since neighborhoods collapse to the origin under scaling, when $\lambda > (d-3)/2$ but is only observable on longer time scales when $\lambda = (d-3)/2$. This suggests that $\lambda = (d-3)/2$ will be the most subtle case to analyze.

Note that the scaling is non-Brownian except when $\lambda = 0$. Since Theorem 1.2 deals with the process observed only at reflection times, the particle's velocity does not contribute to this exponent. That is, the non-Brownian scaling is caused purely by the increasing scattering density. The next result, which provides an invariance principle for $(X_d(t))_{t\geq 0}$, shows that the particle's velocity contributes a further non-Brownian term to the scaling. Our approach uses a time change argument, but the result is somewhat weaker since the time change is degenerate when the limiting process hits 0. Consequently, we must stop the process before it hits 0. Clearly, this is only a meaningful restriction if 0 is recurrent for the limiting process.

Theorem 1.3. *Let* $(\mathbf{X}(t), \mathbf{V}(t))_{t\geq 0}$ *be the Markov process with generator (1.2) started from* $(\mathbf{0}, \mathbf{0})$ *with gravitation* g *and scatterer density* $h(\mathbf{x}) = h(x_d) = c|x_d|^\lambda$, *with* $c > 0$ *and* $\lambda \geq 0$. *Let* $(\mathbf{X}(t))_{t\geq 0} = \{(X_1(t), \ldots, X_d(t))\}_{t\geq 0}$. *Fix* $z < v < 0$. *Let* T_z^n *be the time of the first reflection at which* $X_d < n^{1/(2+2\lambda)} z$ *and let* T_v^n *be the time of the first reflection after* T_z^n *such that* $X_d > n^{1/(2+2\lambda)} v$. *Let* $\mathscr{Z}$ *be a diffusion on* $(-\infty, 0)$ *started from* z *whose generator acts on* $f \in C^2$ *with compact support in* $(-\infty, 0)$ *by*

$$\mathcal{G}^{\lambda,c} f(y) = \frac{2\sqrt{2g}}{dc}|y|^{1/2-\lambda}\left[\frac{1}{2}f''(y) - \left(\frac{d-1-2\lambda}{4|y|}\right)f'(y)\right].$$

As $n \to \infty$ *we have*

$$\left(n^{-\frac{1}{2+2\lambda}} X_d\left(\left(n^{\frac{3+2\lambda}{4+4\lambda}} t + T_z^n\right) \wedge T_v^n\right),\ t \geq 0\right) \to (\mathscr{Z}(t \wedge \tau_{v+}),\ t \geq 0),$$

in distribution in the Skorokhod space $D(\mathbb{R}_+, \mathbb{R})$, *where* $\tau_{v+} = \inf\{t : \mathscr{Z}(t) > v\}$.

This result allows us to interpret the effects of λ. Observe that $\frac{2\sqrt{2g}}{dc}|y|^{1/2-\lambda}$ acts as an overall time change and

$$\bar{\mathcal{G}}^{\lambda,c} f(y) = \frac{1}{2} f''(y) - \left(\frac{d-1-2\lambda}{4|y|} \right) f'(y)$$

is the generator of (-1 times) a Bessel process with dimension $\bar{d} = (d+1-2\lambda)/2$. If $\lambda > (d-1)/2$, the drift is positive and the process is effectively pushed up towards 0, while if $\lambda < (d-1)/2$, the process is pushed down away from 0, but may still be recurrent because of its diffusive nature. Thus increasing λ either strengthens the push towards 0 (large λ) or weakens the pull away from 0 (small λ). This can be understood heuristically by noting that it shows that it is more difficult for the particle to penetrate into areas with a high density of scatterers.

We note that our approach bears some similarities to other work on invariance principles related to anomalous diffusions, see e.g. [21], but our situation is fundamentally different. In the current setting the particle's speed is unbounded so that the waiting time between reflections can be very small and this contributes to the anomalous scaling. However, although the scaling is anomalous, our limiting diffusion is not. This is in contrast to [21] and other work on anomalous diffusion where the anomalous scaling arises because waiting times can be heavy tailed. Since Theorem 1.3 is a result about approximation of a one-dimensional diffusion there are other approaches as well, for example using [15]. The general literature on billiards, billiards with potential, and on Lorentz gas models is huge and we do not feel that we can do justice to this body of research. The articles [7,29] and references therein are a good point of entry to this field.

This article is organized as follows. In Section 2 we consider a simplified model where the particle travels distance exactly one between reflections. The computations in this case are simpler and the model illustrates the approach we take in the general case. Section 3 is devoted to the proofs of Theorems 1.1, 1.2, and 1.3, with Section 3.1 containing technical estimates and Section 4 containing the proofs of the theorems.

2. An overview of the method

Our approach is to employ results developed by Lamperti [16–18]. These papers provide a general framework for establishing recurrence or transience of nonnegative Markov processes. We collect and combine several results of Lamperti in Theorem 2.1.

Given $A \geq 0$, we will say that a non-negative stochastic process $(X_m, m \geq 0)$ is A-recurrent if $\mathbb{P}(X_m \in [0, A] \text{ i.o.}) = 1$.

Theorem 2.1. *Let $(X_m, m \geq 0)$ be a Markov chain on $[0, \infty)$ with transition operator $\mathcal{T}$ and for $\vartheta \in \mathbb{R}$, let*

$$\mu_k^\vartheta(x) = \mathbb{E}\left[\left(X_{n+1}^{(2-\vartheta)/2} - X_n^{(2-\vartheta)/2} \right)^k \;\middle|\; X_n = x \right].$$

When $\vartheta = 0$ we suppress it in the notation. That is, we set $\mu_k = \mu_k^0$. Assume:

1. *There exists $\vartheta < 2$ such that, as $x \to \infty$, $x^{1-\vartheta}\mu_1(x) \to a$, $x^{-\vartheta}\mu_2(x) \to b > 0$ with $2a + b(1-\vartheta) > 0$ and for each fixed $k \in \mathbb{N}$, $\mu_k(x) = O(x^{k\vartheta/2})$.*
2. *$\mathcal{T}$ maps the set $C_0(\mathbb{R}_+, \mathbb{R})$ of continuous functions from $[0, \infty) \to \mathbb{R}$ that vanish at ∞ to itself.*
3. *$\mathbb{P}(\limsup X_n = \infty \mid X_0 = x) = 1$ for all $x \in [0, \infty)$.*

Let

$$c = \frac{b\,(1-\vartheta)+2a}{b\left(1-\frac{\vartheta}{2}\right)}.$$

(a) *Regardless of the distribution of* X_0,

$$\left(\frac{1}{n^{\frac{1}{2-\vartheta}}}X_{[nt]},\ t\ge 0\right)\to_d \left(\rho_c\left(b\left(1-\frac{\vartheta}{2}\right)^2 t\right)^{2/(2-\vartheta)},\ t\ge 0\right),$$

where the convergence is in distribution on the Skorokhod space $D(\mathbb{R}_+,\mathbb{R})$ *and* $(\rho_c(t),\ t\ge 0)$ *is a c-dimensional Bessel process started at* 0.

(b) *If* $2a > b$ *then* $(X_m, m\ge 0)$ *is transient.*

(c) *If* $2a < b$ *then there exists* $A\ge 0$ *such that* $(X_m, m\ge 0)$ *is* A*-recurrent.*

(d) *If, for* ϑ *as in Assumption 1 ,* $2x\mu_1^{\vartheta}(x)-\mu_2^{\vartheta}(x)=O(x^{-\varepsilon})$ *for some* $\varepsilon>0$ *then there exists* $A\ge 0$ *such that* $(X_m, m\ge 0)$ *is* A*-recurrent.*

Remark 2.2. What we are calling A-recurrence is simply called recurrence by Lamperti in [16–18].

Proof. Let $Y_m = X_m^{(2-\vartheta)/2}$. Since $(Y_m, m \ge 0)$ is Markov, [17, Lemma 7.1] shows that the claims of recurrence and transience for $(Y_m, m\ge 0)$ are settled by [16, Theorem 3.2], which establishes parts (b), (c), and (d). Assumptions 1, 2, and 3 and [17, Lemma 7.1] show that the hypotheses of Theorem 4.1 in [18] are satisfied for $(Y_m, m\ge 0)$. Combining the conclusions of [18, Theorem 4.1] with Assumptions 1 and 3 shows that the hypotheses of Theorem 5.1 in [17] are satisfied so part (a) follows from the conclusion of [17, Theorem 5.1] along with translating the results for $(Y_m, m\ge 0)$ back to $(X_m, m\ge 0)$. □

We note that the functional limit theorem of [17, Theorem 5.1] actually pertains to the scaled linearly interpolated process rather than the scaled step process, and convergence in distribution on $C(\mathbb{R}_+,\mathbb{R})$, but the convergence of the scaled step process in distribution on $D(\mathbb{R}_+,\mathbb{R})$ follows immediately.

In our present context, there is no difference between A-recurrence and neighborhood recurrence.

Proposition 2.3. *If* $(X_{k,d})_{k\ge 0}$ *is a Markov process with transition operator* (1.12)*, then* $(X_{k,d})_{k\ge 0}$ *is neighborhood recurrent if and only if* $(|X_{k,d}|)_{k\ge 0}$ *is* A*-recurrent for some* $A\ge 0$.

Proof. Fix $\varepsilon > 0$ and observe from (1.12) that $\min_{0\le x\le A}\mathbb{P}_x(|X_{1,d}|<\varepsilon) > 0$. Combined with the strong Markov property this implies that $\mathbb{P}_0(|X_{k,d}|<\varepsilon\ i.o.) = 1$ since $\mathbb{P}_0(|X_{k,d}|\le A\ i.o.) = 1$. □

With Theorem 2.1 in hand, the conditions that must be checked to prove our results are clear, but the calculations become quite involved in the general case. Thus, before getting into the true model, we show how the method works in a simplified model where $\lambda = 0$ and the particle travels distance exactly equal to one between reflections.

2.1. Motion with deterministic distance between reflections

This section is a warm up, in the sense that we analyze a simplified model, to develop a sense for results that we can expect in a more realistic and hence more complicated situation.

