Available online at www.sciencedirect.com

ScienceDirect

Stochastic Processes and their Applications 121 (2011) 91–108

stochastic processes and their applications

www.elsevier.com/locate/spa

Stationary solutions of the stochastic differential equation $\mathrm{d}V_t = V_{t-}\mathrm{d}U_t + \mathrm{d}L_t$ with Lévy noise

Anita Behme[a,*], Alexander Lindner[a], Ross Maller[b]

[a] Technische Universität Braunschweig, Institut für Mathematische Stochastik, Pockelsstraße 14, D-38106 Braunschweig, Germany

[b] Centre for Mathematical Analysis, and School of Finance & Applied Statistics, Australian National University, Canberra, ACT, Australia

Received 22 January 2010; received in revised form 15 June 2010; accepted 16 September 2010
Available online 22 September 2010

Abstract

For a given bivariate Lévy process $(U_t, L_t)_{t\geq 0}$, necessary and sufficient conditions for the existence of a strictly stationary solution of the stochastic differential equation $\mathrm{d}V_t = V_{t-}\,\mathrm{d}U_t + \mathrm{d}L_t$ are obtained. Neither strict positivity of the stochastic exponential of U nor independence of V_0 and (U, L) is assumed and non-causal solutions may appear. The form of the stationary solution is determined and shown to be unique in distribution, provided it exists. For non-causal solutions, a sufficient condition for U and L to remain semimartingales with respect to the corresponding expanded filtration is given.

Keywords: Stochastic differential equation; Lévy process; Generalized Ornstein–Uhlenbeck process; Stochastic exponential; Stationarity; Non-causal; Filtration expansion

1. Introduction

Let $(\xi, \eta) = (\xi_t, \eta_t)_{t\geq 0}$ be a bivariate Lévy process. The generalised Ornstein–Uhlenbeck process (GOU) associated with (ξ, η) is

$$V_t = \mathrm{e}^{-\xi_t}\left(V_0 + \int_0^t \mathrm{e}^{\xi_{s-}}\mathrm{d}\eta_s\right), \quad t \geq 0, \tag{1.1}$$

where V_0 is a finite random variable, *independent* of (ξ, η). See [14,16] for further information and references on GOUs. In [14], necessary and sufficient conditions for a GOU to be strictly stationary were obtained, and properties of the strictly stationary solution studied.

* Corresponding author. Tel.: +49 531 3917562; fax: +49 531 3917564.
E-mail addresses: a.behme@tu-bs.de (A. Behme), a.lindner@tu-bs.de (A. Lindner), Ross.Maller@anu.edu.au (R. Maller).

0304-4149/$ - see front matter
doi:10.1016/j.spa.2010.09.003

As pointed out in Eq. (15) in [16], the GOU in (1.1) is the unique solution of the stochastic differential equation

$$\mathrm{d}V_t = V_{t-}\mathrm{d}U_t + \mathrm{d}L_t, \quad t \geq 0, \tag{1.2}$$

where (U, L) is another bivariate Lévy process, constructed from (ξ, η) by

$$\begin{pmatrix} U_t \\ L_t \end{pmatrix} = \begin{pmatrix} -\xi_t + \sum_{0<s\leq t} (\mathrm{e}^{-\Delta\xi_s} - 1 + \Delta\xi_s) + t\,\sigma_\xi^2/2 \\ \eta_t + \sum_{0<s\leq t} (\mathrm{e}^{-\Delta\xi_s} - 1)\Delta\eta_s - t\,\sigma_{\xi,\eta} \end{pmatrix}, \quad t \geq 0. \tag{1.3}$$

Here $(\Delta\xi_t, \Delta\eta_t) = (\xi_t - \xi_{t-}, \eta_t - \eta_{t-})$ denotes the jump process of (ξ, η) at time t, and σ_ξ^2 and $\sigma_{\xi,\eta}$ denote the $(1, 1)$ and $(1, 2)$ elements of the Gaussian covariance matrix in the Lévy–Khintchine representation of the characteristic function of (ξ, η). The definition of U in (1.3) is equivalent to saying that $\mathcal{E}(U)_t = \mathrm{e}^{-\xi_t}$, where $\mathcal{E}(U)$ denotes the Doléans–Dade stochastic exponential of U (see [17], Theorem II.37). In general the stochastic exponential may take zero or negative values, but in satisfying $\mathcal{E}(U)_t = \mathrm{e}^{-\xi_t}$, we see that this version of $\mathcal{E}(U)$ must be strictly positive, which is equivalent to the Lévy measure of U having no mass on $(-\infty, -1]$.

The purpose of the present paper is to extend the results of [14] to the more general setting of solutions to the stochastic differential equation (1.2), where (U, L) is an arbitrary bivariate Lévy process. In particular, we do not assume that the Lévy measure Π_U of U is concentrated on $(-1, \infty)$, but also allow jumps of size less than or equal to -1. As a second generalisation, we shall allow possible dependence between the starting random variable V_0 and (U, L). Even in the case when $\Pi_U((-\infty, -1]) = 0$, this represents a sharpening of the results of [14]. As in time series analysis, we will call a solution with V_0 being independent of (U, L) a *causal* or *non-anticipative* solution. We shall see that non-causal solutions can appear in some important cases.

Dealing with the non-causality is non-trivial as it introduces a possible problem regarding the filtration with respect to which the stochastic differential equation (1.2) is defined, such that U still remains a semimartingale. Hence, in the following, possible non-causal solutions (relevant in the case $\Pi_U(\{-1\}) = 0$) will be interpreted in the following sense. First, (1.2) is solved assuming that U is a semimartingale for a suitable filtration to which V is adapted. This is achieved, with the general solution given by (2.7) below. In Eq. (2.7), however, the semimartingale problem is avoided since V_0 enters in an additive fashion there and does not have to be measurable with respect to the filtration for which the stochastic integrals are defined. The problem of finding all stationary solutions is thus reduced to finding all possible choices of V_0, without assuming independence, such that the process given by (2.7) is strictly stationary.

This we do in Theorems 2.1 and 2.2 of the next section. After that, Section 3 sets notation, verifies that the solution to (1.2) is as given in Eqs. (2.3) and (2.7) of Theorems 2.1 and 2.2, and introduces various auxiliary processes used throughout the paper. Also in Section 3 necessary and sufficient conditions for the almost sure convergence of the integrals $\int_0^\infty \mathcal{E}(U)_{s-}\,\mathrm{d}L_s$ and $\int_0^\infty [\mathcal{E}(U)_{s-}]^{-1}\,\mathrm{d}\eta_s$ in terms of the characteristic triplets of the underlying Lévy processes are given. These are essential results for characterising the existence of a stationary solution to (1.2).

Section 4 gives the proofs of Theorems 2.1 and 2.2, and of two useful corollaries also stated in Section 2. The semimartingale problem described above is taken up again in Section 5. In the situation of Theorem 2.1(b), non-causal solutions of (2.7) appear, and Section 5 is concerned with the question of filtration enlargements such that the non-causal solution is adapted and U remains a semimartingale with respect to it. It is shown that absolute continuity of $\int_0^\infty [\mathcal{E}(U)_{s-}]^{-1}\,\mathrm{d}\eta_s$ is a sufficient condition for this to hold and examples when this condition is satisfied are mentioned.

We shall not deal with applications in this paper, but only remark at this stage that the GOU and stationary solutions of the SDE (1.2) are important in the analysis of the COGARCH (continuous time GARCH model) due to Klüppelberg et al. [11]. An option pricing model based on COGARCH, and incorporating the possibility of default, has recently been proposed by Szimayer; see [12]. For the solution of (1.2), in a financial process setting, a jump of U of size -1 can be interpreted as the occurrence of default, and jumps of size less than -1 have interpretations when U describes the value of a certain contract, when a positive value enforces an obligation to pay.

2. Main results

Let (U, L) be a bivariate Lévy process with characteristic triplet $\left(\begin{pmatrix} \sigma_U^2 & \sigma_{U,L} \\ \sigma_{U,L} & \sigma_L^2 \end{pmatrix}, \Pi_{U,L}, \gamma_{U,L}\right)$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$, and correspondingly denote the characteristic triplets of the coordinate processes U and L by $(\sigma_U^2, \Pi_U, \gamma_U)$ and $(\sigma_L^2, \Pi_L, \gamma_L)$, respectively. Here and in the following, the characteristic triplet is taken as in [18], Definition 8.2. To avoid trivialities assume throughout that neither U nor L is the zero Lévy process. Let $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$ be the smallest filtration satisfying the "usual hypotheses" (cf. [17], Section I.1) such that both U and L are adapted. Then U and L are semimartingales with respect to $\mathbb{F}$. Denote by

$$\mathcal{E}(U)_t := \mathrm{e}^{U_t - t\sigma_U^2/2} \prod_{0<s\le t} (1 + \Delta U_s)\mathrm{e}^{-\Delta U_s}, \quad t \ge 0, \tag{2.1}$$

the Doléans–Dade exponential of U (e.g. [17, Theorem II.37]). The exponential $\mathcal{E}(U)$ is the unique semimartingale Z (with respect to $\mathbb{F}$) such that $Z_t = 1 + \int_{(0,t]} Z_{s-}\,\mathrm{d}U_s$. It is strictly positive if and only if $\Pi_U((-\infty, -1]) = 0$, and nowhere zero if and only if $\Pi_U(\{-1\}) = 0$.

The main theorems of this paper give necessary and sufficient conditions for the existence of a strictly stationary solution of (1.2) in all cases, in particular including $\Pi_U((-\infty, -1)) \ge 0$ and $\Pi_U(\{-1\}) \ge 0$. Even in the case $\Pi_U((-\infty, -1]) = 0$ (the only one treated in [14]) they sharpen the results of [14], since independence of V_0 and (U, L) is not assumed *a priori* in our present results, whereas it was a crucial ingredient in [14] for the proof in the oscillating case.

