



Minimal Root's embeddings for general starting and target distributions[☆]

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Abstract

Most results regarding Skorokhod embedding problems SEP so far rely on the assumption that the corresponding stopped process is uniformly integrable, which is equivalent to the convex ordering condition $U^\mu \leq U^\nu$ when the underlying process is a local martingale. In this paper, we study the existence, construction of Root's solutions to SEP, in the absence of this convex ordering condition. We replace the uniform integrability condition by the minimality condition (Monroe, 1972), as the criterion of “good” solutions. A sufficient and necessary condition (in terms of local time) for minimality is given. We also discuss the optimality of such minimal solutions. These results extend the generality of the results given by Cox and Wang (2013) and Gassiat et al. (2015). At last, we extend all the results above to multi-marginal embedding problems based on the work of Cox et al. (2018).

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1. Introduction

Given a stochastic process X on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, and a distribution μ on the state space of X , the Skorokhod embedding problem is to find a stopping time τ such that $X_\tau \sim \mu$. This problem was initially proposed by Skorokhod [28].

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Under the classical setting where X is a Brownian motion starting at 0 and the target distribution μ has zero mean and finite variance, there is a rich literature regarding this problem, for example, [1,5,6,11,24–26,29], etc. We will not state them one by one in details. Instead, we refer curious readers to the survey paper Oblój [22].

Most of the results above can be generalized to the cases where the underlying process is a diffusion process with a general starting distribution. In this paper, we denote such embedding problem by $\text{SEP}(\sigma, \nu, \mu)$:

$$\begin{aligned} \text{Given } X_0 \sim \nu, \text{ to find a stopping time } \tau \text{ such that } X_\tau \sim \mu, \\ \text{where } X \text{ satisfies } dX_t = \sigma(X_t)dW_t. \end{aligned} \quad (\text{SEP})$$

However, the results mentioned above are concerned with the cases where the embeddings are namely *UI stopping times*. Here, a stopping time τ is a UI stopping time if the corresponding stopped process $X^\tau := \{X_{t \wedge \tau}\}_{t \geq 0}$ is uniformly integrable, otherwise we call τ a *non-UI stopping time*.

When the underlying process is a continuous local martingale Oblój [22, Prop. 8.1] shows that there exists a UI embedding for $\text{SEP}(\sigma, \nu, \mu)$ if and only if *the convex ordering condition* holds

$$U^\nu(x) \geq U^\mu(x) > -\infty, \quad \text{for all } x \in \mathbb{R}, \quad (1.1)$$

where the function U^μ is called *the potential of μ* [4]:

$$U^\mu(x) := -\mathbb{E}^{Y \sim \mu} [|Y - x|] = -\int_{\mathbb{R}} |y - x| \mu(dy).$$

We say that $\nu \leq \mu$ in *convex order* if (1.1) holds.

In this paper, we are concerned with $\text{SEP}(\sigma, \nu, \mu)$ in the absence of convex ordering condition (1.1). In such circumstances we cannot expect the corresponding embedding to be a UI stopping time.

For example, suppose that the initial distribution is the Dirac measure $\nu = \delta_0$ and the target is $\mu = \delta_1$. The mean values of ν and μ do not agree, and then (1.1) fails. The hitting time $H_1 = \inf\{t \geq 0 : W_t = 1\}$ is an embedding for $\text{SEP}(\nu, \mu)$ but obviously it is *not* a UI stopping time. Another example is that $\nu = (\delta_1 + \delta_{-1})/2$ and $\mu = \delta_0$. The mean values agree, but (1.1) fails as $U^\nu \leq U^\mu$. The hitting time $H_0 = \inf\{t \geq 0 : W_t = 0\}$ is a non-UI embedding for $\text{SEP}(\nu, \mu)$.

As presented above, in the absence of (1.1), we cannot restrict our attention to UI stopping times for embeddings. Instead, we may pose some other restrictions. For example, Pedersen and Peskir [23] pose an integrability condition on the maximum of the scale function of X as the replacement of UI condition. After that, Cox and Hobson [8] propose another criterion on stopping times, which was initially introduced by Monroe [21]:

Definition 1.1 (Minimal Stopping Time). A stopping time τ for the process X is minimal if whenever $\theta \leq \tau$ is a stopping time such that X_θ and X_τ have the same distribution then $\tau = \theta$, a.s.

According to the definition, minimal stopping times could be a natural choice for “good” solutions of the embedding problem in a general context. For example, as stated in [16, Sect. 4.2], there exists a trivial solution for SEP in the general cases — simply run the process X until it firstly hits the mean of μ , and thereafter can use any regular embedding mentioned above. The embeddings constructed in this way are always minimal stopping times, see [8].

Cox and Hobson [8] have made significant effort in the study of minimal stopping times for the Brownian motion starting at 0. A group of necessary and sufficient conditions for minimality is given. After that, Cox [7] extends the previous results to the cases of general starting distributions. Thanks to these results, some well-known embeddings have been extended to the cases in which (1.1) fails, such as Chacon–Walsh’s embedding, Azéma–Yor’s embedding, and Vallois’ embedding.

In this work we are concerned with embeddings of Root’s type which was initially proposed by Root [25]. Formally, suppose that W is a Brownian motion starting at zero and the target distribution is a centred distribution with finite second moment, the Skorokhod embedding problem admits a solution which is the first hitting time of the joint process (t, W_t) of a called *Root’s barrier*:

Definition 1.2 (Root’s Barrier). A closed subset B of $[0, +\infty] \times [-\infty, +\infty]$ is a *Root’s barrier* if

- (a) $(+\infty, x) \in B$ if $x \in [-\infty, +\infty]$;
- (b) $(t, \pm\infty) \in B$ if $t \in [0, \infty]$;
- (c) if $(t, x) \in B$, then $(s, x) \in B$ whenever $s > t$.

There have been a number of important contributions concerning Root’s barriers (given that (1.1) holds). An immediately subsequent paper Loynes [20] shows some elementary analytical properties of Root’s barriers. Further, by posing the definition *regular barrier*, the uniqueness of Root’s embedding is given in this paper.

Another important paper regarding Root’s construction is Röst [27] which vastly extends the generality of Root’s existence result. More importantly, Röst firstly proved the optimality of Root’s embedding, which was conjectured by Kiefer [18], in the sense of *minimal residual expectation* (m.r.e., for short):

Amongst all solutions of SEP(σ, ν, μ), the Root’s solution
minimises $\mathbb{E}^v[(\tau - t)^+]$ simultaneously for all $t > 0$. (m.r.e.)

Dupire [12] proposes the connections among Root’s embeddings, PDE and robust pricing problem for variance options. Enlightened by his idea, we derive the construction of Root’s embeddings using variational inequalities (given that (1.1) holds) in [10]. We also propose the conjecture that, by slightly changing the terminal condition in our variational inequalities, this construction method could be extended to the cases where (1.1) fails [10, Rmk.4.5]. In the same paper, an alternative proof of m.r.e. property is given, which has an important application for the construction of sub-hedging strategies in the financial context. Later, using PDE techniques, Gassiat et al. [15] describe Root’s embedding in terms of viscosity solutions of obstacle problems, and give a rigorous proof of the existence of Root’s embedding given (1.1); using method from optimal transport, Beiglböck et al. [2] show same existence and optimality results of Root’s barriers. A more recent paper, Cox et al. [9], discusses the multi-marginal SEP, which is to find an increasing sequence of stopping times embedding the given multiple target distributions (in convex order) in sequence. They construct the UI solution of Root’s type to the multi-marginal SEP via iterated optimal stopping problems. The optimality of such solutions is also given in their work.

In this work, we will extend the generality of the construction given by Cox and Wang [10] and Gassiat et al. [15] to the cases without convex ordering condition (1.1). On the other hand, thanks to the rich results given in [8] and [7], it will turn out that we can characterize minimal stopping times by the local times of the corresponding stopped process $(\mathbb{E}^v[L_\tau^x])$. This characterization then ensure that we can construct a minimal Root’s embedding via an obstacle

problem with proper boundary condition. Using the result about minimality, we then can discuss optimality of minimal Root's solutions (among all minimal solutions). After that, based on the work of Cox et al. [9], it turns out that one can construct Root's solution to multi-marginal SEP via iterated obstacle problems even when convex ordering condition fails. Moreover, we define the minimality for a sequence of stopping times, and tell when the solution to a multi-marginal SEP is "minimal".

The paper will therefore proceed as follows: in Section 2, we review some early results about Root's barriers. In Section 3, the existence result and the construction of Root's barrier for general starting and target distributions are given. In Section 4, we study the potentials of the corresponding stopped process (and their limit), and obtain a necessary and sufficient condition for a Root's stopping time to be minimal. In Section 5, we consider the optimality of non-UI Root's embeddings in the sense of *maximal principal expectation*, which can be regarded as the generalization of minimal residual expectation (m.r.e.). In Section 6, we extend all the results (construction, minimality, optimality) to the embedding problems with multi-marginal distributions.

2. Preliminaries: Root's barriers for regular cases

We firstly review the previous results regarding Root's embeddings, which are useful throughout this work.

It was shown in [20, Prop. 3] that the set B defined in Definition 1.2 can be represented as a closed set bounded below by a lower semi-continuous function $R : \mathbb{R} \rightarrow [0, +\infty]$, i.e. $B = \{(t, x) : t \geq R(x)\}$. This representation has been helpful in the characterization of the law of the stopped process X^τ . Additionally, in the rest of this paper, we will say that a barrier is either a closed set described in Definition 1.2, or equivalently its complement:

$$D = \{(t, x) : 0 < t < R(x)\} = (\mathbb{R}_+ \times \mathbb{R}) \setminus B.$$

The corresponding stopping time is denoted by

$$\tau_D := \inf\{t > 0 : (t, X_t) \notin D\} = \inf\{t > 0 : t \geq R(X_t)\}.$$

Moreover, Loynes [20, Prop. 1] says that,

$$\text{for a Root's stopping } \tau_D, \text{ either } \mathbb{P}[\tau_D < \infty] = 1 \text{ or } \mathbb{P}[\tau_D = \infty] = 1.$$

As a straightforward result of this proposition, when $X_{\tau_D} \sim \mu$ where μ is integrable, τ_D is finite almost surely.

The following properties are given in [10], which enable us to characterize the behaviour of the path of corresponding stopped process.

Proposition 2.1. *Suppose X is a continuous process. Given a Root's barrier D and the corresponding stopping time is denoted by τ_D , then*

- (i) *if $(t, x) \in D$, $\mathbb{P}^v[X_{t \wedge \tau_D} \in dx] = \mathbb{P}^v[X_t \in dx, t < \tau_D]$;*
- (ii) *if $(t, x) \notin D$, $\mathbb{P}^v[L_{t \wedge \tau_D}^x = L_{\tau_D}^x] = 1$.*

These properties are local properties and do not rely on the integrability of the stopped process X^τ , so they remain true even when X^τ is not uniformly integrable.

Denote the potential of the stopped process by $u(t, x) := -\mathbb{E}^v|x - X_{t \wedge \tau_D}|$, then, according to Cox and Wang [10] (see also [15]), u is of the class $\mathcal{C}^0(\mathbb{R}_+ \times \mathbb{R}) \cap \mathcal{C}^{2,1}(D)$, and satisfies

$$Lu := \frac{\partial u}{\partial t} - \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{on } D; \quad u(0, \cdot) = U^v(\cdot) \quad \text{on } \mathbb{R}. \quad (2.1a)$$

Moreover, if τ_D is a UI stopping time such that $X_{\tau_D} \sim \mu$, then

$$u(t, x) = U^\mu(x), \text{ if } (t, x) \notin D; \quad u(t, x) \longrightarrow U^\mu(x), \text{ as } t \rightarrow \infty. \quad (2.1b)$$

Note that the UI condition implies that $U^\nu \geq U^\mu$ everywhere on \mathbb{R} .

In [10], we consider Root's embeddings for homogeneous diffusions, i.e. $\sigma(t, x) \equiv \sigma(x)$. Suppose that (1.1) holds, using (2.1a)–(2.1b), we construct a one-to-one correspondence between Root's stopping times and strong solutions to variational inequalities. Later, using the notion of viscosity solutions, Gassiat et al. [15] extend the result to more general cases.