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Specifically, we assume that the distance between any two consecutive reflections measured along the trajectory of the particle is exactly one, which is the expected distance the particle travels between reflections in the true model when $h = 1$. In this model, upon reflection at $\mathbf{x} \in \mathbb{R}^d$, the particle starts its path in a uniform direction $\mathbf{u} \in \mathbf{S}^{d-1}$ and then travels along the parabola (1.10) (with t measuring the time since the last reflection) until it has traveled distance exactly one, at which point it reflects again. Let $(\mathbf{X}(t),\ t \geq 0)$ be the path of such a particle and let the discrete time process $(X_d^*(k), k \in \mathbb{N}_0)$ record the positions of $(X_d(t),\ t \geq 0)$ at the reflection times. Note that this is not the same as sampling of X_d at equal or identically distributed time intervals because the velocity of X_d increases with $|X_d|$ and the times between scattering events become smaller on average. The process $(X_d^*(k),\ k \in \mathbb{N}_0)$ is a Markov chain with transition operator U that acts on C^2 function f with compact support in $(-\infty, 0)$ by

$$(Uf)(y) = \int_{\mathbf{S}^{d-1}} f(\Lambda_d(y\mathbf{e}_d, \mathbf{u}, t(y\mathbf{e}_d, \mathbf{u})))\sigma(d\mathbf{u})$$

where $t(\mathbf{x}, \mathbf{u})$ is the time it takes to travel distance 1 along the parabola in (1.10) with initial position $\mathbf{x}$ and initial velocity in the direction of $\mathbf{u}$. That is, $t(\mathbf{x}, \mathbf{u}) = \inf\{s : \ell(\mathbf{x}, \mathbf{u}, s) > 1\}$ where

$$\ell(\mathbf{x}, \mathbf{u}, t) = \int_0^t \sqrt{2g|x_d|(1 - u_d^2) + \left(\sqrt{2g|x_d|}u_d - gs\right)^2}\, ds.$$

Theorem 2.4. *The process $(X_d^*(m), m \in \mathbb{N}_0)$ is neighborhood recurrent if $d \leq 3$ and transient if $d \geq 4$.*

Proof. We will apply Theorem 2.1 to the process $(|X_d^*(m)|,\ m \in \mathbb{N}_0)$ in place of $(X_m, m \geq 0)$, with $\vartheta = 0$. Condition 2 of Theorem 2.1 follows from the continuity of Λ and ℓ while Condition 3 of the same theorem is a consequence of Markov property and the fact that there for every n, there is a positive probability that the process started in the interval $[-n, 0]$ will leave the interval after $n + 1$ steps. This leaves the problem of finding the limits in Condition 1 of Theorem 2.1. We need to analyze $\mu_1(y) = |y|\mathbb{E}_y(X_d^*(1) - y)$ and $\mu_2(y) = \mathbb{E}_y\left[(X_d^*(1) - y)^2\right]$ as y tends to $-\infty$. The key to doing this is to analyze how $t(y\mathbf{e}_d, \mathbf{u})$, the time between reflections, depends on y. Intuitively, it takes the particle the longest amount of time to travel distance 1 when its direction goes against the pull of gravity and the shortest amount of time to travel distance 1 when it travels with the pull of gravity. Making this rigorous leads to the monotonicity relation $t(\mathbf{x}, -\mathbf{e}_d) \leq t(\mathbf{x}, \mathbf{u}) \leq t(\mathbf{x}, \mathbf{e}_d)$ for all $\mathbf{u} \in \mathbf{S}^{d-1}$. Moreover, assuming $x_d \leq -1$, as we will for the remainder, we can explicitly compute

$$t(\mathbf{x}, \mathbf{e}_d) = \sqrt{\frac{2}{g}}\left(\sqrt{|x_d|} - \sqrt{|x_d| - 1}\right) \quad \text{and} \quad t(\mathbf{x}, -\mathbf{e}_d) = \sqrt{\frac{2}{g}}\left(\sqrt{|x_d| + 1} - \sqrt{|x_d|}\right).$$

From this, one observes that $\sqrt{|y|}t(y\mathbf{e}_d, \pm\mathbf{e}_d) \to (\sqrt{2g})^{-1/2}$ as $y \to -\infty$ and, consequently,

$$\lim_{y \to -\infty} \sqrt{|y|}t(y\mathbf{e}_d, \mathbf{u}) \to \frac{1}{\sqrt{2g}}, \tag{2.1}$$

uniformly in $\mathbf{u}$. Let $\ell_t(\mathbf{x}, \mathbf{u}, t)$, $\ell_{tt}(\mathbf{x}, \mathbf{u}, t)$ and $\ell_{ttt}(\mathbf{x}, \mathbf{u}, t)$ denote the first, second and third partial derivatives, respectively, of $\ell(\mathbf{x}, \mathbf{u}, t)$ in the third variable. We have $\ell_t(\mathbf{x}, \mathbf{u}, 0) = \sqrt{2g|x_d|}$ and $\ell_{tt}(\mathbf{x}, \mathbf{u}, 0) = -gu_d$. It follows from the definition of $t(\mathbf{x}, \mathbf{u})$ that $\ell(\mathbf{x}, \mathbf{u}, t(\mathbf{x}, \mathbf{u})) = 1$. Taylor

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expanding ℓ in the t variable yields

$$\begin{aligned}1 &= \ell(\mathbf{x},\mathbf{u},t(\mathbf{x},\mathbf{u}))\\ &= \sqrt{2g|x_d|}\,t(\mathbf{x},\mathbf{u}) - \frac{gu_d}{2}t(\mathbf{x},\mathbf{u})^2 + \frac{\ell_{ttt}(\mathbf{x},\mathbf{u},\alpha)}{6}t(\mathbf{x},\mathbf{u})^3\end{aligned}$$

for some $\alpha = \alpha(\mathbf{x},\mathbf{u}) \le t(\mathbf{x},\mathbf{e}_d)$. Rearranging, this yields the relation

$$t(\mathbf{x},\mathbf{u}) = \frac{1}{\sqrt{2g|x_d|}}\left(1 + \frac{gu_d}{2}t(\mathbf{x},\mathbf{u})^2 - \frac{\ell_{ttt}(\mathbf{x},\mathbf{u},\alpha)}{6}t(\mathbf{x},\mathbf{u})^3\right). \tag{2.2}$$

We have

$$|y|\mathbb{E}_y(X_d^*(1) - y) = \mathbb{E}_y\left(U_d\sqrt{2g}|y|^{3/2}t(y\mathbf{e}_d,\mathbf{U})\right) + \mathbb{E}_y\left(-\frac{g}{2}|y|t(y\mathbf{e}_d,\mathbf{U})^2\right). \tag{2.3}$$

By (2.1), the second term in (2.3) converges to $-1/4$ as $y \to -\infty$. To analyze the first term, we substitute (2.2) and use $\mathbb{E}_y(U_d) = 0$ to find that

$$\begin{aligned}\mathbb{E}_y\left(U_d\sqrt{2g}|y|^{3/2}t(y\mathbf{e}_d,\mathbf{U})\right) &= \mathbb{E}_y\left(\frac{gU_d^2}{2}|y|t(y\mathbf{e}_d,\mathbf{U})^2\right)\\ &- \mathbb{E}_y\left(U_d\frac{\ell_{ttt}(y\mathbf{e}_d,\mathbf{U},\alpha(y,\mathbf{U}))}{6}|y|t(y\mathbf{e}_d,\mathbf{U})^3\right).\end{aligned} \tag{2.4}$$

From (2.1) we see that

$$\lim_{y\to-\infty}\mathbb{E}_y\left(\frac{gU_d^2}{2}|y|t(y\mathbf{e}_d,\mathbf{U})^2\right) = \frac{1}{4}\mathbb{E}(U_d^2) = \frac{1}{4d}.$$

Furthermore, a somewhat tedious, but ultimately straightforward, calculus exercise shows that $\ell_{ttt}(y\mathbf{e}_d,\mathbf{u},t) = O(|y|^{-1/2})$ as $y \to -\infty$ uniformly in $\mathbf{u}$, and $0 \le t \le t(y\mathbf{e}_d,\mathbf{e}_d)$ which, combined with (2.1), show that the second term in (2.4) converges to 0 as $y \to -\infty$. Therefore

$$-a := \lim_{y\to-\infty}|y|\mu_1(y) = \lim_{y\to-\infty}|y|\mathbb{E}_y(X_d^*(1) - y) = \frac{1}{4d} - \frac{1}{4} = \frac{1-d}{4d}.$$

The negative sign is because Lamperti's processes are positive while ours are negative. Similarly, using (2.1) we see that

$$\begin{aligned}b &:= \lim_{y\to-\infty}\mu_2(y) = \lim_{y\to-\infty}\mathbb{E}_y\left[(X_d^*(1)-y)^2\right]\\ &= \lim_{y\to-\infty}\mathbb{E}_y\left[\left(U_d\sqrt{2g|y|}\,t(y\mathbf{e}_d,\mathbf{U}) - \frac{g}{2}t(y\mathbf{e}_d,\mathbf{U})^2\right)^2\right]\\ &= \frac{1}{d}.\end{aligned}$$

This shows that the limits in Condition 1 of Theorem 2.1 exist. Moreover,

$$2a - b = \frac{d-1}{2d} - \frac{1}{d} = \frac{d-3}{2d}.$$

The claims of Theorem 2.4 can now be read off from Theorem 2.1. Since $2a - b$ is positive if $d \ge 4$, the process is transient in this case. Moreover, $2a - b$ is negative if $d \le 2$ so the process is A-recurrent in this case for some $A \ge 0$. In the case $d = 3$, we have $2a - b = 0$, so this is the critical case. One can verify that when $d = 3$, $2|y|\mu_1(y) - \mu_2(y) = O(|y|^{-\varepsilon})$ for sufficiently small $\varepsilon > 0$ and, consequently, the process is A-recurrent in this case as well. We leave this calculation in the present toy model to the reader since we do the analogous (more difficult) calculation for our main model below. A slight modification of the Markov property argument in Proposition 2.3 shows that A-recurrence for any $A \geq 0$ implies neighborhood recurrence for $(|X^*_d(m)|, m \in \mathbb{N}_0)$. □

3. The general model

In this section we address the general model with generator (1.2) where h is of the form $h(y) = c|y|^\lambda$ for some $\lambda \geq 0$ and $c > 0$. We prove some limit theorems and results on transience and recurrence. Although Section 2 illustrates our methods, the results in this section are technically more difficult because we must control the distance the particle travels between reflections as well as the time between reflections in order to establish our invariance principles.

3.1. Basic estimates

Recall that for $y \leq 0$, we define $N(y, \mathbf{u}) = N(y\mathbf{e}_d, \mathbf{u})$ where $N(\mathbf{x}, \mathbf{u})$ is defined in (1.9) for $\mathbf{x} \in \mathbb{R}^d$.

Lemma 3.1. *For every $t \geq 0$ we have*

$$\lim_{y\to-\infty} \sup_{\mathbf{u}\in\mathbf{S}^{d-1}} \left| \mathbb{P}\left(\sqrt{2g|y|}h(y)N(y,\mathbf{u}) > t\right) - e^{-t} \right| = 0.$$

Proof. We use (1.9) and the substitution $w = \sqrt{2g|y|}h(y)s$ to see that

$$\begin{aligned}
&-\log\left(\mathbb{P}\left(\sqrt{2g|y|}h(y)N(y,\mathbf{u}) > t\right)\right) = -\log\left(\mathbb{P}\left(N(y,\mathbf{u}) > \frac{t}{\sqrt{2g|y|}h(y)}\right)\right)\\
&= \int_0^t \frac{h\left(y + \frac{u_d}{h(y)}w - \frac{1}{4|y|h(y)^2}w^2\right)}{\sqrt{2g|y|}h(y)}\\
&\quad \times \sqrt{2g|y|(1-u_d^2) + \left(\sqrt{2g|y|}u_d - \frac{g}{\sqrt{2g|y|}h(y)}w\right)^2}\, dw\\
&= \int_0^t \frac{h\left(y + \frac{u_d}{h(y)}w - \frac{1}{4|y|h(y)^2}w^2\right)}{h(y)} \sqrt{(1-u_d^2) + \left(u_d - \frac{1}{2|y|h(y)}w\right)^2}\, dw.
\end{aligned}$$

For $h(y) = c|y|^\lambda$, we have

$$\lim_{y\to-\infty} \sup_{(w,\mathbf{u})\in[0,t]\times\mathbf{S}^{d-1}} \left| \frac{h\left(y + \frac{u_d}{h(y)}w - \frac{1}{4|y|h(y)^2}w^2\right)}{h(y)} - 1 \right| = 0,$$

and the lemma follows. □

In fact, this convergence in distribution can be extended to convergence of moments.

Lemma 3.2. *For fixed $p \geq 1$,*

$$\mathbb{E}\left[\left(\max\{\sqrt{2g|y|}h(y), 1\}N(y, \mathbf{u})\right)^p\right]$$

is bounded uniformly in y and $\mathbf{u} \in \mathbf{S}^{d-1}$.