We first deal with the case $\Pi_U(\{-1\}) = 0$. Define an auxiliary process η by

$$\eta_t := L_t - \sum_{\substack{0<s\le t \\ \Delta U_s \ne -1}} \frac{\Delta U_s \Delta L_s}{1 + \Delta U_s} - t\sigma_{U,L}, \quad t \ge 0. \tag{2.2}$$

As will be seen in Proposition 3.2 below, the general solution to (1.2) is given by (2.7), which in the case $\Pi_U(\{-1\}) = 0$ simplifies to (2.3).

Theorem 2.1. *Let (U, L) be a bivariate Lévy process such that $\Pi_U(\{-1\}) = 0$. Let $(V_t)_{t\ge 0}$ be given by*

$$V_t = \mathcal{E}(U)_t \left(V_0 + \int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1} \mathrm{d}\eta_s\right), \quad t \ge 0, \tag{2.3}$$

where the stochastic integral in (2.3) is with respect to $\mathbb{F}$.

(a) *Suppose that* $\lim_{t\to\infty} \mathcal{E}(U)_t = 0$ *a.s. Then a finite random variable V_0 can be chosen such that $(V_t)_{t\ge 0}$ is strictly stationary if and only if $\int_{(0,\infty)} \mathcal{E}(U)_{s-}\mathrm{d}L_s$ converges almost surely. If this* *condition is satisfied, then the strictly stationary solution is unique in distribution when viewed as a random element in* $D[0,\infty)$, *and it is obtained by choosing* V_0 *to be independent of* (U, L) *and to have the same distribution as* $\int_{(0,\infty)} \mathcal{E}(U)_{s-} \mathrm{d}L_s$.

(b) *Suppose that* $\lim_{t\to\infty}[\mathcal{E}(U)_t]^{-1} = 0$ *a.s. Then a finite random variable* V_0 *can be chosen such that* $(V_t)_{t\geq 0}$ *is strictly stationary if and only if* $\int_{(0,\infty)}[\mathcal{E}(U)_{s-}]^{-1}\mathrm{d}\eta_s$ *converges a.s. In this case the stationary solution is unique and given by* $V_t = -\mathcal{E}(U)_t \int_{(t,\infty)}[\mathcal{E}(U)_{s-}]^{-1}\mathrm{d}\eta_s$ *a.s.,* $t \geq 0$.

(c) *Suppose that* $\mathcal{E}(U)_t$ *oscillates in the sense that*

$$0 = \liminf_{t\to\infty} |\mathcal{E}(U)_t| < \limsup_{t\to\infty} |\mathcal{E}(U)_t| = +\infty \quad a.s.$$

Then V_t *admits a strictly stationary solution if and only if there exists* $k \in \mathbb{R}\setminus\{0\}$ *such that* $U = -L/k$. *In this case the strictly stationary solution is indistinguishable from the constant process* $t \mapsto k$.

The possibilities for the asymptotic behaviour of $\mathcal{E}(U)_t$ in (a), (b) and (c) of Theorem 2.1 are mutually exclusive and exhaustive; see Theorem 3.5 in Section 3. Conditions for the almost sure convergence of the integrals $\int_{(0,\infty)} \mathcal{E}(U)_{s-}\,\mathrm{d}L_s$ and $\int_{(0,\infty)}[\mathcal{E}(U)_{s-}]^{-1}\,\mathrm{d}\eta_s$ are given in Theorem 3.6 and Corollary 3.7, respectively. Observe that the solutions obtained in Theorem 2.1(a), (c) are equal in distribution to a causal solution, while the solution in part (b) is purely non-causal.

The case when $\Pi_U(\{-1\}) > 0$ is treated in the next theorem. Again, the solutions turn out to be equal in distribution to a causal solution. We will need some other auxiliary processes:

$$\widetilde{U}_t = U_t - \sum_{\substack{0<s\leq t\\ \Delta U_s=-1}} \Delta U_s \quad \text{and} \quad \widetilde{\eta}_t = \eta_t - \sum_{\substack{0<s\leq t\\ \Delta U_s=-1}} \Delta\eta_s, \quad t\geq 0, \tag{2.4}$$

and

$$K(t) := \text{number of jumps of size } -1 \text{ of } U \text{ in } [0,t], \tag{2.5}$$

$$T(t) := \sup\{s \leq t : \Delta U_s = -1\}, \tag{2.6}$$

all for $t \geq 0$. It is easy to see that (U, L, η, K) is a Lévy process. Also, for $0 \leq s < t$ define

$$\mathcal{E}(U)_{(s,t]} := \mathrm{e}^{(U_t-U_s)-\sigma_U^2(t-s)/2} \prod_{s<u\leq t} (1+\Delta U_u)\mathrm{e}^{-\Delta U_u},$$

$$\mathcal{E}(U)_{(s,t)} := \mathrm{e}^{(U_{t-}-U_s)-\sigma_U^2(t-s)/2} \prod_{s<u<t} (1+\Delta U_u)\mathrm{e}^{-\Delta U_u},$$

while for $s \geq t$ let $\mathcal{E}(U)_{(s,t]} := 1$, and define similar quantifies for $\mathcal{E}(\widetilde{U})$. Recall again that (2.7) gives the general solution of (1.2) as will be seen in Proposition 3.2.

Theorem 2.2. *Let* (U, L) *be a bivariate Lévy process such that* $\Pi_U(\{-1\}) > 0$. *Let* η *and* K *be as defined in* (2.2) *and* (2.5), *respectively, and let* $(V_t)_{t\geq 0}$ *be given by*

$$\begin{aligned} V_t &= \mathcal{E}(U)_t \left(V_0 + \int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1}\mathrm{d}\eta_s\right) \mathbb{1}_{\{K(t)=0\}} \\ &\quad + \mathcal{E}(U)_{(T(t),t]} \left(\Delta L_{T(t)} + \int_{(T(t),t]} [\mathcal{E}(U)_{(T(t),s)}]^{-1}\mathrm{d}\eta_s\right) \mathbb{1}_{\{K(t)\geq 1\}}, \quad t\geq 0, \end{aligned} \tag{2.7}$$

where the stochastic integrals in (2.7) *are with respect to* $\mathbb{F}$. *Then the following hold:*

(a) *A finite random variable* V_0 *can be chosen such that* $(V_t)_{t\geq 0}$ *is strictly stationary. More precisely, with* $\widetilde{U}$ *and* $\widetilde{\eta}$ *as defined in* (2.4), *define*

$$Z_t = \mathcal{E}(\widetilde{U})_t \left(Y + \int_{(0,t]} [\mathcal{E}(\widetilde{U})_{s-}]^{-1} \mathrm{d}\widetilde{\eta}_s \right), \quad t \geq 0, \tag{2.8}$$

where Y *is a random variable, independent of* (U, L), *with distribution*

$$P_Y(\mathrm{d}y) = \frac{\Pi_{U,L}(\{-1\}, \mathrm{d}y)}{\Pi_U(\{-1\})},$$

i.e., Y *has the same distribution as* ΔL_{T_1}, *where* T_1 *denotes the time of the first jump of* U *of size* -1. *Let* τ *be an exponentially distributed random variable with parameter* $\lambda := \Pi_U(\{-1\})$, *independent of* (U, L) *and* Y. *Then if* V_0 *is chosen to be independent of* (U, L) *and to have the same distribution as* Z_τ, *the process* $(V_t)_{t\geq 0}$ *is strictly stationary.*

(b) *Any two strictly stationary solutions* $(V_t)_{t\geq 0}$ *are equal in distribution when viewed as random elements of* $D[0, \infty)$, *having the same distribution as the process specified in* (a).

The necessary and sufficient conditions for strictly stationary solutions of (1.2) in the specific cases can be summarised as follows.

Corollary 2.3. *Let* (U, L) *be a bivariate Lévy process, and let* $(\eta_t)_{t\geq 0}$ *and* $V = (V_t)_{t\geq 0}$ *be defined by* (2.2) *and* (2.7). *Then a finite random variable* V_0 *can be chosen such that* V *is strictly stationary if and only if one of the conditions* (i), (ii) *or* (iii) *below holds:*

(i) *There is a constant* $k \neq 0$ *such that* $U = -L/k$.
(ii) *The integral* $\int_0^t \mathcal{E}(U)_{s-} \, \mathrm{d}L_s$ *converges almost surely to a finite random variable as* $t \to \infty$.
(iii) $\Pi_U(\{-1\}) = 0$ *and the integral* $\int_0^t [\mathcal{E}(U)_{s-}]^{-1} \, \mathrm{d}\eta_s$ *converges almost surely to a finite random variable as* $t \to \infty$.

If one of the conditions (i) *to* (iii) *is satisfied, then the distributions of* V_0 *and of the corresponding strictly stationary process* V *are unique.*

A natural question is that of how close the stationary solution of Theorem 2.2 is to the stationary solution of Theorem 2.1(a) if $\Pi_U(\{-1\})$ is small. The following shows that the stationary marginal distribution of Theorem 2.1 can be obtained as a limit of stationary marginal distributions with $\Pi_U(\{-1\}) > 0$ under certain conditions, and more generally that the corresponding stationary processes converge weakly in the J_1-Skorokhod topology. Recall that this is the unique topology on $D[0, \infty)$ making it a Polish space and such that a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $D[0, \infty)$ converges to $\alpha \in D[0, \infty)$ if and only if there is a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of continuous bijections on $[0, \infty)$ with $\gamma_n(0) = 0$ such that

$$\lim_{n\to\infty} \sup_{s\geq 0} |\gamma_n(s) - s| = 0 \quad \text{and} \quad \lim_{n\to\infty} \sup_{0\leq s\leq N} |\alpha_n(\gamma_n(s)) - \alpha(s)| = 0 \quad \text{for all } N \in \mathbb{N};$$

see e.g. [8], Section VI.1.