Theorem 2.2 ([15]). *Assume that (ν, μ) satisfies (1.1), and σ satisfies that the following regular conditions:*

$$\begin{aligned} &\text{there exists } L > 0, \text{ s.t. } \forall t \geq 0, x, y \in \mathbb{R}, \\ &\quad |\sigma(t, x) - \sigma(t, y)| < L|x - y|, \quad |\sigma(t, x)| < L(1 + |x|); \\ &\text{for each compact } K \subset \{x : U^\nu(x) > U^\mu(x)\}, \\ &\quad \exists C_K > 0, \text{ s.t. } \forall t \geq 0, x \in K, \sigma(t, x) \geq C_K > 0. \end{aligned}$$

Further, let τ_D be a UI Root's solution to $\text{SEP}(\sigma, \nu, \mu)$, and the function $u(t, x)$ be a viscosity solution to the following obstacle problem

$$\min\{Lu, u - U^\mu\} = 0, \quad u(0, \cdot) = U^\nu(\cdot), \quad \lim_{t \rightarrow \infty} u(t, \cdot) = U^\mu(\cdot)$$

Then $u(t, x) = -\mathbb{E}^\nu[x - X_{t \wedge \tau_D}]$ and $D = \{(t, x) : u(t, x) > U^\mu(x)\}$.

As stated in Section 1, τ_D is non-UI when (1.1) fails, and then (2.1b) does not hold any longer. Consequently, the results of Cox and Wang [10] and Gassiat et al. [15] are not available. However, since (2.1a) still holds, in order to construct Root's embedding, we only need to find a more general version of (2.1b) — it is the starting point of this work.

3. Existence and construction of Root's embeddings

Given $U^\mu \leq U^\nu$, $\text{SEP}(\sigma, \nu, \mu)$ admits a UI Root's solution, and we can construct this solution via an obstacle problem (Theorem 2.2). However, when (1.1) fails, we cannot even be sure if the Root's embedding exists. From now on, we are concerned with the existence and construction of Root's embedding in such general cases.

First of all, let ν and μ be two probability distributions on \mathbb{R} , and define

$$\begin{aligned} u_0(x) &= U^\nu(x), \quad \bar{u}(x) = U^\mu(x) - C, \quad \text{for } x \in \mathbb{R}, \\ &\text{where } C > 0 \text{ is a constant s.t. } u_0 \geq \bar{u} \text{ everywhere.} \end{aligned} \quad (3.1)$$

We assume that the diffusion coefficient σ satisfies the regular conditions:

$$\begin{aligned} &\text{there exists } L > 0, \text{ s.t. } \forall x, y \in \mathbb{R}, \\ &\quad |\sigma(x) - \sigma(y)| < L|x - y|, \quad |\sigma(x)| < L(1 + |x|); \end{aligned} \quad (3.2)$$

$$\begin{aligned} &\text{for each compact } K \subset \{x : u_0(x) > \bar{u}(x)\}, \\ &\quad \exists C_K > 0, \text{ s.t. } \forall x \in K, \sigma(x) \geq C_K > 0. \end{aligned} \quad (3.3)$$

Consider the obstacle problem $\text{OBS}(\sigma, u_0, \bar{u})$:

$$\min\{Lu, u - \bar{u}\} = 0, \quad u(0, \cdot) = u_0(\cdot). \quad (\text{OBS})$$

Given (3.1)–(3.3), the existence of viscosity solutions to $\text{OBS}(\sigma, u_0, \bar{u})$ follows from standard results (see e.g. [14]). We then define

$$D = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R} : u(t, x) > \bar{u}(x)\}. \quad (3.4)$$

Obviously, D is an open set since u and \bar{u} are continuous.

Moreover, if $D = \mathbb{R}_+ \times \mathbb{R}$, then $u > \bar{u}$ everywhere. However, since $Lu = 0$ on $D = \mathbb{R}_+ \times \mathbb{R}$, $u(t, x) = -\mathbb{E}^v|x - X_t| = U^v(x) - L_t^x \searrow -\infty$ as $t \rightarrow \infty$, which violates the fact that $u > \bar{u}$ everywhere. Therefore, we have that $D \subsetneq \mathbb{R}_+ \times \mathbb{R}$.

In this section, we will see that D is a Root's barrier such that the first hitting time $\tau_D = \inf\{t > 0 : (t, X_t) \notin D\}$ is a solution for $\text{SEP}(\sigma, v, \mu)$.

The key observation is that the solution $u(t, x)$ has an interpretation in terms of an optimal stopping problem (see [3, Sect. 3.4.9]):

$$u(t, x) = \sup_{\theta \leq t} J_{t,x}(\theta), \quad \text{where } J_{t,x}(\theta) := \mathbb{E}^x[u_0(Y_\theta) + (\bar{u} - u_0)(Y_\theta)\mathbb{1}_{\theta < t}].$$

Here, Y is an independent copy of X , but runs backward from (t, x) . Moreover, according to Cox and Wang [10, Rmk 4.4],

$$u(t, x) = J_{t,x}(\theta_t), \quad \text{where } \theta_t = \inf\{r \geq 0 : (t - r, Y_r) \notin D\} \wedge t.$$

Using this result we firstly verify that the open set D is a Root's barrier.

Lemma 3.1. *Suppose that (3.1)–(3.3) hold, then $u(t, x)$ is non-increasing in t and D is a Root's barrier.*

Proof. For any fixed (t, x) , and a stopping time $\theta \leq t$ and a deterministic time $s \leq t$,

$$\begin{aligned} J_{t,x}(\theta) &= \mathbb{E}^x[u_0(Y_\theta) + (\bar{u} - u_0)(Y_\theta)\mathbb{1}_{\theta < s} + (\bar{u} - u_0)(Y_\theta)\mathbb{1}_{s \leq \theta < t}] \\ &\leq \mathbb{E}^x[u_0(Y_{s \wedge \theta}) + (\bar{u} - u_0)(Y_{s \wedge \theta})\mathbb{1}_{s \wedge \theta < s}] + \mathbb{E}^x[u_0(Y_\theta) - u_0(Y_{s \wedge \theta})] \\ &= J_{s,x}(s \wedge \theta) + \mathbb{E}^x[u_0(Y_\theta) - u_0(Y_{s \wedge \theta})] \end{aligned}$$

where the inequality holds because $\bar{u} \leq u_0$. Then $J_{t,x}(\theta) \leq J_{s,x}(s \wedge \theta)$ by Jensen's inequality since u_0 is concave. It follows that

$$u(t, x) = \sup_{\theta \leq t} J_{t,x}(\theta) \leq \sup_{\theta \leq s} J_{s,x}(\theta) = u(s, x).$$

Thus, $u(t, x)$ is non-increasing in t . It follows that D is a Root's barrier. \square

The non-increase of u in time also can be found in [15, Cor. 1], and they proved the result using PDE theory. The proof we present here is independently derived via the connection between optimal stopping problems and obstacle problems.

Next we will interpret the viscosity solution $u(t, x)$ in a probabilistic viewpoint.

Lemma 3.2. *Suppose that (3.1)–(3.3) hold, then there exists some probability distribution μ_t such that $u(t, \cdot) = U^{\mu_t}$ for all $t \geq 0$.*

Proof. Firstly, the concavity of u in space easily follows from the non-increase of u in time and (3.3).

Noting that $|(u_0)'_-| \leq 1$ and the Radon measure $u_0''(dx) = -2\nu(dx)$, we have, by Itô–Tanaka formula,

$$\begin{aligned} 0 \leq u_0(x) - u(t, x) &\leq u_0(x) - J_{t,x}(t) = u_0(x) - \mathbb{E}^x[u_0(Y_t)] \\ &= -\mathbb{E}^x \left[\int_0^t (u_0)'_-(Y_s) dY_s + \frac{1}{2} \int_{\mathbb{R}} L_t^a u_0''(da) \right] = \int_{\mathbb{R}} \mathbb{E}^x[L_t^a] \nu(da). \end{aligned}$$

Denote the transition density of Y by $p_t^Y(x, y)$. By the symmetry property of density (cf. [17, p.149]; [13, Thm. 2.2]),

$$\mathbb{E}^x[L_t^a] = \int_0^t \sigma^2(a) p_s^Y(x, a) ds = \int_0^t \sigma^2(x) p_s^Y(a, x) ds = \mathbb{E}^a[L_t^x].$$

It then follows from [4, Lem. 2.2] that, as $|x| \rightarrow \infty$,

$$u_0(x) - u(t, x) \leq \mathbb{E}^v[L_t^x] = \mathbb{E}^v[|x - X_t| - |x - X_0|] \rightarrow 0.$$

Thus, we conclude that there exists some probability distribution, denoted by μ_t , such that $u(t, \cdot) = U^{\mu_t} \leq U^v$ (cf. [30, Lem. 2.3.1]). \square

Noting that $u(t, x)$ is non-increasing in t and bounded below by $\bar{u}(x)$, we can define $\hat{u}(x) = \lim_{t \rightarrow \infty} u(t, x)$ for all $x \in \mathbb{R}$. According to Chacon [4, Lem. 2.5 & 2.6], there exists some constant C_L and a measure $\hat{\mu}$ defined on \mathbb{R} such that

$$\mu_t \implies \hat{\mu} \quad \text{and} \quad \hat{u}(x) = U^{\hat{\mu}}(x) - C_L, \quad \forall x \in \mathbb{R}. \quad (3.5)$$

We also define

$$\hat{D} = \{(t, x) : u(t, x) > \hat{u}(x)\} \quad \text{and} \quad \hat{\tau} = \inf\{t > 0 : (t, X_t) \notin \hat{D}\}.$$

Obviously $\hat{D} \subset D$ and $\hat{\tau} \leq \tau_D$. Moreover, we have the following result.

Lemma 3.3. Suppose that (3.1)–(3.3) hold, then $X_{\hat{\tau}} \sim \hat{\mu}$.

Proof. For some fixed time $t > 0$, one can easily check that $u(\cdot \wedge t, \cdot)$ is the viscosity solution of $\text{OBS}(\sigma, U^v, u(t, \cdot))$. Define

$$D_t := \{(s, x) : u(s \wedge t, x) > u(t, x)\}, \quad \tau_t = \inf\{s > 0 : (s, X_s) \notin D_t\}.$$

Then by Theorem 2.2,

$$u(t, x) = -\mathbb{E}^v|x - X_{\tau_t}| \quad \text{and} \quad X_{\tau_t} \sim \mu_t \quad (3.6)$$

Since u is non-increasing in time, it is easy to check that $\{D_t\}_{t>0}$ is a non-decreasing sequence of open sets. Further, since $u(t, x) \geq \hat{u}(x)$, one can check that $D_t \subset \hat{D}$.

Conversely, for any $(t, x) \in \hat{D}$, $u(t, x) > \hat{u}(x)$. Since $\lim_{s \rightarrow \infty} u(s, x) = \hat{u}(x)$, there must be some $T > t$ such that $u(t, x) > u(T, x) > \hat{u}(x)$, i.e. $(t, x) \in D_T$. As conclusion, we have that $D_t \nearrow \hat{D}$. It follows from (3.5) and (3.6) that

$$\tau_t \nearrow \hat{\tau} \text{ as } t \rightarrow \infty, \quad \mathbb{P}^v\text{-a.s.}, \quad \text{and hence,} \quad X_{\hat{\tau}} = \lim_{t \rightarrow \infty} X_{\tau_t} \sim \hat{\mu}. \quad \square \quad (3.7)$$

We then can present the main result of this section, which connects Skorokhod embedding problems to obstacle problems when the convex ordering condition (1.1) fails.

Theorem 3.4. Given (3.1)–(3.3), let τ_D be the stopping time defined in (3.4). Then τ_D is a Root's solution to $\text{SEP}(\sigma, \nu, \mu)$. Moreover, $u(t, x) = -\mathbb{E}^v|x - X_{t \wedge \tau_D}|$ and $u(t, x) \searrow \bar{u}(x)$ as $t \rightarrow \infty$ for all $x \in \mathbb{R}$.