Proof. We handle the cases $y \leq -1$ and $y > -1$ separately. For $-1 \leq y \leq 0$ there is a finite longest time for a parabolic path started with $-1 \leq y \leq 0$ to leave $[-1, 0]$. Outside this interval h is bounded below by a strictly positive constant. Hence, once the particle is outside $[-1, 0]$, it will encounter a scatterer at some strictly positive rate. This implies that all of the $N(y, \mathbf{u})$ with $-1 \leq y \leq 0$ are stochastically dominated by a single random variable with an exponential tail. The lemma easily follows in this case.

We now turn to the case $y \leq -1$. A monotonicity argument shows that

$$\mathbb{P}(N(y, \mathbf{u}) > t) \leq \exp\left[-\int_0^t c\left(\frac{g}{2}s^2 - \sqrt{2g|y|}s - y\right)^\lambda \left|\sqrt{2g|y|} - gs\right| ds\right].$$

Fix $0 < \varepsilon < 1/4$. We need to control the amount of time the particle can spend above ε, since this is where the collision rate is low and $\mathbb{P}(N(y, \mathbf{u}) > t)$ decreases slowly in this region. Define

$$s_-(\mathbf{u}) = \inf\left\{s \geq 0 : y + u_d\sqrt{2g|y|}s - \frac{g}{2}s^2 = -\varepsilon\right\} \tag{3.1}$$

and

$$s_+(\mathbf{u}) = \sup\left\{s \geq 0 : y + u_d\sqrt{2g|y|}s - \frac{g}{2}s^2 = -\varepsilon\right\}.$$

Monotonicity arguments show that

$$s_-(\mathbf{u}) \geq s_-(\mathbf{e}_d) = \sqrt{\frac{2}{g}}\left(\sqrt{|y|} - \sqrt{\varepsilon}\right) \tag{3.2}$$

and

$$s_+(\mathbf{u}) \leq s_+(\mathbf{e}_d) = \sqrt{\frac{2}{g}}\left(\sqrt{|y|} + \sqrt{\varepsilon}\right).$$

To simplify notation, let us use $s_\pm := s_\pm(\mathbf{e}_d)$. This leads to the bounds

$$\begin{aligned}&\mathbb{P}(N^n(y, \mathbf{u}) > t)\\ &\quad\leq \begin{cases}\exp\left[-h(-\varepsilon)t\left(\sqrt{2g|y|} - \frac{g}{2}t\right)\right], & t \leq s_-,\\ \exp\left[-h(-\varepsilon)\left(\sqrt{2g|y|}[s_- + s_+ - t] + \frac{g}{2}[t^2 - s_-^2 - s_+^2]\right)\right], & t \geq s_+.\end{cases}\end{aligned} \tag{3.3}$$

An application of Fubini's theorem shows that $\mathbb{E}(R^p) = p\int_0^\infty t^{p-1}\mathbb{P}(R > t)dt$ for any non-negative random variable R. Using this and (3.3) we find that

$$\begin{aligned}&\mathbb{E}(N(y, \mathbf{u})^p)\\ &\quad\leq p\int_0^{s_-/4} t^{p-1}\exp\left[-\int_0^t c\left(\frac{g}{2}s^2 - \sqrt{2g|y|}s - y\right)^\lambda\left(\sqrt{2g|y|} - gs\right)ds\right]dt\\ &\qquad+ p\int_{s_-/4}^{4s_+} t^{p-1}\exp\left[-h(-\varepsilon)s_-\left(\sqrt{2g|y|} - \frac{g}{8}s_-\right)/4\right]dt\end{aligned}$$

$$+ p \int_{4s_+}^{\infty} t^{p-1} \exp\left[-h(-\varepsilon)\left(\sqrt{2g|y|}\left[s_- + s_+ - t\right] + \frac{g}{2}\left[t^2 - s_-^2 - s_+^2\right]\right)\right] dt. \tag{3.4}$$

The first integral is the most challenging, so we take care of the second and third integrals first. Since

$$\sqrt{2g|y|} - \frac{g}{8}s_- = \sqrt{2g|y|} - \frac{\sqrt{2g}}{8}(\sqrt{|y|} - \sqrt{\varepsilon}) \geq \frac{3}{4}\sqrt{2g|y|},$$

we have

$$\lim_{y\to-\infty} p\left(\sqrt{2g|y|}h(y)\right)^p \int_{s_-/4}^{4s_+} t^{p-1} \exp\left[-h(-\varepsilon)s_-\left(\sqrt{2g|y|} - \frac{g}{8}s_-\right)/4\right] dt = 0 \tag{3.5}$$

because the integral term decays exponentially in $|y|$. Similarly we have

$$\lim_{y\to-\infty} p\left(\sqrt{2g|y|}h(y)\right)^p \times \int_{4s_+}^{\infty} t^{p-1} \exp\left[-h(-\varepsilon)\left(\sqrt{2g|y|}\left[s_- + s_+ - t\right] + \frac{g}{2}\left[t^2 - s_-^2 - s_+^2\right]\right)\right] dt = 0. \tag{3.6}$$

For the first integral in (3.4), use the Mean Value Theorem to see that for $0 \leq t \leq s_-/4$,

$$\begin{aligned}
&-\int_0^t c\left(\frac{g}{2}s^2 - \sqrt{2g|y|}s - y\right)^\lambda \left(\sqrt{2g|y|} - gs\right) ds \\
&= \frac{c}{\lambda+1}\left[\left(\frac{g}{2}t^2 - \sqrt{2g|y|}t - y\right)^{\lambda+1} - |y|^{\lambda+1}\right] \\
&\leq \frac{c}{\lambda+1}\left[\left(-\frac{\sqrt{2g|y|}}{2}t - y\right)^{\lambda+1} - |y|^{\lambda+1}\right] \\
&\leq -c\frac{\sqrt{2g|y|}}{2}t \inf\left\{|z|^\lambda : -\frac{\sqrt{2g|y|}}{2}t - y \leq z \leq -y\right\} \\
&\leq -c\frac{\sqrt{2g|y|}}{2}t\left|-\frac{\sqrt{2g|y|}}{2}s_+/4 - y\right|^\lambda \leq -C\sqrt{2g|y|}\,|y|^\lambda\, t,
\end{aligned}$$

where $C > 0$ is a constant depending on λ but not y. Consequently, we have

$$\begin{aligned}
&p\int_0^{s_-} t^{p-1} \exp\left[-\int_0^t c\left(\frac{g}{2}s^2 - \sqrt{2g|y|}s - y\right)^\lambda \left(\sqrt{2g|y|} - gs\right) ds\right] dt \\
&\qquad \leq p\int_0^\infty t^{p-1}\exp\left(-C\sqrt{2g|y|}\,|y|^\lambda\, t\right) dt = \frac{p!}{C^p(2g|y|)^{p/2}|y|^{p\lambda}}.
\end{aligned}$$

This and (3.4)–(3.6) prove the result for $y \leq -1$. □

The next lemma gives a uniform version of the classical result that convergence in distribution together with bounded moments implies the convergence of moments.

Lemma 3.3. *For every $p \geq 1$ we have*

$$\lim_{y\to-\infty} \sup_{\mathbf{u}\in\mathbf{S}^{d-1}} \left|\mathbb{E}\left[\left(\sqrt{2g|y|}h(y)N(y,\mathbf{u})\right)^p\right] - \int_0^\infty pt^{p-1}e^{-t}dt\right| = 0.$$

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Proof. Let $B(y, \mathbf{u}, r) = \left\{\sqrt{2g|y|}h(y)N(y, \mathbf{u}) \le r\right\}$ and define q by $1/q + p/(p+1) = 1$. Also, let $\eta(y) = \sqrt{2g|y|}h(y)$. Then

$$\begin{aligned}
&\mathbb{E}\left(\left[\eta(y)^p N(y, \mathbf{u})^p\right] - \left[\eta(y)^p N(y, \mathbf{u})^p\right] \wedge r^p\right) \\
&\quad \le \mathbb{E}\left(\left[\eta(y)^p N(y, \mathbf{u})^p\right] \mathbb{1}_{B^c(y,\mathbf{u},r)}\right) \\
&\quad \le \mathbb{E}\left(\eta(y)^{p+1} N(y, \mathbf{u})^{p+1}\right)^{p/(p+1)} \left(\mathbb{P}(\eta(y)N(y, \mathbf{u}) > r)\right)^{1/q} \\
&\quad \le \mathbb{E}\left(\eta(y)^{p+1} N(y, \mathbf{u})^{p+1}\right)^{p/(p+1)} \mathbb{E}(\eta(y)N(y, \mathbf{u}))^{1/q} r^{-1/q}.
\end{aligned}$$

Both expectations are uniformly bounded by Lemma 3.2 so the bound goes uniformly to 0 as $r \to \infty$. The proof is completed by noting that it follows from Lemma 3.1 that for every $r \ge 0$,

$$\lim_{y\to-\infty} \sup_{\mathbf{u}\in\mathbf{S}^{d-1}} \left| \mathbb{E}\left[\left(\sqrt{2g|y|}h(y)N(y, \mathbf{u})\right)^p \wedge r^p\right] - \int_0^r pt^{p-1}e^{-t}dt \right| = 0. \quad \square$$

The next lemma is needed to control lower order fluctuations. This is where the averaging occurs and it becomes important that the scattering distribution has mean 0.

Lemma 3.4. *Let* $\mathbf{U}$ *be uniformly distributed on* $\mathbf{S}^{d-1}$ *and let* $N(y, \mathbf{U})$ *be distributed like* $N(y, \mathbf{u})$ *conditional on* $\mathbf{U} = \mathbf{u}$*. We then have*

$$\lim_{y\to-\infty} \left| \left(h(y)^2\sqrt{2g|y|^3}\right) \mathbb{E}\left[U_d N(y, \mathbf{U})\right] - \frac{1+2\lambda}{2d} \right| = 0.$$

Proof. Let

$$\begin{aligned}
H(y, \mathbf{u}, t) &= h\left(y + u_d\sqrt{2g|y|}t - \frac{g}{2}t^2\right), \\
M(y, \mathbf{u}, t) &= \sqrt{2g|y|(1-u_d^2) + \left(u_d\sqrt{2g|y|} - gt\right)^2}, \\
F(y, \mathbf{u}, t) &= \int_0^t H(y, \mathbf{u}, s)M(y, \mathbf{u}, s)ds.
\end{aligned}$$

Using the change of variables $z = F(y, \mathbf{u}, t)$ and the density of $N(y, \mathbf{u})$ derived from (1.9) one finds that

$$1 = \mathbb{E}[F(y, \mathbf{u}, N(y, \mathbf{u}))] \quad \text{and} \quad 2 = \mathbb{E}[F(y, \mathbf{u}, N(y, \mathbf{u}))^2], \tag{3.7}$$

for all $y < 0$ and $\mathbf{u} \in \mathbf{S}^{d-1}$. Taylor expanding F in t about 0, we find that for $t < (2|y|/g)^{1/2}$,

$$F(y, \mathbf{u}, t) = h(y)\sqrt{2g|y|}t + F''(y, \mathbf{u}, T(y, \mathbf{u}, t))\frac{t^2}{2}, \tag{3.8}$$

for some $0 \le T(y, \mathbf{u}, t) \le t$. Let $B(y, \mathbf{u}) = \{N(y, \mathbf{u}) < 1\}$. We then have

$$1 = \mathbb{E}[F(y, \mathbf{u}, N(y, \mathbf{u}))] = \mathbb{E}[F(y, \mathbf{u}, N(y, \mathbf{u}))\mathbb{1}_B] + \mathbb{E}[F(y, \mathbf{u}, N(y, \mathbf{u}))\mathbb{1}_{B^c}] \tag{3.9}$$

and, by the Cauchy–Schwarz inequality and (3.7),

$$\mathbb{E}[F(y, \mathbf{u}, N(y, \mathbf{u}))\mathbb{1}_{B^c}] \le \sqrt{2\mathbb{P}(N(y, \mathbf{u}) \ge 1)}.$$