Corollary 2.4. *Let* (U, L) *be a bivariate Lévy process with* $\Pi_U(\{-1\}) = 0$ *and such that* $\int_{(0,\infty)} \mathcal{E}(U)_{s-} \, \mathrm{d}L_s$ *converges almost surely. Let* $V = (V_t)_{t\geq 0}$ *be the strictly stationary solution of* (2.3) *specified in* Theorem 2.1(a). *Let* $(\overline{U}^{(n)}, \overline{L}^{(n)})$ *be a sequence of bivariate compound Poisson processes, independent of* (U, L), *with Lévy measure* $\lambda_n\sigma$, *where* σ *is a probability distribution on* $\{-1\} \times \mathbb{R}$ *and* $\lambda_n > 0$ *for each* $n \in \mathbb{N}$ *with* $\lambda_n \to 0$ *as* $n \to \infty$. *Let*

$(U^{(n)}, L^{(n)}) := (U + \overline{U}^{(n)}, L + \overline{L}^{(n)})$*, and let* $V^{(n)} = (V_t^{(n)})_{t\geq 0}$ *be the strictly stationary solution of the process associated with* $(U^{(n)}, L^{(n)})$ *as specified in Theorem 2.2(a). Then* $V^{(n)}$ *converges weakly to* V *as* $n \to \infty$ *when viewed as random elements in* $D[0, \infty)$ *endowed with the* J_1*-Skorokhod topology.*

3. Preliminary results

Throughout the paper, "$\xrightarrow{P}$" and "$\xrightarrow{D}$" will denote convergence in probability and distribution, respectively, while "$\stackrel{D}{=}$" denotes equality in distribution of two random variables.

3.1. Solving the SDE

We begin with the following lemma, which is a generalisation of Proposition 2.3 in [14] and can be proved analogously. As usual, $[\cdot, \cdot]$ denotes the quadratic covariation of two semimartingales, and the integrals and quadratic covariation below are understood with respect to $\mathbb{F}$.

Lemma 3.1. *Let* $(U_t, L_t)_{t\geq 0}$ *be a bivariate Lévy process with* $\Pi_U(\{-1\}) = 0$ *and* $(\eta_t)_{t\geq 0}$ *defined by (2.2). Then for every* $t \geq 0$*, we have*

$$\int_{(0,t]} \mathcal{E}(U)_{s-} \mathrm{d}L_s = \int_{(0,t]} \mathcal{E}(U)_{s-} \mathrm{d}\eta_s + [\mathcal{E}(U), \eta]_t \tag{3.1}$$

and

$$\begin{pmatrix} \mathcal{E}(U)_t \\ \mathcal{E}(U)_t \int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1} \mathrm{d}\eta_s \end{pmatrix} \stackrel{D}{=} \begin{pmatrix} \mathcal{E}(U)_t \\ \int_{(0,t]} \mathcal{E}(U)_{s-} \mathrm{d}L_s \end{pmatrix}. \tag{3.2}$$

We can now verify that (2.3) and (2.7) solve the stochastic differential equation (1.2). For the case when both U and L remain semimartingales for $\mathbb{H}$ in the following proposition, the result can be found in Exercise V.27 of Protter [17], who refers to an unpublished note by Yoeurp and Yor. For the case when additionally $\Pi_U((-\infty, -1]) = 0$, see also Eq. (15) of [16]. Given that U and L are semimartingales and $\Pi_U(\{-1\}) = 0$ the result is also given in [9, Theorem 1]. Since the result is of fundamental importance for this paper, we shall give a short sketch of its proof for the case when both U and L remain semimartingales and then extend it to the case when only U remains a semimartingale.

Proposition 3.2. *Let* V_0 *be a finite random variable and let* $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ *be the smallest filtration satisfying the usual hypotheses which contains* $\mathbb{F}$ *and is such that* V_0 *is* $\mathcal{H}_0$ *measurable. Let* η, K, T *be as defined in (2.2), (2.5) and (2.6), respectively. Assume that* U *remains a semimartingale with respect to* $\mathbb{H}$*. Then the unique adapted càdlàg solution to (1.2), or, equivalently, to the integral equation*

$$V_t = V_0 + L_t + \int_{(0,t]} V_{s-} \mathrm{d}U_s, \quad t \geq 0, \tag{3.3}$$

is given by (2.7). If $\Pi_U(\{-1\}) = 0$*, then the unique solution is given by (2.3).*

Proof. By Theorem V.7 in [17], (3.3) has a unique $\mathbb{H}$-adapted càdlàg solution, so it only remains to show that the process given by (2.7) satisfies (3.3). For that, suppose first that V_0 is $\mathcal{F}_0$-

measurable, so $\mathbb{H} = \mathbb{F}$, in which case the result is known from Exercise V.27 in [17], but again it is useful to give a short sketch: since the solution of (3.3) clearly satisfies $V_t = \Delta L_t$ if $\Delta U_t = -1$, the equation renews itself with starting value ΔL_t whenever a jump in K occurs at time t, so by (2.7) it suffices to consider the case $\Pi_U(\{-1\}) = 0$; thus, $K(t) = 0$. Then writing $A_t = \mathcal{E}(U)_t$ and $B_t = V_0 + \int_{(0,t]} \mathcal{E}(U)_{s-}^{-1}\,\mathrm{d}\eta_s$, the process V given by (2.3) satisfies $V_t = A_t B_t$ and A, B, V are semimartingales with respect to $\mathbb{F}$. Partial integration then gives

$$\begin{aligned} V_t - V_0 &= \int_{(0,t]} A_{s-}\mathrm{d}B_s + \int_{(0,t]} B_{s-}\mathrm{d}A_s + [A, B]_t \\ &= \int_{(0,t]} \mathrm{d}\eta_s + \int_{(0,t]} B_{s-}\mathrm{d}(\mathcal{E}(U)_s) + \int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1}\mathrm{d}([\mathcal{E}(U), \eta]_s) \\ &= \int_{(0,t]} \mathrm{d}L_s + \int_{(0,t]} V_{s-}\mathrm{d}U_s, \end{aligned}$$

where we have used the facts that $\mathrm{d}\mathcal{E}(U)_t = \mathcal{E}(U)_{t-}\,\mathrm{d}U_t$ and $\mathrm{d}[\mathcal{E}(U), \eta]_t = \mathcal{E}(U)_{t-}\mathrm{d}(L_t - \eta_t)$ (the latter follows from (3.1)). Thus (3.3) holds.

Now suppose that V_0 is not necessarily $\mathcal{F}_0$-measurable and that U remains a semimartingale with respect to $\mathbb{H}$. Let V_t be the unique $\mathbb{H}$-adapted càdlàg solution of (3.3) and define a process V' by

$$V'_t := V_t - V_0\mathcal{E}(U)_t\mathbb{1}_{\{K(t)=0\}} = V_t - V_0\mathcal{E}(U)_t, \quad t \ge 0. \tag{3.4}$$

Substituting for V_t in (3.3) gives

$$\begin{aligned} V'_t &= V_0 + L_t + \int_{(0,t]} V'_{s-}\,\mathrm{d}U_s + \int_{(0,t]} V_0\mathcal{E}(U)_{s-}\,\mathrm{d}U_s - V_0\mathcal{E}(U)_t \\ &= L_t + \int_{(0,t]} V'_{s-}\,\mathrm{d}U_s + V_0\left(1 + \int_{(0,t]} \mathcal{E}(U)_{s-}\,\mathrm{d}U_s - \mathcal{E}(U)_t\right) \\ &= L_t + \int_{(0,t]} V'_{s-}\,\mathrm{d}U_s. \end{aligned}$$

Since $V'_0 = 0$ is $\mathcal{F}_0$-measurable it follows from the part already proved that V'_t is of the form (2.7) with $V'_0 = 0$, and (3.4) then shows that V_t satisfies (3.3). □

As already pointed out in Section 1, when seeking stationary solutions of the SDE (1.2), in Theorems 2.1 and 2.2 we more conveniently look for stationary solutions of Eq. (2.7), since no semimartingale problems with respect to $\mathbb{H}$ arise in (2.7), the integrals being defined in terms of $\mathbb{F}$ there. The arising semimartingale problem for the SDE (1.2) for non-causal solutions as in Theorem 2.1(b) is taken up again in Section 5. In the case that V_0 is chosen independent of (U, L), as in Theorem 2.1(a), (c) and Theorem 2.2, there are no problems with the filtration, since then, further, U, L and η all remain semimartingales for $\mathbb{H}$ by Corollary 1 to Theorem VI.11 in [17]. In that case, $(V_t)_{t\ge0}$ is also a time homogeneous Markov process and we give its transition functions in the following lemma. Recall $\widetilde{U}$ and $\widetilde{\eta}$ defined in (2.4).

Lemma 3.3. *Let $(V_t)_{t\ge0}$ be as defined in (2.7) and suppose that V_0 is independent of $(U_t, L_t)_{t\ge0}$. Then $(V_t)_{t\ge0}$ is a time homogeneous Markov process. More precisely, defining*

$$A_{s,t} := \mathcal{E}(\widetilde{U})_{(s,t]}\mathbb{1}_{\{K(t)=K(s)\}} \quad \text{and} \quad B_{s,t} := \mathcal{E}(\widetilde{U})_{(s,t]}\int_{(s,t]} [\mathcal{E}(\widetilde{U})_{(s,u)}]^{-1}\mathrm{d}\widetilde{\eta}_u \tag{3.5}$$

for $0 \le s < t$, *with* $\widetilde{U}$ *and* $\widetilde{\eta}$ *given by* (2.4), *we have*

$$V_t = A_{s,t}V_s + B_{s,t}\mathbb{1}_{\{K(t)-K(s)=0\}} + [A_{T(t),t}\Delta L_{T(t)} + B_{T(t),t}]\mathbb{1}_{\{K(t)-K(s)>0\}}, \tag{3.6}$$

with $(A_{s,t}, B_{s,t}, K(t)-K(s))_{t\ge s}$ *being independent of* $\mathcal{H}_s$ *and*

$$(A_{s,t}, B_{s,t}, K(t)-K(s)) \stackrel{D}{=} (A_{s+h,t+h}, B_{s+h,t+h}, K(t+h)-K(s+h)) \tag{3.7}$$

for every $h \ge 0$ *and* $t \ge s$. *Here,* $\mathcal{H}_s$ *is as defined in Proposition* 3.2.