Proof. First of all, we define $F := \{x \in \mathbb{R} : R(x) < +\infty\}$, $\widehat{F} := \{x \in \mathbb{R} : \widehat{R}(x) < +\infty\}$ where R and \widehat{R} denote the barrier functions of D and \widehat{D} respectively. Because $\widehat{D} \subset D$, we have that $\widehat{R} \leq R$ and $\widehat{F} \supset F$. In addition, F is non-empty since $D \subsetneq \mathbb{R}_+ \times \mathbb{R}$ as mentioned before, and hence both the stopping times τ_D and $\widehat{\tau}$ are non-trivial, i.e. finite almost surely [20, Prop. 1].

For any $x \in \widehat{F} \cap F^\complement$, we have that $\widehat{u}(x) = u(\widehat{R}(x), x) > \bar{u}(x)$. By the continuity of \widehat{u} and \bar{u} , there exists ε such that $u_0(y) \geq \widehat{u}(y) > \bar{u}(y)$ for all $y \in (x - \varepsilon, x + \varepsilon)$, and then $u(t, y) > \bar{u}(y)$ for all $t > 0$. It follows that $(0, +\infty) \times (x - \varepsilon, x + \varepsilon) \subset D$. Since $Lu = 0$ on D , the process $\{u(t - r, Y_r)\}$ is a martingale up to the hitting time $H_{x \pm \varepsilon}$ under \mathbb{P}^x . Therefore, for $t > 0$, since u is non-increasing in t ,

$$u(2t, x) = \mathbb{E}^x[u(2t - t \wedge H_{x \pm \varepsilon}, Y_{t \wedge H_{x \pm \varepsilon}})] \leq \mathbb{E}^x[u(t, Y_{t \wedge H_{x \pm \varepsilon}})].$$

Let $t \rightarrow \infty$, since $u(t, x) \searrow \widehat{u}(x)$, by the concavity of \widehat{u} and Fatou's Lemma,

$$\widehat{u}(x) \leq \mathbb{E}^x[\lim_{t \rightarrow \infty} u(t, Y_{t \wedge H_{x \pm \varepsilon}})] = \mathbb{E}^x[\widehat{u}(Y_{H_{x \pm \varepsilon}})] \leq \widehat{u}(x).$$

Hence $\mathbb{E}^x[\widehat{u}(Y_{H_{x \pm \varepsilon}})] = \widehat{u}(x)$. Since $u_0 > \bar{u}$ on $(x - \varepsilon, x + \varepsilon)$, the process Y is non-degenerate before $H_{x \pm \varepsilon}$ by (3.3), so the concave function \widehat{u} is in fact linear on $(x - \varepsilon, x + \varepsilon)$ (cf. [19, Prop. 3.5.1]). This implies that $\widehat{\mu}(x - \varepsilon, x + \varepsilon) = 0$, and then it follows that $\widehat{\mu}(F) = \widehat{\mu}(\widehat{F})$. Moreover, since $\widehat{R}(X_{\widehat{\tau}}) \leq \widehat{\tau} < \infty$ almost surely, we have that $\widehat{\mu}(F) = \widehat{\mu}(\widehat{F}) = 1$ by

$$\widehat{\mu}(\widehat{F}) = \mathbb{P}^\nu[\widehat{R}(X_{\widehat{\tau}}) < \infty] \geq \mathbb{P}^\nu[\widehat{\tau} < \infty] = 1.$$

Same argument implies $\mu(F) = 1$, and then we have that

$$\mu(F) = \widehat{\mu}(F) = 1, \quad \mu(F^\complement) = \widehat{\mu}(F^\complement) = 0. \quad (3.8)$$

For any $x \in F$, $u(t, x) = u(R(x), x) = \bar{u}(x)$ for all $t \geq R(x)$, and then $\widehat{u}(x) = \lim_{t \rightarrow \infty} u(t, x) = \bar{u}(x)$. Hence we have that, by the continuity of \bar{u} and \widehat{u} ,

$$\bar{u} = U^\mu - C \leq U^{\widehat{\mu}} - C_L = \widehat{u} \quad \text{on } \mathbb{R}, \quad \text{with "=" on } \text{cl}(F),$$

where $\text{cl}(F)$ denotes the closure of F .

Define $x_* = \inf F$, $x^* = \sup F$. Then we have that

$$\bar{u}(x_*) = \widehat{u}(x_*) \quad \text{if } x_* > -\infty; \quad \bar{u}(x^*) = \widehat{u}(x^*) \quad \text{if } x^* < +\infty.$$

For any x such that $-\infty < x < x_*$, since $\mu(F) = \widehat{\mu}(F) = 1$, it is easy to compute that $\bar{u}(x) = -(m_\mu + C) + x$ and $\widehat{u}(x) = -(m_{\widehat{\mu}} + C_L) + x$, where the mean values of μ and $\widehat{\mu}$ are denoted by m_μ and $m_{\widehat{\mu}}$. Let $x \rightarrow x_*$. It follows from the continuity of potential functions and $\bar{u}(x_*) = \widehat{u}(x_*)$ that $-(m_\mu + C) = -(m_{\widehat{\mu}} + C_L) = \bar{u}(x_*) - x_*$, and then \bar{u} and \widehat{u} agree on $(-\infty, x_*)$:

$$\bar{u}(x) = \widehat{u}(x) = \bar{u}(x_*) - (x_* - x) \quad \text{for all } x < x_*.$$

Similarly, we have that \bar{u} and \widehat{u} also agree on $(x^*, +\infty)$:

$$\bar{u}(x) = \widehat{u}(x) = \bar{u}(x^*) - (x - x^*) \quad \text{for all } x > x^*.$$

For the case where $x \in F^\complement$ and there exist $z_1, z_2 \in F$ such that $z_1 < x < z_2$, denote $z_* := \sup\{y \in F : y < x\}$, $z^* := \inf\{y \in F : y > x\}$. Since $(z_*, z^*) \subset F^\complement$, we have that $\mu((z_*, z^*)) = 0$ and $\widehat{\mu}((z_*, z^*)) = 0$ by (3.8), which implies that both \bar{u} and \widehat{u} are linear on (z_*, z^*) . In addition, $\bar{u}(z_*) = \widehat{u}(z_*)$, $\bar{u}(z^*) = \widehat{u}(z^*)$ because $z_*, z^* \in \text{cl}(F)$. Then we can conclude that $\bar{u} = \widehat{u}$ on (z_*, z^*) :

$$\bar{u}(x) = \widehat{u}(x) = \frac{z^* - x}{z^* - z_*} \bar{u}(z_*) + \frac{x - z_*}{z^* - z_*} \bar{u}(z^*), \quad \text{for all } x \in (z_*, z^*)$$

As conclusion, we have that $\bar{u} = \hat{u}$ on \mathbb{R} , and hence μ and $\hat{\mu}$ agree on \mathbb{R} and so $C = C_L$. It then follows from (3.7) that

$$D = \hat{D}, \quad X_{\tau_D} = X_{\hat{\tau}} \sim \mu, \quad u(t, x) \searrow \bar{u}(x).$$

At last we show that $u(t, x) = -\mathbb{E}^v|x - X_{t \wedge \tau_D}|$. Fix some $t \geq 0$, for all $T > t$, one can easily verify that $u(\cdot \wedge T, \cdot)$ is the viscosity solution to $\text{OBS}(\sigma, U^v, u(T, \cdot))$, and then, by Theorem 2.2 and Tanaka's formula,

$$u(t, x) = -\mathbb{E}^v|x - X_{t \wedge \tau_T}| = -|x| - \mathbb{E}^v[L_{t \wedge \tau_T}^x].$$

Since $\tau_T \nearrow \hat{\tau} = \tau_D$ (recall (3.7)), let $T \rightarrow \infty$, the desired result follows from the monotone convergence theorem. \square

Remark 3.5. We prove the existence and construction of Root's solution to SEP for the case where X is a time-homogeneous diffusion. Thanks to the work of Gassiat et al. [15], our proof also works if the diffusion coefficient $\sigma : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies (3.2) and (3.3) uniformly in time t .

4. Minimality of Root's embeddings

In Section 3, we have shown that for any integrable distribution v and μ , even if (1.1) fails, we still can construct a Root's solution to $\text{SEP}(\sigma, v, \mu)$ by solving $\text{OBS}(\sigma, U^v, U^\mu - C)$. It also turns out that there exist infinitely many Root's embeddings for $\text{SEP}(\sigma, v, \mu)$ (dependent on different choices of C in the boundary condition).

For the cases where $U^\mu \leq U^v$, one may think that $C = 0$ is the best choice because such Root's embeddings are UI stopping times. For the general cases where $U^\mu \not\leq U^v$, we have learned that there is no UI solution to $\text{SEP}(\sigma, v, \mu)$. As mentioned in Section 1, now we need the embeddings to be minimal in the sense of Monroe [21]. In this section, we study the minimality of embeddings, and then, we will see how to choose suitable boundary condition in the obstacle problems such that the corresponding Root's embeddings are minimal.

To this end, we firstly recall the following result [7, Thm. 17], which connects the minimality of stopping times to potential functions.

Theorem 4.1. Let T solve $\text{SEP}(v, \mu)$ where v, μ are integrable. Define

$$\begin{aligned} \mathcal{A} &= \{x \in [-\infty, +\infty] : \lim_{y \rightarrow x} (U^\mu - U^v)(y) = C^*\}, \\ \text{where } C^* &:= \sup_{x \in \mathbb{R}} \{U^\mu(x) - U^v(x)\}, \end{aligned} \quad (4.1)$$

$$a_+ = \sup\{x \in \overline{\mathbb{R}} : x \in \mathcal{A}\} \quad \text{and} \quad a_- = \inf\{x \in \overline{\mathbb{R}} : x \in \mathcal{A}\}. \quad (4.2)$$

Moreover, denote the first hitting times of the set \mathcal{A} and the horizontal level γ by $H_{\mathcal{A}}$ and H_γ respectively. Then the following statements are equivalent:

- (i) T is minimal;
- (ii) $T \leq H_{\mathcal{A}}$ and for all stopping times $S \leq T$,

$$\mathbb{E}^v[W_T | \mathcal{F}_S] \leq W_S \text{ on } \{W_0 \geq a_-\}; \quad \mathbb{E}^v[W_T | \mathcal{F}_S] \geq W_S \text{ on } \{W_0 \leq a_+\};$$

- (iii) $T \leq H_{\mathcal{A}}$ and as $\gamma \rightarrow \infty$,

$$\gamma \mathbb{P}^v[T > H_{-\gamma}, W_0 \geq a_-] \rightarrow 0; \quad \gamma \mathbb{P}^v[T > H_{+\gamma}, W_0 \leq a_+] \rightarrow 0.$$

Further, if there exists $a \in \mathbb{R}$ such that $\mathbb{P}^v[T \leq H_a] = 1$, then T is minimal.

The original proof of [Theorem 4.1](#) does not rely on any properties of Brownian motion beyond the strong Markov property and the continuity of paths, so this result can be extended to any continuous strong Markov processes.

Now, let τ be a solution to [SEP](#)(σ, ν, μ) (not necessarily be of Root's type), we denote the potential of the corresponding stopped process by

$$u(t, x) = -\mathbb{E}^\nu |x - X_{t \wedge \tau}|.$$

We are interested in what will happen to $u(t, x)$ as $t \rightarrow \infty$.

If τ is a UI stopping time, we immediately have that $\lim_{t \rightarrow \infty} u(t, x) = U^\mu$. For non-UI cases, we firstly review the examples mentioned in [Section 1](#).

Example 4.2. For some $a > 0$, $H_a = \inf\{t > 0 : W_t = a\}$ is a non-UI solution for [SEP](#)(δ_0, δ_a). Let $u(t, x) = -\mathbb{E}^{\delta_0} |x - W_{t \wedge H_a}|$. One can compute for $x < a$,

$$\begin{aligned} u(t, x) &= x - 2x \cdot \Phi\left(\frac{x}{\sqrt{t}}\right) + 2(x - 2a) \cdot \Phi\left(\frac{x - 2a}{\sqrt{t}}\right) - 2\sqrt{t} \\ &\quad \cdot \left[\phi\left(\frac{x}{\sqrt{t}}\right) - \phi\left(\frac{x - 2a}{\sqrt{t}}\right) \right] \\ &\rightarrow x - 2x \cdot \Phi(0) + (2x - 4a) \cdot \Phi(0) = x - 2a = -|x - a| - a, \end{aligned}$$

where Φ and ϕ denote the CDF and PDF of standard normal distribution respectively. For $x \geq a$, we have that $u(t, x) = -x = -|x - a| - a$. Therefore, $\lim_{t \rightarrow \infty} u(t, x) = U^{\delta_a}(x) - a$ for all $x \in \mathbb{R}$.