By Lemma 3.2 we see that for every $r \ge 0$

$$\lim_{y\to-\infty} \sup_{\mathbf{u}\in\mathbf{S}^{d-1}} |y|^r \mathbb{P}(N(y, \mathbf{u}) \ge 1) = 0.$$

Consequently

$$\lim_{y\to-\infty}\sup_{\mathbf{u}\in\mathbf{S}^{d-1}}|y|^r\mathbb{E}[F(y,\mathbf{u},N(y,\mathbf{u}))\mathbb{1}_{B^c}]=0. \tag{3.10}$$

Similarly, for every $r\geq 0$ and $p\geq 1$ we see that

$$\lim_{y\to-\infty}\sup_{\mathbf{u}\in\mathbf{S}^{d-1}}|y|^r\mathbb{E}\left[N(y,\mathbf{u})^p\mathbb{1}_{B^c}\right]=0. \tag{3.11}$$

For y such that $y\leq -g/2$, substituting (3.8) into the first expectation on the right hand side of (3.9) and solving for $\mathbb{E}(N(y,\mathbf{u})\mathbb{1}_B)$ yields

$$\begin{aligned}
&\mathbb{E}(N(y,\mathbf{u})\mathbb{1}_B)\\
&=\frac{1}{h(y)\sqrt{2g|y|}}\Bigg(1-\mathbb{E}[F(y,\mathbf{u},N(y,\mathbf{u}))\mathbb{1}_{B^c}]\\
&\quad-\frac{1}{2}\mathbb{E}\left[F''(y,\mathbf{u},T(y,\mathbf{u},N(y,\mathbf{u})))N(y,\mathbf{u})^2\mathbb{1}_B\right]\Bigg).
\end{aligned}$$

Conditioning $\mathbb{E}(U_dN(y,\mathbf{U})\mathbb{1}_B)$ on $\{\mathbf{U}=\mathbf{u}\}$ and using the fact that $\mathbb{E}(U_d)=0$, we have

$$\begin{aligned}
h(y)^2\sqrt{2g|y|^3}\mathbb{E}(U_dN(y,\mathbf{U})\mathbb{1}_B)&=-h(y)|y|\mathbb{E}[U_dF(y,\mathbf{U},N(y,\mathbf{U}))\mathbb{1}_{B^c}]\\
&\quad-\frac{h(y)|y|}{2}\mathbb{E}\left[U_dF''(y,\mathbf{U},T(y,\mathbf{U},N(y,\mathbf{U})))N(y,\mathbf{U})^2\mathbb{1}_B\right].
\end{aligned} \tag{3.12}$$

The first term on the right hand side vanishes as $y\to-\infty$ by (3.10). For the second term, observe that

$$\begin{aligned}
F''(y,\mathbf{u},t)&=H'(y,\mathbf{u},t)M(y,\mathbf{u},t)+H(y,\mathbf{u},t)M'(y,\mathbf{u},t)\\
&=\left(u_d\sqrt{2g|y|}-gt\right)h'\left(y+u_d\sqrt{2g|y|}t-\frac{g}{2}t^2\right)M(y,\mathbf{u},t)\\
&\quad-h\left(y+u_d\sqrt{2g|y|}t-\frac{g}{2}t^2\right)\frac{g\left(\sqrt{2g|y|}u_d-gt\right)}{\sqrt{2g|y|(1-u_d^2)+\left(\sqrt{2g|y|}u_d-gt\right)^2}}.
\end{aligned}$$

Elementary calculations show that

$$\lim_{y\to-\infty}\sup_{(t,\mathbf{u})\in[0,1]\times\mathbf{S}^{d-1}}\left|\frac{F''(y,\mathbf{u},t)}{h(y)}-gu_d(-2\lambda-1)\right|=0.$$

Therefore, a combination of Lemma 3.3 and (3.11) shows that

$$\begin{aligned}
&\lim_{y\to\infty}\frac{h(y)|y|}{2}\mathbb{E}\left[U_dF''(y,\mathbf{U},T(y,\mathbf{U},N(y,\mathbf{U})))N(y,\mathbf{U})^2\mathbb{1}_B\right]\\
&=-\frac{1+2\lambda}{2}\mathbb{E}(U_d^2)=-\frac{1+2\lambda}{2d}.
\end{aligned}$$

The lemma follows by combining this with (3.12). □

Proposition 3.5. *Let* $(Y_m,m\geq 0)$ *be the Markov chain with transition operator* (1.12) *and* $h(y)=c|y|^\lambda$. *For* $x\geq 0$, *define*

$$\widehat{\mu}_k(x)=\mathbb{E}\left[(|Y_1|-x)^k\,\middle|\,Y_0=-x\right].$$

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We then have $\sup_x x^{\lambda k}\mu_k(x) < \infty$ *for all* $k \geq 1$ *and*

$$\lim_{x\to\infty} x^{1+2\lambda}\widehat{\mu}_1(x) = \frac{d-1-2\lambda}{2dc^2} \quad \textit{and} \quad \lim_{x\to\infty} x^{2\lambda}\widehat{\mu}_2(x) = \frac{2}{dc^2}.$$

Proof. First note that, by (1.10) and (1.12),

$$\begin{aligned} x^{\lambda k}\widehat{\mu}_k(x) &= \sum_{i=0}^{k}(-1)^{k-i}\left(\frac{g}{2}\right)^i (2gx)^{(k-i)/2}x^{\lambda k}\mathbb{E}\left[U_d^{k-i}N(-x,\mathbf{U})^{i+k}\right] \\ &\leq \sum_{i=0}^{k}\left(\frac{g}{2}\right)^i (2gx)^{(k-i)/2}x^{\lambda k}\mathbb{E}\left[N(-x,\mathbf{U})^{i+k}\right], \end{aligned}$$

which is bounded, when $x \to \infty$, by Lemma 3.2. Observe that

$$x^{1+2\lambda}\widehat{\mu}_1(x) = \mathbb{E}\left(\frac{gxh(-x)^2}{2c^2}N(-x,\mathbf{U})^2 - \frac{U_d\sqrt{2gx^3}h(-x)^2}{c^2}N(-x,\mathbf{U})\right).$$

Lemma 3.3 implies that

$$\lim_{x\to\infty}\mathbb{E}\left((1/2)gxh(-x)^2N(-x,\mathbf{U})^2\right) = 1/2,$$

while Lemma 3.4 shows that

$$\lim_{x\to\infty}\mathbb{E}\left(U_d\sqrt{2gx^3}h(-x)^2N(-x,\mathbf{U})\right) = \frac{1+2\lambda}{2d}.$$

Consequently,

$$\lim_{x\to\infty} x^{1+2\lambda}\widehat{\mu}_1(x) = \frac{1}{2c^2} - \frac{1+2\lambda}{2dc^2} = \frac{d-1-2\lambda}{2dc^2}.$$

Similarly, we see that

$$\begin{aligned} \widehat{\mu}_2(x) &= \mathbb{E}\left[\left(\frac{g}{2}N(-x,\mathbf{U})^2 - U_d\sqrt{2gx}N(-x,\mathbf{U})\right)^2\right] \\ &= \mathbb{E}\left[2gxU_d^2N(-x,\mathbf{U})^2 - gU_d\sqrt{2gx}N(-x,\mathbf{U})^3 + \frac{g^2}{4}N(-x,\mathbf{U})^4\right]. \end{aligned} \tag{3.13}$$

Lemma 3.3 implies that

$$\lim_{x\to\infty}\mathbb{E}\left[-gU_d\sqrt{2gx}N(-x,\mathbf{U})^3 + \frac{g^2}{4}N(-x,\mathbf{U})^4\right] = 0, \tag{3.14}$$

and

$$\lim_{x\to\infty}\mathbb{E}\left[x^{2\lambda}2gxU_d^2N(-x,\mathbf{U})^2\right] = \frac{2}{dc^2}.$$

This, (3.13) and (3.14) yield $x^{2\lambda}\widehat{\mu}_2(x) \to \frac{2}{dc^2}$ as $x \to \infty$. □

The next proposition contains an estimate needed in the case when $d = 3$ and $\lambda = 0$.

Proposition 3.6. *If* $d = 3$ *and* $\lambda = 0$ *then* $2x\widehat{\mu}_1(x) - \widehat{\mu}_2(x) \leq O(x^{-\delta})$ *for some* $\delta > 0$ *as* $x \to \infty$.

Proof. For simplicity, we assume $c = 1$; the proof is similar in other cases. Observe that

$$2x\widehat{\mu}_1(x) - \widehat{\mu}_2(x) = \mathbb{E}\left(gx(1-2U_3^2)N(-x,\mathbf{U})^2\right) - \mathbb{E}\left(U_3\sqrt{8gx^3}N(-x,\mathbf{U})\right) + \mathbb{E}\left(gU_3\sqrt{2gx}N(-x,\mathbf{U})^3\right) - \mathbb{E}\left(\frac{g^2}{4}N(-x,\mathbf{U})^4\right). \quad (3.15)$$

If $\delta < 1$, then

$$\lim_{x\to\infty} x^\delta \mathbb{E}\left(gU_3\sqrt{2gx}N(-x,\mathbf{U})^3\right) = 0 \quad \text{and} \quad \lim_{x\to\infty} x^\delta \mathbb{E}\left(\frac{g^2}{4}N(-x,\mathbf{U})^4\right) = 0, \quad (3.16)$$

by Lemma 3.3.

For the remaining terms we need more careful estimates. Consider any $\varepsilon \in (0, 1/2)$, $r > 1$ and $p \geq 1$. Let q be so large that $-p/2 - q\varepsilon/2 < -r$. By Lemma 3.2, for all $\mathbf{u}$ and large x,

$$\begin{aligned}
\mathbb{E}\left[N(-x,\mathbf{u})^p \mathbb{1}_{\{\sqrt{x}N(-x,\mathbf{u})>x^\varepsilon\}}\right] &\leq \mathbb{E}\left[N(-x,\mathbf{u})^{2p}\right]^{1/2} \mathbb{E}\left[\mathbb{1}_{\{\sqrt{x}N(-x,\mathbf{u})>x^\varepsilon\}}\right]^{1/2} \\
&\leq c_1 x^{-p/2}\mathbb{P}\left((\sqrt{x}N(-x,\mathbf{u}))^q > x^{q\varepsilon}\right)^{1/2} \leq c_1 x^{-p/2}\left(\mathbb{E}\left[(\sqrt{x}N(-x,\mathbf{u}))^q\right]/x^{q\varepsilon}\right)^{1/2} \\
&\leq c_2 x^{-p/2} x^{-q\varepsilon/2}.
\end{aligned}$$

It follows that

$$\lim_{x\to\infty} \sup_{\mathbf{u}\in\mathbf{S}^{d-1}} x^r \mathbb{E}\left[N(-x,\mathbf{u})^p \mathbb{1}_{\{\sqrt{x}N(-x,\mathbf{u})>x^\varepsilon\}}\right] = 0. \quad (3.17)$$

As in the toy model of Section 2.1, one can verify the intuition that it takes the particle the longest amount of time to travel distance 1 when its direction goes against the pull of gravity and the shortest amount of time to travel distance 1 when it travels with the pull of gravity. This leads to the result that for all $\mathbf{u} \in \mathbf{S}^2$ we have

$$\mathbb{P}(N(-x,-\mathbf{e}_3) > t) \leq \mathbb{P}(N(-x,\mathbf{u}) > t) \leq \mathbb{P}(N(-x,\mathbf{e}_3) > t).$$