Proof. These are direct consequences of (2.7) and the strong Markov property of Lévy processes, respectively. □

It should be noted that Eq. (3.6) also holds with $A_{s,t}$, $B_{s,t}\mathbb{1}_{\{K(t)-K(s)=0\}}$, $A_{T(t),t}$ and $B_{T(t),t}\mathbb{1}_{\{K(t)-K(s)>0\}}$ being replaced by the corresponding quantities using (U,η) in the definition of (3.5) rather than $(\widetilde{U},\widetilde{\eta})$, but the advantage of the definition using $(\widetilde{U},\widetilde{\eta})$ in (3.5) is that $B_{s,t}$ can be defined for any $s \le t$ and hence allows a statement like (3.7).

3.2. *Other auxiliary processes and their properties*

In the case that $\Pi_U(\{-1\}) = 0$ it is helpful to introduce the processes $N = (N_t)_{t\ge0}$, $\widehat{U} = (\widehat{U}_t)_{t\ge0}$ and $W = (W_t)_{t\ge0}$ defined by

$$N_t := \text{number of jumps of size } < -1 \text{ of } U \text{ in } [0,t], \tag{3.8}$$

$$\widehat{U}_t := -U_t + \sigma_U^2 t/2 + \sum_{0<s\le t}[\Delta U_s - \log|1+\Delta U_s|], \tag{3.9}$$

$$W_t := -U_t + \sigma_U^2 t + \sum_{0<s\le t}\frac{(\Delta U_s)^2}{1+\Delta U_s}. \tag{3.10}$$

Then $(U, L, \eta, N, \widehat{U}, W)$ is a Lévy process. We are interested in the characteristic triplets of $\widehat{U}$ and W and their expectations when they exist, which appear in Theorem 3.5 and Corollary 3.7, respectively.

Lemma 3.4. *Let* U *have characteristic triplet* $(\sigma_U^2, \Pi_U, \gamma_U)$ *and suppose that* $\Pi_U(\{-1\}) = 0$. *Let* N, $\widehat{U}$ *and* W *be as defined in* (3.8)–(3.10). *Then we have:*

(a) *The process* $\widehat{U}$ *is a Lévy process satisfying*

$$\mathcal{E}(U)_t = (-1)^{N_t}\mathrm{e}^{-\widehat{U}_t}, \quad t \ge 0, \tag{3.11}$$

and the characteristic triplet $(\sigma_{\widehat{U}}^2, \Pi_{\widehat{U}}, \gamma_{\widehat{U}})$ *of* $\widehat{U}$ *has* $\sigma_{\widehat{U}}^2 = \sigma_U^2$, $(\Pi_{\widehat{U}})_{|\mathbb{R}\setminus\{0\}} = X(\Pi_U)_{|\mathbb{R}\setminus\{0\}}$ *and*

$$\gamma_{\widehat{U}} = -\gamma_U + \sigma_U^2/2 + \int_{\mathbb{R}} (x\mathbb{1}_{\{|x|\le1\}} - (\log|1+x|)\mathbb{1}_{\{x\in[-e-1,-1-\mathrm{e}^{-1}]\cup[\mathrm{e}^{-1}-1,e-1]\}})\Pi_U(\mathrm{d}x),$$

where $X(\Pi_U)$ *is the image measure of* Π_U *under the transformation*

$$X : \mathbb{R}\setminus\{-1\} \to \mathbb{R}, \quad x \mapsto X(x) = -\log|1+x|. \tag{3.12}$$

We have $E|\widehat{U}_1| < \infty$ *if and only if*

$$\int_{|x|\geq e} \log|x|\, \Pi_U(\mathrm{d}x) < \infty \quad \text{and} \quad \int_{(-3/2,-1/2)} |\log|1+x||\, \Pi_U(\mathrm{d}x) < \infty, \tag{3.13}$$

in which case

$$E\widehat{U}_1 = -\gamma_U + \sigma_U^2/2 + \int_{\mathbb{R}} (x\mathbb{1}_{\{|x|\leq 1\}} - \log|1+x|)\Pi_U(\mathrm{d}x). \tag{3.14}$$

(b) *The process* W *is a Lévy process satisfying*

$$[\mathcal{E}(U)_t]^{-1} = \mathcal{E}(W)_t, \quad t \geq 0, \tag{3.15}$$

and its characteristic triplet $(\sigma_W^2, \Pi_W, \gamma_W)$ *is given by* $\sigma_W^2 = \sigma_U^2$, $\Pi_W = Y(\Pi_U)$ *for the transformation*

$$Y : \mathbb{R}\setminus\{-1\} \to \mathbb{R}\setminus\{-1\}, \quad x \mapsto Y(x) = \frac{-x}{1+x},$$

and

$$\gamma_W = -\gamma_U + \sigma_U^2 + \int_{\mathbb{R}} \left(x\mathbb{1}_{\{|x|\leq 1\}} - \frac{x}{1+x}\mathbb{1}_{\{x\geq -1/2\}} \right) \Pi_U(\mathrm{d}x).$$

We have $E|W_1| < \infty$ *if and only if*

$$\int_{(-3/2,-1/2)} |1+x|^{-1}\, \Pi_U(\mathrm{d}x) < \infty, \tag{3.16}$$

in which case

$$EW_1 = -\gamma_U + \sigma_U^2 + \int_{[-1,1]} \frac{x^2}{1+x}\Pi_U(\mathrm{d}x) - \int_{|x|>1} \frac{x}{1+x}\Pi_U(\mathrm{d}x). \tag{3.17}$$

Proof. (a) Eq. (3.11) is immediate from (2.1), (3.8) and (3.9). From (3.9) we obtain

$$\Delta\widehat{U}_t = -\log|1+\Delta U_t|, \quad t \geq 0,$$

which implies $(\Pi_{\widehat{U}})_{|\mathbb{R}\setminus\{0\}} = X(\Pi_U)_{|\mathbb{R}\setminus\{0\}}$. The Brownian motion components of $\widehat{U}$ and U satisfy $B_{\widehat{U}_t} = -B_{U_t}$, so $\sigma_{\widehat{U}}^2 = \sigma_U^2$. For the calculation of $\gamma_{\widehat{U}}$, take $\varepsilon > 0$ and let $C_\varepsilon := [-1-\mathrm{e}^{\varepsilon}, -1-\mathrm{e}^{-\varepsilon}]$ and $D_\varepsilon := [-1+\mathrm{e}^{-\varepsilon}, -1+\mathrm{e}^{\varepsilon}]$. Omitting the summation index $0 < s \leq 1$ in the following calculations, it then follows from the Lévy–Itô decomposition ([18], Theorem 19.2) of $\widehat{U}$ that

$$\gamma_{\widehat{U}} + B_{\widehat{U}_1} = \widehat{U}_1 - \lim_{\varepsilon\downarrow 0}\left(\sum_{|\Delta\widehat{U}_s|>\varepsilon} \Delta\widehat{U}_s - \int_{|x|\in(\varepsilon,1]} x\, \Pi_{\widehat{U}}(\mathrm{d}x) \right).$$

By (3.9), the latter is equal to

$$-U_1 + \sigma_U^2/2 + \lim_{\varepsilon\downarrow 0}\left(\sum_{\Delta U_s\in\mathbb{R}} (\Delta U_s - \log|1+\Delta U_s|) - \sum_{|\Delta\widehat{U}_s|>\varepsilon} \Delta\widehat{U}_s + \int_{|x|\in(\varepsilon,1]} x\, \Pi_{\widehat{U}}(\mathrm{d}x) \right).$$

Now, because $\Delta\widehat{U}_s = -\log|1+\Delta U_s|$, we have $|\Delta\widehat{U}_s| \le \varepsilon$ if and only if $\Delta U_s \in C_\varepsilon \cup D_\varepsilon$. Thus

$$\gamma_{\widehat{U}} + B_{\widehat{U}_1} = -U_1 + \sigma_U^2/2 + \lim_{\varepsilon\downarrow 0}\left(\sum_{\Delta U_s \in C_\varepsilon \cup D_\varepsilon} (\Delta U_s - \log|1+\Delta U_s|) + \sum_{\Delta U_s \notin C_\varepsilon \cup D_\varepsilon} \Delta U_s + \int_{|x|\in(\varepsilon,1]} x\, \Pi_{\widehat{U}}(\mathrm{d}x)\right).$$

Since C_ε and D_ε shrink to the points -2 and 0 as $\varepsilon \downarrow 0$, since $\Delta U_s - \log|1+\Delta U_s| = O(\Delta U_s)^2$ for ΔU_s near 0 and $\lim_{\varepsilon\downarrow 0}\sum_{\Delta U_s\in D_\varepsilon}(\Delta U_s)^2 = 0$, this leaves

$$\gamma_{\widehat{U}} + B_{\widehat{U}_1} = -U_1 + \sigma_U^2/2 + \lim_{\varepsilon\downarrow 0}\left(\sum_{\Delta U_s \notin D_\varepsilon} \Delta U_s + \int_{|x|\in(\varepsilon,1]} x\, \Pi_{\widehat{U}}(\mathrm{d}x)\right).$$

Using the Lévy–Itô decomposition ([18], Theorem 19.2) again, but now for the process U, we obtain

$$\gamma_{\widehat{U}} + B_{\widehat{U}_1} = \sigma_U^2/2 - \gamma_U - B_{U_1} + \lim_{\varepsilon\downarrow 0}\left(\int_{|x|\in[-1,1]\setminus D_\varepsilon} x\, \Pi_U(\mathrm{d}x) + \int_{|x|\in(\varepsilon,1]} x\, \Pi_{\widehat{U}}(\mathrm{d}x)\right).$$

Together with $B_{\widehat{U}_1} = -B_{U_1}$ and $\Pi_{\widehat{U}} = X(\Pi_U)$ this implies the given representation for $\gamma_{\widehat{U}}$.