Example 4.3. For some $a > 0$, $H_0 = \inf\{t > 0 : W_t = 0\}$ is a non-UI solution for [SEP](#)($(\delta_a + \delta_{-a})/2, \delta_0$). Then we have that

$$\begin{aligned} u(t, x) &= -(|x| + 2a) + (|x| + a) \cdot \Phi\left(\frac{|x| + a}{\sqrt{t}}\right) \\ &\quad - (|x| - a) \cdot \Phi\left(\frac{|x| - a}{\sqrt{t}}\right) + \sqrt{t} \cdot \left[\phi\left(\frac{|x| + a}{\sqrt{t}}\right) \right. \\ &\quad \left. - \phi\left(\frac{|x| - a}{\sqrt{t}}\right) \right]. \end{aligned}$$

It is easy to verify that $\lim_{t \rightarrow \infty} u(t, x) = -|x| - a = U^{\delta_0}(x) - a$.

By the last line of [Theorem 4.1](#), both embeddings given in the above examples are minimal. Denote the starting and target distributions by ν and μ respectively in these examples, one then can find that (see [Fig. 1](#))

$$\lim_{t \rightarrow \infty} u(t, x) = U^\mu(x) - C^*, \quad \text{where } C^* = \sup_{\mathbb{R}} \{U^\mu - U^\nu\}.$$

This result can be extended to general cases as the following lemma.

Lemma 4.4. Let τ be a solution to [SEP](#)(σ, ν, μ) and $C^* := \sup_{x \in \mathbb{R}} \{U^\mu(x) - U^\nu(x)\}$. Then

$$\begin{aligned} \lim_{t \rightarrow \infty} u(t, x) &= U^\mu(x) - C_L, \quad \text{for all } x \in \mathbb{R}, \\ \text{where } C_L &= C^* + \inf_{x \in \mathbb{R}} \mathbb{E}^\nu [L_\tau^x]. \end{aligned}$$

In particular, $C^* = C_L$, if τ is a minimal stopping time.

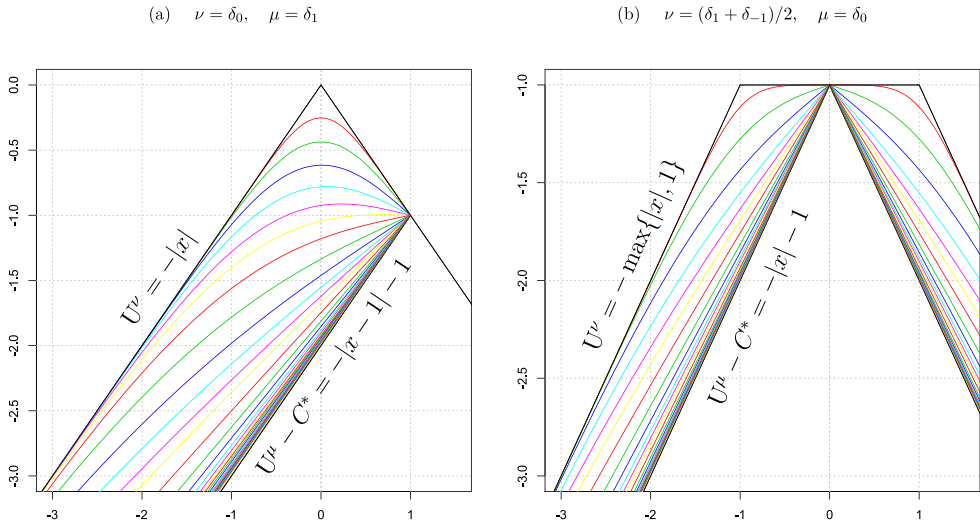


Fig. 1. The evolution of potentials described in Examples 4.2 and 4.3.

Proof. Since $t \wedge \tau \rightarrow \tau$, $X_{t \wedge \tau} \rightarrow X_\tau$ almost surely, and then $\mathcal{L}(X_{t \wedge \tau}) \Rightarrow \mathcal{L}(X_\tau)$. By Chacon [4, Lem. 2.5], there exists a constant C_L such that

$$\lim_{t \rightarrow \infty} u(t, x) = U^\mu(x) - C_L, \quad \text{for all } x \in \mathbb{R}.$$

By martingale property and Tanaka's formula, we have that

$$\begin{aligned} -\infty < U^\mu(x) - u(t, x) &= \mathbb{E}^\nu \left[\int_{t \wedge \tau}^\tau \text{sgn}(x - X_s) dX_s + (L_{t \wedge \tau}^x - L_\tau^x) \right] \\ &= \mathbb{E}^\nu \left[\int_0^\tau \text{sgn}(x - X_s) dX_s + (L_{t \wedge \tau}^x - L_\tau^x) \right]. \end{aligned}$$

Then, by the monotone convergence theorem,

$$C_L \equiv U^\mu(x) - \lim_{t \rightarrow \infty} u(t, x) = \mathbb{E}^\nu \left[\int_0^\tau \text{sgn}(x - X_s) dX_s \right], \quad \forall x \in \mathbb{R}.$$

It follows that, by the definition of C^* ,

$$\begin{aligned} C^* &= \sup_{x \in \mathbb{R}} \{U^\mu(x) - U^\nu(x)\} = \sup_{x \in \mathbb{R}} \mathbb{E}^\nu[|x - X_0| - |x - X_\tau|] \\ &= \sup_{x \in \mathbb{R}} \mathbb{E}^\nu \left[\int_0^\tau \text{sgn}(x - X_s) dX_s - L_\tau^x \right] = C_L - \inf_{x \in \mathbb{R}} \mathbb{E}^\nu[L_\tau^x]. \end{aligned}$$

Now we assume additionally that τ is a minimal stopping time. Consider the following cases dependent on the intersection of \mathbb{R} and \mathcal{A} defined in (4.1).

- The case where $\mathcal{A} \cap \mathbb{R} \neq \emptyset$.

We can pick $y \in \mathcal{A} \cap \mathbb{R}$. Since τ is minimal, by Theorem 4.1, we have that $\tau \leq H_A \leq H_y < \infty$, \mathbb{P}^ν -a.s. It follows that $\mathbb{E}^\nu[L_\tau^y] \leq \mathbb{E}^\nu[L_{H_y}^y] = 0$. Therefore, $\inf_{x \in \mathbb{R}} \mathbb{E}^\nu[L_\tau^x] = 0$.

- The case where $\mathcal{A} \cap \mathbb{R} = \emptyset$.

Without loss of generality, we assume that $+\infty \in \mathcal{A}$. For any $y \in \mathbb{R}$, denoting $a^+ := \max(a, 0) = (|a| + a)/2$, then

$$\begin{aligned}\mathbb{E}^\nu [L_{t \wedge \tau}^y] &= \mathbb{E}^\nu |X_{t \wedge \tau} - y| - \mathbb{E}^\nu |X_0 - y| \\ &= 2\mathbb{E}^\nu [(X_{t \wedge \tau} - y)^+] - 2\mathbb{E}^\nu [(X_0 - y)^+].\end{aligned}\quad (4.3)$$

Since τ is a minimal stopping time, by Jensen's inequality and [Theorem 4.1\(ii\)](#),

$$(X_{t \wedge \tau} - y)^+ \leq \left(\mathbb{E}^\nu [X_\tau - y | \mathcal{F}_{t \wedge \tau}] \right)^+ \leq \mathbb{E}^\nu [(X_\tau - y)^+ | \mathcal{F}_{t \wedge \tau}].$$

Then the process $\{(X_{t \wedge \tau} - y)^+\}$ is uniformly integrable because μ is integrable. Now letting t go to infinity in (4.3), we have that

$$\begin{aligned}\mathbb{E}^\nu [L_\tau^y] &= 2\mathbb{E}^\nu [(X_\tau - y)^+] - 2\mathbb{E}^\nu [(X_0 - y)^+] \\ &= \left\{ \mathbb{E}^\nu |X_\tau - y| + \mathbb{E}^\nu [X_\tau - y] \right\} - \left\{ \mathbb{E}^\nu |X_0 - y| + \mathbb{E}^\nu [X_0 - y] \right\} \\ &= [U^\nu(y) - U^\mu(y)] + (m_\mu - m_\nu),\end{aligned}$$

where m_ν and m_μ denote the mean values of ν and μ respectively. On the other hand, by Chacon [\[4, Lem. 2.2\]](#), we have that

$$\lim_{y \rightarrow +\infty} [U^\nu(y) + (y - m_\nu)] = \lim_{y \rightarrow +\infty} [U^\mu(y) + (y - m_\mu)] = 0,$$

which implies that $U^\mu(y) - U^\nu(y) \rightarrow m_\mu - m_\nu$ as $y \rightarrow +\infty$. We then can conclude that $\mathbb{E}^\nu [L_\tau^y] \rightarrow 0$ as $y \rightarrow +\infty$, and then $\inf_{y \in \mathbb{R}} \mathbb{E}^\nu [L_\tau^y] = 0$. The case where $-\infty \in \mathcal{A}$ is similar. \square

We have seen that $C^* = C_L$ (or equivalently, $\inf_{x \in \mathbb{R}} \mathbb{E}^\nu [L_x^x] = 0$) is a necessary condition for the minimality. However, our aim in this section is to show that the Root's embedding given by [OBS](#)($\sigma, U^\nu, U^\mu - C^*$) is minimal. To this end, next we will see $C^* = C_L$ is also a sufficient condition.

Theorem 4.5. *Under the same assumptions imposed in [Lemma 4.4](#), τ is a minimal stopping time if and only if $\inf_{x \in \mathbb{R}} \mathbb{E}^\nu [L_x^x] = 0$, or equivalently,*

$$\lim_{t \rightarrow \infty} u(t, x) = U^\mu(x) - C^*, \quad \text{for all } x \in \mathbb{R}.$$

Proof. It has been shown in [Lemma 4.4](#) that $\lim_{t \rightarrow \infty} u(t, x) = U^\mu(x) - C^*$ if τ is minimal. It only remains to show the “if” part. Now we suppose that $\lim_{t \rightarrow \infty} u(t, x) = U^\mu(x) - C^*$. Consider the following cases dependent on the intersection of \mathbb{R} and \mathcal{A} defined in (4.1).

- *The case where $\mathcal{A} \cap \mathbb{R} \neq \emptyset$.*

We can pick some $y \in \mathcal{A} \cap \mathbb{R}$. Since potential functions are continuous and $u(t, x) \rightarrow U^\mu(x) - C^*$, we have that $\lim_{t \rightarrow \infty} u(t, y) = U^\nu(y)$. Then by Tanaka's formula and monotone convergence theorem,

$$\mathbb{E}^\nu [L_\tau^y] = \lim_{t \rightarrow \infty} \mathbb{E}^\nu [L_{t \wedge \tau}^y] = U^\nu(y) - \lim_{t \rightarrow \infty} u(t, y) = 0,$$

and hence, $L_\tau^y = 0$, \mathbb{P}^ν -a.s. It follows that $\tau \leq H_y$ almost surely, and then τ is a minimal stopping time by the last line of [Theorem 4.1](#).