Consequently, for sufficiently large x we have

$$\begin{aligned}
\mathbb{E}\left(gxN(-x,\mathbf{U})^2\right) &\leq \mathbb{E}\left(gxN(-x,\mathbf{e}_3)^2\right) \\
&= \mathbb{E}\left(gxN(-x,\mathbf{e}_3)^2\mathbb{1}_{\{\sqrt{x}N(-x,\mathbf{e}_3)\leq x^\varepsilon\}}\right) + \mathbb{E}\left(gxN(-x,\mathbf{e}_3)^2\mathbb{1}_{\{\sqrt{x}N(-x,\mathbf{e}_3)>x^\varepsilon\}}\right) \\
&= 2g\int_0^{x^\varepsilon} t\mathbb{P}(\sqrt{x}N(-x,\mathbf{e}_3) > t)dt + o(x^{-r}) \\
&= 2g\int_0^{x^\varepsilon} t\exp\left(-\int_0^{tx^{-1/2}} \left|\sqrt{2gx} - gs\right| ds\right) dt + o(x^{-r}) \\
&= 2g\int_0^{x^\varepsilon} t\exp\left(-\sqrt{2g}t + \frac{gt^2}{x}\right) dt + o(x^{-r}) \\
&= 2g\int_0^{x^\varepsilon} t\exp\left(-\sqrt{2g}t\right)\left(\exp\left(\frac{gt^2}{x}\right) - 1\right) dt \\
&\quad + 2g\int_0^{x^\varepsilon} t\exp\left(-\sqrt{2g}t\right) dt + o(x^{-r}) \\
&\leq O(x^{-1}) + 1 + o(x^{-r}).
\end{aligned}$$

Similarly, for some $\nu > 0$

$$\begin{aligned}
\mathbb{E}\left(2gxU_3^2 N(-x,\mathbf{U})^2\right) &\geq \frac{2gx}{3}\mathbb{E}\left(N(-x,-\mathbf{e}_3)^2\right) \\
&= \frac{4g}{3}\int_0^\infty t\exp\left(-\sqrt{2g}t - \frac{gt^2}{x}\right)dt \\
&\geq (e^{-gx^{2\varepsilon-1}}-1)\frac{4g}{3}\int_0^{x^\varepsilon} t\exp\left(-\sqrt{2g}t\right)dt \\
&\quad + \frac{4g}{3}\int_0^{x^\varepsilon} t\exp\left(-\sqrt{2g}t\right)dt \\
&\geq O(x^{2\varepsilon-1}) + o(e^{-\nu x^\varepsilon}) + \frac{2}{3}.
\end{aligned}$$

Combining the last two estimates, we obtain,

$$\mathbb{E}\left(gx(1-2U_3^2)N(-x,\mathbf{U})^2\right) \leq \frac{1}{3} + O(x^{2\varepsilon-1}). \tag{3.18}$$

Using (3.17) and arguing as in the proof of Lemma 3.4 (and using the notation there) we have

$$\begin{aligned}
-\mathbb{E}\left(U_3\sqrt{8gx^3}N(-x,\mathbf{U})\right) &= -\sqrt{8gx^3}\mathbb{E}\left(U_3 N(-x,\mathbf{U})\mathbb{1}_{\{\sqrt{x}N(-x,\mathbf{U})\leq x^\varepsilon\}}\right) + o(x^{-r}) \\
&= -x\mathbb{E}\left[U_3 F''(-x,\mathbf{U},T(-x,\mathbf{U},N(-x,\mathbf{U})))N(-x,\mathbf{U})^2\mathbb{1}_{\{\sqrt{x}N(-x,\mathbf{U})\leq x^\varepsilon\}}\right] + o(x^{-r}),
\end{aligned}$$

where $0 \leq T(-x,\mathbf{U},N(-x,\mathbf{U})) \leq x^{\varepsilon-1/2}$ and

$$F''(-x,\mathbf{u},t) = \frac{g\left(u_3 - \frac{gt}{\sqrt{2gx}}\right)}{\sqrt{1-u_3^2+\left(u_3-\frac{gt}{\sqrt{2gx}}\right)^2}} = J\left(\mathbf{u},\frac{gt}{\sqrt{2gx}}\right)$$

with

$$J(\mathbf{u},t) := \frac{g\,(u_3-t)}{\sqrt{1-u_3^2+(u_3-t)^2}}.$$

Note that, for sufficiently small T, $J(\mathbf{u},t)$ is continuously differentiable on $\mathbf{S}^2\times[-T,T]$ and, consequently, there exists a constant C such that

$$|J(\mathbf{u},t) - gu_3| = |J(\mathbf{u},t) - J(\mathbf{u},0)| \leq C|t|.$$

Therefore

$$\begin{aligned}
-\mathbb{E}\left(U_3\sqrt{8gx^3}N(-x,\mathbf{U})\right) &= -gx\mathbb{E}\left[U_3^2 N(-x,\mathbf{U})^2\mathbb{1}_{\{\sqrt{x}N(-x,\mathbf{U})\leq x^\varepsilon\}}\right] + O(x^{\varepsilon-1}) + o(x^{-r}) \\
&\leq -\frac{1}{3} + O(x^{2\varepsilon-1}) + o(e^{-\nu x^\varepsilon}) + O(x^{\varepsilon-1}) + o(x^{-r}).
\end{aligned}$$

This, (3.15), (3.16) and (3.18) imply that $2x\widehat{\mu}_1(x) - \widehat{\mu}_2(x) \leq O(x^{-\delta})$ for every $0 < \delta < 1$. □

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4. Power function scatterer density: proofs of the main results

Proof of Theorem 1.1. Let $(\mathbf{X}_k)_{k\geq 0} = \{(X_{1,k}, \dots, X_{d-1,k}, X_{d,k})\}_{k\geq 0}$ be the Markov chain with transition operator (1.8) started from $\mathbf{0}$, with gravitation g and scatterer density $h(\mathbf{x}) = h(x_d) = c|x_d|^\lambda$, with $c > 0$ and $\lambda \geq 0$. Theorem 2.1, Propositions 2.3, 3.5, and 3.6 imply that

(i) if $1 \leq d \leq 3$ then $(X_{d,k})_{k\geq 0}$ is neighborhood recurrent, and

(ii) if $d \geq 4$ then $(X_{d,k})_{k\geq 0}$ is transient if $\lambda < (d-3)/2$ and neighborhood recurrent if $\lambda > (d-3)/2$.

Let $(\mathbf{X}(t), \mathbf{V}(t))_{t\geq 0}$ be the Markov process with generator (1.2) started from $(\mathbf{0}, \mathbf{0})$ with gravitation g and scatterer density $h(\mathbf{x}) = h(x_d) = c|x_d|^\lambda$, with $c > 0$ and $\lambda \geq 0$ and let $(\mathbf{X}(t))_{t\geq 0} = \{(X_1(t), \dots, X_d(t))\}_{t\geq 0}$. The process $(\mathbf{X}_k)_{k\geq 0}$ can be constructed as $(\mathbf{X}(t))_{t\geq 0}$ sampled at some random times. Hence, if $(X_{d,k})_{k\geq 0}$ visits an interval $[y, 0]$ infinitely often, so does $(X_d(t))_{t\geq 0}$. In other words, if $(X_{d,k})_{k\geq 0}$ is neighborhood recurrent then $(X_d(t))_{t\geq 0}$ is neighborhood recurrent. Next suppose that $(X_d(t))_{t\geq 0}$ is neighborhood recurrent and fix any $y < 0$. A consequence of this is that $(X_d(t))_{t\geq 0}$ will visit $(2y, y)$ infinitely often, a.s. The random flight construction shows that there exists $p > 0$, depending on y, such that if $X_d(0) \in (2y, y)$ then with probability greater than p there will be a scattering event at a location such that $X_d(t) \in (y, y/2)$ before X_d hits $3y$. A standard argument based on the strong Markov property then shows that there will be infinitely many scattering events with $X_d(t) \in (y, y/2)$, a.s. and it follows that $(X_{d,k})_{k\geq 0}$ is neighborhood recurrent. We conclude that $(X_{d,k})_{k\geq 0}$ is neighborhood recurrent if and only if $(X_d(t))_{t\geq 0}$ is neighborhood recurrent.

It remains to show that $(X_d(t))_{t\geq 0}$ is recurrent only in the case $d = 1$. The continuity of $(X_d(t))_{t\geq 0}$ implies that neighborhood recurrence implies that all $y < 0$ are visited infinitely often, a.s. The energy of the particle is preserved forever, so if $(\mathbf{X}(0), \mathbf{V}(0)) = (\mathbf{0}, \mathbf{0})$ then we may have $X_d(t_1) = 0$ for some t_1 only if $\mathbf{V}(t_1) = \mathbf{0}$. But if $d \geq 2$ then after every scattering event, the first coordinate of $\mathbf{V}$ is a non-zero constant until the next scattering event, a.s. This shows that $X_d(t) \neq 0$ for all $t > 0$, a.s.

If $d = 1$ and $(X_d(t))_{t\geq 0}$ is neighborhood recurrent then the process will visit some interval $[y, 0]$ infinitely often, a.s., and, because of the claim (i) for $(X_{d,k})_{k\geq 0}$, it will scatter within this interval. After the scattering event, it will travel upwards with probability 1/2 and reach 0 with probability $p_1 > 0$, depending on y. A standard argument based on the strong Markov property shows that $(X_d(t))_{t\geq 0}$ will hit 0 infinitely often, a.s. □

Proof of Theorem 1.2. This follows from 2.1 and Proposition 3.5. □

We now turn to the proof of Theorem 1.3. The idea is to augment the result of Theorem 1.2 to include information on the time between reflections and then make a time change argument using the continuity properties of the Skorokhod topology.

To simplify notation, let

$$(Z_t^\lambda,\ t \geq 0) =_d \left(-\rho_{d'}\left(\frac{2}{dc^2}(1+\lambda)^2 t \right)^{1/(1+\lambda)},\ t \geq 0 \right) \tag{4.1}$$

where $d' = (d+1+2\lambda)/(2+2\lambda)$ and $(\rho_{d'}(t),\ t \geq 0)$ is a d'-dimensional Bessel process. Note that Z^λ is a Feller process whose generator acts on $f \in C^2(-\infty, 0)$ with compact support by

$$\mathcal{A}^\lambda f(y) = \frac{2}{dc^2}|y|^{-2\lambda}\left[\frac{1}{2} f''(y) - \left(\frac{d-1-2\lambda}{4|y|} \right) f'(y) \right]. \tag{4.2}$$

Theorem 4.1. *Consider the Markov chain* $((Y_m, \Delta_m), m \geq 0)$ *started from* $(0, 0)$ *with transition operator*

$$\widetilde{\mathfrak{U}} f(y, z) = \mathbb{E}\left[f\left(y + U_d \sqrt{2g|y|} N(y, \mathbf{U}) - \frac{g}{2} N(y, \mathbf{U})^2,\ N(y, \mathbf{U}) \right) \right].$$

Fix $\varepsilon > 0$ *and for* $m \in \mathbb{N}$, *let* $T_m^{n,\varepsilon} = \sum_{j=1}^m \Delta_j \mathbb{1}_{\{Y_{j-1} \leq -\varepsilon n^{1/(2+2\lambda)}\}}$. *We extend* $T^{n,\varepsilon}$ *to* $\mathbb{R}_+$ *by linear interpolation. We have the joint convergence in distribution*

$$\left(\left(n^{-1/(2+2\lambda)} Y_{[sn]}, n^{-\frac{3+2\lambda}{4+4\lambda}} T_{nt}^{n,\varepsilon} \right), s, t \geq 0 \right) \to_d \left(\left(Z_s^\lambda, \Phi_\varepsilon(Z^\lambda)_t \right),\ s, t \geq 0 \right)$$

in $D(\mathbb{R}_+, \mathbb{R}) \times D(\mathbb{R}_+, \mathbb{R})$, *where* $\Phi_\varepsilon : D(\mathbb{R}_+, \mathbb{R}) \to D(\mathbb{R}_+, \mathbb{R})$ *is defined by*

$$\Phi_\varepsilon(f)_t = \int_0^t \frac{\mathbb{1}(f(s) \leq -\varepsilon)}{c\sqrt{2g}\,|f(s)|^{\lambda+1/2}} ds. \tag{4.3}$$