Next, observe that $E|\widehat{U}_1| < \infty$ if and only if $\int_{|x|>1}|x|\,\Pi_{\widehat{U}}(\mathrm{d}x) < \infty$ ([18], Example 25.12), which is equivalent to (3.13) since $\Pi_{\widehat{U}} = X(\Pi_U)$ on $\mathbb{R}\setminus\{0\}$. Eq. (3.14) then follows from the representation of $\gamma_{\widehat{U}}$ and the fact that $E\widehat{U}_1 = \gamma_{\widehat{U}} + \int_{|x|>1} x\, \Pi_{\widehat{U}}(\mathrm{d}x)$.

(b) Eq. (3.15) was obtained by Léandre [13] and detailed proofs can be found in [10, Theorem 1] or [9, Lemma A.1].

The remaining assertions follow similarly to the ones proved in (a). □

Similarly, it can be shown that the Lévy measure of η as defined in (2.2) is the restriction to $\mathbb{R}\setminus\{0\}$ of the image measure of $\Pi_{U,L}$ under the mapping $(\mathbb{R}\setminus\{-1\})\times\mathbb{R}\to\mathbb{R}$, $(x,y)\mapsto \frac{y}{1+x}$, and moment conditions for η can be expressed in terms of the characteristic triplet of (U, L). We omit further details here.

3.3. *Convergence of $\mathcal{E}(U)_t$ and integrals involving it*

In the case $\Pi_U(\{-1\}) = 0$ the characterisation of the existence of stationary solutions in Section 4 will be achieved in terms of the almost sure convergence of $\int_0^\infty \mathcal{E}(U)_{s-}\,\mathrm{d}L_s$ and $\int_0^\infty [\mathcal{E}(U)_{s-}]^{-1}\,\mathrm{d}\eta_s$. So, finally in this section, we obtain necessary and sufficient conditions for convergence of these integrals, which are also interesting in their own right.

We need also necessary and sufficient conditions for a Lévy process to drift to $\pm\infty$ in terms of its characteristic triplet. The following is a reformulation of a result of Doney and Maller (see Theorem 4.4 in [3]) for the process $\widehat{U}$ in terms of the characteristics of U. In the case when $E|\widehat{U}_1| = \infty$, it describes in particular how the large time behaviour of $\widehat{U}$ is determined by the behaviour of Π_U around -1 and for large values.

Theorem 3.5. *Let U be a non-zero Lévy process with $\Pi_U(\{-1\}) = 0$, let $\widehat{U}$ be defined by (3.9), and recall (3.11).*

(a) *The following are equivalent:*
(i) $\mathcal{E}(U)_t$ *converges almost surely to* 0 *as* $t \to \infty$.
(ii) $\widehat{U}_t$ *converges almost surely to* ∞ *as* $t \to \infty$.
(iii) $0 < E\widehat{U}_1 \le E|\widehat{U}_1| < \infty$, *or* $\int_{(-3/2,-1/2)} |\log|1+x||\, \Pi_U(\mathrm{d}x) = \infty$ *and*

$$\int_{\mathbb{R}\setminus[-e,e]} \frac{\log|x|\, \Pi_U(\mathrm{d}x)}{1 + \int_{1/|x|}^{1/e} \Pi_U((-1-z,-1+z))z^{-1}\mathrm{d}z} < \infty.$$

(b) *The following are equivalent:*
(i) $[\mathcal{E}(U)_t]^{-1}$ *converges almost surely to* 0 *as* $t \to \infty$.
(ii) $\widehat{U}_t$ *converges almost surely to* $-\infty$ *as* $t \to \infty$.
(iii) $0 < -E\widehat{U}_1 \le E|\widehat{U}_1| < \infty$, *or* $\int_{|x|\ge e} \log|x|\, \Pi_U(\mathrm{d}x) = \infty$ *and*

$$\int_{(-1-e^{-1},-1+e^{-1})} \frac{-\log|1+x|\, \Pi_U(\mathrm{d}x)}{1 + \int_e^{1/|1+x|} \Pi_U(\mathbb{R}\setminus[1-z,z-1])z^{-1}\mathrm{d}z} < \infty.$$

(c) *If none of the conditions in* (a) *or* (b) *are satisfied, then* $\widehat{U}$ *oscillates, equivalently,*

$$0 = \liminf_{t\to\infty} |\mathcal{E}(U)_t| < \limsup_{t\to\infty} |\mathcal{E}(U)_t| = +\infty.$$

Proof of Theorem 3.5. Let us prove (a). The equivalence of (i) and (ii) is clear from (3.11). Further, by Theorem 4.4 in [3], $\widehat{U}_t$ converges almost surely to ∞ if and only if $0 < E\widehat{U}_1 \le E|\widehat{U}_1| < \infty$, or

$$\lim_{x\to\infty} A^+_{\widehat{U}}(x) = \infty \quad \text{and} \quad \int_{-\infty}^{-1} \frac{|x|\, \Pi_{\widehat{U}}(\mathrm{d}x)}{A^+_{\widehat{U}}(|x|)}\, \mathrm{d}x < \infty,$$

where

$$A^+_{\widehat{U}}(x) := 1 + \int_1^x \Pi_{\widehat{U}}((y,\infty))\, \mathrm{d}y, \quad x \ge 1.$$

Using $\Pi_{\widehat{U}} = X(\Pi_U)$ (cf. (3.12)), it is then easy to see that this is equivalent to the condition (iii). The proof of (b) is similar, and assertion (c) is well known (e.g. [18], Theorem 48.1). □

The following is a version for the stochastic exponential of Theorem 2 of [4], who characterised almost sure convergence of the integral $\int_0^\infty e^{-\zeta_{s-}}\mathrm{d}\chi_s$ for a bivariate Lévy process (ζ,χ).

Theorem 3.6. *Let* (U,L) *be a bivariate Lévy process such that* $\Pi_U(\{-1\}) = 0$. *Then the following are equivalent:*

(i) $\int_0^t \mathcal{E}(U)_{s-}\mathrm{d}L_s$ *converges almost surely to a finite random variable as* $t \to \infty$.
(ii) $\int_0^t \mathcal{E}(U)_{s-}\mathrm{d}L_s$ *converges in distribution to a finite random variable as* $t \to \infty$.
(iii) $\mathcal{E}(U)_t$ *converges almost surely to* 0 *as* $t \to \infty$ *and*

$$I_{U,L} := \int_{\mathbb{R}\setminus[-e,e]} \frac{\log|y|\, \Pi_L(\mathrm{d}y)}{1 + \int_{1/|y|}^{1/e} \Pi_U((-1-z,-1+z))z^{-1}\mathrm{d}z} < \infty. \tag{3.18}$$

In the case of divergence, we have: if $\lim_{t\to\infty} \mathcal{E}(U)_t = 0$ *a.s. but* $I_{U,L} = +\infty$, *then*

$$\left| \int_{(0,t]} \mathcal{E}(U)_{s-}\mathrm{d}L_s \right| \xrightarrow{P} \infty, \quad t \to \infty, \tag{3.19}$$

and if $\mathcal{E}(U)_t$ does not tend to 0 a.s. as $t \to \infty$, then (3.19) holds or there exists $k \in \mathbb{R} \setminus \{0\}$ such that

$$P\left(\int_{(0,t]} \mathcal{E}(U)_{s-} \mathrm{d}L_s = k(1 - \mathcal{E}(U)_t) \ \forall t \geq 0\right) = 1. \tag{3.20}$$

Proof of Theorem 3.6. Using $\mathcal{E}(U)_t = (-1)^{N_t} \mathrm{e}^{-\widehat{U}_t}$, it follows in complete analogy to the proof of Erickson and Maller [4] that $\int_0^t (-1)^{N_{s-}} \mathrm{e}^{-\widehat{U}_{s-}} \, \mathrm{d}L_s$ converges almost surely to a finite random variable if and only if $\widehat{U}_t$ converges almost surely to $+\infty$ as $t \to \infty$ and

$$\int_{\mathbb{R}\setminus[-e,e]} \left(\frac{\log |y|}{1 + \int_1^{\log |y|} \Pi_{\widehat{U}}((x, \infty)) \, \mathrm{d}x} \right) \Pi_L(\mathrm{d}y) < \infty,$$

which by Lemma 3.4 can be seen to be equivalent to (iii). The remaining assertions follow like in [4]. □

Corollary 3.7. *Let (U, L) be a bivariate Lévy process such that $\Pi_U(\{-1\}) = 0$. Let (W, η) be defined by (3.10) and (2.2). Then the following are equivalent:*

(i) $\int_0^t [\mathcal{E}(U)_{s-}]^{-1} \mathrm{d}\eta_s$ *converges almost surely to a finite random variable as $t \to \infty$.*
(ii) $\int_0^t [\mathcal{E}(U)_{s-}]^{-1} \mathrm{d}\eta_s$ *converges in distribution to a finite random variable as $t \to \infty$.*
(iii) $[\mathcal{E}(U)_t]^{-1}$ *converges almost surely to 0 as $t \to \infty$ and $I_{W,\eta} < \infty$, where $I_{W,\eta}$ is defined similarly to (3.18), with Π_L being replaced by Π_η and Π_U by Π_W.*

Proof. This is an immediate consequence of Theorem 3.6 since $[\mathcal{E}(U)_t]^{-1} = \mathcal{E}(W)_t$ for every $t \geq 0$ by (3.15). □

4. Proofs of main results

Proof of Theorem 2.1. (a) Suppose that $\widehat{U}_t \to \infty$ a.s. as $t \to \infty$. Then $\mathcal{E}(U)_t V_0$ converges a.s. to 0. Thus if a stationary solution $(V_t)_{t\geq 0}$ exists, $\mathcal{E}(U)_t \int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1} \mathrm{d}\eta_s$ tends to V_0 in distribution as $t \to \infty$. By (3.2) this means that $\int_{(0,t]} \mathcal{E}(U)_{s-} \mathrm{d}L_s \xrightarrow{D} V_0$ as $t \to \infty$ and hence $\int_0^\infty \mathcal{E}(U)_{s-} \, \mathrm{d}L_s$ converges almost surely by Theorem 3.6. Let $n \in \mathbb{N}$ and $h_1, \ldots, h_n \geq 0$. Since $\lim_{t\to\infty} \mathcal{E}(U)_t = 0$ a.s., and since

$$(V_{h_1}, \ldots, V_{h_n}) \stackrel{D}{=} (V_{t+h_1}, \ldots, V_{t+h_n}), \quad t \geq 0,$$

an application of Slutsky's Lemma shows that $(V_{h_1}, \ldots, V_{h_n})$ has the same distribution as the distributional limit as $t \to \infty$ of

$$\left(\mathcal{E}(U)_{t+h_1} \int_0^{t+h_1} [\mathcal{E}(U)_{s-}]^{-1} \, \mathrm{d}\eta_s, \ldots, \mathcal{E}(U)_{t+h_n} \int_0^{t+h_n} [\mathcal{E}(U)_{s-}]^{-1} \, \mathrm{d}\eta_s \right).$$

This does not depend on V_0. Hence any two stationary solutions have the same finite dimensional distributions and hence the same distributions when viewed as random elements in $D[0, \infty)$.