- *The case where $\mathcal{A} \cap \mathbb{R} = \emptyset$.*

Suppose that $+\infty \in \mathcal{A}$. Since $C^* = \lim_{y \rightarrow +\infty} [U^\mu(y) - U^\nu(y)] = m_\mu - m_\nu$ (see the proof of Lemma 4.4) and $u(t, x) \rightarrow U^\mu(x) - C^*$, we have that

$$\begin{aligned} 2\mathbb{E}^\nu[X_\tau^+ - X_{t \wedge \tau}^+] &= \mathbb{E}^\nu[|X_\tau| - |X_{t \wedge \tau}|] + \mathbb{E}^\nu[X_\tau - X_{t \wedge \tau}] \\ &= [u(t, 0) - U^\mu(0)] + (m_\mu - m_\nu) \\ &\longrightarrow (m_\mu - m_\nu) - C^* = 0, \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Then, by Scheffé's Lemma, $\{X_{t \wedge \tau}^+\}$ is uniformly integrable. Therefore, as $\gamma \rightarrow +\infty$,

$$\begin{aligned} \gamma \mathbb{P}^\nu[\tau > H_\gamma] &= \gamma \cdot \mathbb{P}^\nu[\tau > H_\gamma, X_0 \geq \gamma] + \gamma \cdot \mathbb{P}^\nu[\tau > H_\gamma, X_0 < \gamma] \\ &\leq \gamma \cdot \nu([\gamma, \infty)) + \gamma \cdot \mathbb{P}^\nu[\tau > H_\gamma, X_0 < \gamma] \\ &\leq \gamma \cdot \nu([\gamma, \infty)) + \gamma \cdot \mathbb{P}^\nu[X_{H_\gamma \wedge \tau} \geq \gamma, X_0 < \gamma] \\ &\leq \mathbb{E}^\nu[X_0; X_0 \geq \gamma] + \mathbb{E}^\nu[X_{H_\gamma \wedge \tau}; X_{H_\gamma \wedge \tau} \geq \gamma] \longrightarrow 0. \end{aligned}$$

Then it follows from Theorem 4.1(iii) that τ is minimal. The case in which $-\infty \in \mathcal{A}$ is similar. \square

Thanks to Theorem 4.5, we can directly tell if the Root's embedding given by Theorem 3.4 is minimal or not.

Theorem 4.6. *For integrable probability distributions ν and μ on \mathbb{R} , assume that $C^* = \sup_{x \in \mathbb{R}} \{U^\mu(x) - U^\nu(x)\}$ and σ satisfies (3.2) and (3.3). Let $u(t, x)$ be the viscosity solution to OBS($\sigma, U^\nu, U^\mu - C^*$), and D be the set defined in (3.4). Then τ_D is a minimal solution to SEP(σ, ν, μ). Moreover, we have the presentation that $u(t, x) = -\mathbb{E}^\nu[x - X_{t \wedge \tau_D}]$.*

5. Optimality of minimal Root's embeddings

As well-known, the UI embedding of Root's type is remarkable because it is of minimal residual expectation (m.r.e.). A natural question now arises: can we generalize this optimality result to non-UI Root's embeddings? When the stopped process X^τ is not uniformly integrable, we cannot expect that $\mathbb{E}^\nu[(\tau - t)^+]$ is finite. Thus, we study the quantity $\mathbb{E}^\nu[\tau \wedge t] = \mathbb{E}^\nu[\tau - (\tau - t)^+]$ instead. We conjecture that the minimal Root's embedding τ is of maximal principal expectation, that is,

Amongst all minimal solutions of SEP(σ, ν, μ), the Root's solution maximises $\mathbb{E}^\nu[\tau \wedge t]$ simultaneously for all $t > 0$.

For SEP(1, ν, μ), given that (1.1) holds, this statement holds obviously since the minimal Root's solution is UI and then is of m.r.e..

For general cases in which (1.1) fails, we suppose that τ is a minimal embedding for SEP(σ, ν, μ), and the stopped potential is denoted by $u^\tau(t, x) = -\mathbb{E}^\nu[x - X_{t \wedge \tau}]$.

It is obvious that $u^\tau(t, x)$ is non-increasing in t . According to Theorem 4.5, $u^\tau(t, x) \rightarrow U^\mu(x) - C^*$, and hence $u^\tau(t, x) \geq U^\mu(x) - C^*$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$. On the other hand,

since $u^\tau(t, x) = U^v(x) - \mathbb{E}^v [L_{t \wedge \tau}^x]$, we expect that, in the sense of distribution,¹

$$\begin{aligned} u^\tau(t + \delta, x) - u^\tau(t, x) &= - \int_t^{t+\delta} \frac{\sigma^2(x) \mathbb{P}[X_s \in dx, s < \tau]}{dx} ds \\ &\geq - \int_t^{t+\delta} \frac{\sigma^2(x) \mathbb{P}[X_{s \wedge \tau} \in dx]}{dx} ds = \frac{\sigma^2(x)}{2} \int_t^{t+\delta} \frac{\partial^2}{\partial x^2} u^\tau(s, x) ds. \end{aligned} \quad (5.1)$$

It then follows that u^τ is a viscosity supersolution of $\text{OBS}(\sigma, U^v, U^\mu - C^*)$, while u^{τ_D} is a viscosity solution of $\text{OBS}(\sigma, U^v, U^\mu - C^*)$. According to Gassiat et al. [15, Thm. 5], we then have the following result as an extension of Gassiat et al. [15, Thm. 3].

Proposition 5.1. *Assume that σ satisfies (3.1)–(3.3). Let τ_D and τ be minimal solutions to $\text{SEP}(\sigma, v, \mu)$, among which τ_D is of Root's type. Then for any $t \geq 0$, $u^{\tau_D}(t, \cdot) \leq u^\tau(t, \cdot)$ on \mathbb{R} , or equivalently, $\mathcal{L}(X_{t \wedge \tau}) \leq \mathcal{L}(X_{t \wedge \tau_D})$ in convex order.*

Beyond the optimality result above, we are also interested in deriving a pathwise inequality which encodes the maximal principal expectation in the sense that, for a non-decreasing concave function $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $F(0) = 0$, we can find a supermartingale G_t and a function $H(x)$ such that $F(t) \leq G_t + H(X_t)$, and such that the equality holds when $t = \tau_D$, and $\{G_{t \wedge \tau_D}\}$ is a martingale.

Different from our work in [10], here we do not assume distributions are in convex order (or equivalently, the embeddings are UI stopping times), so we cannot take limit on $G_{t \wedge \tau}$ as before. Instead, in the following proof, we will see the limit of $G_{t \wedge \tau_D} - G_{t \wedge \tau}$ does exist even if the embeddings are not UI.

Theorem 5.2. *Suppose that $\sigma(\cdot)$ satisfies (3.1)–(3.3) and*

$$\mathbb{E}^v \left[\int_0^{X_0} dy \int_0^y \frac{dz}{\sigma^2(z)} + \int_0^T \left(\int_0^{X_s} \frac{\sigma(X_s)}{\sigma(z)} dz \right)^2 ds \right] < \infty, \quad \text{for all } T > 0. \quad (5.2)$$

Let τ and τ_D be two minimal solutions to $\text{SEP}(\sigma, v, \mu)$, among which τ_D is a Root's stopping time. Then $\mathbb{E}^v[F(\tau_D)] \geq \mathbb{E}^v[F(\tau)]$ for all non-decreasing concave function F .

Proof. Without loss of generality, we always assume that $F(0) = 0$. Let $f(t) = F'_+(t)$ be the right derivative of F . Furthermore we may assume that f is bounded and vanishes after some time:

$$\exists N > 0, \quad \text{s.t.} \quad \text{(i). } f(0) \leq N; \quad \text{(ii). } \forall t > N, \quad f(t) = 0. \quad (5.3)$$

Define $M(t, x) := \mathbb{E}^{(t, x)}[f(\tau_D)]$. Since f is non-increasing, $M(t, x) \leq f(t)$ with equality for $t \geq R(x)$, where R denotes the boundary function of D . Hence

$$\begin{aligned} \int_0^t M(s, x) ds + \int_0^{R(x)} [f(s) - M(s, x)] ds \\ = \int_0^{R(x)} f(s) ds - \int_t^{R(x)} M(s, x) ds \geq F(t), \quad \text{with “=” if } t \geq R(x). \end{aligned} \quad (5.4)$$

¹ We have to mention that the argument here is just an intuitive illustration without technique details. We shall refer readers to Gassiat et al. [15, Thm. 1] for a rigorous proof of $Lu^\tau \geq 0$.

Similar as in our previous work [10, Sect. 5], we define

$$G(t, x) := \int_0^t M(s, x) ds - Z(x), \quad \text{where } Z(x) := \int_0^x dy \int_0^y \frac{2M(0, z)}{\sigma^2(z)} dz.$$

Since $M(0, z) \leq f(0) \leq N$, it follows from (5.2) that

$$\mathbb{E}^\nu \left[Z(X_0) + \int_0^T Z'(X_s)^2 \sigma(X_s)^2 ds \right] < \infty, \quad \text{for all } T > 0.$$

The integrability of $Z(X_t)$ then follows from

$$\begin{aligned} \mathbb{E}^\nu [Z(X_t)] &= \mathbb{E}^\nu \left[Z(X_0) + \int_0^t Z'(X_s) \sigma(X_s) dW_s + \int_0^t M(0, X_s) ds \right] \\ &\leq \mathbb{E}^\nu [Z(X_0)] + Nt < \infty, \end{aligned}$$

and the integrability of $G(t, X_t)$ follows from $\int_0^t M(s, x) ds < N^2$ in addition.

Further, note that we do not need the assumption that τ_D is UI in the proof of Cox and Wang [10, Lem. 5.2]. Thus, similarly (with a simple modification because the function f is non-increasing here), one can find that $\{G(t, X_t)\}$ is a \mathbb{P}^ν -supermartingale and a \mathbb{P}^ν -martingale on $[0, \tau_D]$. It then follows from the definition of G that

$$\mathbb{E}^\nu \left[\int_0^{t \wedge \tau_D} M(s, X_{t \wedge \tau_D}) ds - \int_0^{t \wedge \tau} M(s, X_{t \wedge \tau}) ds \right] \geq \mathbb{E}^\nu [Z(X_{t \wedge \tau_D}) - Z(X_{t \wedge \tau})]. \quad (5.5)$$

Since $Z(0) = Z'(0) = 0$, it follows from integration by parts that

$$\begin{aligned} Z(x) &= \int_{\mathbb{R}_+} (x - y)^+ Z''(y) dy + \int_{\mathbb{R}_-} (y - x)^+ Z''(y) dy \\ &= \int_{\mathbb{R}_+} \frac{|x - y| + (x - y)}{2} Z''(y) dy + \int_{\mathbb{R}_-} \frac{|y - x| + (y - x)}{2} Z''(y) dy. \end{aligned}$$

Then for any probability distribution λ with mean m_λ , we have that (assume integrability)

$$\mathbb{E}^{Y \sim \lambda} [Z(Y)] = \int_{\mathbb{R}_+} \frac{m_\lambda - y - U^\lambda(y)}{2} Z''(y) dy + \int_{\mathbb{R}_-} \frac{y - m_\lambda - U^\lambda(y)}{2} Z''(y) dy.$$

Thus, by the fact that $\mathbb{E}^\nu [X_{t \wedge \tau_D}] = \mathbb{E}^\nu [X_{t \wedge \tau}] = \mathbb{E}^\nu [X_0]$, we deduce that

$$\mathbb{E}^\nu [Z(X_{t \wedge \tau_D}) - Z(X_{t \wedge \tau})] = \int_{\mathbb{R}} \frac{u^\tau(t, y) - u^{\tau_D}(t, y)}{2} Z''(y) dy. \quad (5.6)$$

By minimality, $u^\theta(t, \cdot) \searrow U^\mu - C^*$ for $\theta = \tau_D, \tau$, then

$$(U^\mu - C^*) - u^{\tau_D} \leq u^\tau - u^{\tau_D} \leq u^\tau - (U^\mu - C^*).$$

Thus it follows from monotone convergence and squeeze theorem that

$$\lim_{t \rightarrow \infty} \mathbb{E}^\nu [Z(X_{t \wedge \tau_D}) - Z(X_{t \wedge \tau})] = 0.$$

Together with (5.5) and the fact that $\int_0^t M(s, x) ds \leq N^2$ for all (t, x) (by (5.3)), it follows from dominated convergence theorem that

$$\mathbb{E}^\nu \left[\int_0^{\tau_D} M(s, X_{\tau_D}) ds - \int_0^\tau M(s, X_\tau) ds \right] \geq 0. \quad (5.7)$$