Proof. Note that the map Φ_ε is continuous in the Skorokhod topology at all continuous functions f such that $\mathrm{Leb}(\{s : f(s) = -\varepsilon\}) = 0$, where Leb stands for Lebesgue measure. In particular, it is almost surely continuous at $(Z_t^\lambda,\ t \geq 0)$. Hence, we conclude from Theorem 1.2 that for every $\varepsilon > 0$ we have the joint convergence in distribution in $D(\mathbb{R}_+, \mathbb{R}) \times D(\mathbb{R}_+, \mathbb{R})$,

$$\left(\left(n^{-1/(2+2\lambda)} Y_{[sn]}, \Phi_\varepsilon\left(n^{-1/(2+2\lambda)} Y_{[tn]} \right) \right),\ s, t \geq 0 \right) \to_d \left(\left(Z_s^\lambda, \Phi_\varepsilon(Z^\lambda)_t \right),\ s, t \geq 0 \right). \tag{4.4}$$

Let $\mathcal{F}_m = \sigma((Y_j, \Delta_j), 0 \leq j \leq m)$ and consider the martingale with respect to the filtration $(\mathcal{F}_m)_{m \geq 0}$ given by

$$W_m := \sum_{j=1}^m \left(\Delta_j - \mathbb{E}\left[\Delta_j \mid \mathcal{F}_{j-1} \right] \right), \quad m \geq 0.$$

Define $\phi(y) = \mathbb{E}(N(y, \mathbf{U}))$. By the Markov property we see that $\mathbb{E}\left[\Delta_j \mid \mathcal{F}_{j-1}\right] = \phi\left(Y_{j-1}\right)$.

By Lemma 3.2 we see that $\sup_y \phi(y) < \infty$ and

$$\xi := \sup_m \mathbb{E}\left[\left(\Delta_m - \mathbb{E}\left[\Delta_m \mid \mathcal{F}_{m-1} \right] \right)^2 \right] < \infty.$$

By Chebyshev's and Doob's maximal inequalities we see that for every $\varepsilon > 0$ and integer $k \geq 1$,

$$\begin{aligned} &\mathbb{P}\left(\sup_{1 \leq m \leq kn} |W_m| > \varepsilon n^{\frac{3+2\lambda}{4+4\lambda}} \right) \\ &\quad \leq \frac{1}{\varepsilon^2 n^{(3+2\lambda)/(2+2\lambda)}} \mathbb{E}\left[\left(\sup_{1 \leq m \leq kn} |W_m| \right)^2 \right] \leq \frac{4}{\varepsilon^2 n^{(3+2\lambda)/(2+2\lambda)}} \mathbb{E}\left[|W_{kn}|^2 \right] \\ &\quad \leq \frac{4k\xi}{\varepsilon^2 n^{1/(2+2\lambda)}}, \end{aligned}$$

from which it follows that $\sup_{1 \leq m \leq kn} \left| n^{-\frac{3+2\lambda}{4+4\lambda}} W_m \right|$ converges to 0 in probability as $n \to \infty$. Similarly, if for $\varepsilon > 0$ we define

$$W_m^{n,\varepsilon} = \sum_{j=1}^m \left(\Delta_j - \mathbb{E}\left[\Delta_j \mid \mathcal{F}_{j-1} \right] \right) \mathbb{1}_{\{Y_{j-1} \leq -\varepsilon n^{1/(2+2\lambda)}\}}$$

$$= \sum_{j=1}^{m} \left(\Delta_j - \phi\left(Y_{j-1}\right)\right) \mathbb{1}_{\{Y_{j-1} \le -\varepsilon n^{1/(2+2\lambda)}\}}, \quad m \ge 0,$$

we find that $\sup_{1\le m\le kn} \left| n^{-\frac{3+2\lambda}{4+4\lambda}} W_m^{n,\varepsilon} \right|$ converges to 0 in probability as $n \to \infty$. We record this for future reference as

$$\sup_{1\le m\le kn} \left| n^{-\frac{3+2\lambda}{4+4\lambda}} \sum_{j=1}^{m} \left(\Delta_j - \phi\left(Y_{j-1}\right)\right) \mathbb{1}_{\{Y_{j-1} \le -\varepsilon n^{1/(2+2\lambda)}\}} \right| \to 0, \tag{4.5}$$

in probability as $n \to \infty$.

Lemma 3.3 implies that for every $\varepsilon > 0$,

$$\begin{aligned}
&\limsup_{n\to\infty} n^{\frac{1+2\lambda}{4+4\lambda}} \sup_{y \le -\varepsilon n^{1/(2+2\lambda)}} \left| \phi(y) - \frac{1}{c\sqrt{2g}|y|^{\lambda+1/2}} \right| \\
&\quad = \limsup_{n\to\infty} \varepsilon^{-\lambda-1/2} \inf_{z \le -\varepsilon n^{1/(2+2\lambda)}} |z|^{\lambda+1/2} \sup_{y \le -\varepsilon n^{1/(2+2\lambda)}} \left| \phi(y) - \frac{1}{c\sqrt{2g}|y|^{\lambda+1/2}} \right| \\
&\quad \le \limsup_{n\to\infty} \varepsilon^{-\lambda-1/2} \frac{1}{c\sqrt{2g}} \sup_{y \le -\varepsilon n^{1/(2+2\lambda)}} c\sqrt{2g}|y|^{\lambda+1/2} \left| \phi(y) - \frac{1}{c\sqrt{2g}|y|^{\lambda+1/2}} \right| \\
&\quad = \limsup_{n\to\infty} \varepsilon^{-\lambda-1/2} \frac{1}{c\sqrt{2g}} \sup_{y \le -\varepsilon n^{1/(2+2\lambda)}} \left| c\sqrt{2g}|y|^{\lambda+1/2}\phi(y) - 1 \right| = 0.
\end{aligned}$$

This implies that for every integer $k \ge 1$, a.s.,

$$\begin{aligned}
&\limsup_{n\to\infty} \sup_{1\le m\le kn} n^{-\frac{3+2\lambda}{4+4\lambda}} \sum_{j=1}^{m} \left| \phi\left(Y_{j-1}\right) - \frac{1}{\sqrt{2g\left|Y_{j-1}\right| h\left(Y_{j-1}\right)}} \right| \mathbb{1}_{\{Y_{j-1} \le -\varepsilon n^{1/(2+2\lambda)}\}} \\
&\quad \le \limsup_{n\to\infty} n^{-\frac{3+2\lambda}{4+4\lambda}} kn \sup_{y \le -\varepsilon n^{1/(2+2\lambda)}} \left| \phi(y) - \frac{1}{c\sqrt{2g}|y|^{\lambda+1/2}} \right| = 0.
\end{aligned} \tag{4.6}$$

Note that,

$$\begin{aligned}
\Phi_\varepsilon\left(n^{-1/(2+2\lambda)} Y_{[\cdot n]}\right)_{m/n} &= \frac{1}{n} \sum_{j=1}^{m} \frac{\mathbb{1}_{\{n^{-1/(2+2\lambda)} Y_{j-1} \le -\varepsilon\}}}{\sqrt{2g\left|n^{-1/(2+2\lambda)} Y_{j-1}\right| h\left(n^{-1/(2+2\lambda)} Y_{j-1}\right)}} \\
&= n^{-\frac{3+2\lambda}{4+4\lambda}} \sum_{j=1}^{m} \frac{\mathbb{1}_{\{Y_{j-1} \le -\varepsilon n^{1/(2+2\lambda)}\}}}{\sqrt{2g\left|Y_{j-1}\right| h\left(Y_{j-1}\right)}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sup_{1\le m\le kn} \left| \Phi_\varepsilon\left(n^{-1/(2+2\lambda)} Y_{[\cdot n]}\right)_{m/n} - n^{-\frac{3+2\lambda}{4+4\lambda}} T_m^{n,\varepsilon} \right| \\
&\quad = \sup_{1\le m\le kn} \left| \Phi_\varepsilon\left(n^{-1/(2+2\lambda)} Y_{[\cdot n]}\right)_{m/n} - n^{-\frac{3+2\lambda}{4+4\lambda}} \sum_{j=1}^{m} \mathbb{1}_{\{Y_{j-1} \le -\varepsilon n^{1/(2+2\lambda)}\}} \Delta_j \right| \\
&\quad = \sup_{1\le m\le kn} \left| n^{-\frac{3+2\lambda}{4+4\lambda}} \sum_{j=1}^{m} \frac{\mathbb{1}_{\{Y_{j-1} \le -\varepsilon n^{1/(2+2\lambda)}\}}}{\sqrt{2g\left|Y_{j-1}\right| h\left(Y_{j-1}\right)}} - n^{-\frac{3+2\lambda}{4+4\lambda}} \sum_{j=1}^{m} \mathbb{1}_{\{Y_{j-1} \le -\varepsilon n^{1/(2+2\lambda)}\}} \Delta_j \right|
\end{aligned}$$

$$\le \sup_{1\le m\le kn}\left|n^{-\frac{3+2\lambda}{4+4\lambda}}\sum_{j=1}^{m}\left(\Delta_j-\phi\left(Y_{j-1}\right)\right)\mathbb{1}_{\{Y_{j-1}\le -\varepsilon n^{1/(2+2\lambda)}\}}\right|$$
$$+\sup_{1\le m\le kn} n^{-\frac{3+2\lambda}{4+4\lambda}}\sum_{j=1}^{m}\left|\phi\left(Y_{j-1}\right)-\frac{1}{\sqrt{2g\left|Y_{j-1}\right|h\left(Y_{j-1}\right)}}\right|\mathbb{1}_{\{Y_{j-1}\le -\varepsilon n^{1/(2+2\lambda)}\}}.$$

This, (4.5) and (4.6) imply that for fixed $\varepsilon > 0$ and k,

$$\sup_{1\le m\le kn}\left|\Phi_\varepsilon\left(n^{-1/(2+2\lambda)}Y_{[\,\cdot\, n]}\right)_{m/n}-n^{-\frac{3+2\lambda}{4+4\lambda}}T_m^{n,\varepsilon}\right|\to 0,$$

in probability, as $n\to\infty$. It follows from this and (4.4) that for every $\varepsilon > 0$ we have the joint convergence in distribution

$$\left(\left(n^{-1/(2+2\lambda)}Y_{[sn]}, n^{-\frac{3+2\lambda}{4+4\lambda}}T_{nt}^{n,\varepsilon}\right),\ s,t\ge 0\right)\to_d\left(\left(Z_s^\lambda,\ \Phi_\varepsilon(Z^\lambda)_t\right),\ s,t\ge 0\right)$$

in $D(\mathbb{R}_+,\mathbb{R})\times D(\mathbb{R}_+,\mathbb{R})$, and the result follows. □

Remark 4.2. We conjecture that the convergence in Theorem 4.1 can be extended to include the case $\varepsilon = 0$. One reason to believe this is that the limiting process is still well defined. From the basic properties of Bessel processes it follows that for every fixed $t^*\ge 0$ we have $\lim_{\varepsilon\to 0} m(\{s\le t^* : Z_s^\lambda\ge -\varepsilon\}) = 0$ almost surely. Consequently, we have that

$$\lim_{\varepsilon\to 0}\Phi_\varepsilon(Z^\lambda)=\left(\int_0^t\frac{1}{c\sqrt{2g}|Z_s^\lambda|^{\lambda+1/2}}ds,\ t\ge 0\right)\equiv\Phi(Z^\lambda),\quad \text{a.s.}\tag{4.7}$$

A standard occupation density computation for Bessel processes shows that $\Phi(Z^\lambda)_t<\infty$ a.s., for every $t\ge 0$. The problem comes in controlling the amount of time spent between collisions when Y is near 0, which contribute constant order time. We note that the same difficulty arises in the periodic Galton Board model, studied in [7], where the authors avoided this complication by assuming the particle had a sufficiently large initial velocity and was reflected down at the corresponding level. In [29] the authors considered a model similar to ours when $h\equiv 1$ and, in that setting, were able to overcome this difficulty through different methods.