Conversely, suppose that $\int_0^\infty \mathcal{E}(U)_{s-} \, \mathrm{d}L_s$ converges almost surely to a finite random variable and take V_0 independent of (U, L) and with the same distribution as $\int_0^\infty \mathcal{E}(U)_{s-} \, \mathrm{d}L_s$. Then, by (3.2), V_t converges in distribution to V_0 as $t \to \infty$, since $\lim_{t\to\infty} \mathcal{E}(U)_t = 0$. Together with Lemma 3.3 this shows that

$$V_t = A_{t-h,t} V_{t-h} + B_{t-h,t} \xrightarrow{D} A_{0,h} V_0 + B_{0,h} = V_h, \quad t \to \infty,$$

for every $h \geq 0$. Since also $V_t \xrightarrow{D} V_0$ as $t \to \infty$ it follows that $V_h \stackrel{D}{=} V_0$. Since $(V_t)_{t\geq 0}$ is a Markov process by Lemma 3.3, this implies strict stationarity of $(V_t)_{t\geq 0}$.

(b) Suppose that $\widehat{U}_t \to -\infty$ and hence $[\mathcal{E}(U)_t]^{-1} \to 0$ as $t \to \infty$. Then if $(V_t)_{t\geq 0}$ is a strictly stationary solution, it follows that

$$V_0 + \int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1} \mathrm{d}\eta_s = [\mathcal{E}(U)_t]^{-1} V_t \xrightarrow{P} 0, \quad t \to \infty.$$

Hence $-\int_0^\infty [\mathcal{E}(U)_{s-}]^{-1}\,\mathrm{d}\eta_s$ converges almost surely to V_0 by Corollary 3.7, and this immediately yields $V_t = -\mathcal{E}(U)_t \int_{(t,\infty)} [\mathcal{E}(U)_{s-}]^{-1} \mathrm{d}\eta_s$ a.s.

Conversely, if $\int_{(0,\infty)} [\mathcal{E}(U)_{s-}]^{-1} \mathrm{d}\eta_s$ converges a.s., let $V_0 := -\int_{(0,\infty)} [\mathcal{E}(U)_{s-}]^{-1} \mathrm{d}\eta_s$. Then

$$V_t = -\mathcal{E}(U)_t \int_{(t,\infty)} [\mathcal{E}(U)_{s-}]^{-1} \mathrm{d}\eta_s = \int_{(t,\infty)} (-1)^{(N_{s-}-N_t)} \mathrm{e}^{\widehat{U}_{s-}-\widehat{U}_t}\,\mathrm{d}\eta_s, \quad t \geq 0,$$

which is strictly stationary since $(N, \widehat{U}, \eta)$, as a Lévy process, has stationary increments.

(c) Suppose that $\widehat{U}_t$ oscillates and let $(V_t)_{t\geq 0}$ be a strictly stationary solution of (2.3). By Theorem 3.6 this implies that (3.19) or (3.20) must hold. Suppose first that (3.19) holds. Together with (3.2) this gives $|\mathcal{E}(U)_t \int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1} \mathrm{d}\eta_s| \xrightarrow{P} \infty$ as $t \to \infty$. Since V_t is strictly stationary this and (2.3) imply that $|V_0 \mathcal{E}(U)_t|$ and thus $|\mathcal{E}(U)_t|$ tend to ∞ in probability, too. Hence $V_0 + \int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1} \mathrm{d}\eta_s = [\mathcal{E}(U)_t]^{-1} V_t$ converges to 0 in probability, and hence in distribution, so $\int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1} \mathrm{d}\eta_s \xrightarrow{D} -V_0$ as $t \to \infty$, contradicting Corollary 3.7 because $[\mathcal{E}(U)_t]^{-1}$ does not converge to 0. Hence (3.19) cannot occur.

Now suppose that (3.20) holds, i.e. there is a constant $k \in \mathbb{R} \setminus \{0\}$ such that for all $t > 0$ we have $\int_{(0,t]} \mathcal{E}(U)_{s-} \mathrm{d}L_s = k(1 - \mathcal{E}(U)_t)$ a.s., or equivalently $1 + \int_{(0,t]} \mathcal{E}(U)_{s-} \mathrm{d}(-L_s/k) = \mathcal{E}(U)_t$ a.s. But since the unique adapted càdlàg solution to the stochastic differential equation $1 + \int_{(0,t]} Z_{s-} \mathrm{d}(-L_s/k) = Z_t$ is given by $Z_t = \mathcal{E}(-L/k)_t$, we see that (3.20) is equivalent to $\mathcal{E}(U) = \mathcal{E}(-L/k)$ and hence to $U = -L/k$. From (2.2) and (3.10), this implies

$$\eta_t = -kU_t + \sum_{0<s\leq t} \frac{k\Delta U_s^2}{1+\Delta U_s} + kt\sigma_U^2 = kW_t,$$

so

$$\begin{aligned}\int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1} \mathrm{d}\eta_s &= k \int_{(0,t]} \mathcal{E}(W)_{s-} \mathrm{d}W_s = k(-1 + \mathcal{E}(W)_t) \\ &= (-k)(1 - [\mathcal{E}(U)_t]^{-1}) \quad \text{a.s.}\end{aligned}$$

by (3.15). We conclude that

$$V_t = \mathcal{E}(U)_t \left(\int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1} \mathrm{d}\eta_s + V_0 \right) = \mathcal{E}(U)_t (V_0 - k) + k, \quad t \geq 0, \tag{4.1}$$

so $V_t - k = \mathcal{E}(U)_t (V_0 - k)$ a.s. Since V_t was assumed to be strictly stationary this yields $|V_0 - k| \stackrel{D}{=} |\mathcal{E}(U)_t||V_0 - k| = \mathrm{e}^{-\widehat{U}_t}|V_0 - k|$, because $\mathcal{E}(U)_t = (-1)^{N_t} \mathrm{e}^{-\widehat{U}_t}$. Since $|\widehat{U}_t| \xrightarrow{P} \infty$, we get $V_0 - k = 0$ a.s. and hence $V_t = k$ a.s. for all $t \geq 0$. So V is indistinguishable from the constant process, since it has càdlàg paths.

Conversely, if there is a $k \in \mathbb{R} \setminus \{0\}$ such that $U = -L/k$, and $V_0 := k$, then it follows from (4.1) that $V_t = k$ for all $t \geq 0$, which is a strictly stationary solution. $\square$

Proof of Theorem 2.2. (a) Choose V_0 to be independent of (U, L) with $V_0 \overset{D}{=} Z_\tau$. Then $(V_t)_{t\geq 0}$ is a Markov process by Lemma 3.3; hence it suffices to show that $V_t \overset{d}{=} V_0$ for every $t > 0$. Fix $t > 0$ and for $k \in \mathbb{N}_0$ let $p_k := P(K(t) = k)$ and let T_k be the time of the kth jump of size -1 of U. Then by (2.7) we get, for $x \in \mathbb{R}$,

$$\begin{aligned}
P(V_t \leq x) &= p_0 P\left(\mathcal{E}(U)_t \left(V_0 + \int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1} \mathrm{d}\eta_s\right) \leq x | K(t) = 0\right) \\
&\quad + \sum_{k\geq 1} p_k P\left(\mathcal{E}(U)_{(T_k,t]} \left(\Delta L_{T_k} + \int_{(T_k,t]} [\mathcal{E}(U)_{(T_k,s)}]^{-1} \mathrm{d}\eta_s\right) \leq x | K(t) = k\right) \\
&=: A(x) + B(x), \quad \text{say.}
\end{aligned}$$

By (2.4), $U = \widetilde{U}$ and $\eta = \widetilde{\eta}$ on $\{K(t) = 0\}$. Thus

$$A(x) = p_0 P\left(\mathcal{E}(\widetilde{U})_t \left(V_0 + \int_{(0,t]} [\mathcal{E}(\widetilde{U})_{s-}]^{-1} \mathrm{d}\widetilde{\eta}_s\right) \leq x\right).$$

Since τ and $(\widetilde{U}, \widetilde{\eta})$ are independent, an application of the strong Markov property to the Lévy process $(\widetilde{K}, \widetilde{U}, \widetilde{\eta})$, where $\widetilde{K}$ is a Poisson process with parameter λ, independent of $(\widetilde{U}, \widetilde{\eta})$ and first jump time τ, shows that $(\widetilde{U}_{t+\tau}, \widetilde{\eta}_{t+\tau})_{t\geq 0}$ is a Lévy process with the same distribution as $(\widetilde{U}_t, \widetilde{\eta}_t)_{t\geq 0}$, independent of Z_τ and V_0. Together with $V_0 \overset{d}{=} Z_\tau$ this shows

$$\mathcal{E}(\widetilde{U})_t \left(V_0 + \int_{(0,t]} [\mathcal{E}(\widetilde{U})_{s-}]^{-1} \mathrm{d}\widetilde{\eta}_s\right) \overset{D}{=} \mathcal{E}(\widetilde{U})_{(\tau,t+\tau]} \left(Z_\tau + \int_{(\tau,t+\tau]} [\mathcal{E}(\widetilde{U})_{(\tau,s)}]^{-1} \mathrm{d}\widetilde{\eta}_s\right).$$