On the other hand, since $\tau_D \geq R(X_{\tau_D})$, the inequality (5.4) implies that

$$\begin{aligned} F(\tau_D) - F(\tau) &\geq \int_0^{\tau_D} M(s, X_{\tau_D}) ds - \int_0^{\tau} M(s, X_{\tau}) ds \\ &\quad + \int_0^{R(X_{\tau_D})} [f(s) - M(s, X_{\tau_D})] ds - \int_0^{R(X_{\tau})} [f(s) - M(s, X_{\tau})] ds. \end{aligned}$$

Because $\mathcal{L}(X_{\tau_D}) = \mathcal{L}(X_{\tau})$ and all the terms above are integrable by (5.3),

$$\mathbb{E}^v[F(\tau_D) - F(\tau)] \geq \mathbb{E}^v \left[\int_0^{\tau_D} M(s, X_{\tau_D}) ds - \int_0^{\tau} M(s, X_{\tau}) ds \right] \geq 0.$$

To observe that the result still holds when (5.3) does not hold, we define $F_N(t) = \min\{Nt, F(t \wedge N)\}$. Then F_N is non-decreasing, concave function satisfying (5.3). Hence $\mathbb{E}^v[F_N(\tau_D)] \geq \mathbb{E}^v[F_N(\tau)]$. Then it follows from the monotone convergence theorem that

$$\mathbb{E}^v[F(\tau_D)] = \lim_{N \rightarrow \infty} \mathbb{E}^v[F_N(\tau_D)] \geq \lim_{N \rightarrow \infty} \mathbb{E}^v[F_N(\tau)] = \mathbb{E}^v[F(\tau)]. \quad \square$$

Remark 5.3. One may find that $\mathbb{E}^v[Z(X_{t \wedge \tau_D}) - Z(X_{t \wedge \tau})] \geq 0$ because Z is a convex function and $\mathcal{L}(X_{t \wedge \tau}) \leq \mathcal{L}(X_{t \wedge \tau_D})$ in convex order (Proposition 5.1), and then (5.7) holds by dominated convergence. However, here we show (5.7) by showing that $\mathbb{E}^v[Z(X_{t \wedge \tau_D}) - Z(X_{t \wedge \tau})]$ vanishes as t goes to infinity. The chief reason we adopt such a proof is that the comparison between viscosity (super-)solutions is not sufficient to show the optimality of Root's solutions to multi-marginal SEP, while the proof presented here still works under such cases (see Section 6.3).

6. Multi-marginal Skorokhod embedding problem

In this section, we will extend our results to multi-marginal Skorokhod embedding problems. Thanks to a very recent paper, Cox et al. [9], and the arguments presented in previous sections of this work, it is not difficult to construct Root's embeddings to such multi-marginal embedding problems.

6.1. Construction of Root's embeddings to multi-marginal SEP

Cox et al. [9] study the long-standing question of a multi-marginal Skorokhod embedding problem SEP($\sigma, \mu_0, \mathbf{\mu}$) where $\mathbf{\mu}$ is a sequence of integrable probability measures $\mu_1, \mu_2, \dots, \mu_n$:

$$\begin{aligned} \text{Given } X_0 \sim \mu_0, \text{ to find stopping times } \tau_1 \leq \tau_2 \leq \dots \leq \tau_n, \\ \text{such that } X_{\tau_1} \sim \mu_1, X_{\tau_2} \sim \mu_2, \dots, X_{\tau_n} \sim \mu_n. \end{aligned} \quad (\text{SEP})$$

Given that μ_0 and $\mathbf{\mu} = \{\mu_k\}_{k=1, \dots, n}$ is of convex ordering:

$$U^{\mu_0}(x) \geq U^{\mu_1}(x) \geq \dots \geq U^{\mu_{n-1}}(x) \geq U^{\mu_n}(x) > -\infty, \quad \text{for all } x \in \mathbb{R}, \quad (6.1)$$

we consider the following iterated optimal stopping problems:

$$u_0(t, x) = U^{\mu_0}(x), \quad u_k(t, x) = \sup_{\theta \leq t} J_{t,x}^k(\theta), \quad \text{for } k = 1, 2, \dots, n,$$

$$\text{where } J_{t,x}^k(\theta) := \mathbb{E}^x[u_{k-1}(t - \theta, Y_{\theta}) + (U^{\mu_k} - U^{\mu_{k-1}})(Y_{\theta})\mathbf{1}_{\theta < t}].$$

Using the solutions $\{u_k\}$, one can define

$$\text{for } k = 1, 2, \dots, n, \quad \tau_0 := 0, \quad \tau_k := \inf\{t > \tau_{k-1} : (t, X_t) \notin D_k\},$$

$$\text{where } D_k = \{(t, x) : u_k(t, x) - u_{k-1}(t, x) > U^{\mu_k}(x) - U^{\mu_{k-1}}(x)\},$$

and then, Cox et al. [9, Thm. 3.1] say that,

$\{D_k\}$ are Root's barriers, $\{\tau_k\}$ is a UI solution to [SEP](#)($\sigma, \mu_0, \mathbf{\mu}$),
moreover, $u_k(t, x) = -\mathbb{E}^{\mu_0}[x - X_{t \wedge \tau_k}]$ for all $k = 1, 2, \dots, n$.

Inspired by the result, we are going to consider the multi-marginal [SEP](#) when the convex ordering (6.1) fails.

Example 6.1. We shall begin with a simple example [SEP](#)($\delta_0, \{\delta_1, \delta_{-1}\}$). Obviously, a solution of this problem is given by $\tau_1 = H_1$, $\tau_2 = \inf\{t \geq H_1 : X_t = -1\}$. Same as in Section 3, we are interested in the limit of $u_j(t, x) := -\mathbb{E}^0[x - W_{t \wedge \tau_j}]$ as $t \rightarrow \infty$. For $j = 1$, according to [Theorem 4.6](#) (or [Example 4.2](#)), $u_1(t, x) \rightarrow U^{\delta_1}(x) - 1$. For $j = 2$,

$$\begin{aligned} u_2(t, x) &= -\mathbb{E}^0[|x - W_{t \wedge \tau_2}| \mathbb{1}_{t \geq \tau_1}] - \mathbb{E}^0[|x - W_t| \mathbb{1}_{t < \tau_1}] \\ &= u_1(t, x) - \mathbb{E}^0[|x - W_{t \wedge \tau_2}| \mathbb{1}_{t \geq \tau_1}] + \mathbb{E}^0[|x - W_{\tau_1}| \mathbb{1}_{t \geq \tau_1}] \\ &= u_1(t, x) - \int_0^t \mathbb{E}^0\left[|x - 1 - \tilde{W}_{(t-s) \wedge \tilde{H}_{-2}}|\right] \mathbb{P}^0[H_1 \in ds] + |x - 1| \cdot \mathbb{P}^0[H_1 \leq t] \\ &\rightarrow (-|x - 1| - 1) + (-|(x - 1) - (-2)| - 2) + |x - 1| = U^{\delta_{-1}}(x) - 3. \end{aligned}$$

Here we can interpret the constant $C_2 = 3$ as

$$C_2 = \sup_{\mathbb{R}}\{U^{\delta_{-1}} - (U^{\delta_1} - C_1)\}, \quad \text{where } C_1 = \sup_{\mathbb{R}}\{U^{\delta_1} - U^{\delta_0}\} = 1.$$

We also have to mention that τ_2 is not a minimal stopping time. In fact, $W_{H_{-1}} = W_{\tau_2} = -1$ and $H_{-1} \leq \tau_2$ almost surely, and $\mathbb{P}^0[H_{-1} < \tau_2] > 0$. We will see later that the sequence $\{\tau_1, \tau_2\}$ is “minimal” in some other sense.

Inspired by [Example 6.1](#), when the convex ordering (6.1) fails, we may define

$$\begin{aligned} U_0(x) &= U^{\mu_0}(x), \quad U_k(x) := U^{\mu_k}(x) - C_k, \quad \text{for } k = 1, \dots, n, \\ &\quad \text{where } C_k := \sup_{x \in \mathbb{R}}\{U^{\mu_k}(x) - U_{k-1}(x)\}. \end{aligned} \tag{6.2}$$

Same as before, we assume that the diffusion coefficient σ is of linear growth and Lipschitz continuous:

$$\begin{aligned} &\text{there exists } L > 0, \quad \text{s.t. } \forall x, y \in \mathbb{R}, \\ &\quad |\sigma(x) - \sigma(y)| < L|x - y|, \quad |\sigma(x)| < L(1 + |x|); \end{aligned} \tag{6.3}$$

Moreover, we impose the assumption, similar as (3.3),

$$\begin{aligned} &\text{for each compact } K \subset \{x : \exists k = 1, \dots, n, \text{ s.t. } U_{k-1}(x) > U_k(x)\}, \\ &\quad \text{there exists } C_K > 0, \text{ s.t. } \forall x \in K, \quad \sigma(x) \geq C_K > 0. \end{aligned} \tag{6.4}$$

Consider the iterated obstacle problems [OBS](#)($\sigma, U_0, u_{k-1} + U_k - U_{k-1}$) as follows, for $k = 1, \dots, n$,

$$\min\left\{Lu_k, (u_k - u_{k-1}) - (U_k - U_{k-1})\right\} = 0, \quad u_k(0, \cdot) = U_0(\cdot) \tag{OBS}$$

where $u_0(t, \cdot) := U_0(\cdot)$ for all $t \geq 0$, and the operator L is defined as before.

Given the viscosity solutions u_k to [OBS](#)($\sigma, U_0, u_{k-1} + U_k - U_{k-1}$) for $k = 1, \dots, n$, one can define

$$\begin{aligned} \tau_0 &:= 0, \quad \tau_k := \inf\{t \geq \tau_{k-1} : (t, X_t) \notin D_k\}, \quad \text{for } k = 1, 2, \dots, n, \\ &\quad \text{where } D_k := \{(t, x) : u_k(t, x) - u_{k-1}(t, x) > U_k(x) - U_{k-1}(x)\}. \end{aligned} \tag{6.5}$$

As in Section 3, since $u_{k-1}(0, \cdot) = U_0$, we firstly note that the viscosity solution u_k to this OBS is also the value function of the following optimal stopping problem (cf. [3, Sect. 3.4.9]):

$$\begin{aligned} \text{Given that } dY_t &= \sigma(Y_t)dW_t, \quad u_k(t, x) = \sup_{\theta \leq t} J_{t,x}^k(\theta), \\ \text{where } J_{t,x}^k(\theta) &:= \mathbb{E}^x[U_0(Y_\theta)\mathbb{1}_{\theta=t} + (u_{k-1}(t-\theta, \cdot) + U_k - U_{k-1})(Y_\theta)\mathbb{1}_{\theta < t}] \\ &= \mathbb{E}^x[u_{k-1}(t-\theta, Y_\theta) + (U_k(Y_\theta) - U_{k-1}(Y_\theta))\mathbb{1}_{\theta < t}]. \end{aligned}$$

In this section, we will generalize Theorem 3.4 as follows.

Theorem 6.2. Suppose that (6.2)–(6.4) hold, let the sequence of stopping times $\{\tau_k\}$ be given by (6.5). Then, for all $k = 1, \dots, n$, we have that

$$X_{\tau_k} \sim \mu_k, \quad u_k(t, x) = -\mathbb{E}^{\mu_0} |x - X_{t \wedge \tau_k}|, \quad u_k(t, \cdot) \searrow U_k \text{ as } t \rightarrow \infty. \quad (6.6)$$

Obviously, the desired result directly follows from Theorem 3.4 for $k = 1$. Next we will prove Theorem 6.2 by induction. Firstly, using this connection between obstacle problems and optimal stopping problems, we will generalize Lemmas 3.1 and 3.2 as follows.

Lemma 6.3. Suppose that (6.2)–(6.4) hold, and moreover, (6.6) holds for some $k = 1, \dots, n-1$, then

- (i) $u_{k+1}, u_{k+1} - u_k$ are non-increasing in t , and D_{k+1} is a Root's barrier;
- (ii) there exist probability distributions μ_{k+1}^t such that $u_{k+1}(t, \cdot) = U^{\mu_{k+1}^t}$.