Theorem 4.1 allows us to obtain a scaling limit for the continuous time particle path (away from 0). In addition to keeping track of time we need to keep track of the direction of reflection. That is, we consider the Markov chain $((Y_m,\Delta_m,\mathbf{U}^m), m\ge 0)$ with transition operator

$$\widehat{\mathcal{U}}f(y,z,w)=\mathbb{E}\left[f\left(y+U_d\sqrt{2g|y|}N(y,\mathbf{U})-\frac{g}{2}N(y,\mathbf{U})^2,\ N(y,\mathbf{U}),\mathbf{U}\right)\right],$$

started from $(0,0,(0,\ldots,0,-1))$. Let $T_m=\sum_{j=0}^m\Delta_j$. The dth component of the path of the particle is then given by

$$Y(t)=Y_{m-1}+U_d^m\sqrt{2g|Y_{m-1}|}(t-T_{m-1})-\frac{g}{2}(t-T_{m-1})^2$$
$$\text{on } T_{m-1}\le t<T_m,\quad m\ge 1.\tag{4.8}$$

The following lemma is likely to be known but we could not find a reference.

Let $\mathbb{R}^*=\mathbb{R}\cup\{\infty\}$ and $\mathbb{R}_+^*=\mathbb{R}_+\cup\{\infty\}$. By convention, $\inf\emptyset=\infty$ and for any function f, $f(\infty)=\infty$. For $f\in D(\mathbb{R}_+,\mathbb{R}_+)$, define $\Psi : D(\mathbb{R}_+,\mathbb{R}_+)\to D(\mathbb{R}_+,\mathbb{R}_+^*)$ by $\Psi(f)(t)=\inf\{s : f(s)>t\}$.

Lemma 4.3. *If $f \in D(\mathbb{R}_+, \mathbb{R}_+)$ is continuous and strictly increasing with $\lim_{t\to\infty} f(t) = \infty$, then $\Psi(f) \in D(\mathbb{R}_+, \mathbb{R}_+)$ and Ψ is continuous at f.*

Proof. First we prove that for any $h \in D(\mathbb{R}_+, \mathbb{R}_+)$, the function $\Psi(h)$ is in $D(\mathbb{R}_+, \mathbb{R}_+^*)$. It is clear that $\Psi(h)$ is a non-decreasing function. Since the function $\Psi(h)$ is monotone, it has left and right limits at every point. It remains to show that it is right-continuous. Since $\Psi(h)$ is non-decreasing, we have $\lim_{s\downarrow t} \Psi(h)(s) \geq \Psi(h)(t)$ for every t. Consider any t and an arbitrarily small $\delta > 0$, and let $b = \Psi(h)(t)$. If $h(b) \leq t$ then there must exist $b_1 \in (b, b+\delta)$ and $t_1 > t$ such that $h(b_1) = t_1$. This claim holds also in the case $h(b) > t$, by the right-continuity of h. For all $s \in (t, t_1)$ we have $\Psi(h)(s) \leq b_1 < b + \delta$. Since $\delta > 0$ is arbitrarily small, this implies that $\lim_{s\downarrow t} \Psi(f)(s) \leq \Psi(h)(t)$. In view of the previously proved opposite inequality, we conclude that $\Psi(h)$ is right continuous at t. This completes the proof that $\Psi(h) \in D(\mathbb{R}_+, \mathbb{R}_+^*)$.

Now suppose that f satisfies the hypotheses of the lemma and that $f_n \in D(\mathbb{R}_+, \mathbb{R}_+)$ is a sequence converging to f. Since f is continuous and strictly increasing, the function $\Psi(f)$ is also continuous and strictly increasing. Fix any $T < \infty$. It suffices to show that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} |\Psi(f_n)(t) - \Psi(f)(t)| = 0.$$

Suppose otherwise. Then there exist $\varepsilon > 0$, a subsequence n_k and a sequence t_{n_k} of points in $[0, T]$, such that $|\Psi(f_{n_k})(t_{n_k}) - \Psi(f)(t_{n_k})| > \varepsilon$ for all k. By compactness, we may suppose that $t_{n_k} \to t_\infty \in [0, T]$ as $k \to \infty$. We will assume that $t_\infty \in (0, T)$. The argument requires only small modifications when t_∞ is 0 or T.

Let $s_\infty = \Psi(f)(t_\infty)$ and

$$\delta = \min(f(s_\infty - \varepsilon/4) - f(s_\infty - \varepsilon/2), f(s_\infty + \varepsilon/2) - f(s_\infty + \varepsilon/4)).$$

Since $f(s_\infty) = t_\infty$, f is strictly increasing and $t_{n_k} \to t_\infty$, there exists k_1 such that for all $k \geq k_1$,

$$f(s_\infty - \varepsilon/4) \leq t_{n_k} \leq f(s_\infty + \varepsilon/4). \tag{4.9}$$

Since f is continuous, $f_n \to f$ uniformly on compact sets. Let $k_2 \geq k_1$ be so large that for $k \geq k_2$.

$$|\Psi(f)(t_\infty) - \Psi(f)(t_{n_k})| < \varepsilon/4, \tag{4.10}$$

$$\sup_{t\in[0,s_\infty-\varepsilon/2]} |f_{n_k}(t) - f(t)| < \delta/4, \tag{4.11}$$

$$\sup_{t\in[s_\infty+\varepsilon/2,T]} |f_{n_k}(t) - f(t)| < \delta/4.$$

It follows from the definition of δ and (4.11) that

$$\sup_{t\in[0,s_\infty-\varepsilon/2]} f_{n_k}(t) < f(s_\infty - \varepsilon/4).$$

This, (4.9), the definition of s_∞ and (4.10) imply that

$$\begin{aligned}\Psi(f_{n_k})(t_{n_k}) &\geq \Psi(f_{n_k})(f(s_\infty - \varepsilon/4)) \geq s_\infty - \varepsilon/2 = \Psi(f)(t_\infty) - \varepsilon/2 \\ &\geq \Psi(f)(t_{n_k}) - 3\varepsilon/4.\end{aligned} \tag{4.12}$$

The following estimates can be obtained in an analogous way,

$$\begin{aligned}\Psi(f_{n_k})(t_{n_k}) &\le \Psi(f_{n_k})(f(s_\infty+\varepsilon/4)) \le s_\infty+\varepsilon/2 = \Psi(f)(t_\infty)+\varepsilon/2\\ &\le \Psi(f)(t_{n_k})+3\varepsilon/4.\end{aligned}$$

We combine this with (4.12) to obtain $|\Psi(f_{n_k})(t_{n_k}) - \Psi(f)(t_{n_k})| \le 3\varepsilon/4$. This contradicts the definition of the sequence t_{n_k}. This contradiction completes the proof. □

In order to apply this lemma, we need the following proposition. Let $\tau_{v+} = \inf\{t : Z^\lambda(t) > v\}$.

Proposition 4.4. *For all $y < v < 0$,*

$$\mathbb{P}_y\left(\lim_{t\to\infty}\int_0^t \frac{1}{c\sqrt{2g}\,|Z^\lambda(s\wedge\tau_{v+})|^{1/2+\lambda}}ds = \infty\right) = 1.$$

Proof. The result is trivial on the set where $(Z^\lambda(t\wedge\tau_{v+}),\ t\ge 0)$ is absorbed at v. It follows from (4.2) that the scale function G and speed measure m for Z^λ are given by

$$G(y) = \int_{-1}^y |u|^{\lambda-(d-1)/2}du \quad\text{and}\quad m(dy) = \frac{dc^2}{2}|y|^{\lambda+(d-1)/2}dy.$$

If $G(-\infty) = -\infty$, then $(Z^\lambda(t\wedge\tau_{v+}),\ t\ge 0)$ is absorbed at v with probability 1, so we may assume that $G(-\infty)$ is finite. Note that this implies that $d > 3$. In this case there exists $C > 0$ such that for all $y < -1$

$$(G(y)-G(-\infty))\left(\frac{1}{c\sqrt{2g}\,|y|^{1/2+\lambda}}\right)\frac{dm}{dy}(y) \ge C\sqrt{|y|}.$$

Since $\int_{-\infty}^y \sqrt{|u|}du = \infty$ for all $y\in\mathbb{R}$, the result is an application of [22, Theorem 2.11]. □

Theorem 4.5. *Fix $y < v < 0$ and define $\tau^n_{y-} = \inf\{m : Y_m \le n^{1/(2+2\lambda)}y\}$ and $\tau^n_{v+} = \inf\{m > \tau^n_{y-} : Y_m \ge n^{1/(2+2\lambda)}v\}$. For $(Y(t), t\ge 0)$ as defined in (4.8) and $y < v < 0$ we have the following convergence in distribution on $D(\mathbb{R}_+,\mathbb{R})$,*

$$\left(n^{-\frac{1}{2+2\lambda}}Y\left(\left(n^{\frac{3+2\lambda}{4+4\lambda}}t + T_{\tau^n_{y-}}\right)\wedge T_{\tau^n_{v+}}\right),\ t\ge 0\right) \to (Z^\lambda(A(t)\wedge\tau_{v+}),\ t\ge 0),$$

where Z^λ is the diffusion (4.1) started from y and

$$A(t) = \Psi\left(\Phi\left(Z^\lambda(\cdot\wedge\tau_{v+})\right)\right).$$

Remark 4.6. The theorem remains true replacing A with $\Psi\left(\Phi_\varepsilon(Z^\lambda(\cdot\wedge\tau_{v+}))\right)$ for any $0<\varepsilon<|v|$, where Φ_ε is defined in (4.3).

Proof of Theorem 4.5. Fix $0<\varepsilon<|v|$ and define $\lambda' = (3+2\lambda)/(4+4\lambda)$. Recall $T^{n,\varepsilon}_m$ from Theorem 4.1 and let

$$\begin{aligned}A_n(t) &= \Psi\left(n^{-\lambda'}\left(T_{(\tau^n_{y-}+n\,\cdot)\wedge\tau^n_{v+}} - T_{\tau^n_{y-}}\right) + \left(\cdot - n^{-1}\left(\tau^n_{v+}-\tau^n_{y-}\right)\right)^+\frac{1}{\sqrt{2g|v|}\,h(v)}\right)(t)\\ &= \Psi\left(n^{-\lambda'}\left(T^{n,\varepsilon}_{(\tau^n_{y-}+n\,\cdot)\wedge\tau^n_{v+}} - T^{n,\varepsilon}_{\tau^n_{y-}}\right) + \left(\cdot - n^{-1}\left(\tau^n_{v+}-\tau^n_{y-}\right)\right)^+\frac{1}{\sqrt{2g|v|}\,h(v)}\right)(t).\end{aligned}$$

Using Theorem 4.1, Lemma 4.3, and the Skorokhod-continuity of composition with a continuous function (see e.g. [3, Section 17]), we have that

$$\left(n^{-\frac{1}{2+2\lambda}} Y_{[nA_n(t)]\wedge(\tau^n_{v+}-\tau^n_{y-})},\ t \ge 0\right) \to (Z^\lambda(A(t)\wedge\tau_{v+}),\ t\ge 0).$$

Observe that for all $0 \le m \le \tau^n_{v+} - \tau^n_{y-}$ we have $nA_n\left(n^{-\lambda'}\left(T^{n,\varepsilon}_{\tau^n_{y-}+m} - T^{n,\varepsilon}_{\tau^n_{y-}}\right)\right) = m$ and, as a result, if $T^{n,\varepsilon}_{\tau^n_{y-}+m-1} - T^{n,\varepsilon}_{\tau^n_{y-}} \le n^{\lambda'}t < T^{n,\varepsilon}_{\tau^n_{y-}+m} - T^{n,\varepsilon}_{\tau^n_{y-}}$ then $m-1 \le nA_n(t) < m$.