Hence we obtain for $A(x)$, recalling that $p_0 = \mathrm{e}^{-\lambda t}$,

$$\begin{aligned}
&p_0 P\left(\mathcal{E}(\widetilde{U})_{(\tau,t+\tau]} \left(\mathcal{E}(\widetilde{U})_\tau \left(Y + \int_{(0,\tau]} [\mathcal{E}(\widetilde{U})_{s-}]^{-1} \mathrm{d}\widetilde{\eta}_s\right)\right.\right. \\
&\quad \left.\left. + \int_{(\tau,t+\tau]} [\mathcal{E}(\widetilde{U})_{(\tau,s)}]^{-1} \mathrm{d}\widetilde{\eta}_s\right) \leq x\right) \\
&= p_0 P\left(\mathcal{E}(\widetilde{U})_{t+\tau} \left(Y + \int_{(0,t+\tau]} [\mathcal{E}(\widetilde{U})_{s-}]^{-1} \mathrm{d}\widetilde{\eta}_s\right) \leq x\right) \\
&= p_0 P(Z_{t+\tau} \leq x) \\
&= \mathrm{e}^{-\lambda t} \int_{(0,\infty)} P(Z_{t+y} \leq x) \, \mathrm{d}P_\tau(y) \\
&= \lambda \int_{(t,\infty)} P(Z_y \leq x) \mathrm{e}^{-\lambda y} \mathrm{d}y.
\end{aligned}$$

For $B(x)$, recall that the times of jumps of size -1 on an interval $[0, t]$ of the Lévy process U given the value of $K(t) = k$ have the same distribution as the order statistics of k uniformly distributed random variables on $[0, t]$. In particular, $P(T_k \leq y | K(t) = k) = (y/t)^k$ for all $0 \leq y \leq t$. Defining a random variable $\upsilon(k)$ with this distribution, independent of (U, L), we conclude, recalling that $p_k = \mathrm{e}^{-\lambda t} (\lambda t)^k / k!$,

$$
\begin{aligned}
B(x) &= \sum_{k\ge 1} p_k P\left(\mathcal{E}(\widetilde{U})_{(T_k,t]}\left(Y + \int_{(T_k,t]} [\mathcal{E}(\widetilde{U})_{(T_k,s)}]^{-1}\mathrm{d}\widetilde{\eta}_s\right) \le x | K(t) = k\right)\\
&= \sum_{k\ge 1} p_k P\left(\mathcal{E}(\widetilde{U})_{t-\upsilon(k)}\left(Y + \int_{(0,t-\upsilon(k)]} [\mathcal{E}(\widetilde{U})_{s-}]^{-1}\mathrm{d}\widetilde{\eta}_s\right) \le x\right)\\
&= \sum_{k\ge 1} p_k P(Z_{t-\upsilon(k)} \le x)\\
&= \sum_{k\ge 1} \frac{\mathrm{e}^{-\lambda t}(\lambda t)^k}{k!} \int_{(0,t]} P(Z_{t-y} \le x)\mathrm{d}(y/t)^k\\
&= \lambda \mathrm{e}^{-\lambda t} \int_{(0,t]} P(Z_{t-y} \le x)\mathrm{e}^{\lambda y}\mathrm{d}y\\
&= \lambda \int_{(0,t]} P(Z_y \le x)\mathrm{e}^{-\lambda y}\mathrm{d}y.
\end{aligned}
$$

Summing $A(x)$ and $B(x)$ we obtain

$$
P(V_t \le x) = \lambda \int_{(0,\infty)} P(Z_y \le x)\mathrm{e}^{-\lambda y}\mathrm{d}y = P(Z_\tau \le x) = P(V_0 \le x),
$$

so $V_t \stackrel{d}{=} V_0$, giving strict stationarity of $(V_t)_{t\ge 0}$.

(b) Let $(V_t)_{t\ge 0}$ be a strictly stationary solution of (2.7). Then for any $n \in \mathbb{N}$ and $h_1, \ldots, h_n \ge 0$ we have

$$
(V_{t+h_1}, \ldots, V_{t+h_n}) \xrightarrow{D} (V_{h_1}, \ldots, V_{h_n}), \quad t \to \infty,
$$

and since $K(t) \to +\infty$ a.s. as $t \to \infty$, it can be seen from (2.7) that the last expression does not depend on V_0. Hence any two strictly stationary solutions have the same finite dimensional distributions and hence are equal as random elements in $D[0, \infty)$. $\square$

Proof of Corollary 2.3. To show sufficiency of each of the conditions (i)–(iii), it is enough to suppose $\Pi_U(\{-1\}) = 0$, since otherwise a strictly stationary solution exists by Theorem 2.2. Then by Theorem 3.6 and Corollary 3.7, convergence of $\int_0^\infty \mathcal{E}(U)_{s-}\,\mathrm{d}L_s$ and that of $\int_0^\infty [\mathcal{E}(U)_{s-}]^{-1}\,\mathrm{d}\eta_s$ imply $\lim_{t\to\infty} \mathcal{E}(U)_t = 0$ a.s. and $\lim_{t\to\infty} [\mathcal{E}(U)_t]^{-1} = 0$, respectively, so Theorem 2.1(a), (b) shows sufficiency of conditions (ii) and (iii). By Theorem 2.1(c), condition (i) is sufficient if $\hat{U}$ oscillates, but its proof shows that (i) is sufficient whenever $\Pi_U(\{-1\}) = 0$, since $U = -L/k$ clearly implies Eq. (4.1) by the same argument. The uniqueness assertion is clear from Theorems 2.1 and 2.2.

To see that the existence of a strictly stationary solution implies at least one of the conditions (i)–(iii), observe that this is clear from Theorem 2.1 if $\Pi_U(\{-1\}) = 0$. In the case that $\Pi_U(\{-1\}) > 0$, denote by T_1 the time of the first jump of U of size -1. Then T_1 is finite almost surely and it is the case that $\mathcal{E}(U)_t = 0$ for $t \ge T_1$. Hence the integral $\int_0^\infty \mathcal{E}(U)_{s-}\,\mathrm{d}L_s$ converges almost surely, which is condition (ii). $\square$

Proof of Corollary 2.4. In the following we denote the quantities corresponding to $(U^{(n)}, L^{(n)})$ as needed in Theorem 2.2(a) by $\widetilde{\eta}^{(n)}$, $T_1^{(n)}$, $\tau^{(n)}$, etc. Observe that $\widetilde{U}^{(n)} = U$ and $\widetilde{\eta}^{(n)} = \eta$. Further observe that convergence of $\int_{(0,\infty)} \mathcal{E}(U)_{s-}\,\mathrm{d}L_s$ implies that $\mathcal{E}(U)_t \to 0$ a.s. as $t \to \infty$ by

Theorem 3.6. But since the distribution of $\Delta L^{(n)}_{T_1^{(n)}}$ is σ for each n, it follows that $\mathcal{E}(U)_{\tau^{(n)}} Y^{(n)} \stackrel{P}{\to} 0$ as $n \to \infty$ since $\lambda_n \to 0$. Next, observe that

$$\mathcal{E}(U)_{\tau^{(n)}} \int_{(0,\tau^{(n)}]} [\mathcal{E}(U)_{s-}]^{-1} \, \mathrm{d}\eta_s \stackrel{D}{=} \int_{(0,\tau^{(n)}]} \mathcal{E}(U)_{s-} \, \mathrm{d}L_s,$$

which follows from (3.2) by conditioning on $\tau^{(n)} = t$ and using that $\tau^{(n)}$ is independent of (U, L). This, together with $\mathcal{E}(U)_{\tau^{(n)}} Y^{(n)} \stackrel{P}{\to} 0$ implies that

$$V_0^{(n)} \stackrel{D}{=} Z^{(n)}_{\tau^{(n)}} \stackrel{D}{\to} \int_{(0,\infty)} \mathcal{E}(U)_{s-} \, \mathrm{d}L_s \stackrel{D}{=} V_0, \quad n \to \infty,$$

so the marginal stationary distributions converge weakly. By Skorokhod's theorem we can then assume that $V_0^{(n)}$ and V_0 are additionally chosen such that $V_0^{(n)} \to V_0$ a.s. as $n \to \infty$, since this does not alter the distributions of the processes $V^{(n)}$ and V, respectively, and we are only concerned with weak convergence. But since $\lambda_n \to 0$ we have $K_t^{(n)} \stackrel{P}{\to} 0$ as $n \to \infty$ for fixed $t \geq 0$, and hence it follows from (2.3) and (2.7), for any $t > 0$ and $\varepsilon > 0$, that

$$\lim_{n\to\infty} P(\sup_{0\le s\le t} |V_s - V_s^{(n)}| > \varepsilon) = 0,$$

giving weak convergence of $V^{(n)}$ to V in the J_1-Skorokhod topology (cf. [8], Lemma VI.3.31, p. 352). □

5. Filtration expansions

Having determined all strictly stationary solutions of (2.7), it is natural to ask whether the strictly stationary process $(V_t)_{t\ge0}$ still satisfies (3.3) for the smallest filtration $\mathbb{H} = (\mathcal{H}_t)_{t\ge0}$ containing $\mathbb{F}$, satisfying the usual hypotheses and which is such that V_0 is $\mathcal{H}_0$-measurable. In other words, we pose the question: does U at least remain a semimartingale with respect to $\mathbb{H}$?

In the causal cases described in Theorem 2.1(a), (c) and Theorem 2.2, this is indeed the case, as a consequence of Jacod's criterion ([6], Théorème (1.1); see also [17], Corollary 1 to Theorem VI.11). For the non-causal cases, this is not at all evident. Clearly, if U is of bounded variation, then U remains an $\mathbb{H}$-semimartingale, but the general case is not clear. The following theorem presents a sufficient condition for all $\mathbb{F}$-semimartingales to remain $\mathbb{H}$-semimartingales. The proof is along the lines of Theorem 3.6 of [7], who considered the case $U_t = \lambda t$ with $\lambda > 0$ below, in which case the distribution of V_0 is either degenerate, or absolutely continuous.