Proof. For $s \leq t$, define $\tilde{u}_k^t(r, x) := u_k(t-r, x)$ and $\tilde{w}_k^{t,s}(t, x) := (\tilde{u}_k^t - \tilde{u}_k^s)(r, x)$. Given the assumption that (6.6) holds for k , it follows from [9, Lem. 5.2] that both $\{\tilde{u}_k^t(r, Y_r)\}_{r \leq t}$ and $\{\tilde{w}_k^{t,s}(r, Y_r)\}_{r \leq s}$ are supermartingales where the process Y is an independent copy of X .

For any stopping time $\theta \leq t$, we then have that

$$\begin{aligned} J_{t,x}^{k+1}(\theta) - J_{s,x}^{k+1}(s \wedge \theta) &= \mathbb{E}^x[\tilde{u}_k^t(\theta, Y_\theta) - \tilde{u}_k^s(s \wedge \theta, Y_{s \wedge \theta})] + \mathbb{E}^x[(U_{k+1}(Y_\theta) - U_k(Y_\theta))\mathbb{1}_{s \leq \theta < t}] \\ &\leq \mathbb{E}^x[\tilde{w}_k^{t,s}(s \wedge \theta, Y_{s \wedge \theta})] \leq \tilde{w}_k^{t,s}(0, x) = u_k(t, x) - u_k(s, x), \end{aligned}$$

where the inequalities hold because $\{\tilde{u}_k^t(r, X_r)\}$ and $\{\tilde{w}_k^{t,s}(r, X_r)\}$ are supermartingales, and $U_{k+1} \leq U_k$. Taking supremum over $\theta \leq t$, we conclude the non-increase of $u_{k+1} - u_k$ in t :

$$\begin{aligned} u_{k+1}(t, x) - u_k(t, x) &= \sup_{\theta \leq t} J_{t,x}^{k+1}(\theta) - u_k(t, x) \\ &\leq \sup_{\theta \leq t} J_{s,x}^{k+1}(s \wedge \theta) - u_k(s, x) \leq u_{k+1}(s, x) - u_k(s, x). \end{aligned} \quad (6.7)$$

Therefore D_{k+1} is a Root's barrier, and u_{k+1} inherits from u_k the non-increase in t . At last, (ii) follows from a similar proof as Lemma 3.2. \square

Now we can prove the main result of this section.

Proof of Theorem 6.2. As mentioned above, the desired results hold for $k = 1$. For general $k = 1, \dots, n-1$, suppose that (6.6) holds for k .

Since $u_{k+1} - u_k \geq U_{k+1} - U_k$ and $u_k \geq U_k$, we have that $u_{k+1} \geq U_{k+1}$. Then we can define $\hat{U}_{k+1}(x) := \lim_{t \rightarrow \infty} u_{k+1}(t, x) \geq U_{k+1}(x)$, and hence, there exists some constant C_L and

a measure $\widehat{\mu}_{k+1}$ such that $\mu_{k+1}' \implies \widehat{\mu}_{k+1}$ and $\widehat{U}_{k+1} = U^{\widehat{\mu}_{k+1}} - C_L$ on \mathbb{R} . We also define

$$\widehat{D}_{k+1} := \{(t, x) : u_{k+1}(t, x) - u_k(t, x) > \widehat{U}_{k+1}(x) - U_k(x)\} \subset D_{k+1},$$

$$\text{and } \widehat{\tau}_{k+1} := \inf\{t > \tau_k : (t, X_t) \notin \widehat{D}_{k+1}\} \leq \tau_{k+1}.$$

Fix some $t > 0$. Let $s = 0$ in (6.7), we have that $u_{k+1}(t, x) \leq u_k(t, x)$. Moreover, according to Lemma 6.3, we then have that $\mu_k^t = \mathcal{L}(X_{t \wedge \tau_k})$ and $\mu_k^t \leq \mu_{k+1}^t$ in convex order. Define $v_j(s, x) := u_j(s \wedge t, x)$ for $j = k, k+1$. One can verify that

$$\min\{(Lv_{k+1})(s, x), (v_{k+1} - v_k)(s, x) - (u_{k+1} - u_k)(t, x)\} = 0,$$

which implies that

$$u_{k+1}(s \wedge t, x) = \sup_{\theta \leq s} \mathbb{E}^x[u_k((s - \theta) \wedge t, Y_\theta) + (u_{k+1}(t, Y_\theta) - u_k(t, Y_\theta))\mathbb{1}_{\theta < s}].$$

Define $\tau_{k+1}^t := \inf\{s \geq t \wedge \tau_k : (s, X_s) \notin D_{k+1}^t\}$, where

$$D_{k+1}^t := \{(s, x) : u_{k+1}(s \wedge t, x) - u_k(s \wedge t, x) > u_{k+1}(t, x) - u_k(t, x)\}.$$

According to Cox et al. [9, Thm.4.1], we have that $\tau_{k+1}^t \geq t \wedge \tau_k$, $X_{\tau_{k+1}^t} \sim \mu_{k+1}^t$. Similar as (3.7), we can show that, as $t \rightarrow \infty$, $D_{k+1}^t \nearrow \widehat{D}_{k+1}$, and $\tau_{k+1}^t \nearrow \widehat{\tau}_{k+1}$, and hence, $X_{\widehat{\tau}_{k+1}} = \lim_{t \rightarrow \infty} X_{\tau_{k+1}^t} \sim \widehat{\mu}_{k+1}$. Then, by a simple modification of the proof of Theorem 3.4, we can conclude that (6.6) also holds for $k+1$. This completes our proof. \square

6.2. Minimality of Root's embeddings to multi-marginal SEP

Let $\tau = \{\tau_1, \dots, \tau_n\}$ be the sequence given by (6.5). When the convex ordering condition (6.1) holds, we have that $C_k = 0$ for all k in (6.2). According to Cox et al. [9, Thm. 3.1], τ_k are UI stopping times. However, as mentioned before, we cannot expect so in the absence of (6.1). Same as Section 4, we now consider the minimality of our solution to $\text{SEP}(\sigma, \mu_0, \mathbf{\mu})$.

The first hitting time τ_1 is a minimal Root's stopping time under \mathbb{P}^{μ_0} (Theorem 4.6). However, as seen in Example 6.1, the subsequent stopping times do not inherit the minimality (unless $\mu_0 \leq \dots \leq \mu_{k-1}$ in convex order, or equivalently, $\tau_1, \dots, \tau_{k-1}$ are UI stopping times). Hence, we focus on the “minimality” of a sequence of stopping times in some other sense.

Definition 6.4 (Minimal Sequence of Stopping Times). A non-decreasing sequence of stopping times $\tau = \{\tau_k\}_{k=1, \dots, n}$ for the process X is minimal if whenever $\theta = \{\theta_k\}_{k=1, \dots, n}$ is a non-decreasing sequence of stopping times such that $\theta_k \leq \tau_k$ and $\mathcal{L}(X_{\theta_k}) = \mathcal{L}(X_{\tau_k})$ for all k then $\tau = \theta$ almost surely.

Moreover we say that τ is a minimal solution to $\text{SEP}(\sigma, \mu_0, \mathbf{\mu})$ if τ is a minimal sequence and a solution to $\text{SEP}(\sigma, \mu_0, \mathbf{\mu})$ simultaneously.

Proposition 6.5. Denote $\tau_0 = 0$. A non-decreasing sequence τ is a minimal sequence of stopping times if and only if

$$\text{whenever } \theta \text{ is a stopping time such that } \tau_{k-1} \leq \theta \leq \tau_k$$

$$\text{and } \mathcal{L}(X_\theta) = \mathcal{L}(X_{\tau_k}) \text{ for some } k \text{ then } \theta = \tau_k. \quad (6.8)$$

Proof. Firstly suppose that the sequence τ satisfies (6.8), and θ is a sequence such that $\theta_k \leq \tau_k$ and $\mathcal{L}(X_{\theta_k}) = \mathcal{L}(X_{\tau_k})$. Obviously $\theta_1 = \tau_1$ since τ_1 is a minimal stopping time in the sense of Monroe. It then follows from (6.8) and induction that $\theta_k = \tau_k$ for all k .

Conversely, suppose that τ is a minimal sequence. For any $k^* \in \{1, \dots, n\}$, let θ be a stopping time such that $\tau_{k^*-1} \leq \theta \leq \tau_{k^*}$ and $\mathcal{L}(X_\theta) = \mathcal{L}(X_{\tau_{k^*}})$. Replacing τ_{k^*} by θ in τ , one then have another sequence which is also non-decreasing and embeds same marginal distributions as $\{\tau_k\}$ does. It then follows that $\theta = \tau_{k^*}$ since τ is a minimal sequence. \square

Now we focus on the property described in (6.8). Given a pair of stopping times $S \leq T$, we say that T is *minimal with respect to S* if whenever R is a stopping time s.t. $S \leq R \leq T$ and $\mathcal{L}(X_R) = \mathcal{L}(X_T)$ then $R = T$ a.s. (as described in (6.8)). By a similar proof as in [21, Prop. 2], for any stopping time $R \geq S$ there is a stopping time $T \leq R$ which is minimal with respect to S and embeds same distribution as R does. Further, by a careful review and simple modification of the arguments in [8, Sect. 2], [7, Appx. A, B] and Section 4 of this work, one can generalize Theorems 4.1 and 4.5 as follows:

Proposition 6.6. *Suppose that $S \leq T$ are stopping times such that $X_S \sim \nu$, $X_T \sim \mu$ under some probability measure \mathbb{P} . The set \mathcal{A} and its upper/lower bound a_\pm are defined as in (4.1) and (4.2). Moreover, for the set \mathcal{A} and some horizontal level γ , denote the first hitting times after S by $H_{\mathcal{A}}^S$ and H_γ^S :*

$$H_{\mathcal{A}}^S = \inf\{t \geq S : X_t \in \mathcal{A}\} \quad \text{and} \quad H_\gamma^S = \inf\{t \geq S : X_t = \gamma\}.$$

Then the following statements are equivalent:

- (i) T is minimal with respect to S ;
- (ii) $T \leq H_{\mathcal{A}}^S$ and for all stopping times R such that $S \leq R \leq T$,

$$\mathbb{E}[X_T | \mathcal{F}_R] \leq X_R \text{ on } \{X_S \geq a_-\}; \quad \mathbb{E}[X_T | \mathcal{F}_R] \geq X_R \text{ on } \{X_S \leq a_+\};$$

- (iii) $T \leq H_{\mathcal{A}}^S$ and as $\gamma \rightarrow \infty$,

$$\gamma \mathbb{P}[T > H_{-\gamma}^S, X_S \geq a_-] \longrightarrow 0; \quad \gamma \mathbb{P}[T > H_{+\gamma}^S, X_S \leq a_+] \longrightarrow 0;$$

- (iv) $\inf_{x \in \mathbb{R}} \mathbb{E}[L_T^x - L_S^x] = 0$.

Further, if $\exists a \in \mathbb{R}$ s.t. $\mathbb{P}[T \leq H_a^S] = 1$, T is minimal with respect to S .

Given a solution to SEP($\sigma, \mu_0, \mathbf{\mu}$), denoted by $\theta = \{\theta_k\}$, same as in the proof of Lemma 4.4 we have that there exists $\{c_k\}$ such that $u_k^\theta = -\mathbb{E}^{\mu_0}[x - X_{t \wedge \theta_k}] \rightarrow U^{\mu_k}(x) - c_k$ and

$$c_k \equiv \mathbb{E}^{\mu_0} \left[\int_0^{\theta_k} \text{sgn}(x - X_s) dX_s \right], \quad \text{for all } x \in \mathbb{R}.$$

And then it follows from the definition of $\{C_k\}$ (recall (6.2))

$$\begin{aligned} C_k - C_{k-1} &= \sup_{x \in \mathbb{R}} \{U^{\mu_k}(x) - U^{\mu_{k-1}}(x)\} \\ &= (c_k - c_{k-1}) - \inf_{x \in \mathbb{R}} \mathbb{E}^{\mu_0}[L_{\theta_k}^x - L_{\theta_{k-1}}^x]. \end{aligned}$$

Noting that $C_0 = c_0 = 0$, by Propositions 6.5 and 6.6 we then have that

$$\begin{aligned} C_k = c_k, \quad \text{for all } k &\iff \inf_{x \in \mathbb{R}} \mathbb{E}^{\mu_0}[L_{\theta_k}^x - L_{\theta_{k-1}}^x] = 0, \quad \text{for all } k \\ &\iff \theta \text{ is a minimal sequence of stopping times.} \end{aligned}$$

This result extends Theorem 4.6 to the multi-marginal SEP as follows.