Define $\widehat{T}^n_m = T_{\tau^n_{y+}+m}$, fix $S > 0$ and observe that

$$\begin{aligned}
&\sup_{0\le t\le S} n^{-\frac{1}{2+2\lambda}} \left| Y\left(\left(n^{\lambda'}t + T_{\tau^n_{y-}}\right)\wedge T_{\tau^n_{v+}}\right) - Y_{[nA_n(t)]\wedge(\tau^n_{v+}-\tau^n_{y-})}\right| \\
&\quad\le \sup_{m\le\left\lceil nA_n\left(S\wedge n^{-\lambda'}(T_{\tau^n_{v+}}-T_{\tau^n_{y-}})\right)\right\rceil} \ \sup_{\widehat{T}^n_{m-1}\le t\le \widehat{T}^n_m} \\
&\qquad \times \left\{\left| U_d^{\tau^n_{y+}+m-1}\sqrt{\frac{2g|Y_{\tau^n_{y-}+m-1}|}{n^{1/(2+2\lambda)}}}\, n^{-\frac{1}{4+4\lambda}}(t-\widehat{T}^n_{m-1}) - \frac{g}{2n^{1/(2+2\lambda)}}(t-\widehat{T}^n_{m-1})^2\right|\right\} \\
&\quad\le \sup_{m\le\left\lceil nA_n\left(S\wedge n^{-\lambda'}(T_{\tau^n_{v+}}-T_{\tau^n_{y-}})\right)\right\rceil} \sqrt{\frac{2g|Y_{\tau^n_{y-}+m-1}|}{n^{1/(2+2\lambda)}}}\, n^{-\frac{1}{4+4\lambda}}(\widehat{T}^n_m - \widehat{T}^n_{m-1}) \\
&\qquad + \sup_{m\le\left\lceil nA_n\left(S\wedge n^{-\lambda'}(T_{\tau^n_{v+}}-T_{\tau^n_{y-}})\right)\right\rceil} \frac{g}{2n^{1/(2+2\lambda)}}(\widehat{T}^n_m - \widehat{T}^n_{m-1})^2. \qquad (4.13)
\end{aligned}$$

Since Z^λ almost surely fluctuates across levels, the convergence in Theorem 4.1 occurs jointly with the hitting time of v, so that

$$\begin{aligned}
&\left(\left(n^{-\frac{1}{2+2\lambda}} Y_{[nA_n(t)]\wedge(\tau^n_{v+}-\tau^n_{y-})},\ A_n(t),\ n^{-1}(\tau^n_{v+}-\tau^n_{y-})\right),\ t\ge 0\right) \\
&\quad \xrightarrow{d} ((Z^\lambda(A(t)\wedge\tau_{v+}), A(t), \tau_{v+}), t \ge 0). \qquad (4.14)
\end{aligned}$$

Let

$$\begin{aligned}
B_n = \Bigg\{ &\sup_{m\le\left\lceil nA_n\left(S\wedge n^{-\lambda'}(T_{\tau^n_{v+}}-T_{\tau^n_{y-}})\right)\right\rceil} \frac{1}{n^{2+2\lambda}}|Y_{\tau^n_{y-}+m}| \le M, \\
&A_n\left(S\wedge n^{-\lambda'}(T_{\tau^n_{v+}}-T_{\tau^n_{y-}})\right) \le M,\ Y_{\tau^n_{v+}} < n^{1/(2+2\lambda)}(v+\delta)\Bigg\}.
\end{aligned}$$

It follows from (4.14) that for every $p_1 < 1$ there exist $M > 0$ and $0 < \delta < |v|$ such that for large n, $\mathbb{P}(B_n) > p_1$. We use (4.13) to conclude that for $\varepsilon \in (0,1)$ there exist $C_1, C_2 > 0$ such that

$$\mathbb{P}\left(\sup_{0\le t\le S}\left|Y\left(\left(n^{\lambda'}t+T_{\tau^n_{y-}}\right)\wedge T_{\tau^n_{v+}}\right)-Y_{[nA_n(t)]\wedge(\tau^n_{v+}-\tau^n_{y-})}\right|>\varepsilon,\,B_n\right)$$
$$\le C_1 n\sup_{y\le n^{1/(2+2\lambda)}(v+\delta),\mathbf{U}\in\mathbf{S}^{d-1}}\mathbb{P}(N(y,\mathbf{U})>C_2\varepsilon).$$

The right hand side goes to 0 by Lemma 3.2, applied with a large enough value of p, and using Markov's inequality. Since $\mathbb{P}(B_n)\to 1$, it follows from (4.14) that

$$\left(\frac{1}{n^{\frac{1}{2+2\lambda}}}Y\left(\left(n^{\frac{3+2\lambda}{4+4\lambda}}t+T_{\tau^n_{y-}}\right)\wedge T_{\tau^n_{v+}}\right),\ t\ge 0\right)\to (Z^\lambda(A(t)\wedge\tau_{v+}),\ t\ge 0).\qquad\square$$

Proof of Theorem 1.3. This result is a consequence of Theorem 4.5 and a standard time change computation [33]. $\square$

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Appendix. Reflection direction

This short section presents an elementary fact about the classical (specular) reflection. The claim is known in dimension $d=3$ (see, for example, the discussion of the so-called hard-sphere scattering in [13, Sect. 4.8]) but we could not find a reference for the analogous result in all dimensions $d\ge 2$.

Suppose that $d\ge 2$. Let $\mathbf{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$ and let $\mathbf{e}_1,\dots,\mathbf{e}_d$ be the standard basis for $\mathbb{R}^d$. Let $\mathbf{B}^{d-1}=\{(0,x_2,\dots,x_d)\in\mathbb{R}^d : x_2^2+\dots+x_d^2\le 1\}$. Let $\mathbf{b}$ be a random vector with the uniform distribution in $\mathbf{B}^{d-1}$ and let $\mathcal{L}$ be the random straight line $\{\mathbf{b}+a\mathbf{e}_1, a\in\mathbb{R}\}$. Suppose that a light ray starts from the point $\mathbf{b}+2\mathbf{e}_1$ and travels along $\mathcal{L}$ in the direction of the point $\mathbf{b}-2\mathbf{e}_1$. Now suppose that this random light ray reflects from $\mathbf{S}^{d-1}$ according to the classical law of specular reflection, i.e., the angle of reflection is equal to the angle of incidence. Let $\mathbf{v}\in\mathbf{S}^{d-1}$ be the vector representing the direction of the reflected ray, i.e., the reflected light ray travels along a straight line of the form $\{\mathbf{w}+a\mathbf{v}, a\in\mathbb{R}\}$ for some vector $\mathbf{w}\in\mathbb{R}^d$.

Proposition A.1. *The distribution of $\mathbf{v}$ is uniform on $\mathbf{S}^{d-1}$ if and only if $d=3$.*

Proof. Let $\mathbf{n}$ be the outer normal vector to the sphere $\mathbf{S}^{d-1}$ at the point where the light ray hits the sphere. If $|\mathbf{b}|=r_1$ and the angle between $\mathbf{e}_1$ and $\mathbf{n}$ is α_1 then $r_1=\sin\alpha_1$. Let Θ be the angle between $\mathbf{v}$ and $\mathbf{e}_1$. The specular law of reflection implies that the angle between $\mathbf{v}$ and $\mathbf{n}$ is α_1 so $\Theta=2\alpha_1$. Hence, for a given $r\in(0,1)$, we have $|\mathbf{b}|\le r$ if and only if $\Theta\le 2\alpha$, where $r=\sin\alpha$. Let $\beta=2\alpha$ so that $r=\sin(\beta/2)$. We obtain

$$\mathbb{P}(\Theta\le\beta)=\mathbb{P}(|\mathbf{b}|\le r)=r^{d-1}=(\sin\alpha)^{d-1}=(\sin(\beta/2))^{d-1}.$$

Let A_β be the spherical cap with the angle β, i.e., the set of points $x\in\mathbf{S}^{d-1}$ such that the angle between the vector $\overrightarrow{0x}$ and $\mathbf{e}_1$ is smaller than or equal to β. Let μ be the uniform probability measure on $\mathbf{S}^{d-1}$. It suffices to show that $\mu(A_\beta)=\mathbb{P}(\Theta\le\beta)$ for all $\beta\in(0,\pi)$ if and only if $d=3$.

The following formulas for the area of A_β and $\mathbf{S}^{d-1}$ are taken from [19]. The area of A_β is equal to $(2\pi^{(d-1)/2}/\Gamma((d-1)/2))\int_0^\beta \sin^{d-2}\gamma d\gamma$. The area of $\mathbf{S}^{d-1}$ is $2\pi^{d/2}/\Gamma(d/2)$. It follows that

$$\mu(A_\beta) = \frac{\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)}\int_0^\beta \sin^{d-2}\gamma d\gamma.$$

For $d = 3$ and all $\beta \in (0, \pi)$,

$$\mathbb{P}(\Theta \le \beta) = (\sin(\beta/2))^2 = \frac{1}{2}(1 - \cos\beta) = \frac{\Gamma(3/2)}{\sqrt{\pi}\Gamma(1)}\int_0^\beta \sin\gamma d\gamma = \mu(A_\beta),$$

so the proposition is proved for $d = 3$.

For all $d \ge 2$ and $\beta \in (0, \pi)$,

$$f(\beta) := \frac{\partial}{\partial\beta}\mathbb{P}(\Theta \le \beta) = \frac{\partial}{\partial\beta}(\sin(\beta/2))^{d-1} = \frac{d-1}{2}(\sin(\beta/2))^{d-2}\cos(\beta/2),$$

$$g(\beta) := \frac{\partial}{\partial\beta}\mu(A_\beta) = \frac{\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)}\sin^{d-2}\beta.$$

This implies that

$$\frac{f(\pi/2)}{g(\pi/2)}\frac{g(\pi/4)}{f(\pi/4)} = 2^{(3/2)-d}\sec(\pi/8)(\sin(\pi/8))^{2-d} = (2\sin(\pi/8))^{3-d}.$$

The last quantity is not equal to 1 for $d \ne 3$ so the functions f and g are not identically equal to each other. Hence, for $d \ne 3$, it is not true that $\mathbb{P}(\Theta \le \beta) \equiv \mu(A_\beta)$. □

Since $d = 3$ is the dimension of our physical space, this justifies the choice of the uniform direction of reflection in this paper. In other dimensions, we also assume that the direction of reflection is uniform, for several reasons. The first is mathematical convenience. Second, the assumption of the uniform angle of reflection allows us to use a Markov model for the process of locations of consecutive scattering events. Finally, we believe that due to mixing (in the probabilistic sense of the word), our results would remain unchanged, in the qualitative sense, if we incorporated the true distribution of reflection in dimensions $d \ne 3$.

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