Theorem 5.1. *Let (U, L) be a bivariate Lévy process such that $\Pi_U(\{-1\}) = 0$ and suppose that $\lim_{t\to\infty}[\mathcal{E}(U)_t]^{-1} = 0$ a.s. and that $V_0 := -\int_{(0,\infty)}[\mathcal{E}(U)_{s-}]^{-1} \, \mathrm{d}\eta_s$ converges a.s., where η is defined by (2.2). Denote by $V_t = -\int_{(t,\infty)}[\mathcal{E}(U)_{(t,s)}]^{-1} \, \mathrm{d}\eta_s$, $t \geq 0$, as in Theorem 2.1(b), the unique solution of (2.3), and suppose that the distribution of V_0 is absolutely continuous or a Dirac measure. Then every $\mathbb{F}$-semimartingale is also an $\mathbb{H}$-semimartingale. In particular, U and L are $\mathbb{H}$-semimartingales and $(V_t)_{t\ge0}$ solves (1.2) when considered as an SDE with respect to the filtration $\mathbb{H}$ and is an $\mathbb{H}$-semimartingale.*

Proof of Theorem 5.1. First observe that

$$V_0 = -\int_{(0,\infty)} [\mathcal{E}(U)_{s-}]^{-1} \, \mathrm{d}\eta_s = -\int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1} \, \mathrm{d}\eta_s + [\mathcal{E}(U)_t]^{-1} V_t, \tag{5.1}$$

so $(V_t)_{t\geq 0}$ is clearly adapted to $\mathbb{H}$, and if V_0 is a constant random variable, then $\mathbb{F} = \mathbb{H}$ and there is nothing to prove. So suppose that the law μ of V_0 is absolutely continuous. Since $\int_{(0,t]}[\mathcal{E}(U)_{s-}]^{-1}\,\mathrm{d}\eta_s$ and $[\mathcal{E}(U)_t]^{-1}$ are measurable with respect to $\mathcal{F}_t$ but $V_t = -\int_{(t,\infty)}[\mathcal{E}(U)_{(s,t)}]^{-1}\,\mathrm{d}\eta_s$ is independent of $\mathcal{F}_t$, and has distribution μ by stationarity of V, (5.1) shows that the regular conditional distribution of V_0 given $\mathcal{F}_t$ is given by

$$P(V_0 \in B|\mathcal{F}_t)(\omega) = \mu(\mathcal{E}(U)_t(\omega)B + \mathcal{E}(U)_t(\omega)\int_{(0,t]}[\mathcal{E}(U)_{s-}]^{-1}\,\mathrm{d}\eta_s(\omega))$$

for every Borel set B in $\mathbb{R}$ and $\omega \in \Omega$. Hence if the Lebesgue measure of B is zero, the Lebesgue measure of $\mathcal{E}(U)_t(\omega)B + \mathcal{E}(U)_t(\omega)\int_{(0,t]}[\mathcal{E}(U)_{s-}]^{-1}\,\mathrm{d}\eta_s(\omega)$ is zero as well, and since μ is absolutely continuous it follows that $P(V_0 \in B|\mathcal{F}_t)(\omega) = 0$. But this means that the regular conditional distribution of V_0 given $\mathcal{F}_t$ is almost surely absolutely continuous, and hence by Jacod's criterion ([6], Théorème (1.1); see also [17], Theorem VI.10) every $\mathbb{F}$-semimartingale is an $\mathbb{H}$-semimartingale. That then also V is an $\mathbb{H}$-semimartingale follows from Theorem V.7 in [17]. □

The problem of characterising when the law μ of $V_0 := -\int_0^\infty[\mathcal{E}(U)_{s-}]^{-1}\,\mathrm{d}\eta_s$ appearing in Theorem 5.1 is absolutely continuous is an open question. As pointed out by Watanabe [19], it follows from Theorem 1.3 in [1] that μ is either absolutely continuous, continuous singular, or a Dirac measure, i.e. a pure types theorem holds for μ. Watanabe's proof is based on the fact that, by (5.1), $V_0 \stackrel{D}{=} V_t \stackrel{D}{=} \mu$ satisfies a distributional fixed point equation $V_t \stackrel{D}{=} V_0 = M_tV_t + Q_t$, with V_t being independent of (M_t, Q_t) and $P(M_t = 0) = 0$, for which Theorem 1.3 in [1] applies. The same pure types theorem holds by the same argument for the causal solutions of Theorem 2.1(a).

While it follows from the arguments of Theorem 2.2 in [2] that V_0 as defined in Theorem 2.1(b) is constant if and only if $U = kL$ for some constant $k \neq 0$ (or equivalently that $W = -k\eta$ as seen in the proof of Theorem 2.1(c)), the question of when this law is absolutely continuous or continuous singular is much more involved. Lindner and Sato [15] investigate the distribution $-\int_{(0,\infty)}[\mathcal{E}(U)_{s-}]^{-1}\,\mathrm{d}\eta_s$ when $U_t = (c^{-1}-1)R_t$ for a constant $c > 1$ and independent Poisson processes R and η, showing that the distribution can be absolutely continuous or continuous singular, depending in an intrinsic way on c and the ratio of the rates of the Poisson processes R and η.

We conclude by mentioning that if $\Pi_U((-\infty,-1]) = 0$ and $\Pi_U \neq 0$, U and L are independent with L being of bounded variation with non-zero drift term, and $V_0 = -\int_0^\infty[\mathcal{E}(U)_{s-}]^{-1}\,\mathrm{d}\eta_s$ converges almost surely, then it follows from Theorem 3.9 in [2] that V_0 is absolutely continuous. Further examples for absolutely continuous V_0 with independent U and L can be found in [5], covering also cases when U is Brownian motion with drift.

Acknowledgements

We would like to thank Professor Ken-iti Sato for communicating Professor Toshiro Watanabe's result on pure types of the stationary distribution to us, and a referee for careful and constructive reading. Support from an NTH-grant of the state of Lower Saxony as well as from ARC grant DP1092502 are gratefully acknowledged.

References

[1] G. Alsmeyer, A. Iksanov, U. Rösler, On distributional properties of perpetuities, J. Theoret. Probab. 22 (2009) 666–682.

[2] J. Bertoin, A. Lindner, R. Maller, On continuity properties of the law of integrals of Lévy processes, in: C. Donati-Martin, M. Émery, A. Rouault, C. Stricker (Eds.), Séminaire de Probabilités XLI, in: Lecture Notes in Mathematics, vol. 1934, Springer, Berlin, 2008, pp. 137–159.
[3] R. Doney, R.A. Maller, Stability and attraction to normality for Lévy processes at zero and at infinity, J. Theoret. Probab. 15 (2002) 751–792.
[4] K.B. Erickson, R.A. Maller, Generalised Ornstein–Uhlenbeck processes and the convergence of Lévy integrals, in: M. Émery, M. Ledoux, M. Yor (Eds.), Séminaire de Probabilités XXXVIII, in: Lecture Notes in Mathematics, vol. 1857, Springer, Berlin, 2005, pp. 70–94.
[5] H.K. Gjessing, J. Paulsen, Present value distributions with applications to ruin theory and stochastic equations, Stochastic Processes Appl. 71 (1997) 123–144.
[6] J. Jacod, Grossissement initial, hypothèse (H') et théorème de Girsanov, in: Th. Jeulin, M. Yor (Eds.), Grossissements de filtrations: exemples et applications, in: Lecture Notes in Mathematics, vol. 1118, Springer, Berlin, 1985, pp. 15–35.
[7] J. Jacod, Grossissement de filtration et processus d'Ornstein Uhlenbeck généralisé, in: Th. Jeulin, M. Yor (Eds.), Grossissements de filtrations: exemples et applications, in: Lecture Notes in Mathematics, vol. 1118, Springer, Berlin, 1985, pp. 36–44.
[8] J. Jacod, A.N. Shiryaev, Limit Theorems for Stochastic Processes, second edition, Springer, Berlin, 2003.
[9] S. Jaschke, A note on the inhomogeneous linear stochastic differential equation, Insur. Math. Econ. 32 (2003) 461–464.
[10] R.L. Karandikar, Multiplicative decomposition of nonsingular matrix valued semimartingales, in: J. Azema, P.A. Meyer, M. Yor (Eds.), Séminaire de Probabilités XXV, in: Lecture Notes in Mathematics, vol. 1485, Springer, Berlin, 1991, pp. 262–269.
[11] C. Klüppelberg, A. Lindner, R. Maller, A continuous time GARCH process driven by a Lévy process: stationarity and second order behaviour, J. Appl. Probab. 41 (2004) 601–622.
[12] C. Klüppelberg, R.A. Maller, A. Szimayer, The COGARCH: a review, with news on option pricing and statistical inference, 2010, available at SSRN: http://ssrn.com/abstract=1538115.
[13] R. Léandre, Flot d'une équation différentielle stochastique avec semi-martingale directrice discontinue, in: J. Azema, M. Yor (Eds.), Séminaire de Probabilités XIX, in: Lecture Notes in Mathematics, vol. 1123, Springer, Berlin, 1985, pp. 271–274.
[14] A. Lindner, R. Maller, Lévy integrals and the stationarity of generalised Ornstein–Uhlenbeck processes, Stochastic Processes Appl. 115 (2005) 1701–1722.
[15] A. Lindner, K. Sato, Continuity properties and infinite divisibility of stationary distributions of some generalised Ornstein–Uhlenbeck processes, Ann. Probab. 37 (2009) 250–274.
[16] R.A. Maller, G. Müller, A. Szimayer, Ornstein–Uhlenbeck processes and extensions, in: T. Andersen, R.A. Davis, J.-P. Kreiß, T. Mikosch (Eds.), Handbook of Financial Time Series, Springer, Berlin, 2009, pp. 421–437.
[17] P.E. Protter, Stochastic Integration and Differential Equations, second edition, Springer, Berlin, 2005, Version 2.1.
[18] K. Sato, Lévy Processes and Infinitely Divisible Distributions, Cambridge University Press, Cambridge, 1999.
[19] T. Watanabe, Private communication, 2009.