Theorem 6.7. Suppose that (6.2)–(6.4) hold, then

- (i) The sequence τ given by (6.5) is a minimal solution to $\text{SEP}(\sigma, \mu_0, \mathbf{\mu})$.
- (ii) Let θ be a minimal solution to $\text{SEP}(\sigma, \mu_0, \mathbf{\mu})$, then the potential process $u_k^\theta(t, x) \searrow U_k(x)$ as $t \rightarrow \infty$ for all $k = 1, \dots, n$.

6.3. Optimality of Root's embeddings to multi-marginal SEP

At last, we consider the optimality obtained in Section 5 to the multi-marginal distributions cases. Suppose that $\tau = \{\tau_k\}$ and $\theta = \{\theta_k\}$ are minimal solutions to $\text{SEP}(\sigma, \mu_0, \mathbf{\mu})$, among which τ is given by (6.5). As before, define the potential processes

$$u_k^\theta(t, x) = -\mathbb{E}^{\mu_0} |x - X_{t \wedge \theta_k}|, \quad u_k^\tau(t, x) = -\mathbb{E}^{\mu_0} |x - X_{t \wedge \tau_k}|.$$

According to Theorem 6.7, $u_j^\theta(t, x) \searrow U^{j-1}(x) - C_j = U_j(x)$ as $t \rightarrow \infty$ for $j = k - 1, k$, where C_j and U_j are as defined in (6.2). Moreover, one can easily verify that, for $t \geq s \geq 0$,

$$\begin{aligned} [u_k^\theta(t, x) - u_{k-1}^\theta(t, x)] - [u_k^\theta(s, x) - u_{k-1}^\theta(s, x)] \\ = -\mathbb{E}^{\mu_0} [(L_{t \wedge \theta_k}^x - L_{s \wedge \theta_k}^x) - (L_{t \wedge \theta_{k-1}}^x - L_{s \wedge \theta_{k-1}}^x)] \leq 0. \end{aligned}$$

It follows that $u_k^\theta(t, x) - u_{k-1}^\theta(t, x) \searrow U_k(x) - U_{k-1}(x)$ as $t \rightarrow \infty$, and hence,

$$\begin{aligned} \min \{ (Lu_k^\theta)(t, x), (u_k^\theta - u_{k-1}^\theta)(t, x) - (U_k - U_{k-1})(x) \} \\ \geq 0 = \min \{ (Lu_k^\tau)(t, x), (u_k^\tau - u_{k-1}^\tau)(t, x) - (U_k - U_{k-1})(x) \} \end{aligned}$$

where $Lu_k^\theta \geq 0$ follows from Gassiat et al. (see (5.1) & the footnote on p. 17).

Suppose that $u_{k-1}^\tau(t, x) \leq u_{k-1}^\theta(t, x)$ for all (t, x) . Since $u_k^\theta, u_k^\tau, u_{k-1}^\tau, U_k, U_{k-1}$ are all Lipschitz continuous in x (uniformly in t), a slight extension of Gassiat et al. [15, Thm. 5] implies that $u_k^\tau \leq u_k^\theta$. Since $u_1^\tau \leq u_1^\theta$ (Proposition 5.1), we have the following result by induction.

Proposition 6.8. Suppose that (6.2)–(6.4) hold. Let τ and θ be minimal solutions to $\text{SEP}(\sigma, \mu_0, \mathbf{\mu})$, among which τ is given by (6.5). Then $u_k^\tau(t, x) \leq u_k^\theta(t, x)$ for all $(k, t, x) \in \{1, \dots, n\} \times [0, +\infty) \times \mathbb{R}$.

Now we follow the work of Cox et al. [9] and Section 5 of this work to get the multi-marginal analogue of Theorem 5.2.

Theorem 6.9. Suppose that (5.2) and (6.2)–(6.4) hold. Let τ and θ be the minimal solutions to $\text{SEP}(\sigma, \mu_0, \mathbf{\mu})$, among which τ is given by (6.5). Then $\mathbb{E}^{\mu_0}[F(\tau_n)] \geq \mathbb{E}^{\mu_0}[F(\theta_n)]$ for all non-decreasing concave function F .

Proof. Without loss of generality, we always assume that $F(0) = 0$. In addition, we may firstly suppose that (5.3) holds for the right derivative $f = F'_+$, and the general case follows from monotone convergence theorem (see the last paragraph of the proof of Theorem 5.2).

Let $\{D_k\}$ be the sequence of barriers given by (6.5). For $(k, t, x) \in \{1, \dots, n\} \times [0, +\infty) \times \mathbb{R}$, we define stopping times $\eta_k = \inf\{s \geq t : (s, X_s) \notin D_k\}$ under $\mathbb{P}^{(t, x)}$. Further, define $M_{n+1}(t, x) := f(t)$ and

$$M_k(t, x) := \mathbb{E}^{(t, x)}[M_{k+1}(\eta_k, X_{\eta_k})], \quad \text{for } k = 1, \dots, n.$$

Similar as in the proof of Cox et al. [9, Lem.3.4], $M_k(t, x) = \mathbb{E}^{(t,x)}[f(\zeta^k)]$ where ζ^k is the first time we exit D_n , having previously exited D_k, \dots, D_{n-1} in sequence. Hence ζ^k is non-increasing in k , and M_k is non-decreasing in k because f is non-increasing. Moreover, for any k , if $(t, x) \notin D_k$, then $\zeta^k = \zeta^{k+1}$, $\mathbb{P}^{(t,x)}$ -a.s. As conclusion,

$$M_k(t, x) \leq M_{k+1}(t, x), \quad \text{for all } (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \quad \text{with “=” if } t \geq R_k(x),$$

where R_k denotes the boundary function of D_k . Hence, given $(t_k, x_k)_{0 \leq k \leq n}$ with $0 = t_0 \leq t_1 \leq \dots \leq t_n$, we deduce that

$$\begin{aligned} & \int_0^{t_k} M_k(s, x_k) ds + \int_0^{R_k(x_k)} (M_{k+1} - M_k)(s, x_k) ds - \int_0^{t_{k-1}} M_k(s, x_{k-1}) ds \\ &= \int_0^{R_k(x_k)} M_{k+1}(s, x_k) ds - \int_{t_k}^{R_k(x_k)} M_k(s, x_k) ds - \int_0^{t_{k-1}} M_k(s, x_{k-1}) ds \\ &\geq \int_0^{t_k} M_{k+1}(s, x_k) ds - \int_0^{t_{k-1}} M_k(s, x_{k-1}) ds, \quad \text{with “=” if } t_k \geq R_k(x_k). \end{aligned}$$

Taking sum for $k = 1, \dots, n$ and noting $F(0) = 0$ and $t_0 = 0$, we have that

$$\begin{aligned} & \sum_{k=1}^n \left\{ \int_0^{t_k} M_k(s, x_k) ds + \int_0^{R_k(x_k)} (M_{k+1} - M_k)(s, x_k) ds - \int_0^{t_{k-1}} M_k(s, x_{k-1}) ds \right\} \\ &\geq \int_0^{t_n} M_{n+1}(s, x_k) ds - \int_0^{t_0} M_1(s, x_0) ds \quad (\text{with “=” if } t_k \geq R_k(x_k) \text{ for all } k) \\ &= F(t_n) \end{aligned}$$

Since $\mathcal{L}(X_{\tau_k}) = \mathcal{L}(X_{\theta_k}) = \mu_k$ and $\tau_k \geq R(X_{\tau_k})$, we then have that

$$\begin{aligned} \mathbb{E}^{\mu_0}[F(\tau_n)] - \mathbb{E}^{\mu_0}[F(\theta_n)] &\geq \sum_{k=1}^n \left\{ \mathbb{E}^{\mu_0} \left[\int_0^{\tau_k} M_k(s, X_{\tau_k}) ds - \int_0^{\tau_{k-1}} M_k(s, X_{\tau_{k-1}}) ds \right] \right. \\ &\quad \left. - \mathbb{E}^{\mu_0} \left[\int_0^{\theta_k} M_k(s, X_{\theta_k}) ds - \int_0^{\theta_{k-1}} M_k(s, X_{\theta_{k-1}}) ds \right] \right\}. \end{aligned}$$

Thus, to see $\mathbb{E}^{\mu_0}[F(\tau_n) - F(\theta_n)] \geq 0$, it is sufficient to show that, for all k ,

$$\begin{aligned} & \mathbb{E}^{\mu_0} \left[\int_0^{\tau_k} M_k(s, X_{\tau_k}) ds - \int_0^{\tau_{k-1}} M_k(s, X_{\tau_{k-1}}) ds \right] \\ &\geq \mathbb{E}^{\mu_0} \left[\int_0^{\theta_k} M_k(s, X_{\theta_k}) ds - \int_0^{\theta_{k-1}} M_k(s, X_{\theta_{k-1}}) ds \right]. \end{aligned} \tag{6.9}$$

We are going to show (6.9) in the same manner as proving (5.7). Define

$$G_k(t, x) = \int_0^t M_k(s, x) ds - Z_k(x), \quad \text{where } Z_k(x) = \int_0^x dy \int_0^y \frac{2M_k(0, z)}{\sigma^2(z)} dz.$$

Same as in the proof of Theorem 5.2, one can show that $Z_k(X_t)$ and $G_k(t, X_t)$ are integrable according to (5.2) and (5.3). Then a simple modification of the proof of Cox et al. [9, Lem. A.1] says that $\{G_k(t, X_t)\}$ is a \mathbb{P}^{μ_0} -supermartingale and a \mathbb{P}^{μ_0} -martingale on $[\tau_{k-1}, \tau_k]$. Thus, for the

sequences τ and θ , we deduce that (recall (5.6))²

$$\begin{aligned} \mathbb{E}^{\mu_0} \left[\int_0^{t \wedge \tau_k} M_k(s, X_{t \wedge \tau_k}) ds - \int_0^{t \wedge \tau_{k-1}} M_k(s, X_{t \wedge \tau_{k-1}}) ds \right] \\ - \mathbb{E}^{\mu_0} \left[\int_0^{t \wedge \theta_k} M_k(s, X_{t \wedge \theta_k}) ds - \int_0^{t \wedge \theta_{k-1}} M_k(s, X_{t \wedge \theta_{k-1}}) ds \right] \\ \geq \int_{\mathbb{R}} \frac{(u_k^\theta - u_{k-1}^\theta)(t, y) - (u_k^\tau - u_{k-1}^\tau)(t, y)}{2} Z_k''(y) dy. \end{aligned} \quad (6.10)$$

By minimality, $u_k^\rho(t, \cdot) - u_{k-1}^\rho(t, \cdot) \searrow U_k - U_{k-1}$ for $\rho = \tau, \theta$, and then,

$$\begin{aligned} (U_k - U_{k-1}) - (u_k^\tau - u_{k-1}^\tau) &\leq (u_k^\theta - u_{k-1}^\theta) - (u_k^\tau - u_{k-1}^\tau) \\ &\leq (u_k^\theta - u_{k-1}^\theta) - (U_k - U_{k-1}). \end{aligned}$$

It follows from monotone convergence and squeeze theorem that the RHS of (6.10) vanishes as $t \rightarrow \infty$. Then (6.9) follows from dominated convergence on (6.10) (similar as in the proof of Theorem 5.2). \square

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² We have mentioned in Remark 5.3 that the comparison $u_k^\tau \leq u_k^\theta$ alone is not sufficient to show (6.9). In more detail, one may find that we need a stronger result than Proposition 6.8, that is $u_k^\tau - u_{k-1}^\tau \leq u_k^\theta - u_{k-1}^\theta$ everywhere.

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